UNIVERSITÉ D'AIX-MARSEILLE
UNIVERSITEIT VAN AMSTERDAM
ÉCOLE DOCTORALE EN MATHÉMATIQUES ET INFORMATIQUE
INSTITUT DE MATHÉMATIQUES DE MARSEILLE

Thèse présenté pour obtenir le grade universitaire de docteur
Discipline : Mathématiques

Marcelo Gonçalves DE MARTINO

On the unramified spherical automorphic spectrum

Soutenue le 02/06/2016 devant le jury :

Anne-Marie AUBERT	Inst. Mathématiques de Jussieu	Rapporteur
Gordan SAVIN	University of Utah	Rapporteur
Erik VAN DEN BAN	Universiteit Utrecht	Examinateur
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ON THE UNRAMIFIED SPHERICAL AUTOMORPHIC SPECTRUM

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad van doctor

aan de Universiteit van Amsterdam

op gezag van de Rector Magnificus

prof. dr. D.C. van den Boom

ten overstaan van een door het College voor Promoties

ingestelde commissie,

in het openbaar te verdedigen in de Agnietenkapel

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MARCELO GONÇALVES DE MARTINO

geboren te São Paulo, Brazilië

Promotiecommissie

Promotores:	Prof. dr. E.M. Opdam	Universiteit van Amsterdam
	Prof. dr. V.J. Heiermann	Université d'Aix Marseille
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	Prof. dr. J.P. Labesse	Université d'Aix-Marseille
	Prof. dr. D. Prasad	Tata Inst. of Fund. Research

Dit proefschrift is tot stand gekomen met als doel het behalen van een gezamenlijk doctoraat. Het proefschrift is voorbereid in de Faculteit der Natuurwetenschappen, Wiskunde en Informatica van de Universiteit van Amsterdam en in de Faculté des Science van de Université d'Aix Marseille.

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ON THE UNRAMIFIED SPHERICAL AUTOMORPHIC SPECTRUM

Marcelo Gonçalves De Martino

To Jaqueline

Contents

According to the Doctorate Regulations of the University of Amsterdam, the following list and explanation is provided.

This thesis is based on many intensive discussions between the author and his supervisors. Moreover, Chapters 3 and 4 were based on the joint pre-prints:

(i) De Martino, M., Opdam, E., *Limit transition between the spherical spectrum of graded and affine Hecke algebras.* In preparation.

(ii) De Martino, M., Heiermann, V., Opdam, E., *On the unramified spherical automorphic spectrum.* arXiv:1512.08566.

Regarding co-authorship, in these pre-prints, all authors have equally contributed to all obtained results.

Chapter 1

Introduction

1.1 Overview

This thesis is the result of a journey of the author in the area of harmonic analysis of reductive groups under the supervision of professors E. Opdam and V. Heiermann. To state the problems we were interested in the past years, consider $\underline{G}$ a connected reductive linear algebraic group defined over a field F, which we will assume to be semisimple for simplicity in this introductory part. Suppose first that $\underline{G}$ is a Chevalley group and F is a global field. As usual, for a place v of F we let F_v be the local completion of F at v. For each non-Archimedean local place v, we obtain a maximal compact (and open) subgroup $K_v := \underline{G}(\mathfrak{o}_v)$ of $G_v := \underline{G}(F_v)$, where $\mathfrak{o}_v$ denotes the ring of integers of F_v. If v is an Archimedean place, we fix a maximal compact (and closed) subgroup K_v of G_v. Let $\underline{G}(\mathbb{A})$ be the group of adelic points of $\underline{G}$. It is constructed as the restricted direct product of the collection $\{G_v \mid v \text{ is a place of } F\}$ with respect to the collection $\{K_v \mid v \text{ is non-Archimedean}\}$. This is a locally compact group and the homogeneous space $\underline{G}(F)\backslash\underline{G}(\mathbb{A})$ has finite volume. The first part of the thesis is devoted to the following classical theme in automorphic representation theory:

(a) for F a global field, let S be a finite set of places *not* including the Archimedean ones. If $v \notin S$ let $K'_v = K_v$ as above. If $v \in S$ let $K'_v \subseteq K_v$ be a compact-open subgroup. For example, we could take $K'_v = I_v$, the Iwahori subgroup of G_v. Consider the compact subgroup $\mathbf{K}' = \prod_v K'_v$ of $\underline{G}(\mathbb{A})$. We wish to study the space $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}'}$ of $\mathbf{K}'$-invariant square integrable functions on $\underline{G}(F)\backslash\underline{G}(\mathbb{A})$ as a module for a global Hecke algebra $\mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K}')$, as well as the $\underline{G}(\mathbb{A})$-space generated by these $\mathbf{K}'$-invariants. In special, we wish to determine the representations occurring discretely in these spaces.

As for our second problem, we drop the assumption that $\underline{G}$ is a Chevalley group but we still ask for $\underline{G}$ to be connected. We discuss:

(b) for F a local non-Archimedean field, we wish to construct an explicit contraction of its Bruhat-Tits building $\mathscr{B}(\underline{G}, F)$ by means of piecewise linear Morse theory.

The first problem is inserted in a more general framework where $\underline{G}$ is not necessarily split over the global field F. The space $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))$ of square-integrable automorphic forms is a central object in the theory of automorphic forms and its relation to representation theory and number theory via the Langlands program. Langlands [55] gave an abstract decomposition of this space in terms of classes of cuspidal data $\mathfrak{X}$ (see Section 2.7.6, below), in which each $\mathfrak{X} := [\underline{M}, \mathfrak{P}]$ is an equivalence class of pairs $(\underline{M}, \mathfrak{P})$, called cuspidal datum, with $\underline{M}$ an F-Levi subgroup of $\underline{G}$ and $\mathfrak{P}$ an orbit of certain character twists of a cuspidal automorphic representation of $\underline{M}$:

$$L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A})) = \hat{\oplus}_{\mathfrak{X}}\, L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}. \tag{1.1}$$

These spaces are acted upon by the group $\underline{G}(\mathbb{A})$ by means of the right regular representation. A fundamental problem in the theory of automorphic forms is to decompose further each summand $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}$ (we will define this space below, in the end of Section 2.7.9) as a Hilbert integral of irreducible, unitary $\underline{G}(\mathbb{A})$-representations. If we fix a compact subgroup $\mathbf{K}' \subseteq \underline{G}(\mathbb{A})$ as above, in the decomposition of $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}$, one can study the subset of $\mathbf{K}'$-spherical representations occurring in the support of the spectral measure for the decomposition of this $\underline{G}(\mathbb{A})$-space. This is done by considering the space $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}^{\mathbf{K}'}$, which is a module for a global Hecke algebra associated to the pair $(\underline{G}(\mathbb{A}), \mathbf{K}')$.

The irreducible closed subspaces of $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))$ are called discrete series representations. Among the discrete series representations are the cuspidal automorphic representations (those whose space contains a cuspidal automorphic form of $\underline{G}(\mathbb{A})$, that is, they appear in an $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}$ with $\mathfrak{X} = [\underline{G}, \mathfrak{P}]$ and $\mathfrak{P}$ a singleton) and the residual spectrum (that is, those discrete series representations which appear in an $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{[\underline{M},\mathfrak{P}]}$ with $\underline{M} \neq \underline{G}$). One of the main achievements of Langlands was to show that the representations in the residual spectrum correspond to certain residues of Eisenstein series via a complicated contour shift, which was the most difficult part in [55].

A discrete series representation π of $\underline{G}(\mathbb{A})$ (and more generally any irreducible automorphic representation) admits a decomposition $\otimes_v \pi_v$ into local components (this infinite tensor product is in the sense of [29]). If $(\underline{M}, \mathfrak{P})$ is the class of cuspidal data corresponding to π, and σ is a base point of $\mathfrak{P}$, then there is an element λ in the complex vector space $a^*_{M,\mathbb{C}} = \mathbb{C} \otimes X^*(\underline{M})$, where

$X^*(\underline{M})$ is the character lattice of $\underline{M}$, such that π is a subquotient of $I(\sigma_\lambda)$, the normalized induced representation to $\underline{G}(\mathbb{A})$ from the character σ_λ of $\underline{P}(\mathbb{A})$. For a given base point σ in $\mathfrak{P}$, the contour shift procedure established by Langlands computes, among other things, those λ such that $I(\sigma_\lambda)$ has a discrete series subquotient.

If a local component σ_v of σ in a finite place v is a supercuspidal representation, then it is a striking observation in known cases (cf. [51, Conjecture 8.7] and [65, Conjecture p. 817]) that the λ such that the automorphic representation $I(\sigma_\lambda)$ has a discrete series subquotient correspond to those λ for which the induced representation $I_v(\sigma_{\lambda,v})$ of the local p-adic group $G_v := \underline{G}(F_v)$ has a square-integrable subquotient. In [37], using at a crucial step a trace argument inspired from [73] (and in addition a nonvanishing result of the residues of the μ function therein), it has been shown that such λ in the context of p-adic groups correspond to residue points of Harish-Chandra's μ-function.

The idea that global information can be obtained in terms of local considerations is an inspiring principle used in the study of problem (a). A known example of this local-global phenomenon was given by A. Prasad in [79], [80] and [81]. Suppose that $F = \mathbb{F}_q(t)$ is the field of rational functions on the projective line $\mathbb{P}^1(\mathbb{F}_q)$ over the finite field with q elements and that $\underline{G}$ is a Chevalley group. In this situation, all places are non-Archimedean and the local completions F_v of F are of the type $\mathbb{F}_{q^n}((\varpi_v))$, the field of Laurent series on a parameter ϖ_v with coefficients in $\mathbb{F}_{q^n}$ for suitable n. For the rational places of F, we have $n = 1$. Put $F_\bullet := \mathbb{F}_q((\varpi))$, for an indeterminate ϖ and $G_\bullet := \underline{G}(F_\bullet)$. Let $\mathfrak{o}_\bullet := \mathbb{F}_q[[\varpi]] \subseteq F_\bullet$ be the ring of Taylor series on ϖ. Let also $K_\bullet := \underline{G}(\mathfrak{o}_\bullet)$ and $I_\bullet \subseteq K_\bullet$ be the Iwahori subgroup, given as the inverse image of a fixed Borel subgroup $\underline{B}(\mathbb{F}_q)$ under the reduction modulo ϖ map $K_\bullet \to \underline{G}(\mathbb{F}_q)$. Consider the compact subgroup $\mathbf{K}' = \prod_v K'_v$ in which $K'_v = I_v$ for $v = \infty, 0$ and $K'_v = K_v$ for all other v. For each place, we have a Hecke algebra $\mathcal{H}(G_v, K'_v)$, which is an Iwahori-Hecke algebra for $v = \infty, 0$ and is a spherical Hecke algebra for all other places. A. Prasad gave a description of the space $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A})/\mathbf{K}')$ as a module for the Hecke algebra $\mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K}') = \otimes_v \mathcal{H}(G_v, K'_v)$ in the following way. First, he showed ([79] and [80]) that essentially there is an equivariant (up to a twist by suitable involutions) isometric isomorphism

$$L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A})/\mathbf{K}') \cong L^2(G_\bullet\backslash(G_\bullet \times G_\bullet)/(I_\bullet \times I_\bullet)), \tag{1.2}$$

where the action on the left-hand side is of $\mathcal{H}(G_\infty, I_\infty) \otimes \mathcal{H}(G_0, I_0)$ and the action on the right-hand side is of $\mathcal{H}(G_\bullet, I_\bullet) \otimes \mathcal{H}(G_\bullet, I_\bullet)$. Note that, via the bijective map $G_\bullet\backslash(G_\bullet \times G_\bullet)/(I_\bullet \times I_\bullet) \to I_\bullet\backslash G_\bullet/I_\bullet$ given by sending the coset of (g_1, g_2) to the coset of $g_2^{-1}g_1$, the space in the right-hand side of (1.2) contains a dense subspace isomorphic to $C_c^\infty(I_\bullet\backslash G_\bullet/I_\bullet)$. Using this, A. Prasad effectively reduced the problem of decomposing the automorphic space $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A})/\mathbf{K}')$ in the problem of decomposing the left-and-right regular representation of the

Hecke algebra $\mathcal{H}(G_\bullet, I_\bullet)$ on the space $L^2(I_\bullet\backslash G_\bullet/I_\bullet, dg)$, where dg is the Haar measure on $G_\bullet$ that assigns measure one to the Iwahori subgroup $I_\bullet$. This local problem was extensively studied, for example, in [48] and in [73]. In particular, one obtains the representations of $\mathcal{H}(G_\infty, I_\infty)\otimes\mathcal{H}(G_0, I_0)$ occurring discretely in the automorphic space. Later, in [81], A. Prasad completes the picture by showing that if a module (π, V) of $\mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K}')$ occurs discretely on $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A})/\mathbf{K}')$, then its local constituents for the remaining $v \neq \infty, 0$ are completely determined by the eigencharacters of the center of $\mathcal{H}(G_\infty, I_\infty)$.

In broad-strokes, to obtain the isometry in (1.2), let $\mathbf{K} = \prod_v K_v$ (where we recall $K_v = \underline{G}(\mathfrak{o}_v)$, for all v), so that $\mathbf{K}' \subseteq \mathbf{K}$. Let also $\underline{T}$ be a split maximal torus of $\underline{G}$, W_0 be the Weyl group of $(\underline{G}, \underline{T})$ and $X_*(\underline{T})$ be the group of cocharacters of $\underline{T}$. Starting from the known bijection between $\underline{G}(F)\backslash\underline{G}(\mathbb{A})/\mathbf{K}$ and the dominant cocharacters $X_*(\underline{T})^+$, given by sending $y \in X_*(\underline{T})^+$ to the coset $\underline{G}(F)y(\varpi_\infty)\mathbf{K}$, where $y(\varpi_\infty) \in \underline{G}(\mathbb{A})$ is the adelic element that is the identity 1_v for all $v \neq \infty$ and is the image $y(\varpi_\infty) \in G_\infty$, A. Prasad shows that the fibers of the natural map $\underline{G}(F)\backslash\underline{G}(\mathbb{A})/\mathbf{K}' \to \underline{G}(F)\backslash\underline{G}(\mathbb{A})/\mathbf{K}$ are in bijection with $W_0 \times W_0 y$, the fibers of the map $W_0 \ltimes X_*(\underline{T}) \to X_*(\underline{T})$ given by sending $(w, y) \mapsto y^+$, where y^+ is the unique dominant element in the orbit $W_0 y$. One then obtains a bijection

$$W := W_0 \ltimes X_*(\underline{T}) \to \underline{G}(F)\backslash\underline{G}(\mathbb{A})/\mathbf{K}', \tag{1.3}$$

given by sending (w, y) to the coset $\underline{G}(F)\dot{w}y(\varpi_\infty)\mathbf{K}$, where $\dot{w}$ is a representative of the Weyl group element in $\underline{G}(\mathbb{F}_q)$ seen as element of G_∞, $y(\varpi_\infty)$ is as before and everywhere else we have 1_v. Using this bijection, there is a linear isomorphism between the spaces in (1.2), and the delicate matters proven by A. Prasad is that the actions of the Hecke algebras in question match up, and one obtains the desired result.

In the picture sketched above, a natural question that arises is what happens if, instead of Iwahori ramifications on two places, one allows Iwahori ramifications on three or more places. Is there a description similar to the one above still available in this case? Or, can we use A. Prasad's description in the two-point case to infer results about the three-point case? Unfortunately, this thesis will not provide answers to these interesting questions. Rather, we will briefly describe some superficial difficulties that one is confronted with in this problem. A naïve approach would be to consider $\mathbf{K}'' = \prod_v K_v$, where now, say, $K_v' = I_v$ for $v = \infty, 0, 1$ and the maximal compact everywhere else. Then, we could consider the spaces

$$L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A})/\mathbf{K}'') \quad \text{and} \quad L^2(G_\bullet\backslash G_\bullet^3/I_\bullet^3)$$

(where H^n means the Cartesian product of n copies of the group H). To explain some of the difficulties involved, consider the diagram

$$\begin{tikzcd} \underline{G}(F)\backslash\underline{G}(\mathbb{A})/\mathbf{K}'' \arrow[r, dashed, dash, "?"] \arrow[d, two heads] & G_\bullet\backslash G_\bullet^3/I_\bullet^3 \arrow[r, two heads] \arrow[d, two heads] & W\times W \arrow[d, two heads] \\ \underline{G}(F)\backslash\underline{G}(\mathbb{A})/\mathbf{K}' \arrow[r, equal] & G_\bullet\backslash G_\bullet^2/I_\bullet^2 \arrow[r, equal] & W, \end{tikzcd}$$

where in the bottom row, we have the bijection given by (1.3) and the affine Bruhat decomposition. On the one hand, in Corollary 6.0.3, we parametrize the double coset space $G_\bullet\backslash G_\bullet^3/I_\bullet^3$. The answer we provide is the union of all fibers of the surjection $G_\bullet\backslash G_\bullet^3/I_\bullet^3 \to (I_\bullet\backslash G_\bullet/I_\bullet)\times(I_\bullet\backslash G_\bullet/I_\bullet)\cong W\times W$ given by sending the coset of (g_1,g_2,g_3) to the coset of $(g_3^{-1}g_1, g_3^{-1}g_2)$. This shows that the map $G_\bullet\backslash G_\bullet^3/I_\bullet^3 \to G_\bullet\backslash G_\bullet^2/I_\bullet^2$ given by taking the projection onto the two first coordinates has fibers of infinite cardinality. On the other hand, the natural surjective map $\underline{G}(F)\backslash\underline{G}(\mathbb{A})/\mathbf{K}'' \to \underline{G}(F)\backslash\underline{G}(\mathbb{A})/\mathbf{K}'$ has fibers of finite cardinality (this cardinality is bounded by $|\underline{G}(\mathbb{F}_q)|/|\underline{B}(\mathbb{F}_q)|$). Thus, we do not expect to find a linear isomorphism between $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A})/\mathbf{K}'')$ and $L^2(G_\bullet\backslash G_\bullet^3/I_\bullet^3)$. But it may be the case that one can identify $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A})/\mathbf{K}'')$ with a subspace of $L^2(G_\bullet\backslash G_\bullet^3/I_\bullet^3)$, although we did not push these ideas any further.

1.2 Statement of main results

After giving the inspiring example of a local-global phenomenon provided by A. Prasad and taking some time to describe what was *not* done in this thesis, let us now describe some actual results that we found and at the same time describe the contents of the subsequent chapters. Still regarding the problem (a), we take a step back. Consider now F a number field, $\underline{G}$ a Chevalley group and let $\mathbf{K}'=\mathbf{K}=\prod_v K_v$, with $K_v=\underline{G}(\mathfrak{o}_v)$ for all non-Archimedean places v. In this situation, consider the class of cuspidal data $\mathfrak{X}=[\underline{T},\imath]$ where $\underline{T}$ is a fixed F-split maximal torus and $\imath$ denotes the orbit of the trivial character of $\underline{T}(F)\backslash\underline{T}(\mathbb{A})$. We are interested in the residual spectrum associated with $[\underline{T},\imath]$. Let $X^*(\underline{T})$ denote the character lattice of $\underline{T}$ and let a_T^* denote the space $\mathbb{R}\otimes X^*(\underline{T})$, with the root system $\Phi_0=\Phi(\underline{G},\underline{T})\subseteq a_T^*$. Let also $a_{T,\mathbb{C}}^*$ be its complexification and W_0 be the Weyl group of Φ_0. The main results concerning problem (a) are the subject of Chapter 4. They are also about the decomposition of an automorphic space, in our case $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}}_{[\underline{T},\imath]}$, in terms of the decomposition of a local module. In more of details, let ${}^LG$ be the complex (connected) dual group of $\underline{G}$ and let ${}^L\mathfrak{g}$ be its Lie algebra. Fix a Cartan involution θ and a corresponding maximal compact ${}^LK$ of ${}^LG$ in such a way that ia_T^* is the Lie algebra of a maximal torus of a compact form of ${}^L\mathfrak{g}$. We then have that θ acts as the identity on ia_T^* and as minus the identity on a_T^*, where $a_{T,\mathbb{C}}^*=a_T^*+ia_T^*$. Now consider the space P^{su}, which we have reasons to call the space of "spherical unramified parameters", that consists of ${}^LG$-conjugacy classes of pairs $\{(\phi, \lambda)\}$ satisfying the properties (1) $\phi : \mathfrak{sl}_2 \to {}^L\mathfrak{g}$ is a Lie algebra homomorphism, (2) $\lambda \in {}^L\mathfrak{g}$ is a semisimple element, (3) $[\lambda, \phi(x)] = 0$ for all $x \in \mathfrak{sl}_2$, (4) $\theta(\lambda) = \lambda$ and (5) $\theta \circ \phi \circ \theta_{\mathfrak{sl}_2} = \phi$, where $\theta_{\mathfrak{sl}_2}$ is the Cartan involution $x \mapsto -x^T$ of $\mathfrak{sl}_2$. In Proposition 2.5.6 we argue that this space is in bijection with a union of affine subspaces of $a^*_{T,\mathbb{C}}$ modulo the action of W_0 (we mean $(\cup_L L^t)/W_0$, where $L^t \subseteq a_{T,\mathbb{C}}$ are called residual subspaces (see Definition 2.4.3 and Section 2.5.1)). In (4.2), we endow the space P^{su} with an explicit measure μ_{aut}. We then have:

Theorem A. *We have an isometric isomorphism*

$$L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}}_{[\underline{T},\mathbf{1}]} \cong L^2(P^{\mathrm{su}}, \mu_{\mathrm{aut}}). \tag{1.4}$$

Moreover, this isomorphism is explicitly given in terms of a Fourier transform which diagonalizes the natural action of the global spherical Hecke algebra $\mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K})$, the tensor product of the local spherical Hecke algebras $\mathcal{H}(G_v, K_v)$ in each place v.

To obtain the above isomorphism, we must deal with a contour shift as the one considered by Langlands in [55]. We are forced to take into consideration poles induced by zeroes of the Dedekind zeta function, which is a global ingredient. However, in the present unramified and **K**-spherical case, we can reduce the problem to a residue calculus in a local context, namely that of the spectral decomposition of the Yang system, a module over the graded affine Hecke algebra. This was studied by G. Heckman and E. Opdam in [36].

The space $P^{\mathrm{su}}_{\mathrm{dist}}$, of isolated points of P^{su}, consists of the pairs $(\phi, 0)$ in which ϕ is the Jacobson-Morozov $\mathfrak{sl}_2$-homomorphism of a distinguished orbit $\mathbf{o}$ of ${}^LG$. It follows that the discrete part of $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}}_{[\underline{T},\mathbf{1}]}$ is parametrized by those distinguished unipotent orbits of the dual group. Given such an orbit, let $2\lambda(\mathbf{o}) \in a^*_T$ be the corresponding weighted Dynkin diagram (see (2.56)). To each local place v, let $\pi_{v,\lambda(\mathbf{o})}$ be the unique irreducible spherical subquotient of the unramified minimal principal series of G_v at the parameter $\lambda(\mathbf{o})$ (see Definition 2.7.19) and put $\pi_{\lambda(\mathbf{o})} = \otimes_v \pi_{v,\lambda(\mathbf{o})}$. We are able to conclude from Theorem A that:

Theorem B. *Each $\underline{G}(\mathbb{A})$-representation $\pi_{\lambda(\mathbf{o})}$ occurs in the discrete spectrum of $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{[\underline{T},\mathbf{1}]}$ with multiplicity one.*

Theorem C. *The spherical irreducible representations $\pi_{v,\lambda(\mathbf{o})}$ are unitarizable, for each local place v.*

Also, now using the isometry of (1.4) we can compare norms of vectors on both sides to obtain some nontrivial identities. Among them, we recover a result of Langlands [54] on the volume of $\underline{G}(F)\backslash\underline{G}(\mathbb{A})$ as well as the square norm of some **K**-spherical vectors given as the square integrable residues of

Eisenstein series at the points $\lambda(\mathbf{o})$ mentioned above (see Corollaries 4.1.6 and 4.1.7). We remark that in many special classes of reductive groups similar results on the **K**-spherical automorphic spectrum have been obtained [46], [55], [49], [63], [65] and [66]. (We remark further that the results on exceptional groups in [63] are partly based on computer assisted computations). We present here a new uniform and conceptual approach. Such results are all in accordance with Arthur's conjectures [2].

In order for (1.4) to yield meaningful identities when comparing norms on both sides, one needs to make explicit all the constants involved in the spectral measures. In particular, we need to compute the precise constants of ν_P, the Plancherel measure of the Yang system [36] (see (2.50) and (4.43), below), which were not yet available at the time that G. Heckman and E. Opdam wrote their article but which can be derived by a limit procedure from the explicit formal degree formulas for discrete series characters of Iwahori-Hecke algebras [84, (0.3)] and [21]. These explicit formulas constitute a special case of the proof of the conjecture [40, Section 3.4] for unipotent representations of semisimple groups of adjoint type over a non-Archimedean local field [76, Section 4.6]. The limit procedure and the computation of the relevant constants in the present case of graded Hecke algebras is carried out in Chapter 3 and we obtain (see Theorem 3.6.1 and cf. [36, Example 3.16]). The next result is crucial for the expression of the measure μ_{aut} of Theorem A.

Theorem D. *For each distinguished orbit* $\mathbf{o}$*, the Plancherel measure of the Yang system satisfies*

$$\nu_P(\{\lambda(\mathbf{o})\}) = \left| \frac{1}{\mathrm{vol}(X_*(\underline{T}))} \frac{|W_{\lambda(\mathbf{o})}|}{|A_{\mathbf{o}}|} \frac{\prod'_{\alpha\in\Phi_0}\langle\alpha^\vee, \lambda(\mathbf{o})\rangle}{\prod'_{\alpha\in\Phi_0}\langle\alpha^\vee, \lambda(\mathbf{o})\rangle + 1} \right|.$$

Here, $A_{\mathbf{o}}$ *is the component group of the centralizer of an element of* $\mathbf{o}$ *and* $W_{\lambda(\mathbf{o})} \subseteq W_0$ *is the stabilizer of* $\lambda(\mathbf{o})$*.*

In Chapter 5, we discuss the problem (b). The building of a reductive group $\underline{G}$ over a local non-Archimedean field F is an important object used to understand the representation theory of $G := \underline{G}(F)$. It is also an object of study per se due to its many symmetries and geometrical properties. Amongst them, it is a cell complex (whose cells are polysimplices, that is, a finite product of simplices) endowed with a natural action of G and a contractible, complete metric space. Exploring these properties, P. Schneider and U. Stuhler studied homological resolutions for smooth representations of G in [87] and also in [86], in the special case of GL_n. Further, E. Opdam and M. Solleveld studied resolutions for tempered representations in [71] using a explicit contraction of the Coxeter complex they obtained in [70].

Following an idea of G. Savin and M. Bestvina (unpublished manuscript, [6]) for the building of SL_n, we propose to study Euclidean buildings, and in particular the complexes above, by means of piecewise linear Morse theory. Let $\mathscr{B} := \mathscr{B}(\underline{G}, F)$ be the Bruhat-Tits building of a connected reductive algebraic group defined over a local non-Archimedean field F. Given a (poly)simplex σ of $\mathscr{B}$, the star of σ, denoted $\mathrm{St}(\sigma)$, is the union of all (poly)simplices of $\mathscr{B}$ that contain σ, together with their faces. We show in Theorem 5.3.16 that:

Theorem E. *One can endow $\mathscr{B}$ with a Morse function $\varphi : \mathscr{B} \to \mathbb{R}$ that has a unique global minimum and satisfies the following property: for each (poly)simplex σ of $\mathscr{B}$, any local minimum of φ on $\mathrm{St}(\sigma)$ is the unique global minimum of φ on $\mathrm{St}(\sigma)$.*

The property mentioned in the statement of Theorem C allow us to uniquely define the direction of steepest descent around any cell of $\mathscr{B}$ towards a fixed special point in the building. We would later like to apply this construction given by Morse theory to revisit some results of P. Schneider and U. Stuhler, [87], and E. Opdam and M. Solleveld, [71]. However, in order to do so, we must have that the contraction we construct here have good bounds in its coefficients. We, unfortunately, leave this as a conjecture for now (see Conjecture 5.4.4) and thank G. Savin for pointing out a mistake in an earlier version of this manuscript. We mention that G. Savin and M. Bestvina have pushed further their ideas in the SL_n case to a more general setting as was done here in an independent way in [7], also an unpublished work. In Section 5.5 we make a few comments comparing their constructions with the ones given here.

Finally, Chapter 2 is supposed to provide prerequisites for the problem (a) on the spectral decomposition, and in Chapter 6, we discuss some small remarks we could make while working with triple spaces.

Chapter 2

Preliminaries

This chapter is intended as literature review concerning the topics related to this thesis. Our intention is not to make a self-contained exposition of the ideas involved, but we intent to collect some facts and set up the notations.

2.1 Algebraic groups

We assume the basic notions of algebraic groups over a field F, specially the notion of a split reductive group (and the notion of a Chevalley group [20], by which we mean a semisimple group defined over $\mathbb{Z}$) and standard notations such as $\mathbb{G}_m$ for the multiplicative group. Some references to this subject are [43], [93] and [94].

In this (very) brief section, we wish to establish the following notational convention to be used throughout the whole thesis. First, an algebraic group defined over a field will be denoted with an underline. Thus, for example, we will write $\underline{G}$ defined over F. For any F-algebra A, we will write $\underline{G}(A)$ to denote the set of A-points. When $F = \overline{F}$ is algebraically closed, we will omit the underline.

Next, given a reductive group $\underline{H}$ defined over F, we will always denote by $X^*(\underline{H}) := \mathrm{Hom}(\underline{H}, \mathbb{G}_m)$ the group of F-rational characters of $\underline{G}$, and by $X_*(\underline{H}) := \mathrm{Hom}(\mathbb{G}_m, \underline{H})$ the group of F-rational cocharacters of $\underline{H}$. The bilinear pairing $\langle \cdot, \cdot \rangle$ is the integer one gets by composing $x \circ y \in \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$, for $x \in X^*(\underline{G})$ and $y \in X_*(\underline{G})$.

2.2 Root data

We describe the notion of a root datum and point out some facts that will be relevant to this work. The exposition is based on [92] and [57].

Definition 2.2.1. A **root datum** is a quadruple $\mathcal{R} = (\mathcal{X}, \Psi_0, \mathcal{X}^\vee, \Psi_0^\vee)$ consisting of two lattices of finite rank, in duality by a pairing $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X}^\vee \to \mathbb{Z}$, and finite subsets $\Psi_0 \subseteq \mathcal{X}$ and $\Psi_0^\vee \subseteq \mathcal{X}^\vee$ in bijective correspondence $\alpha \mapsto \alpha^\vee$. Elements in Ψ_0 rare called roots and those in $\Psi_0^\vee$ coroots. Each $\alpha \in \Psi_0$ defines endomorphisms

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha, \quad s_\alpha(y) = y - \langle \alpha, y \rangle \alpha^\vee,$$

and these data are subject to the axioms

(A1) For all $\alpha \in \Psi_0$, we have $\langle \alpha, \alpha^\vee \rangle = 2$

(A2) For all $\alpha \in \Psi_0$ it holds that $s_\alpha(\Psi_0) \subseteq \Psi_0$ and $s_\alpha(\Psi_0^\vee) \subseteq \Psi_0^\vee$.

Given a root datum $\mathcal{R}$ as above, the **dual root datum**, denoted by $\mathcal{R}^\vee$, is the root datum $(\mathcal{X}^\vee, \Psi_0^\vee, \mathcal{X}, \Psi_0)$.

We extend the pairing $\langle \cdot, \cdot \rangle$ linearly to the vector spaces $V^* = \mathbb{R} \otimes \mathcal{X}$ and $V = \mathbb{R} \otimes \mathcal{X}^\vee$, so that they are dual to each other. Viewed as a subset of V^*, we have that $\Psi_0 \cong 1 \otimes \Psi_0$ is a root system and the group generated by $\{s_\alpha \mid \alpha \in \Psi_0\}$ is identified with the Weyl group of Ψ_0. We will denote by V_0^* the subspace spanned by the roots, that is, $V_0^* := \mathbb{R} \otimes Q \subseteq V^*$. If Π_0 is a basis of Ψ_0, we have the finer notion of a **based root datum**; a quadruple $(\mathcal{X}, \Pi_0, \mathcal{X}^\vee, \Pi_0^\vee)$ satisfying similar properties as above [92, 1.9]. Both notions of root data will be denoted by $\mathcal{R}$. The basis Π_0 defines a subset of V called fundamental (vectorial[1]) chamber of V, denoted V_+ and given as $\{\lambda \in V \mid \langle \alpha, \lambda \rangle > 0 \text{ for all } \alpha \in \Pi_0\}$. Likewise for $(V_+^*, \Pi_0^\vee)$. We recall that a root system is called **reduced** if, for every $\alpha \in \Psi_0$, the only multiples of α that are in Ψ_0 are α and $-\alpha$. A root datum is reduced if its root system is reduced. Throughout this thesis, we will assume that the root data are reduced.

Suppose we are give a root datum $\mathcal{R}$ as in Definition 2.2.1. Whenever $\Psi_0^\perp := \{y \in \mathcal{X}^\vee \mid \langle \alpha, y \rangle = 0, \text{ for all } \alpha \in \Psi_0\}$ is equal to $\{0\}$, the root datum is called **semisimple** and if $\Psi_0 = \emptyset$, it is called **toral**. Assume $\mathcal{R}$ is not toral. The groups $Q \subseteq \mathcal{X}$ and $Q^\vee \subseteq \mathcal{X}^\vee$ generated by Ψ_0 and $\Psi_0^\vee$ are called the **root lattice** and **coroot lattices**. Their duals, $P^\vee = \mathrm{Hom}(Q, \mathbb{Z})$ and $P = \mathrm{Hom}(Q^\vee, \mathbb{Z})$, which by using $\langle \cdot, \cdot \rangle$ can be viewed as lattices on $\mathcal{X}^\vee$ and $\mathcal{X}$, are called the **coweight** and **weight lattices**, respectively. We thus have $Q \subseteq \mathcal{X} \subseteq P$, similarly $Q^\vee \subseteq \mathcal{X}^\vee \subseteq P^\vee$ and we remark that when $\mathcal{R}$ is semisimple these inclusions are of finite index. Moreover, in the semisimple case:

Lemma 2.2.2. *We have* $[\mathcal{X}^\vee : Q^\vee] = [P : \mathcal{X}]$.

Proof. The diagram

[1] In Chapter 5, we will make a distinction between two types of fundamental chambers. The one we use here will be called vectorial, there. Until then, we omit the *vectorial* and write only fundamental chamber.

$$
\begin{tikzcd}
& & & 0 \arrow[d] & \\
& & & \mathrm{Hom}(\mathcal{X}^\vee/Q^\vee, \mathbb{C}^*) \arrow[d] \arrow[dddll, rounded corners] & \\
0 \arrow[r] & \mathcal{X} \arrow[r] \arrow[d] & \mathbb{C}\otimes\mathcal{X} \arrow[r, "id\otimes\exp"] \arrow[d] & \mathbb{C}\otimes\mathcal{X}^* \cong \mathrm{Hom}(\mathcal{X}^\vee, \mathbb{C}^*) \arrow[r] \arrow[d, "\mathrm{res}"] & 1 \\
0 \arrow[r] & P \arrow[r] \arrow[d] & \mathbb{C}\otimes P \arrow[r, "id\otimes\exp"] & \mathbb{C}\otimes P^* \cong \mathrm{Hom}(Q^\vee, \mathbb{C}^*) \arrow[r] & 1 \\
& P/\mathcal{X} \arrow[d] & & & \\
& 0 & & &
\end{tikzcd}
$$

has commutative squares and exact rows and columns, so we apply the snake lemma to obtain a homomorphism $\mathrm{Hom}(\mathcal{X}^\vee/Q^\vee, \mathbb{C}^*) \to P/\mathcal{X}$, which is in fact an isomorphism since the middle map $\mathbb{C}\otimes\mathcal{X} \to P\otimes\mathbb{C}$ is bijective. $\square$

Still in the semisimple case, we say that a root system is **simply connected** if $\mathcal{X} = P$, or equivalently $Q^\vee = \mathcal{X}^\vee$ and **adjoint** if $Q = \mathcal{X}$, or equivalently $\mathcal{X}^\vee = P^\vee$. The nomenclature above comes from the fact that, if $F = \overline{F}$ is an algebraically closed field, G is a reductive algebraic group over F, $T \subseteq G$ is a maximal torus and $\Phi_0 = \Phi(G,T)$ is the set of roots of T on the Lie algebra of G, then

$$\mathcal{R}(G,T) := (X^*(T), \Phi_0, X_*(T), \Phi_0^\vee) \tag{2.1}$$

is a root data as above. In this situation, a group is simply connected or adjoint if and only if the corresponding root data has the same property. On the other hand, we have [92, Theorem 2.9]:

Proposition 2.2.3. *Let $F = \overline{F}$ be an algebraically closed field. For every root datum $\mathcal{R} = (\mathcal{X}, \Psi_0, \mathcal{X}^\vee, \Psi_0^\vee)$ there is a reductive group G and a maximal torus T, both defined F, such that $\mathcal{R} = \mathcal{R}(G,T)$. Such pair is unique up to isomorphism.*

In the above proposition, by an isomorphism of (G,T), we mean an isomorphism of algebraic groups $G \to G'$ that maps T onto T'.

Note that the choice of a Borel subgroup $B \subseteq T$ induces a notion of positive roots $\Phi_0^+ \subseteq \Phi(G,T)$ and thus a set of simple roots Δ_0. We then have a based root datum $(X^*(T), \Delta_0, X_*(T), \Delta_0^\vee)$, which will be denoted by $\mathcal{R}(G,B,T)$. If a based root datum is given, similar to the Proposition 2.2.3, there exists a triple (G,B,T) of algebraic groups defined over F, unique up to isomorphism, associated to the given based root datum.

Given a subgroup $A \subseteq \mathcal{X}$, we let $A^\perp := \{y \in \mathcal{X}^\vee \mid \langle A, y\rangle = 0\} \subseteq \mathcal{X}^\vee$. Similarly, we define $B^\perp \subseteq \mathcal{X}$ for any subgroup $B \subseteq \mathcal{X}^\vee$. Given any root datum

$\mathcal{R}$, we obtain a semisimple and a toral root data via

$$\begin{aligned} \mathcal{R}_{\mathrm{ss}} &:= (\mathcal{X}/(Q^\vee)^\perp, \Psi_0, ((Q^\vee)^\perp)^\perp, \Psi_0^\vee), \\ \mathcal{R}_{\mathrm{to}} &:= (\mathcal{X}/(Q^\perp)^\perp, \emptyset, Q^\perp, \emptyset). \end{aligned} \tag{2.2}$$

Moreover, we have that $\mathcal{R}$ is isogenous[2] to $\mathcal{R}_{\mathrm{ss}} \oplus \mathcal{R}_{\mathrm{to}}$. When $\mathcal{R} = \mathcal{R}(G,T)$ for G reductive over an algebraically closed field and $T \subseteq G$ a maximal torus, reflecting the decomposition as an almost product $G = G' \cdot \mathrm{Rad}(G)$ with G' the derived group and $\mathrm{Rad}(G)$ the radical of G, $\mathcal{R}_{\mathrm{ss}}$ corresponds to the root datum of (G', T_1), where $T_1 \subseteq T$ is the subgroup generated by the image of all $\alpha^\vee$ with $\alpha \in \Psi_0$ and $\mathcal{R}_{\mathrm{to}}$ to the one of $\mathrm{Rad}(G)$ [94, p.135].

2.3 Affine Hecke algebras

We describe the notion of an affine Hecke algebra. The material of this section is inspired by [57], [35], [36] and [72].

2.3.1 Weyl groups

In this thesis we will find ourselves typically in the situation in which the vector spaces V^* and V of Section 2.2 have more structure. Namely, let $(V^*, (\cdot,\cdot))$ be a finite dimensional Euclidean space and $\Psi_0 \subseteq V^*$ be a root system. Let $\mathcal{X}$ be a lattice of full rank in V^* with $\Psi_0 \subseteq \mathcal{X}$. Suppose we are given a based root datum

$$\mathcal{R} = (\mathcal{X}, \Pi_0, \mathcal{X}^\vee, \Pi_0^\vee), \tag{2.3}$$

in which $\mathcal{X}^\vee = \mathrm{Hom}(\mathcal{X}, \mathbb{Z}) \subseteq V$ is dual to $\mathcal{X}$, contains the dual root system $\Psi_0^\vee$ and Π_0 is a set of simple roots. The canonical pairing between V and V^* is denoted by $\langle \cdot, \cdot \rangle$ and $V^*_{\mathbb{C}}, V_{\mathbb{C}}$ are the complexifications of V^* and V, respectively. We fix an inner product on V by transport of structure from $(V^*, (\cdot,\cdot))$, via the canonical isomorphism $V^* \to V$ associated with $(\cdot,\cdot)$. Thus, this maps becomes an isometry, and for each $\alpha \in \Psi_0$, the coroot $\alpha^\vee \in V$ is given as the image of $2(\alpha,\alpha)^{-1}\alpha \in V^*$.

To this data we associate a triple of groups (W, W_Q, W_0) in the following way. First, to the pair (Ψ_0, Π_0) we have W_0, the finite Weyl group generated by the reflections s_α over the hyperplanes $H_{\alpha^\vee} \subseteq V^*$ consisting of all elements $x \in V^*$ which are orthogonal to $\alpha^\vee$ with respect to $\langle \cdot, \cdot \rangle$. This group is a Coxeter group with set of generators S_0 that is in bijection with Π_0 via $\alpha \mapsto s_\alpha$. Each s_α is an orthogonal transformation of $(V^*, (\cdot,\cdot))$. We view the finite Weyl group and the translation group on each lattice as subgroups $W_0, t(\mathcal{X}), t(Q) \subseteq \mathrm{Aff}(V^*)$

[2] The (important) notion of isogeny will not play a role in this thesis so it will not be defined, nor will the direct sum of root data. For details, see [92, 1.7] and [94, Section 8.1].

of the group of affine linear transformations on V^* and define

$$\begin{aligned} W &:= W_0 \ltimes t(\mathcal{X}) \\ W_Q &:= W_0 \ltimes t(Q). \end{aligned}$$

Let $\Psi = \Psi_0^\vee \times \mathbb{Z} \subseteq \mathcal{X}^\vee \times \mathbb{Z}$ be the set of affine roots which can be viewed as a subset of all affine linear maps from $V^* \to \mathbb{R}$. Each affine root $a = (\alpha^\vee, n) \in \Psi$ defines an affine linear map $s_a(\xi) = \xi - (\langle \xi, \alpha^\vee \rangle + n)\alpha = s_\alpha t_{n\alpha}(\xi)$, with $\xi \in V^*$, in which $t_{n\alpha}$ denotes the translation $\xi \mapsto \xi + n\alpha$. If Ψ_{m} denotes the set of $\beta \in \Psi$ such that $\beta^\vee$ is a minimal element in the dominance ordering of $\Psi^\vee$ and we define

$$\Pi = \{(\beta^\vee, 1) \mid \beta \in \Psi_{\mathrm{m}}\} \cup \{(\alpha^\vee, 0) \mid \alpha \in \Pi_0\},$$

then the set $S = \{s_a \mid a \in \Pi\}$ is a set of generators for W_Q that gives a Coxeter presentation of W_Q [42, Theorem 4.6]. The set Π is the set of simple affine roots in the sense that any affine root $a \in \Psi$ can be written uniquely as a linear combination of elements in Π with either all coefficients non-negative or all non-positive. This defines a decomposition $\Psi = \Psi^+ \cup \Psi^-$, with

$$\Psi^+ = (\Psi_0^+ \times \{0\}) \cup (\Psi_0^\vee \times \mathbb{Z}_{>0}),$$

and $\Psi^- = -\Psi^+$, in which Ψ_0^+ is the set of positive roots corresponding to the choice of simple roots Π_0.

An element $w = ut_x$ in W acts on $\mathcal{X}^\vee \times \mathbb{Z}$ by $ut_x(y, n) = (uy, n - \langle x, y \rangle)$ and we define the length function $\ell : W \to \mathbb{N}$ by the formula

$$\ell(w) = |\Psi^+ \cap w^{-1}\Psi^-|.$$

By restriction, this defines length functions on W_Q and on W_0, which coincides with their naturally defined length functions, coming from the fact that (W_0, S_0) and (W_Q, S) are Coxeter systems. Let $\Omega = \{w \in W \mid \ell(w) = 0\}$. This finitely generated abelian group can be viewed as the quotient $\mathcal{X}/Q$ and it is a complement to the group $W_Q \subseteq W$ so that one can write

$$W = \Omega \ltimes W_Q. \tag{2.4}$$

If the the root data is semisimple, then $Q \otimes \mathbb{R} = V^* = \mathbb{R} \otimes \mathcal{X}$ and the group Ω is finite. We call W the **Weyl group**, W_0 the **finite Weyl group** and W_Q the **affine Weyl group** of the root data $\mathcal{R}$. Recall that $V_0^* = \mathbb{R} \otimes Q$. Let $(V^*)^0$ denote its orthogonal (with respect to the Euclidean product) complement, (which can be written as $(\mathbb{R} \otimes Q^\vee)^\perp$). Thus, any element $\xi \in V^*$ is uniquely written as $\xi = \xi_0 + \xi^0$. For each $w \in W$, define the norm function on W

$$\mathcal{N}(w) := \ell(w) + \|w(0)^0\| \tag{2.5}$$

(see [73, (2.5)] and [25, (2.1)]).

We end by highlighting that the objects $V^*, (\cdot,\cdot), V, \langle\cdot,\cdot\rangle, \mathcal{R}$ and W_0 discussed here will be used quite often in this thesis. Recall that this means that from the based root datum $\mathcal{R}$, we have $\mathbb{R}\otimes\mathcal{X} = V^* \supseteq \Psi_0$, $\mathbb{R}\otimes\mathcal{X}^\vee = V \supseteq \Psi_0^\vee$, with the canonical pairing $\langle\cdot,\cdot\rangle$. If $\mathcal{R}$ is a based root datum we have $\Pi_0 \subseteq \Psi_0$ and the notion of positive roots Ψ_0^+. In addition, we are given a W_0-invariant inner product $(\cdot,\cdot)$ on V^*. We equip V with the inner product for which the map $V^* \to V$ induced by $(\cdot,\cdot)$ is an isometry. Then, the canonical bijection $\Psi_0 \leftrightarrow \Psi_0^\vee$ is given by the map sending $\alpha \in \Psi_0$ to the image of $2(\alpha,\alpha)^{-1}\alpha$ under $V^* \to V$, for each $\alpha \in \Psi_0$.

2.3.2 Presentation of Hecke algebras

The root data $\mathcal{R} = (\mathcal{X}, \Pi_0\mathcal{X}^\vee, \Pi_0^\vee)$ defines groups W, W_Q, W_0. To each of these groups there correspond a suitable algebra that goes by the name Hecke algebra which we describe below. Fix $q \in \mathbb{C}^\times$, called a parameter. Define $\mathcal{H}(\mathcal{R}, q)$ as the unique unital associative algebra over $\mathbb{C}$ generated by the elements $\{T_w \mid w \in W\}$ and subject to the relations

$$\begin{aligned} (T_s - q)(T_s + 1) &= 0, && \text{for all } s \in S, \\ T_w T_{w'} &= T_{ww'}, && \ell(ww') = \ell(w) + \ell(w'). \end{aligned} \tag{2.6}$$

Note that the algebra $\mathcal{H}(\mathcal{R}, q)$ is entirely determined by W and q, so we may occasionally write $\mathcal{H}(W, q)$ instead of $\mathcal{H}(\mathcal{R}, q)$. Similarly, we define the subalgebras $\mathcal{H}_Q = \mathcal{H}(W_Q, q)$ and $\mathcal{H}_0 = \mathcal{H}(W_0, q)$ generated by $\{T_w \mid w \in W_Q\}$ and $\{T_w \mid w \in W_0\}$, respectively, with similar relations as in (2.6). The subspace generated by the elements $\{T_\omega \mid \omega \in \Omega\}$ generates yet another subalgebra isomorphic to $\mathbb{C}[\Omega]$ and the semidirect product (2.4) yields

$$\mathcal{H}(\mathcal{R}, q) \cong \mathbb{C}[\Omega] \otimes \mathcal{H}_Q. \tag{2.7}$$

We call $\mathcal{H}(\mathcal{R}, q)$[3] the **Hecke algebra**, $\mathcal{H}_Q$ the **affine Hecke algebra** and $\mathcal{H}_0$ the **finite Hecke algebra** of the pair $(\mathcal{R}, q)$. Because of (2.7), $\mathcal{H}(\mathcal{R}, q)$ is also known as the extended affine Hecke algebra (inasmuch as W is also sometimes referred to as the extended affine Weyl group).

The Hecke algebra $\mathcal{H}(\mathcal{R}, q)$ has another very useful presentation, due to Bernstein and introduced by Lusztig in [57], that reflects the semidirect product $W = W_0 \ltimes \mathcal{X}$ and we describe in what follows. An element $x \in \mathcal{X}$ is called **dominant** if $x \in \overline{V_+^*}$, where V_+^* is the fundamental chamber with respect to $\Pi_0^\vee$. We will denote by $\mathcal{X}_+$ to the set of dominant elements of $\mathcal{X}$. Given a dominant x, it can be checked that

$$\ell(t_x) = \langle x, 2\varrho^\vee\rangle, \tag{2.8}$$

[3] Due to the abundance of "Hecke algebras" present in this world, we will refrain from using the short-hand notation $\mathcal{H}$ for $\mathcal{H}(\mathcal{R}, q)$, even when the pair $(\mathcal{R}, q)$ is fixed and clear from the context. We will reserve $\mathcal{H}$ to the definition of the global Hecke algebra, below.

with $2\varrho^\vee = \sum \alpha^\vee$, the sum over all positive coroots. Given any $x \in \mathcal{X}$, write $x = x_1 - x_2$ for a choice of $x_1, x_2 \in \mathcal{X}_+$ and define $\theta_x = q^{-\langle x, \varrho^\vee \rangle} T_{t_{x_1}} (T_{t_{x_2}})^{-1}$. Using (2.8) and the second relation in (2.6), one can show that $\theta_{x+x'} = \theta_x \theta_{x'}$ and that the definition of each θ_x is independent of the chosen x_1, x_2. Denote by Θ the commutative subalgebra generated by $\{\theta_x \mid x \in \mathcal{X}\}$. The assignment $x \mapsto \theta_x$ yields an isomorphism

$$\mathbb{C}[\mathcal{X}] \cong \Theta, \tag{2.9}$$

in which the finite Weyl group W_0 acts on Θ via $w\theta_x = \theta_{wx}$. The Hecke algebra $\mathcal{H}(\mathcal{R}, q)$ can be viewed as the $\mathbb{C}$-vector space

$$\mathcal{H}(\mathcal{R}, q) \cong \mathcal{H}_0 \otimes \Theta \tag{2.10}$$

with the structure of a unital associative algebra over $\mathbb{C}$ generated by the elements $\{T_w \theta_x \mid w \in W_0, x \in \mathcal{X}\}$ and subject to the relations

$$\begin{array}{rcll} (T_s - q)(T_s + 1) & = & 0, & \text{for all } s \in S_0, \\ T_w T_{w'} & = & T_{ww'}, & \ell(ww') = \ell(w) + \ell(w'), \\ \theta_x T_s - T_s \theta_{s(x)} & = & (q-1)\frac{\theta_x - \theta_{s(x)}}{1 - \theta_{-\alpha}}, & x \in \mathcal{X}, s \in S_0, \end{array} \tag{2.11}$$

for all $w, w' \in W_0$. An important consequence of this presentation is the determination of the center of $\mathcal{H}(\mathcal{R}, q)$ (cf. [57, Proposition 3.11]):

Proposition 2.3.1. *The center $Z(\mathcal{H}(\mathcal{R}, q))$ of $\mathcal{H}(\mathcal{R}, q)$ is Θ^{W_0}.*

Remark. More generally, a parameter $q = (q_\alpha)_{\alpha \in \Psi_0}$ is a collection of complex numbers satisfying $q_{w\alpha} = q_\alpha$ for all $w \in W_0$ (or equivalently, when s_α and s_β are conjugate in the Weyl group). The relations in (2.6) and (2.11) are slightly more complicated in that case. However, our interest here is in the *equal parameter case* and thus the choice of a single q.

2.3.3 Representations of $\mathcal{H}(\mathcal{R}, q)$

Let us recall some facts about the well-know representation theory of $\mathcal{H}(\mathcal{R}, q)$ (see [35], [73] and [48]). Let $\mathcal{T} := \mathrm{Hom}(\mathcal{X}, \mathbb{C}^\times) \cong \mathbb{C}^\times \otimes \mathcal{X}^\vee$. We denote by $\mathcal{H}(\mathcal{R}, q)^\sqcap$ the set of equivalence classes of irreducible modules for $\mathcal{H}(\mathcal{R}, q)$. By a well-known version of the Schur Lemma, the center of $\mathcal{H}(\mathcal{R}, q)$ acts by a scalar and there is a decomposition $\mathcal{H}(\mathcal{R}, q)^\sqcap = \coprod \mathcal{H}(\mathcal{R}, q)^\sqcap_{W_0 s}$ running over $W_0 s \in W_0 \backslash \mathcal{T}$. A way to produce representations for $\mathcal{H}(\mathcal{R}, q)$ is via an induction procedure. The irreducible representations of the algebra $\Theta \cong \mathbb{C}[\mathcal{X}]$ are parametrized by the torus $\mathcal{T}$ and to each $s \in \mathcal{T}$, the principal series $(\pi(s), \mathcal{I}(s))$ with parameter s is defined to be

$$\mathcal{I}(s) := \mathrm{Ind}_\Theta^{\mathcal{H}(\mathcal{R}, q)} \mathbb{C}_s = \mathcal{H}(\mathcal{R}, q) \otimes_\Theta \mathbb{C}_s,$$

in which $\mathbb{C}_s$ is the representation of $\mathbb{C}[\mathcal{X}]$ corresponding to s. From $\mathcal{H}(\mathcal{R}, q) \cong \mathcal{H}_0 \otimes \Theta$, it follows that $\mathcal{I}(s)$ is isomorphic to $\mathcal{H}_0$ as a vector space, so $\{T_w \mid w \in W_0\}$ is a basis of $\mathcal{I}(s)$. The features we highlight are that (see [72, Proposition 2.3] and [35, (1.26)]):

(a) the center acts by the character $W_0 s$ on $\mathcal{I}(s)$,

(b) each irreducible module $\pi \in \mathcal{H}(\mathcal{R}, q)^{\sqcap}$ is a quotient of some $\mathcal{I}(s)$,

(c) $\mathcal{I}(s)$ is irreducible if and only if $\prod_{\alpha \in \Psi_0} (1 - q^{-1}\theta_{-\alpha})(s) \neq 0$.

An (equivalence class of an) irreducible representation $\pi \in \mathcal{H}(\mathcal{R}, q)^{\sqcap}$ is called **spherical** if $[\pi|_{\mathcal{H}_0} : \mathrm{triv}] = 1$ in which $\mathrm{triv} : \mathcal{H}_0 \to \mathbb{C}$ is the representation

$$\mathrm{triv}(T_w) = q^{\ell(w)}, \tag{2.12}$$

for all $w \in W_0$. Similarly, π is called **antispherical** if $[\pi|_{\mathcal{H}_0} : \mathrm{sign}] = 1$ in which $\mathrm{sign} : \mathcal{H}_0 \to \mathbb{C}$ is the representation

$$\mathrm{sign}(T_w) = (-1)^{\ell(w)}. \tag{2.13}$$

The set of all spherical and antispherical irreducible modules are denoted, respectively, by $\mathcal{H}(\mathcal{R}, q)^{\sqcap}_{\mathrm{sph}}$ and $\mathcal{H}(\mathcal{R}, q)^{\sqcap}_{\mathrm{antisph}}$. With these notations, the last fact we highlight is

(d) given $s \in \mathcal{T}$ there is a unique element in $\mathcal{H}(\mathcal{R}, q)^{\sqcap}_{W_0 s} \cap \mathcal{H}(\mathcal{R}, q)^{\sqcap}_{\mathrm{sph}}$ and a unique one in $\mathcal{H}(\mathcal{R}, q)^{\sqcap}_{W_0 s} \cap \mathcal{H}(\mathcal{R}, q)^{\sqcap}_{\mathrm{antisph}}$.

These unique elements will be denoted by $\mathcal{U}(s)$ and $\mathcal{U}_{-}(s)$. For the next result, see [35, (3.8) and (3.9)].

Lemma 2.3.2. *The map $\iota : \mathcal{H}(\mathcal{R}, q^{-1}) \to \mathcal{H}(\mathcal{R}, q)$ between Hecke algebras with different parameter given by $\iota(T'_w) := (-q^{-1})^{\ell(w)} T_w$, $\iota(\theta'_x) := \theta_x$, in which $\{T'_w, \theta'_x\}$ indicates the basis of $\mathcal{H}(\mathcal{R}, q^{-1})$, is an isomorphism that induces a bijection between*

$$\mathcal{H}(\mathcal{R}, q^{-1})^{\sqcap}_{\mathrm{sph}} \longleftrightarrow \mathcal{H}(\mathcal{R}, q)^{\sqcap}_{\mathrm{antisph}}$$

and preserves the central character $W_0 s$.

Proof. The map ι is a linear isomorphism of the underlying vector spaces with inverse $T_w \mapsto (-q)^{\ell(w)} T'_w$, $\theta_x \mapsto \theta'_x$. It is straightforward to check that ι preserves the relations (2.11) and the central character. As for the bijection, note that ι interchanges the triv and sign characters of $\mathcal{H}_0(\mathcal{R}, q^{-1})$: using (2.12) and (2.13)

$$\begin{array}{rclclcl}
\mathrm{triv}\,\iota(T'_w) & = & (-q^{-1})^{\ell(w)}\,\mathrm{triv}(T_w) & = & (-1)^{\ell(w)} & = & \mathrm{sign}(T'_w) \\
\mathrm{sign}\,\iota(T'_w) & = & (-q^{-1})^{\ell(w)}\,\mathrm{sign}(T_w) & = & (q^{-1})^{\ell(w)} & = & \mathrm{triv}(T'_w),
\end{array}$$

and the lemma is proved. $\square$

Given a module $\mathcal{M}$ for the Hecke algebra $\mathcal{H}(\mathcal{R}, q)$ and an element $s \in \mathcal{T}$, the generalized s-weight space is defined by

$$\mathcal{M}(s) := \{m \in \mathcal{M} \mid \text{there exists } k \in \mathbb{N} \text{ such that } (\theta - s(\theta))^k m = 0 \text{ for all } \theta \in \Theta\}.$$

The set of weights of a module $\mathcal{M}$ is $\{s \in \mathcal{T} \mid \mathcal{M}(s) \neq 0\}$. In the special but important case of the principal series, the weights of $\mathcal{I}(s)$ are contained in $\{ws \mid w \in W_0\}$, and the weight spaces are denoted $\mathcal{I}(s, ws)$. Similarly we denote the weight spaces of $\mathcal{U}(s)$ and $\mathcal{U}_-(s)$ by $\mathcal{U}(s, ws)$ and $\mathcal{U}_-(s, ws)$. Whenever $\mathcal{I}(s)$ is irreducible, all the weights $\{ws \mid w \in W_0\}$ are present in the weight space decomposition of $\mathcal{I}(s)$. In the case $\theta_\alpha(s) = q^{-1}$ for some roots $\alpha \in \Psi_0$, M. Reeder determined when the weight spaces of the unique antispherical module $\mathcal{U}_-(s, ws) \neq 0$. Let $\mathfrak{g}$ be the Lie algebra of the complex Lie group associated to the root data $\mathcal{R}$ of (2.3) and $\mathfrak{q} \subseteq \mathfrak{g}$ be the subspace $\{x \in \mathfrak{g} \mid \mathrm{Ad}(s)x = q^{-1}x\}$. Denote by $\mathfrak{q}_0 \subseteq \mathfrak{q}$ the unique open dense adjoint orbit of the action of C, the connected component of the centralizer of s in the complex Lie group associated to $\mathcal{R}$, on $\mathfrak{q}$.

Proposition 2.3.3 ([83], Corollary 5.13)**.** *The weight space $\mathcal{U}_-(s, ws) \neq 0$ if and only if*

$$\mathfrak{q}_0 \cap \mathrm{Ad}(w^{-1})\mathfrak{n} \neq \emptyset, \tag{2.14}$$

in which $\mathfrak{n}$ is the nilpotent radical of the Borel algebra $\mathfrak{b} = \mathfrak{b}(\Pi_0)$ corresponding to simple system Π_0.

Proposition 2.3.3 will be important in our results about the decomposition of the automorphic space (see Proposition 4.6.2). It is one of the goals of this preliminary part (see Proposition 2.5.5) to determine when will $\mathcal{U}_-(s)$ be a so-called discrete series module, a notion which we now recall. These definitions is a particular instance of Casselman's criterion (see [73, Section 2.7]) and also [19, Lemma 4.4.1]):

Definition 2.3.4. Let $\mathcal{M}$ be a finite dimensional module for $\mathcal{H}(\mathcal{R}, q)$.

(i) We call $\mathcal{M}$ **tempered** if all weights s of $\mathcal{M}$ satisfy $|s(x)| \leq 1$, for all $x \in \mathcal{X} \cap \overline{V_+^*}$.

(ii) We call $\mathcal{M}$ a **discrete series** if all weights s of $\mathcal{M}$ satisfy $|s(x)| < 1$, for all $0 \neq x \in \mathcal{X} \cap \overline{V_+^*}$.

2.3.4 Star, trace and completions

The affine Hecke algebra is equipped with a natural antilinear anti-involution $T_w^* = T_{w^{-1}}$ and a positive and central trace τ (see [72, Proposition 1.8]) given by $\tau(\sum_w c_w T_w) := c_1$, which defines a Hermitian product in $\mathcal{H}(\mathcal{R}, q)$ via

$$(h', h) = \tau(h' h^*). \tag{2.15}$$

With respect to this Hermitian product, the basis $\{T_w \mid w \in W\}$ is orthogonal, with $(T_w, T_w) = \tau(T_w T_{w^{-1}}) = q^{\ell(w)}$. So replacing T_w by $N_w := q^{-\ell(w)/2} T_w$, we obtain an orthonormal basis of $\mathcal{H}(\mathcal{R}, q)$. Let $L^2(\mathcal{H}(\mathcal{R}, q))$ denote the Hilbert space completion. Then, multiplication by elements in $\mathcal{H}(\mathcal{R}, q)$ induces bounded linear operators on this Hilbert space [73, Lemma 2.3]. The completion of $\mathcal{H}(\mathcal{R}, q)$ with respect to the operator norm is called the **reduced C^*-algebra of $\mathcal{H}(\mathcal{R}, q)$**, denoted $C^*(\mathcal{H}(\mathcal{R}, q))$. We have yet another completion of the affine Hecke algebra. This time, with respect to the family of norms indexed by $n \in \mathbb{Z}_{\geq 0}$

$$p_n(h) := \max_{w \in W} (N_w, h)(1 + \mathcal{N}(w))^n$$

(see [73, Section 6.2], [25, Section 2.8]). This algebra is called the **Schwartz algebra of $\mathcal{H}(\mathcal{R}, q)$**, denoted $S(\mathcal{H}(\mathcal{R}, q))$. This is a nuclear, unital Fréchet algebra. The trace τ and the star operation $*$ extends continuously to both completions (for the Schwartz completion, see [73, Theorem 6.5]). Finally, we denote by $\mathcal{H}(\mathcal{R}, q)^*$ the linear dual of $\mathcal{H}(\mathcal{R}, q)$ endowed with the weak-star topology. This space is identified with the direct product $\prod_{w \in W} \mathbb{C} T_w$ with the product topology. An element in the latter is an arbitrary linear combination with complex coefficients in the basis $\{T_w \mid w \in W\}$. In this identification, an element $\phi = \sum_{w \in W} c_w T_w$ yields the linear functional in $\mathcal{H}(\mathcal{R}, q)^*$, also denoted by ϕ, defined for all $h \in \mathcal{H}(\mathcal{R}, q)$ by

$$\phi(h) := (h, \phi^*) = \tau(h\phi). \tag{2.16}$$

We have inclusions as dense subspaces with respect to the indicated topologies

$$\mathcal{H}(\mathcal{R}, q) \subseteq S(\mathcal{H}(\mathcal{R}, q)) \subseteq C^*(\mathcal{H}(\mathcal{R}, q)) \subseteq L^2(\mathcal{H}(\mathcal{R}, q)) \subseteq \mathcal{H}(\mathcal{R}, q)^*.$$

We note that, from the list above, only $L^2(\mathcal{H}(\mathcal{R}, q))$ is *not* an algebra, as the multiplication is not closed for this completion.

2.3.5 The spherical subalgebra

There is another subalgebra of $\mathcal{H}(\mathcal{R}, q)$ that will be important in this thesis. Let

$$T_0^+ := \frac{1}{P_{W_0}(q)} \sum_{w \in W_0} T_w \in \mathcal{H}_0 \tag{2.17}$$

be the idempotent corresponding to the trivial representation of $\mathcal{H}_0$ and let $\mathcal{H}^+ := T_0^+ \mathcal{H}(\mathcal{R}, q) T_0^+$ be the **spherical Hecke algebra**. This subalgebra has a basis $\{\theta_x^+ \mid x \in \mathcal{X}_+\}$ indexed by the dominant cocharacters, where $\theta_x^+ = T_0^+ \theta_x T_0^+$. The following important result goes by the name of Satake isomorphism, [59, Theorem 4.1.2]:

Proposition 2.3.5. *The map $z \mapsto T_0^+ z$ induces an isomorphism between $\mathcal{Z}$ and $\mathcal{H}^+$ whose inverse $\mathcal{S}$ is*

$$\theta_x^+ \mapsto \mathcal{S}(\theta_x^+) = P_W(q^{-1}) \sum_{w \in W} w(c(\cdot, q)\theta_x),$$

where the c-function given by $c(\cdot, q) := \prod_{\alpha \in \Psi_0^+} c_\alpha(\cdot, q)$, with

$$c_\alpha(\cdot, q) := \frac{1 - q^{-1}\theta_{-\alpha}}{1 - \theta_{-\alpha}} \in \mathbb{C}[\mathcal{X}] \subseteq \mathcal{H}(\mathcal{R}, q). \tag{2.18}$$

Proposition 2.3.6. *The restriction of $*$ and $(\cdot, \cdot)$ to $\mathcal{H}^+$ satisfies*

(a) $(\theta_x^+)^* = \theta_{-w_0(x)}^+$

(b) $(\theta_x^+, \theta_{x'}^+) = \tau(\theta_x^+(\theta_{x'}^+)^*) = \dfrac{q^{\ell(w^x)} P_{W_x}(q)}{P_{W_0}(q)^2} \delta_{x,x'}$,

where W_x is the stabilizer of x, w_x is the longest element of W_x and $w^x \in W_0$ is such that $w_0 = w_x w^x$.

Proof. For (a), from [72, Proposition 1.12] we have $\theta_x^* = T_{w_0}\theta_{-w_0(x)}T_{w_0}^{-1}$. On the one hand, $T_0^+ T_w = q^{\ell(w)} T_0^+$ for all $w \in W_0$. On the other, note that for a simple reflection we have $T_s^{-1} = q^{-1}T_s + (q^{-1} - 1)$ so that

$$T_s^{-1}T_0^+ = T_0^+ + (q^{-1} - 1)T_0^+ = q^{-1}T_0^+.$$

It follows that $T_w^{-1}T_0^+ = q^{-\ell(w)}T_0^+$. Hence,

$$(\theta_x^+)^* = (T_0^+)^* \theta_x^* (T_0^+)^* = T_0^+ T_{w_0}\theta_{-w_0(x)}T_{w_0}^{-1}T_0^+,$$

and thus the result. Part (b) is [72, Proposition 2.31]. □

Corollary 2.3.7. *We have $(\theta_x^+, \theta_x^+) = ((\theta_x^+)^*, (\theta_x^+)^*)$, for all $x \in \mathcal{X}_+$.*

If $s \in \mathcal{T}$, define the **elementary spherical function for $\mathcal{H}(\mathcal{R}, q)$ with spectral parameter** s by

$$\varphi_s(h) := \mathrm{Tr}(\pi_s(h)) \tag{2.19}$$

for all $h \in \mathcal{H}^+$, in which π_s is the unique irreducible subquotient of the minimal principal series at $s \in \mathcal{T}$. Such function is a matrix coefficient of the irreducible representation π_s (see [72, Sections 2.4 and 2.5]).

Proposition 2.3.8. *The spherical vector φ_s satisfies:*

$$\varphi_s(\theta_x^+) = P_{W_0}(q^{-1})^{-1} \sum_{w \in W_0} c(ws, q) ws(\theta_x).$$

Proof. This is [72, Theorem 2.28] and is due to MacDonald [59]. □

As in Section 2.3.4, we consider the linear dual $(\mathcal{H}^+)^*$ of $\mathcal{H}^+$. Using the non-degenerate bilinear form $(h', h) = \tau(h'h^*)$ of (2.15) on $\mathcal{H}^+$, we can identify the linear dual, $(\mathcal{H}^+)^*$, of $\mathcal{H}^+$ (with its weak-star topology) with the direct product $\prod_{y\in\mathcal{X}_+^\vee} \mathbb{C}\theta_y^+$ (with its product topology). One can write φ_s as an element in the linear dual of $\mathcal{H}^+$.

Proposition 2.3.9. *We have*

$$\varphi_s = \sum_{x\in\mathcal{X}_+} \frac{(\varphi_s, \theta_x^+)}{(\theta_x^+, \theta_x^+)}\theta_x^+.$$

Proof. Formally, we write $\varphi_s = \sum_{z\in\mathcal{X}_+} c_z\theta_z^+$. All we need to do is determine the coefficients c_z. From Proposition 2.3.6(b) the basis $\{\theta_z^+ \mid z \in \mathcal{X}_+\}$ is orthogonal, so

$$c_x = \frac{(\varphi_s, \theta_x^+)}{(\theta_x^+, \theta_x^+)},$$

as required. □

2.3.6 Graded affine Hecke algebras

To the objects $V^*, (\cdot,\cdot), V, \langle\cdot,\cdot\rangle, \mathcal{R}$, and W_0 as in Section 2.3.1, we will associate an algebra of operators on the space of smooth function on V^*, following [36]. Fix a parameter $k \in \mathbb{C}$ (or, more generally, fix $(k_\alpha)_{\alpha\in\Psi_0}$ with $k_{w\alpha} = k_\alpha$, for all $w \in W_0$). Given $\alpha \in \Psi_0$ consider the operator $I(\alpha) : C^\infty(V^*) \to C^\infty(V^*)$ defined by

$$(I(\alpha)\phi)(\xi) := \int_0^{\langle\xi,\alpha^\vee\rangle} \phi(\xi - t\alpha)dt, \tag{2.20}$$

for all $\phi \in C^\infty(V^*)$. For example, given $\lambda \in V_\mathbb{C}$, let $e^\lambda \in C^\infty(V^*)$ denote the function $e^\lambda(\xi) := \exp(\langle\xi, \lambda\rangle)$. One has

$$I(\alpha)e^\lambda = \frac{e^\lambda - e^{s_\alpha\lambda}}{\langle\alpha, \lambda\rangle}. \tag{2.21}$$

Further, an element $w \in W_0$ acts on $\phi \in C^\infty(V^*)$ by means of $w\phi(\xi) = \phi(w^{-1}\xi)$ and we define an operator on $C^\infty(V^*)$ via $Q(s, k) := s + kI(\alpha)$, for each $s = s_\alpha \in S_0$. Given an element $w \in W_0$ written in some reduced expression as $w = s_1 \cdots s_n$, define $Q(w, k) = Q(s_1, k)\cdots Q(s_n, k)$. It is a theorem due to Yang and Gutkin (see [36, Section 2] and the references [105] and [32] therein) that $Q(w, k)$ is independent of the choice of reduced expression used to represent w and thus these operators yield an action of W_0 on $C^\infty(V^*)$. Moreover, If we let $\partial(\xi)\phi$ denote the derivative of a function $\phi \in C^\infty(V^*)$ in the direction of $\xi \in V^*$, and $t_s \in \mathbb{C}[W_0]$ with $s \in S_0$ denote a generator of $\mathbb{C}[W_0]$, it follows that we have actions of $\mathbb{C}[W_0]$ and of $\mathrm{Sym}(V_\mathbb{C}^*)$ on $C^\infty(V^*)$ given by the assignment

$$t_s \mapsto Q(s, k) \quad \text{and} \quad \xi \mapsto \partial(\xi), \tag{2.22}$$

for all $s \in S_0$ and $\xi \in V_{\mathbb{C}}^*$. These actions satisfy the commutation relation

$$Q(s,k)\partial(\xi) - \partial(s\xi)Q(s,k) = k\langle \xi, \alpha^\vee \rangle. \tag{2.23}$$

Definition 2.3.10. The **graded affine Hecke algebra**[4], $\mathbb{H}(\mathcal{R},k)$, is defined as the vector space $\mathrm{Sym}(V_{\mathbb{C}}^*)\otimes\mathbb{C}[W_0]$ equipped with the unique complex associative algebra structure in which $\mathrm{Sym}(V_{\mathbb{C}}^*)\otimes 1 \cong \mathrm{Sym}(V_{\mathbb{C}}^*)$, $1\otimes\mathbb{C}[W_0]\cong\mathbb{C}[W_0]$ and

$$t_s\xi - s(\xi)t_s = k\langle \xi, \alpha^\vee \rangle,$$

for all $\xi \in V_{\mathbb{C}}^*$, $s = s_\alpha \in S_0$.

It follows from (2.23) that $C^\infty(V^*)$ is a module for $\mathbb{H} = \mathbb{H}(\mathcal{R},k)$. Denote the action (2.22) by π_k. Further, whenever convergent, the expression below, defined for all $\phi,\psi \in C^\infty(V^*)$

$$(\phi,\psi)_k := \sum_{w\in W_0}\int_{V_+^*} Q(w,k)\phi(\xi)\overline{Q(w,k)\psi(\xi)}\, d\mu^E(\xi), \tag{2.24}$$

where μ^E will always denotes the Euclidean-Lebesgue measure on the relevant Euclidean space and V_+^* is the fundamental chamber, defines an inner product on $C^\infty(V^*)$, which turns the space

$$C(V^*,k) := \{\phi \in C^\infty(V^*) \mid (\partial(p)\phi, \partial(p)\phi)_k < \infty, \text{ for all } p \in \mathrm{Sym}(V_{\mathbb{C}}^*)\} \tag{2.25}$$

into a pre-Hilbert space. Denote its Hilbert space completion by $L^2(V^*,k)$. With respect to the star operation

$$t_w^* := t_{w^{-1}} \quad \text{and} \quad \xi^* := -t_{w_0}w_0(\xi)t_{w_0}, \tag{2.26}$$

where w_0 denote the longest element of W_0, we have that $(\pi_k, L^2(V^*,k))$ is a unitary module for $\mathbb{H}(\mathcal{R},k)$ [36, Theorem 2.4]. The spectral decomposition of this $\mathbb{H}(\mathcal{R},k)$-module is one of the main themes discussed in [36] and will be recalled later in Chapter 3. It will turn out to be an important ingredient in our study of the unramified spherical automorphic spectrum.

We close this subsection with an explicit determination of the center of $\mathbb{H}$. Similarly to $\mathcal{H}(\mathcal{R},q)$, we have [57, Proposition 4.5]:

Proposition 2.3.11. *The center $Z(\mathbb{H})$ of $\mathbb{H}$ is $\mathrm{Sym}(V_{\mathbb{C}}^*)^{W_0}$.*

2.3.7 Representations of $\mathbb{H}(\mathcal{R},k)$

Similar to the case of the extended affine Hecke algebra, one can realize the irreducible representations for $\mathbb{H}$ as quotients of some induced modules. In the

[4] Since the symbol $\mathbb{H}$ is exclusive for the GAHA, we will occasionally use the short-hand notation $\mathbb{H}$ for $\mathbb{H}(\mathcal{R},k)$, when the pair $(\mathcal{R},k)$ is fixed and clear from the context, differently than for $\mathcal{H}(\mathcal{R},q)$.

affine case, the parameter space was the torus $\mathcal{T}$. In the graded affine case, we consider $V_{\mathbb{C}} = \mathbb{C} \otimes \mathcal{X}^\vee$ as the Lie algebra of $\mathcal{T} = \mathbb{C}^\times \otimes \mathcal{X}^\vee$. Given $\lambda \in V_{\mathbb{C}}$, let $\mathbb{C}_\lambda$ be the one dimensional module of $\mathrm{Sym}(V_{\mathbb{C}}^*)$ such that $\xi \cdot e = \langle \xi, \lambda \rangle e$, for all $\xi \in V_{\mathbb{C}}^*$ and $e \in \mathbb{C}_\lambda$. For example, the element $e^\lambda \in C^\infty(V^*)$ is such that

$$\partial(\xi) e^\lambda = \langle \xi, \lambda \rangle e^\lambda,$$

and hence $\mathbb{C}_\lambda \cong \mathbb{C} e^\lambda \subseteq C^\infty(V^*)$. Let also

$$E(\lambda) \quad = \quad \{\phi \in C^\infty(V^*) \mid \partial(p)\phi = p(\lambda)\phi \;\text{ for all }\; p \in \mathrm{Sym}(V_{\mathbb{C}}^*)^{W_0}\}$$

be the submodule of $C^\infty(V^*)$ called the eigenspace representation. It is a theorem of Steinberg (see [96] and [39, Theorem 3.13]) that $\dim E(\lambda)$ is independent of the parameter λ and in fact $\dim E(\lambda) = |W_0|$. One can show that the eigenspace representation is isomorphic to $\mathrm{Ind}_{\mathrm{Sym}(V_{\mathbb{C}}^*)}^{\mathbb{H}} \mathbb{C}_\lambda$. Similarly to the principal series for the affine Hecke algebra, the representations $E(\lambda)$ enjoy the properties [53, Section 2]:

(a) the center acts by the character $W_0\lambda$ on $E(\lambda)$,

(b) each irreducible module for $\mathbb{H}$ is a subquotient of some $E(\lambda)$,

(c) $E(\lambda)$ is irreducible if and only if $\prod_{\alpha \in \Psi_0} (\langle \alpha, \lambda \rangle + k) \neq 0$,

(d) for each λ, there exists a unique irreducible module that contains the trivial representation of $\mathbb{C}[W]$ as a summand and a unique that contains the sign.

We will denote the unique spherical module by $U(\lambda, k)$. This $\mathbb{H}$-module can be described explicitly as follows. Consider the c-function for the graded affine Hecke algebra with parameter k, given by $c_{\mathbb{H}}(\lambda, k) := \prod_{\alpha \in \Psi_0^+} \gamma_\alpha(\lambda, k)$, with

$$\gamma_\alpha(\lambda, k) := \frac{\langle \alpha, \lambda \rangle + k}{\langle \alpha, \lambda \rangle}. \tag{2.27}$$

Definition 2.3.12. The elementary spherical function for $\mathbb{H}$ with spectral parameter λ is the element of $C^\infty(V^*)$ defined by

$$\phi(\lambda, k; \cdot) := \frac{1}{|W_0|} \sum_{w \in W_0} c_{\mathbb{H}}(w\lambda, k) e^{w\lambda}.$$

For λ regular one verifies that $\phi(\lambda, k; \cdot) = |W_0|^{-1} \sum_{w \in W_0} Q(w, k) e^\lambda$. It is therefore the unique $\phi \in E(\lambda)$ that satisfies $Q(w, k)\phi = \phi$ and $\phi(0) = 1$. Let $\tilde{U}(\lambda, k)$ denote the $\mathbb{H}$-module spanned by the elementary spherical vector $\phi(\lambda, k; \cdot)$.

Proposition 2.3.13. *The module $\tilde{U}(\lambda, k)$ is the unique irreducible submodule of $E(\lambda)$. Hence, $\tilde{U}(\lambda, k) = U(\lambda, k)$.*

Proof. The assertion follows from the claim that any submodule of $E(\lambda)$ necessarily contains a multiple of $\phi(\lambda, k; \cdot)$. Indeed, if $M \subseteq E(\lambda)$ is a submodule for $\mathbb{H}$ and $0 \neq f \in M$, from $f(\eta + \xi) = (\exp(\partial(\xi))f)(\eta)$ and $\exp(\partial(\xi))f \in M$, we can assume $f(0) \neq 0$. Hence, the element

$$\phi(\xi) = \frac{1}{|W_0|} \sum_{w \in W} Q(w, k) f(\xi) \in M$$

satisfies $f(0) = \phi(0) \neq 0$ and is $Q(w, k)$-invariant, so a multiple of $\phi(\lambda, k; \cdot)$. $\square$

We will be interested in changing the parameter k of the graded Hecke algebra to $-k$. In view of the symmetries

$$\gamma_\alpha(\lambda, k) = \frac{\langle \alpha, \lambda \rangle + k}{\langle \alpha, \lambda \rangle} = \frac{\langle \alpha, -\lambda \rangle - k}{\langle \alpha, -\lambda \rangle} = \gamma_\alpha(-\lambda, -k), \tag{2.28}$$

we have $c_{\mathbb{H}}(\lambda, k) = c_{\mathbb{H}}(-\lambda, -k)$ and $\phi(\lambda, k; \xi) = \phi(-\lambda, -k; -\xi)$, for all $\xi \in V^*$.

2.3.8 Affine and graded affine Hecke algebras

In this brief subsection, we recall the construction of Lusztig [57]. See also [4] and [90]. Let $\mathfrak{A} = \mathbb{C}[v, v^{-1}]$, where v is an indeterminate, denote the coordinate ring of the torus $\mathbb{C}^\times$. Given the root data $\mathcal{R}$, one can consider the more general Hecke algebra $\underline{\mathcal{H}} = \underline{\mathcal{H}}(\mathcal{R})$ as the unital associative algebra over the ring $\mathfrak{A}$ generated by $\{T_w \mid w \in W_0\}$ and $\{\theta_x \mid x \in \mathcal{X}\}$ verifying the Bernstein relations (2.11)

$$\begin{array}{rcll} (T_s - v^2)(T_s + 1) & = & 0, & \text{for all } s \in S_0, \\ T_w T_{w'} & = & T_{ww'}, & \ell(ww') = \ell(w) + \ell(w'), \\ \theta_x T_s - T_s \theta_{s(x)} & = & (v^2 - 1) \frac{\theta_x - \theta_{s(x)}}{1 - \theta_{-\alpha}}, & x \in \mathcal{X}, s \in S_0. \end{array} \tag{2.29}$$

The algebra $\underline{\Theta} = \mathfrak{A} \otimes \Theta$ can be viewed as $\mathbb{C}[\mathbb{C}^\times \times \mathcal{T}]$, the ring of regular functions of the variety $\mathbb{C}^\times \times \mathcal{T}$, in which $v^n \otimes \theta_x(z, t) = z^n \theta_x(t)$, for all $(z, t) \in \mathbb{C}^\times \times \mathcal{T}$. Let $\mathcal{I}$ be the maximal ideal of $\underline{\Theta}$ given by

$$\mathcal{I} := \{p \in \underline{\Theta} \mid p(1, 1) = 0\}, \tag{2.30}$$

in which the 1's above are the respective identity element of the tori $\mathbb{C}^\times$ and $\mathcal{T}$, respectively. For any $p \in \underline{\Theta}$, we will write $d(p)$ to the image of $(p - p(1, 1))$ in $\mathcal{I}/\mathcal{I}^2$.

The $\mathcal{I}$-adic filtration on the commutative algebra $\underline{\Theta}$ yields a commutative graded algebra $\underline{\mathbb{A}} = \oplus_{n \geq 0} G(\underline{\Theta})^n$ with $G(\underline{\Theta})^n = \mathcal{I}^n / \mathcal{I}^{n+1}$ (where $\mathcal{I}^0 = \underline{\Theta}$). Note that $G(\underline{\Theta})^1 \cong \mathbb{C} \oplus V_{\mathbb{C}}^*$, is isomorphic to the cotangent space at $(1, 1)$ of the torus $\mathbb{C}^\times \times \mathcal{T}$. Let $r = d(v) \in G(\underline{\Theta})^1$ and denote by $\mathbb{C}[r]$ the polynomial algebra on the indeterminate r. The graded algebra $\underline{\mathbb{A}}$ is isomorphic to

$$\underline{\mathbb{A}} \cong \mathrm{Sym}(V_{\mathbb{C}}^*) \otimes \mathbb{C}[r], \tag{2.31}$$

in which $\mathrm{Sym}(V_{\mathbb{C}}^*)$ is the symmetric algebra of $V_{\mathbb{C}}^*$. The ideal $\mathcal{I}$ satisfies $\underline{\mathcal{H}}\mathcal{I} = \mathcal{I}\underline{\mathcal{H}}$ and the $\mathcal{I}$-adic filtration of the $\underline{\Theta}$-module $\underline{\mathcal{H}}$

$$\underline{\mathcal{H}} = \underline{\mathcal{H}}^0 \supseteq \underline{\mathcal{H}}^1 \supseteq \underline{\mathcal{H}}^2 \supseteq \dots \tag{2.32}$$

with $\underline{\mathcal{H}}^n = \underline{\mathcal{H}}\mathcal{I}^n$ yields an associated graded algebra $\mathbb{H} = \oplus_{n\geq 0} G(\underline{\mathcal{H}})^n$, with $G(\underline{\mathcal{H}})^n = \underline{\mathcal{H}}^n/\underline{\mathcal{H}}^{n+1}$. As a $\mathbb{C}$-vector space, this graded object is given by

$$\mathbb{H} \cong \mathbb{C}[W_0] \otimes \mathrm{Sym}(V_{\mathbb{C}}^*) \otimes \mathbb{C}[r]. \tag{2.33}$$

Mapping $T_w \mapsto t_w \in \underline{\mathcal{H}}/\underline{\mathcal{H}}^1$ and $\theta_x \mapsto d(\theta_x) \in G(\underline{\Theta})^1$ (more generally, let $\xi = d(\sum_i z_i\theta_{x_i})$, which is in $V_{\mathbb{C}}^*$ under (2.31)), we obtain that the analogous to the relations (2.29) in the algebra $\mathbb{H}$ are

$$\begin{array}{rcll} t_s^2 &=& 1, & \text{for all } s \in S_0, \\ t_w t_{w'} &=& t_{ww'}, & \ell(ww') = \ell(w) + \ell(w'), \\ \xi t_s - t_s s(\xi) &=& 2r\langle \xi, \alpha^\vee\rangle, & \xi \in V_{\mathbb{C}}^*, s \in S_0. \end{array} \tag{2.34}$$

Our interest is in the specialization $q = v^2$ for $q > 1$ where we have the algebra $\mathcal{H}(\mathcal{R}, q)$. With this parameter, $k = 2r = \log q > 0$ and we obtain $\mathbb{H}(\mathcal{R}, k)$ as in Section 2.3.6. Note that if $z \in \mathbb{C}^\times$, then there is an isomorphism $\mathbb{H}(\mathcal{R}, zk) \cong \mathbb{H}(\mathcal{R}, k)$ given by $t_s \mapsto t_s$ for all $s \in S_0$ and $\xi \mapsto z\xi$. We may, therefore, scale the parameter $k \in \mathbb{C}$, if necessary.

The representation theory of the algebras $\mathcal{H}(\mathcal{R}, q)$ and $\mathbb{H}(\mathcal{R}, \log q)$ are closely related by the so-called Lusztig's Reduction Theorems [57]. To state this relationship, we recall that the set of weights of a module M for the graded affine Hecke algebra is defined analogously to the case of the affine Hecke algebra as in Section 2.3.3. Namely, it is the set of elements $\lambda \in V_{\mathbb{C}} = \mathbb{C} \otimes \mathcal{X}^\vee$ for which the generalized weight space $M(\lambda) \neq 0$. The version we state here can be found in [90, Theorem 6.1].

Proposition 2.3.14. *Let $\lambda \in V_{\mathbb{C}}$ be such that $\langle \alpha, \lambda\rangle$ and $\langle \alpha, \lambda\rangle + k$ are not in $\pi i\mathbb{Z} \setminus \{0\}$. Then, the exponential map $V_{\mathbb{C}} \to \mathcal{T}$ induces an equivalence of categories between*

$$\left\{ \begin{array}{c} \textit{Finite dim. } \mathbb{H}\textit{-modules} \\ \textit{whose weights } \nu \in W_0\lambda \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Finite dim. } \mathcal{H}(\mathcal{R}, q)\textit{-modules} \\ \textit{whose weights } s \in W_0 \exp(\lambda) \end{array} \right\}.$$

2.4 The Yang system

This section is a brief review of [36]. The aim is to introduce the spectral problem of the Yang system, as well as describe its Y-functional and introduce the notion of residual subspaces that will be important later on in our description of the unramified spherical automorphic spectrum.

We assume we have the objects $V^*, (\cdot,\cdot), V, \langle\cdot,\cdot\rangle, \mathcal{R}$, and W_0 as in Section 2.3.1. Assume further that the root datum $\mathcal{R}$ is semisimple. This is not a fundamental assumption, but it will imply the existence of residual points, below. Given $\alpha \in \Psi_0$, we let $H_\alpha \subseteq V$ denote the hyperplane of elements which are orthogonal to α with respect to $\langle\cdot,\cdot\rangle$. Denote by $\mathrm{Sym}(V_{\mathbb{C}}^*)$ the symmetric algebra of $V_{\mathbb{C}}^*$. This algebra is identified with the algebra of polynomial functions on $V_{\mathbb{C}}$. Any such polynomial function p defines an element $\partial(p) \in \mathrm{Sym}(V_{\mathbb{C}}^*)$ that acts on smooth functions on V^* as a constant coefficient differential operator. For example, if α is the polynomial $\lambda \mapsto \langle\alpha,\lambda\rangle$, we have that $\partial(\alpha)\phi$ is the derivative of $\phi \in C^\infty(V^*)$ in the direction of α. A coupling parameter $k = (k_\alpha)_{\alpha\in\Psi_0}$ is a collection of numbers $k_\alpha \in \mathbb{R}$, for each $\alpha \in \Psi_0$, satisfying $k_{w\alpha} = k_\alpha$, for all $w \in W_0$. Our interest is for equal parameters $k_\alpha = k$ for all $\alpha \in \Psi_0$. A spectral parameter is an element $\lambda \in V_{\mathbb{C}}$.

The Yang system for Ψ_0 with coupling parameter k and spectral parameter λ is the boundary value problem on V^* given by the differential equations

$$(\partial(p)\phi)(\xi) = p(\lambda)\phi(\xi), \tag{2.35}$$

for all $p \in \mathrm{Sym}(V_{\mathbb{C}}^*)^{W_0} = Z(\mathbb{H})$, $\xi \in V^*$, together with the boundary conditions

$$\phi(\xi+0\alpha) = \phi(\xi-0\alpha) \quad \text{and} \quad \partial(\alpha)\phi(\xi+0\alpha) - \partial(\alpha)\phi(\xi-0\alpha) = 2k\phi(\xi), \tag{2.36}$$

for all $\alpha \in \Psi_0$ and $\xi \in \cup_\alpha H_\alpha$, where $\phi(\xi+0\alpha)$ is a short-hand notation for $\lim_{t\to 0^+}\phi(\xi+t\alpha)$ (similarly for $\phi(\xi-0\alpha)$). The solutions to this boundary value problem with parameters (λ,k) are in $E(\lambda)$. As explained in [36, Theorem 1.2], up to a constant, the unique W_0-invariant solution to this boundary value problem is the spherical functions $\phi(\lambda,k;\cdot)$ discussed in Definition 2.3.12. Moreover, any W_0-invariant function $f \in C_c^\infty(V^*)$ can be written as a superposition of the W_0-invariant functions $\phi(\lambda,k;\cdot)$ [36, Theorems 1.3 and 1.5]. We will make the precise statement below, in the case $k<0$, which is of interest to us. The upshot is that the decomposition of the W_0-invariant elements of the unitary module $(\pi_k, L^2(V^*,k))$ for $\mathbb{H}(\mathcal{R},k)$ is given in terms of solutions of the Yang system. The decomposition of $L^2(V^*,k)^{W_0}$ constitutes the spectral problem of the Yang system. Here and everywhere else in this thesis, the superscript W_0 means W_0-invariance.

To prove these results, one introduces the spherical Fourier transform (also called the Fourier-Yang transform) $\mathcal{F}(k) : C_c^\infty(V^*)^{W_0} \to PW(V_{\mathbb{C}})^{W_0}$, where $PW(V_{\mathbb{C}})$ denotes the space of Paley-Wiener functions on $V_{\mathbb{C}}$, and the inverse operator $\mathcal{I}(k)$. We remark that the W_0-invariance in the left hand side is with respect to the action of $Q(w,k)$, whereas on the right hand side it is the usual action $w\cdot f(\lambda) = f(w^{-1}\lambda)$. For $\psi \in C_c^\infty(V^*)^{W_0}$ and $f \in PW(V_{\mathbb{C}})^{W_0}$, these operators are given by

$$\mathcal{F}(k)\psi(\lambda) := \int_{V^*} \psi(\xi)\phi(-\lambda,k;\xi)\, d\mu^E(\xi), \tag{2.37}$$

where $d\mu^E$ denotes the Euclidean-Lebesgue measure on V^* and

$$\mathcal{I}(k)f(\xi) := (2\pi)^{-r} \int_{p_V + iV} f(\lambda) e^{\langle \xi, \lambda \rangle} \frac{d\mu^E(\operatorname{Im}\lambda)}{c_{\mathbb{H}}(-\lambda, k)}. \tag{2.38}$$

In (2.38), p_V is an element deep enough in V_-, the negative chamber of V, so that it satisfies $\langle \alpha, p_V \rangle < -|k|$ for all $\alpha \in \Psi_0^+$. In the case where $k < 0$, when we shift p_V to the origin, we must cross the singularities of the function $1/c_{\mathbb{H}}(-\lambda, k)$ along the hyperplanes $\{\lambda \in \mathbb{C} \mid \langle \alpha, \lambda \rangle = k\}$, and a residue calculus is needed. This was done in [36] and in [74]. In [101] a residue calculus for root system is also carefully developed.

2.4.1 The Residue Lemma

Let $(V, (\cdot, \cdot))$ be a real Euclidean r-dimensional vector space, $V_{\mathbb{C}}$ is its complexification, V^* its dual and $\langle \cdot, \cdot \rangle : V_{\mathbb{C}}^* \times V_{\mathbb{C}} \to \mathbb{C}$ the bilinear pairing. Suppose that ω is a rational r-form on $V_{\mathbb{C}}$ written as

$$\omega(\lambda) = \frac{P(\lambda)}{Q(\lambda)} d\mu^E(\operatorname{Im}\lambda),$$

in which P, Q are functions on $V_{\mathbb{C}}$ of the type

$$P(\lambda) = \prod_{m' \in \mathcal{M}'} (\langle \alpha_{m'}, \lambda \rangle + k_{m'}) \quad \text{and} \quad Q(\lambda) = \prod_{m \in \mathcal{M}} (\langle \alpha_m, \lambda \rangle + k_m), \tag{2.39}$$

with $\mathcal{M}, \mathcal{M}'$ finite index sets in which there are maps $\mathcal{M}, \mathcal{M}' \to V^* \times \mathbb{R}$, given by $m \mapsto (\alpha_m, k_m)$ and $m' \mapsto (\alpha_{m'}, k_{m'})$. Thus, the rational form ω defines an arrangement of affine hyperplanes on V, denoted $\mathcal{P} = \mathcal{P}(\omega)$ and given by

$$\mathcal{P} := \{H_m \subseteq V \mid m \in \mathcal{M}\} \cup \{H_{m'} \subseteq V \mid m \in \mathcal{M}'\},$$

such that the H_m are given by $H_m := \{\lambda \in V \mid \langle \alpha_m, \lambda \rangle + k_m = 0\}$, and similar for $H_{m'}$. Let $\mathcal{L}(\omega)$ be the lattice of intersections of elements from $\mathcal{P}$, ordered by inclusion. If $L \in \mathcal{L}(\omega)$, denote by V^L the vector part of L and define c_L as the element in L with shortest distance to the origin, called the **center** of L so that $L = c_L + V^L$. Denote by $\mathcal{C}(\omega) := \{c_L \mid L \in \mathcal{L}(\omega)\}$ the set of centers of elements in $\mathcal{L}(\omega)$. To each $L \in \mathcal{L}(\omega)$, let $L_{\mathbb{C}} := c_L + V_{\mathbb{C}}^L$ denote its complexification. The **tempered form** of a subspace $L \in \mathcal{L}(\omega)$ is given by $L^t := c_L + iV^L \subseteq L_{\mathbb{C}}$. For any $L \in \mathcal{L}(\omega)$, let $\mathcal{M}_L$, $\mathcal{M}'_L$ be defined by

$$\mathcal{M}_L := \{m \in \mathcal{M} \mid L \subseteq H_m\},$$

and similar for $\mathcal{M}'_L$. Define the **order of** ω **along** L as $i_L^\omega := |\mathcal{M}_L| - |\mathcal{M}'_L|$.

Definition 2.4.1. An affine subspace $L \in \mathcal{L}(\omega)$ is called ω-**residual** if it satisfies the condition

$$o_L^\omega := i_L^\omega - \operatorname{codim}(L) \geq 0. \tag{2.40}$$

Denote by $\mathcal{L}^\omega \subseteq \mathcal{L}(\omega)$ the set of ω-residual subspaces and by $\mathcal{C}^\omega \subseteq \mathcal{C}(\omega)$ the set of centers of elements $L \in \mathcal{L}^\omega$.

Denote the complement of the union of all affine hyperplanes $H \in \mathcal{P}$ by V_{reg}. Let $\gamma \in V_{\text{reg}}$. Consider a linear functional on the space of Paley-Wiener functions on $V_{\mathbb{C}}$ of the type $\mathfrak{X}_{V,\gamma} : PW(V_{\mathbb{C}}) \to \mathbb{C}$,

$$\mathfrak{X}_{V,\gamma}(f) = \int_{\gamma+iV} f(\lambda)\omega(\lambda). \tag{2.41}$$

The Residue Lemma [36, Lemma 3.1] (see also [74, Lemma 3.4]) states that:

Proposition 2.4.2 (Residue Lemma)**.** *There exists a unique collection of tempered distributions* $\{\mathfrak{X}_c \mid c \in \mathcal{C}^{\omega}\}$ *on* iV *such that*

(1) $\operatorname{supp}(\mathfrak{X}_c) \subset \cup\, iV^L$*, union over* $L \in \mathcal{L}^{\omega}$ *such that* $c_L = c$*,*

(2) $\mathfrak{X}_{V,\gamma}(f) = \sum_{c\in\mathcal{C}^{\omega}} \mathfrak{X}_c\Big(f(c+\ \cdot)\Big)$*, for all* $F \in PW(V_{\mathbb{C}})$*.*

The distributions $\{\mathfrak{X}_c \mid c \in \mathcal{C}^{\omega}\}$ are called **local contributions**.

2.4.2 Residual subspaces

To study the inversion operator (2.38) in the case $k < 0$, consider the rational r-forms on $V_{\mathbb{C}}$

$$\omega(\lambda) = \frac{d\mu^E(\operatorname{Im}\lambda)}{c_{\mathbb{H}}(-\lambda,k)} \quad \text{and} \quad \eta(\lambda) = \frac{d\mu^E(\operatorname{Im}\lambda)}{c_{\mathbb{H}}(-\lambda,k)c_{\mathbb{H}}(\lambda,k)}. \tag{2.42}$$

We define functionals on the space $PW(V_{\mathbb{C}})$ of Paley-Wiener functions by

$$X_{V,p_V}(f) := \int_{p_V+iV} f(\lambda)\omega(\lambda) \quad \text{and} \quad Y_{V,p_V}(f) := \int_{p_V+iV} f(\lambda)\eta(\lambda). \tag{2.43}$$

We will refer to them as the X- and Y-functionals of the Yang system. Note that for $f \in PW(V_{\mathbb{C}})^{W_0}$ and $\xi \in V_+^*$, we have $2\pi^r\mathcal{I}(k)f(\xi) = X_{V,p_V}(fe^{\xi})$, with $e^{\xi}(\lambda) = e^{\langle\xi,\lambda\rangle}$. These functionals define collections of affine hyperplanes $\mathcal{P}(\omega)$ and $\mathcal{P}(\eta)$ along which the rational forms ω and η of (2.42) have singularities or zeros. These singularities or zeros corresponds to the zeros or poles of the rational functions $c_{\mathbb{H}}(-\lambda,k)$ and $c_{\mathbb{H}}(-\lambda,k)c_{\mathbb{H}}(\lambda,k)$, respectively. These forms gives rise to collections of ω- and η-residual subspaces $\mathcal{L}^{\omega}$ and $\mathcal{L}^{\eta}$ as described in Definition 2.4.1.

We can therefore apply the Residue Lemma to study the functionals in (2.43). This was done in [36] and we will describe it in the next subsection as this same path will be the main tool to obtain our main result regarding the unramified spherical automorphic spectrum on Chapter 4, below. We will end this subsection with a more careful description of the set of η-residual subspaces, which will give rise to *the* set of residual subspaces that will be considered in this thesis.

Given $\alpha \in V^*$ and an affine subspace $L \subseteq V$, we will write $\langle\alpha, L\rangle = s$, for $s \in \mathbb{R}$, to indicate that α has the constant value s on L. An affine subspace of

V defines a subset $\Psi_L \subseteq \Psi_0$ consisting of all roots $\alpha \in \Psi_0$ that are constant on L. If we denote by $V_L \subseteq V$ the span of $\Psi_L^\vee$, then Ψ_L is a parabolic root subsystem of Ψ_0. We can thus write the condition (2.40) for a subspace $L \subseteq V$ to be η-residual as

$$o_L^\eta = |\{\alpha \in \Psi_0 \mid \langle \alpha, L\rangle = k\}| - |\{\alpha \in \Psi_0 \mid \langle \alpha, L\rangle = 0\}| - \operatorname{codim}(L) \geq 0.$$

However, from [74, Theorems 1.1 and 7.1] it holds that $o_L^\eta \leq 0$ and the inequality above is actually an equality. Due to its importance, we will highlight the definition of the η-residual subspaces and give it a special notation.

Definition 2.4.3. An affine subspace $L \subseteq V$ is called **residual**[5] if

$$|\{\alpha \in \Psi_0 \mid \langle \alpha, L\rangle = k\}| = |\{\alpha \in \Psi_0 \mid \langle \alpha, L\rangle = 0\}| + \operatorname{codim}(L). \tag{2.44}$$

We will denote the set of all residual subspaces by $\mathcal{L}$ and the set of centers of residual subspaces by $\mathcal{C}$.

The fact that $\mathcal{R}$ is semisimple implies the existence of singletons $\{c\} \subseteq V$ which are residual subspaces. They are called **residual points**. For example, when $k = 1$, the Weyl (co)vector $\varrho^\vee = \frac{1}{2}\sum_{\alpha\in\Psi_0} \alpha^\vee$ is a residual point, as one can write $\varrho^\vee = \sum_{\alpha\in\Pi_0} \omega_\alpha^\vee$, in which $\{\omega_\alpha^\vee \mid \alpha \in \Pi_0\} \subseteq P^\vee$ denote the set of fundamental coweights, that satisfy $\langle \beta, \omega_\alpha^\vee\rangle = \delta_{\alpha,\beta}$, for all $\alpha, \beta \in \Pi_0$. It follows that, if $\operatorname{ht}(\alpha)$ denotes the height of a root $\alpha \in \Psi_0$, then $\langle \alpha, \varrho^\vee\rangle = \operatorname{ht}(\alpha)$ for every $\alpha \in \Psi_0^+$, and condition (2.44) is readily verified. More generally, for any $k = (k_\alpha)_{\alpha\in\Psi_0}$, the vector $\varrho(k)^\vee = \frac{1}{2}\sum_{\alpha\in\Psi_0} k_\alpha \alpha^\vee$ is a residual point.

Remark. Note that V is trivially residual and its center is the origin. Further, the notion of residual subspace depends on (V, Ψ_0, k). In particular, the definition above implies that, if $L = c_L + V^L$ is a residual subspace of V, then c_L is a residual point with respect to (V_L, Ψ_L, k_L). To us, $k_L = k$ as we are in the equal parameter case, but in general we have $k_L = (k_\alpha)_{\alpha\in\Psi_L}$. Also, because each $H_{(\alpha,k)} \in \mathcal{P}(\eta)$ is residual, we have $\mathcal{L} \subseteq \mathcal{L}(\eta)$ and $\mathcal{C} \subseteq \mathcal{C}(\eta)$.

The set of residual subspaces enjoys the important property [36, Theorem 3.10]:

Proposition 2.4.4. *If $L = c_L + V^L$ is residual then $-c_L \in W(\Psi_L)c_L$.*

2.4.3 Results for the Yang system

When we apply the Residue Lemma to the functionals X_{V,p_V} and Y_{V,p_V} we obtain the collections $\{X_c \mid c \in \mathcal{C}^\omega\}$ and $\{Y_c \mid c \in \mathcal{C}^\eta\}$ of local contributions. Note that the index sets $\mathcal{M}$ and $\mathcal{M}'$ (see (2.39)) corresponding to η are both the set of roots Ψ_0 whereas for ω both $\mathcal{M}$ and $\mathcal{M}'$ are Ψ_0^+. Therefore, we have

[5]This definition differs from the one in [36]. The proof of the equivalence between the definitions is in [74, Corollary 7.2].

$\mathcal{C}(\omega) \subseteq \mathcal{C}(\eta)$. It follows that both $\mathcal{C}^\omega, \mathcal{C}^\eta$ are subsets of $\mathcal{C}(\eta)$. Thus, by declaring $X_c = 0$ and $Y_c = 0$ if $c \in \mathcal{C}(\eta)$ but not in $\mathcal{C}^\omega$ and $\mathcal{C}^\eta$, respectively, we can assume that both collections are indexed by $\mathcal{C}(\eta)$. We will state a precise formula linking the local contributions X_c and Y_c for $c \in \mathcal{C}(\eta)$, below. But for that we need more notation. Given any $c \in V$, let W_c denote the isotropy group of c and define the **averaging operator** $A(k)_c$ acting on meromorphic functions on $V_\mathbb{C}$ via

$$A(k)_c(f)(\lambda) := \frac{1}{|W_c|} \sum_{w \in W_c} c_\mathbb{H}(w\lambda, k) f(w\lambda). \tag{2.45}$$

These operators were introduced in [36, (3.10)] and satisfy the property that, if f is holomorphic on a tubular neighborhood $U + iV$ of $c + iV$, then $A(k)_c(f)$ is also holomorphic. Our interest is when $c \in \mathcal{C}(\eta)$. For example, when $c = 0$ (the center of the trivially residual subspace V) and for the exponential function e^ξ, for $\xi \in V^*$, we obtain

$$A(k)_0(e^\xi)(\lambda) = \frac{1}{|W_0|} \sum_{w \in W_0} c_\mathbb{H}(w\lambda, k) e^{\langle w\xi, \lambda \rangle},$$

which is the spherical function $\phi(\lambda, k; \xi)$. We have [36, Proposition 3.6]:

Proposition 2.4.5. *Let $c \in \mathcal{C}(\eta) \cap \overline{V_-}$. Then for all $w \in W_0$, it holds*

$$X_{wc} = Y_c \circ w^{-1} \circ A(k)_{wc}. \tag{2.46}$$

Further, for each residual subspace $L \in \mathcal{L}$, from $V = V_L \oplus V^L$, we let p_L denote the orthogonal projection of p_V onto V_L. We have a lower rank functional Y_{V_L, p_L} and we denote by Y_{Ψ_L, c_L} its local contribution at the (V_L, Ψ_L, k)-residual point $c_L \in V_L$.

Proposition 2.4.6. *For $c \in \mathcal{C}(\eta) \cap \overline{V_-}$, the local contribution Y_c of Y_{V,p_V} admits a decomposition*

$$Y_c(f(c + \cdot)) = \sum_{\{L \in \mathcal{L}(\eta) \mid c_L = c\}} Y_L(f(c + \cdot)), \tag{2.47}$$

where the functional Y_L is a positive measure on iV with support on iV^L that satisfies $Y_L = 0$ unless L is residual. More precisely, Y_L is defined by

$$Y_L(g) := Y_{\Psi_L, c_L}(\{c_L\}) \int_{\lambda^L \in iV^L} g(\lambda^L) d\mu^L(\operatorname{Im} \lambda^L),$$

for any $g \in PW(V_\mathbb{C})$, in which $Y_{\Psi_L, c_L}(\{c_L\})$ denotes the total mass of Y_{Ψ_L, c_L} and for $\lambda \in V^L$,

$$d\mu^L(\lambda) := \prod_{\alpha \in \Psi^+ \setminus \Psi_L} \frac{\langle \alpha, c_L \rangle^2 + \langle \alpha, \lambda \rangle^2}{(\langle \alpha, c_L \rangle - k)^2 + \langle \alpha, \lambda \rangle^2} \, d\mu^E(\lambda) \tag{2.48}$$

Moreover, in each W_0-orbit of residual subspaces there exists at least one element L for which $Y_{\Psi_L, c_L}(\{c_L\}) > 0$.

Proof. This is [36, Theorem 3.13]. The moreover is [74, Theorem 6.1(A)]. □

Corollary 2.4.7. *The local contributions of the functionals X and Y of (2.43) satisfy $X_c = 0 = Y_c$ unless $c = c_L$ with L residual.*

Remark. We remark that if we denote by $c^L_{\mathbb{H}}(\lambda, k)$ the product of the factors $\gamma_\alpha(\lambda)$ for $\alpha \in \Psi_0^+ \setminus \Psi_L$, then the definition of the measure $d\mu^L(\lambda)$ in (2.48) is the working out of the expression $\frac{d\mu^E(\operatorname{Im}\lambda)}{c^L_{\mathbb{H}}(\lambda,k)c^L_{\mathbb{H}}(-\lambda,k)}$, for $\lambda \in L^t$, as was done in the proof of [36, Theorem 3.13].

Definition 2.4.8. For any residual $L \in \mathcal{L}$, we define the measure ν_L as the unique measure on $V_{\mathbb{C}}$ with support contained in L^t characterized by the equation $\int_{L^t} f \; d\nu_L = (2\pi)^{-r} Y_L(f(c_L + \cdot))$ for $c_L \in \overline{V_-}$, $f \in PW(V_{\mathbb{C}})$ and the requirement that

$$\nu_{W_0 L} := \sum_{L' \subseteq W_0 L} \nu_{L'} \tag{2.49}$$

is a W_0-invariant measure. We define the **Plancherel measure of the Yang system** to be $\nu_P := \sum_{L \in \mathcal{L}} \nu_L$.

It follows that if $L \in \mathcal{L}$, and $c = c_L$ is its center, then the measure ν_L is the push-forward of a positive measure on iV^L via the translation map $\lambda \mapsto c + \lambda$. More precisely, if $\lambda = c + i\lambda^L \in L^t$, we have

$$d\nu_L(\lambda) = C_{W_0 L} d\mu^L(\lambda^L), \tag{2.50}$$

with $d\mu^L$ as in (2.48).

Remark. From the objects $V^*, (\cdot,\cdot), V, \langle\cdot,\cdot\rangle, \mathcal{R}$ and W_0, we stress that what was essential for the Yang system so far was V^*, Φ_0, W_0 and $(\cdot,\cdot)$. The lattices of the root datum were not used. However, these lattices will be important in the computation of the constants $C_{W_0 L} > 0$, as we will see later in Chapter 3.

The last facts of the survey of [36] that we highlight are:

Proposition 2.4.9. *Suppose that the parameter $k < 0$.*

(1) Any $f \in C_c^\infty(V^*_{\mathrm{reg}})^{W_0}$ is decomposed as a superposition of spherical waves

$$f(\cdot) = \sum_{L \in \mathcal{L}} \int_{\lambda \in L^t} \left\{ \int_{\xi \in V^*} f(\xi)\phi(-\lambda, k; \xi) \, d\mu^E(\xi) \right\} \phi(\lambda, k; \cdot) \, d\nu_L(\lambda).$$

(2) The Fourier-Yang transform yields a unitary injection of Hilbert spaces

$$\mathcal{F}(k) : L^2(V^*, k)^{W_0} \to L^2(V_{\mathbb{C}}, \nu_P)^{W_0} = \left(\bigoplus_{L \in \mathcal{L}} L^2(L^t, \nu_L) \right)^{W_0}.$$

(3) If c is a residual point and $\nu_{\{c\}} \neq 0$, then $\phi(c,k;\cdot) \in L^2(V^*,k)$ and

$$\sum_{d \in W_0 c} \nu_{\{d\}}(\{d\}) = \frac{1}{\|\phi(c,k;\cdot)\|^2}.$$

These results are, respectively, [36, Theorem 1.5, (3.28) and Theorem 3.20]. We will adapt the result (2) to the case of the unramified spherical automorphic spectrum later on Chapter 4.

2.5 Bala-Carter theory

Details on the discussion about unipotent conjugacy classes in a connected reductive complex algebraic group can be found in [16], [45] and [24]. All the groups in this subsection will be defined over $\mathbb{C}$. We will write G instead $\underline{G}$. This discussion can be reduced to the case in which G is semisimple and whose center Z_G is trivial, as explained in [16, Proposition 5.1.1]:

Proposition 2.5.1. *The natural homomorphism $G \to G/Z_G$ induces a bijection between the unipotent conjugacy classes of G and those of G/Z_G.*

Further, in view of the bijection between unipotent conjugacy classes of G and nilpotent $\mathrm{Ad}(G)$-orbits (a result of Springer-Steinberg, see [28, Chapter E]) on the Lie algebra $\mathfrak{g}$, we can focus on the latter. So consider G a semisimple adjoint group over $\mathbb{C}$ and let B, T be a fixed Borel subgroup and maximal torus of G, respectively. Denote by $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$ the corresponding Lie algebras over $\mathbb{C}$ and let $\mathfrak{n}$ denote the nilpotent radical of $\mathfrak{b}$. Let $\Phi_0 = \Phi(G,T)$ be the set of roots of T on $\mathfrak{g}$, Φ_0^+ the choice of positive roots corresponding to B and $\Delta_0 \subseteq \Phi_0^+$ the corresponding set of simple roots. We thus have a decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \overline{\mathfrak{n}}, \tag{2.51}$$

where $\overline{\mathfrak{n}}$ is the nilpotent radical of the Borel subalgebra opposite to $\mathfrak{b}$.

Recall that a nilpotent element $n \in \mathfrak{g}$ is called **distinguished** if it does not commute with any non-zero semisimple element of $\mathfrak{g}$ and that a parabolic Lie subalgebra $\mathfrak{p} \supseteq \mathfrak{b}$ is called **distinguished** if $\dim \mathfrak{l} = \dim(\mathfrak{u}/[\mathfrak{u}\ \mathfrak{u}])$, in which $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ is a Levi decomposition of $\mathfrak{p}$. The Bala-Carter theorem (cf. [16, Theorem 5.9.5]) states that there is a bijective correspondence between

$$\left\{ \begin{array}{c} \text{Distinguished nilpotent} \\ \mathrm{Ad}(G)\text{-orbits of } \mathfrak{g} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} G\text{-conjugacy classes of distinguished} \\ \text{parabolic subalgebras of } \mathfrak{g} \end{array} \right\}, \tag{2.52}$$

and also one between

$$\left\{ \begin{array}{c} \text{Nilpotent} \\ \mathrm{Ad}(G)\text{-orbits of } \mathfrak{g} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} G\text{-conjugacy classes} \\ \text{of pairs } (\mathfrak{m}, \mathfrak{p}) \text{ of } \mathfrak{g} \end{array} \right\}, \tag{2.53}$$

in which $\mathfrak{m}$ is a Levi factor of a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{p} \subseteq \mathfrak{m}'$ is a distinguished parabolic subalgebra of the semisimple part of $\mathfrak{m}$. We will denote a nilpotent orbit or a unipotent conjugacy class both by the symbol $\mathbf{o}$.

This is well-known, but we discuss it to set some notation. Roughly, the correspondences goes as follows. Given a non-zero nilpotent element $n \in \mathfrak{g}$, let $\{e, h, f\}$ denote the standard basis of the $\mathfrak{sl}_2$ Lie algebra. One may regard $\mathfrak{g}$ as an $\mathfrak{sl}_2$-module via a Jacobson-Morozov Lie algebra homomorphism $\varphi : \mathfrak{sl}_2 \to \mathfrak{g}$ which, modulo conjugation, satisfies $\varphi(e) = n \in \mathfrak{n}$ and $\varphi(h) = \gamma$ is in the dominant chamber of $\mathfrak{t}$. It is a theorem of Dynkin ([26, Theorem 8.3] see also [44, Proposition 7.6]) that, in this case, $\langle \alpha, \gamma \rangle \in \{0, 1, 2\}$ for every simple root $\alpha \in \Delta_0$. Such an element $\gamma = \phi(h)$ will be called a **weighted Dynkin diagram**. By $\mathfrak{sl}_2$-theory, this yields a grading $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\ \mathfrak{g}(i)$, in which $\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid \mathrm{ad}(\gamma)(x) = ix\}$, $[\mathfrak{g}(i)\ \mathfrak{g}(j)] \subseteq \mathfrak{g}(i+j)$ and $n \in \mathfrak{g}(2)$. All this said, let

$$\begin{aligned} \mathfrak{p} = \mathfrak{p}(\gamma) &= \oplus_{i\geq 0}\ \mathfrak{g}(i) \\ \mathfrak{u} &= \oplus_{i>0}\ \mathfrak{g}(i) \\ \mathfrak{l} &= \mathfrak{g}(0). \end{aligned} \tag{2.54}$$

The Lie subalgebra $\mathfrak{p}$ contains $\mathfrak{b}$ and is thus a parabolic subalgebra whose Levi decomposition is $\mathfrak{p} = \mathfrak{l}\oplus\mathfrak{u}$. On the other hand, starting with a standard parabolic subalgebra $\mathfrak{p}_J$, for $J \subseteq \Delta_0$, let $\eta_J : \Phi_0 \to \mathbb{Z}$ be a function defined first on Δ_0 as twice the indicator function of J and then extended linearly to all roots. Then, one also gets a grading

$$\mathfrak{g} = \bigoplus_{i\geq 0} \mathfrak{g}_J(i), \tag{2.55}$$

by declaring $\mathfrak{g}_J(0) = \mathfrak{t} \oplus \sum_{\eta_J(\alpha)=0} \mathfrak{g}_\alpha$ and otherwise $\mathfrak{g}_J(i) = \sum_{\eta_J(\alpha)=i} \mathfrak{g}_\alpha$. In this situation, $\mathfrak{p}_J = \oplus_{i\geq 0}\ \mathfrak{g}_J(i)$ and its nilpotent radical is $\mathfrak{n}_J = \oplus_{i>0}\ \mathfrak{g}_J(i)$. We have (cf. [16], Corollaries 5.7.5, 5.8.3, 5.8.9 and Proposition 5.7.6)

Proposition 2.5.2. *The standard parabolic subalgebra $\mathfrak{p}_J$ is distinguished if and only if $\dim \mathfrak{g}_J(0) = \dim \mathfrak{g}_J(2)$. In this case, if n is any element in the unique open orbit of the parabolic subgroup P_J on its nilpotent radical $\mathfrak{n}_J$, then the parabolic subalgebra associated to n as in (2.54) equals to $\mathfrak{p}_J$.*

Proposition 2.5.3. *A nilpotent element $n \in \mathfrak{g}$ is distinguished if and only if $\dim \mathfrak{g}(0) = \dim \mathfrak{g}(2)$. Moreover, if n is distinguished then $\dim \mathfrak{g}(1) = 0$.*

Assume n distinguished. Proposition 2.5.3 implies that the parabolic $\mathfrak{p}$ is a standard parabolic $\mathfrak{p}_J$ for $J = \{\alpha \in \Delta_0 \mid \mathfrak{g}_\alpha \subseteq \mathfrak{g}(2)\}$, which is distinguished by Proposition 2.5.2, so the map

$$n \mapsto \mathfrak{p},$$

induces the first bijective correspondence. Note that the assignment of $\mathfrak{p}$ is equivalent to datum of an *even* weighted Dynkin diagram $\gamma = \phi(h)$ for the semisimple algebra $\mathfrak{g}$.

As for the second bijection, one can choose a minimal Levi subalgebra $\mathfrak{m}$ containing n (cf. [16, Proposition 5.9.3]), which modulo conjugation, can be assumed to be a Levi factor of a parabolic subalgebra containing $\mathfrak{b}$. By minimality of $\mathfrak{m}$, it follows that $n \in \mathfrak{m}' = [\mathfrak{m}\ \mathfrak{m}]$ is a distinguished nilpotent element of $\mathfrak{m}'$, so that, by (2.52), there is a distinguished parabolic $\mathfrak{p} \subseteq \mathfrak{m}'$ corresponding to n. We can construct a map on one direction induced by

$$n \mapsto (\mathfrak{m}, \mathfrak{p}).$$

On the other direction, one associates to a conjugacy class of the pair $(\mathfrak{m}, \mathfrak{p})$ the orbit $\mathrm{Ad}(G)n$, in which $n \in \mathfrak{n}_{\mathfrak{p}}$ is any element of the unique dense adjoint orbit of P on $\mathfrak{n}_{\mathfrak{p}}$, with the latter being the nilradical of $\mathfrak{p}$ and P the parabolic subgroup of G associated to $\mathfrak{p}$.

2.5.1 Nilpotent orbits and residual subspaces

Residual subspaces were discussed in Section 2.4 (see Definition 2.4.1). They turn out to be intimately related to nilpotent orbits. Let $\underline{G}$ be a Chevalley group (semisimple) and let $\underline{T} \subseteq \underline{B}$ be a maximal split torus and a Borel subgroup. We get a root datum $\mathcal{R}(\underline{G}, \underline{B}, \underline{T})$ as in Section 2.2. By reversing the roles of $X^*(\underline{T})$ and $X_*(\underline{T})$, we obtain a new root datum, $\mathcal{R}^\vee = (X_*(\underline{T}), \Delta_0^\vee, X^*(\underline{T}), \Delta_0)$. Let $({}^LG, {}^LB, {}^LT)$ be a triple consisting of a semisimple group, a Borel subgroup and a maximal torus, all defined over $\mathbb{C}$, such that $\mathcal{R}^\vee = \mathcal{R}({}^LG, {}^LB, {}^LT)$. The group ${}^LG$ is the so-called **dual group**[6]. Denote the respective Lie algebras by ${}^L\mathfrak{g}, {}^L\mathfrak{b}$ and ${}^L\mathfrak{t}$. Consider the objects $V^*, (\cdot,\cdot), V, \langle\cdot,\cdot\rangle, \mathcal{R}^\vee$, and W_0 as in Section 2.3.1, with $\Phi_0^\vee \subseteq V^*$ (note that $\Phi_0^\vee$ takes the role of Ψ_0) and where the W_0-invariant inner products may be obtained from the Killing form. Observe that from ${}^LT = \mathbb{C}^\times \otimes X_*({}^LT) \cong \mathbb{C}^\times \otimes X^*(\underline{T})$, the commutative Lie algebra ${}^L\mathfrak{t}$ is identified with $\mathbb{C} \otimes X^*(\underline{T}) = V_{\mathbb{C}}$. Fix the parameter $k_\alpha = 1$ for all $\alpha \in \Phi_0$. We have:

Proposition 2.5.4. *(a) There is a bijective correspondence* $\mathbf{o}_{W_0L} \leftrightarrow W_0L$ *between the set of nilpotent orbits of the Langlands dual Lie algebra* ${}^L\mathfrak{g}$ *and the set of* W_0*-orbits of* $(V, \Phi_0^\vee, 1)$*-residual subspaces.*

(b) There is a bijective correspondence between the set of W_0*-orbits of residual subspaces and the set of* W_0*-orbits of centers* $c = c_L$ *of residual subspaces. In particular, distinct* W_0*-orbits of tempered residual subspaces are disjoint.*

[6] In the literature, this group is often denoted by $G^\vee$. When the field of definition comes to play, ${}^LG$ typically means the semidirect product $\Gamma \ltimes G^\vee$, with Γ the finite Galois group of some extension E/F. In the split case, these two coincide.

(c) *For each residual $L \subseteq V$ with center c_L there exists a Lie algebra homomorphism $\varphi : \mathfrak{sl}_2 \to {}^L\mathfrak{g}$ with $\varphi(e) = n \in \mathbf{o}_{W_0 L}$ and $\varphi(h) = 2c_L$, with $\{e, h, f\}$ the standard basis of $\mathfrak{sl}_2$.*

(d) *If $c = c_L$ then $\langle \alpha^\vee, 2c\rangle \in \mathbb{Z}$ for all $\alpha \in \Phi_0$, and $\langle \alpha^\vee, c\rangle \in \mathbb{Z}$ for all $\alpha \in \Phi_L$.*

(e) *If we choose $c = c_L$ in its Weyl group orbit so that c is dominant, then $2c$ is the weighted Dynkin diagram associated to $\mathbf{o}_{W_0 L}$.*

Proof. This is rather well-known. It is discussed with details in [73, Appendices A and B] (see also [74, Remark 7.5] and [38, Proposition 6.2]). ☐

The content of items (a) and (e) of the previous Proposition will be useful later, so we give some details of the correspondence. The idea is to attach, to each $\mathrm{Ad}({}^LG)$-conjugacy class of pairs $({}^L\mathfrak{m}, {}^L\mathfrak{p})$ a W_0-orbit of a residual subspace of V and vice-versa. First, let $({}^L\mathfrak{m}, {}^L\mathfrak{p})$ be a representative of a class, for which ${}^L\mathfrak{m} = {}^L\mathfrak{g}$ and ${}^L\mathfrak{p} \subseteq {}^L\mathfrak{g}$ is standard parabolic and distinguished. We have a corresponding distinguished nilpotent orbit $\mathbf{o}$. As discussed around equation (2.54), the data of ${}^L\mathfrak{p}$ is equivalent to the assignment of an even weighted Dynkin diagram

$$\gamma := 2\lambda(\mathbf{o}). \tag{2.56}$$

Since we have $\dim \mathfrak{g}(0) = \mathrm{rk}({}^LT) + |\{\alpha \in \Phi_0 \mid \langle \alpha^\vee, 2\lambda(\mathbf{o})\rangle = 0\}|$ and also $\dim \mathfrak{g}(2) = |\{\alpha \in \Phi_0 \mid \langle \alpha^\vee, 2\lambda(\mathbf{o})\rangle = 2\}|$, it follows from Proposition 2.5.3 that $\lambda(\mathbf{o}) \in V$ is a $(V, \Phi_0^\vee, 1)$-residual point. One notes that the definition of $\lambda(\mathbf{o})$ depends on the choice of positive roots induced by ${}^LB$. A different choice yields a different element on the same W_0-orbit. The upshot is that for $\mathbf{o}$ distinguished

$$\mathbf{o} \mapsto W_0\lambda(\mathbf{o}), \tag{2.57}$$

fixes the bijective correspondence in the case of distinguished orbits. Next, let $({}^L\mathfrak{m}, {}^L\mathfrak{p})$ be a representative an $\mathrm{Ad}({}^LG)$-conjugacy class of pairs as in (2.53) corresponding to a nilpotent orbit $\mathbf{o}$. Note that we likewise have a conjugacy class of the pair $({}^LM, {}^LP)$ of subgroups. We will typically interchange between these two notions. By suitable conjugation, we assume that $V_\mathbb{C} = {}^L\mathfrak{t} \subseteq {}^L\mathfrak{m}$. The Levi subalgebra defines a root subsystem $\Phi_M \subseteq \Phi_0$, corresponding to the Levi subgroup $\underline{M} \subseteq \underline{G}$ which, in its turn, correspond to ${}^L\mathfrak{m}$ and ${}^LM \subseteq {}^LG$. We fix the set of simple roots $\Delta_M \subseteq \Phi_M$ induced by the Borel subalgebra ${}^L\mathfrak{t} \cap {}^L\mathfrak{m}$. Let V_M be the subspace spanned by Φ_M. Its orthogonal complement V^M is such that $V_\mathbb{C}^M$ is the center of ${}^L\mathfrak{m}$, and the distinguished parabolic ${}^L\mathfrak{p}$ of $[{}^L\mathfrak{m}\ {}^L\mathfrak{m}]$ determines an even weighted Dynkin diagram $2\lambda_M(\mathbf{o}) \in V_M$. We can thus define an affine subspace

$$L := \lambda_M(\mathbf{o}) + V^M, \tag{2.58}$$

which is a $(V, \Phi_0^\vee, 1)$-residual subspace (as mentioned before, details can be found in [73, Appendix B]). We note again that the definition of $\lambda_M(\mathbf{o})$ depended on the choice of Δ_M. Any other choice yield a different element on the same $W(\Phi_M)$-orbit, where $W(\Phi_M) \subseteq W_0$ is the Weyl group of Φ_M. Since V^M is perpendicular to every root in Φ_M, this subspace is pointwise fixed by $W(\Phi_M)$. Now, the orbit $\mathbf{o}$ is associated to the whole conjugacy class of $({}^L\mathfrak{m}, {}^L\mathfrak{p})$. The upshot is that for a nilpotent orbit $\mathbf{o}$, the assignment

$$\mathbf{o} \mapsto W_0 L = W_0(\lambda_M(\mathbf{o}) + V^M) \tag{2.59}$$

fixes the correspondence.

Remark. As mentioned in the beginning of this section, the assumption $\underline{G}$ semisimple is not essential for this discussion. When $\underline{G}$ is reductive, and in (2.58) for $\underline{M} = \underline{G}$, or equivalently, when ${}^L\mathfrak{m} = {}^L\mathfrak{g}$, the space V^G is spanned by the lattice of characters of the center of $\underline{G}$. In this case, we will omit the index when indicating the weighted Dynkin diagram. For example, we would simply write $L = \lambda(\mathbf{o}) + V^G$. For $\underline{G}$ semisimple $V^G = \{0\}$.

In the setting of Proposition 2.5.4, it will be convenient to fix, once and for all, a set of representatives for the ${}^LG$-conjugacy classes of pairs as in (2.52), (2.53) as well as a set of representatives of the W_0-orbit of residual subspaces. We assume that each representative $({}^LM, {}^LP)$ satisfies ${}^LT \subseteq {}^LM$ and that the center $c = c_L$ of the representative L of the W_0-orbit of residual subspace is in the dominant chamber. In this way, we have bijections

$$\mathbf{o} \leftrightarrow ({}^LM_{\mathbf{o}}, {}^LP_{\mathbf{o}}) \leftrightarrow L_{\mathbf{o}}. \tag{2.60}$$

Before we move on, we return to the problem of determining when the module $\mathcal{U}_-(s)$ discussed in Proposition 2.3.3 is a discrete series. Using the language of nilpotent orbits and the celebrated ideas of Kazhdan and Lusztig [48, Section 8], which are of fundamental importance to us. In the following proposition, $\{e, h, f\}$ is the standard basis of $\mathfrak{sl}_2$. We have the following:

Proposition 2.5.5. *Let $\mathbf{o}$ be a distinguished nilpotent orbit and $s_{\mathbf{o}} := q^{-\lambda(\mathbf{o})}$. Then, the unique anti-spherical module $\mathcal{U}_-(s)$ is a discrete series if and only if $W_0 s = W_0 s_{\mathbf{o}}$.*

Proof. Given a distinguished nilpotent orbit $\mathbf{o}$, from Proposition 2.5.4, there is an $\mathfrak{sl}_2$-homomorphism such that $\varphi(h) = 2\lambda(\mathbf{o})$. We then have a homomorphism $\phi : SL_2 \to {}^LG$ such that $\phi\left(\begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}\right) = s_{\mathbf{o}} \in {}^LT$. Let $M \subseteq {}^LG$ be the connected centralizer of $s_{\mathbf{o}}$, and let $\mathfrak{q}$ be the q^{-1}-eigenspace of $s_{\mathbf{o}}$ in ${}^L\mathfrak{g}$ ($\mathfrak{q}$ is the space $\mathfrak{g}(2)$ of (2.54)). Because $s_{\mathbf{o}}$ is antidominant, we have $\mathfrak{q} \subseteq {}^L\mathfrak{n}$. Notice that M acts on $\mathfrak{q}$. Let $\mathfrak{q}_0$ be the unique dense M-orbit in $\mathfrak{q}$. Since in our case $\mathbf{o}$ is distinguished, Richardson's Dense Orbit Theorem [16, Section 5.2] implies that $\mathbf{o} \cap \mathfrak{q} = \mathfrak{q}_0$. It then follows that $n := \varphi(e) \in \mathfrak{q}_0$, and by composing ϕ with elements $\mathrm{Ad}(m)$ for $m \in M$, we assume n is any element of $\mathfrak{q}_0$.

Now let $\lambda^\vee \in V$ be any dominant weight of ${}^L G$. We must show the condition of Definition 2.3.4. Let $(\pi_{\lambda^\vee}, X_{\lambda^\vee})$ the irreducible highest weight representation of ${}^L\mathfrak{g}$ with highest weight vector $v_{\lambda^\vee}$ of weight $\lambda^\vee$. M. Reeder (Proposition 2.3.3) tells us exactly for which $w \in W_0$ we much test Casselman's criterion for discrete series. Given w as in (2.14), consider the $\mathfrak{sl}_2$-module $\pi_w := \pi_{\lambda^\vee} \circ \mathrm{Ad}(w) \circ \phi : \mathfrak{sl}_2 \to \mathfrak{gl}(X_{\lambda^\vee})$. Since we may assume $n = \varphi(e) \in \mathrm{Ad}(w^{-1})\mathfrak{n}$, it follows that the vector $v_{\lambda^\vee}$ is a highest weight vector for π_w. But then it follows, from $\mathfrak{sl}_2$-representation theory, that $\langle \lambda^\vee, w\lambda(\mathbf{o})\rangle \geq 0$. We have therefore that $\mathcal{U}_-(s_\mathbf{o})$ is a tempered module, according to Definition 2.3.4. But, from [25, Theorem 3.2.2], the central character of a tempered module, if not a discrete series, is contained in a residual subspace of positive dimension. Since, for any residual subspace $M \neq L$ we have $M^t \not\subseteq L^t$ [74, Theorem 6.1(C)], (and $\{\lambda(\mathbf{o})\}$ is residual) it follows that $\mathcal{U}_-(s_\mathbf{o})$ is a discrete series. For the converse, we refer to [73, Lemma 3.29]. There it is shown that the central character of any discrete series must correspond to a residual coset[7]. □

With the theory of Bala-Carter in place, we also have enough information to explain the set P^{su} appearing in the statement of Theorem A of Section 1.2. We recall that ${}^L\mathfrak{t} = \mathbb{C} \otimes X^*(\underline{T}) = V_\mathbb{C}$ which in the Introduction we called $a^*_{T,\mathbb{C}}$. In what follows, we will be concerned with the Lie algebra aspect, as well as the residual subspace aspect. So we will not use the notation $a^*_{T,\mathbb{C}}$. We will write ${}^L\mathfrak{t}_\mathbb{R}$, for the real form $\mathbb{R} \otimes X^*(\underline{T})$, so that ${}^L\mathfrak{t} = {}^L\mathfrak{t}_\mathbb{R} + i({}^L\mathfrak{t}_\mathbb{R})$.

We recall that we have fixed a Cartan involution θ and a corresponding maximal compact subgroup ${}^L K \subseteq {}^L G$ in such a way that $i({}^L\mathfrak{t}_\mathbb{R})$ is the Lie algebra of a maximal torus of a compact form of ${}^L\mathfrak{g}$. We then have that θ acts as the identity on $i({}^L\mathfrak{t}_\mathbb{R})$ and as minus the identity on ${}^L\mathfrak{t}_\mathbb{R}$. Also, P^{su} consists of the ${}^L G$-conjugacy classes of pairs $\{(\phi, \lambda)\}$ satisfying the properties (1) $\phi : \mathfrak{sl}_2 \to {}^L\mathfrak{g}$ is a Lie algebra homomorphism, (2) $\lambda \in {}^L\mathfrak{g}$ is a semisimple element, (3) $[\lambda, \phi(x)] = 0$ for all $x \in \mathfrak{sl}_2$, (4) $\theta(\lambda) = \lambda$ and (5) $\theta \circ \phi \circ \theta_{\mathfrak{sl}_2} = \phi$, where $\theta_{\mathfrak{sl}_2}$ is the Cartan involution $x \mapsto -x^T$ of $\mathfrak{sl}_2$. In the next proposition, $\cup_\mathbf{o} W_0 L^t_\mathbf{o} \subseteq V_\mathbb{C}$. We will write $(\cup_\mathbf{o} W_0 L^t_\mathbf{o})/W_0 \subseteq V_\mathbb{C}$ as well, by which we mean that we have chosen a fundamental domain for the action of W_0. We have:

Proposition 2.5.6. *The set P^{su} is in bijection with $(\cup_\mathbf{o} W_0 L^t_\mathbf{o})/W_0 \subseteq V_\mathbb{C}$, with $L_\mathbf{o}$ as in (2.60). In particular, $P^{\mathrm{su}}_{\mathrm{dist}}$ and $\{\lambda(\mathbf{o}) \mid \mathbf{o}$ is distinguished$\}$ are in a bijective correspondence.*

Proof. The in particular is immediate from the main statement in view of the discussion around (2.57). As for the statement, we will only sketch the ideas.

[7] The language in [73], [74] and [25] is in terms of residual cosets, while we refer here to residual subspaces. But we can use these results in our setting.

First, the ${}^LG$ action is by means of $g \cdot (\phi, \lambda) = (\mathrm{Ad}(g) \circ \phi, \mathrm{Ad}(g)\lambda)$. Given a pair (ϕ, λ) satisfying (1) – (5), we would like to associate its ${}^LG$-conjugacy class with a W_0-orbit of residual subspaces. Our point is that with conditions (1) – (5), it can be shown that by only using conjugations with respect to ${}^LK$ (we use arguments of [24, Section 9.4]), we can bring (ϕ, λ) to the form (ϕ', λ') where:

(a) there exists a Levi subalgebra ${}^L\mathfrak{m} \subseteq {}^L\mathfrak{g}$ containing $\phi'(\mathfrak{sl}_2)$, and minimal with this property,

(b) ${}^L\mathfrak{t}_\phi = {}^L\mathfrak{m} \cap {}^L\mathfrak{t}$ is the center of ${}^L\mathfrak{m}$,

(c) $\phi'(h/2) \in \mathfrak{t}_\mathbb{R}$ and $\lambda \in i({}^L\mathfrak{t}_\mathbb{R}) \cap {}^L\mathfrak{t}_\phi$.

Granted a representative of ${}^LG \cdot (\phi, \lambda)$ as above, the bijection is induced by the map

$${}^LG \cdot (\phi, \lambda) \mapsto W_0(\phi'(h/2) + \lambda').$$

Condition (a) implies $\phi'(h/2)$ is the weighted Dynkin diagram of a distinguished orbit of ${}^L\mathfrak{m}'$ (the semisimple part of ${}^L\mathfrak{m}$). So it is the center of a residual subspace L. Condition (b) then says that $\lambda' \in V_\mathbb{C}^L$ and by (c), it is actually in iV^L (recall that ${}^L\mathfrak{t}_\mathbb{R} = V$). □

Example 2.5.7. In a rank one case (say, $\underline{G} = SL_2$), we have $V = \mathbb{R}$, the root system is $\{-1, 1\} \subseteq V^*$ and the action of the Weyl group sends $\lambda \mapsto -\lambda$. The coroots are $\{-2, 2\} \subseteq V$. With respect to condition (2.44), there are only two orbits of residual subspaces of V. Namely, V itself and the orbit of $\{1\} \subseteq V$. This is half the weighted Dynkin diagram of the unique nonzero unipotent orbit. Therefore, in this case:

$$\begin{aligned} \cup_\mathbf{o} W_0 L_\mathbf{o}^t &= i\mathbb{R} \cup \{-1, 1\} \subseteq \mathbb{C} \\ (\cup_\mathbf{o} W_0 L_\mathbf{o}^t)/W_0 &= i\mathbb{R}_{\geq 0} \cup \{1\} \subseteq \mathbb{C}. \end{aligned}$$

2.5.2 More Weyl groups

We retain the notations used in this section so far. We will discuss some Weyl groups that are associated to the objects occurring in the second and third column of (2.60). We start by discussing the pair $({}^LM_\mathbf{o}, {}^LP_\mathbf{o})$. To ease the notations, we will first drop the index $\mathbf{o}$. To define the **Weyl group of** $({}^LM, {}^LP)$, denoted $W({}^LM, {}^LP)$, the fact that ${}^LG$ is the dual group of $\underline{G}$ is immaterial. So suppose that $(\mathcal{G}, \mathcal{B}, \mathcal{T})$ is any triple consisting of a reductive group, a Borel and a maximal torus over $\mathbb{C}$ and that $(\mathcal{M}, \mathcal{P})$ is a pair as in the Bala-Carter correspondence (2.52) and (2.53). We define

$$W(\mathcal{M}, \mathcal{P}) := \{w \in \mathrm{Norm}_\mathcal{G}(\mathcal{M})/\mathcal{M} \mid w\mathcal{P}w^{-1} \sim_\mathcal{M} \mathcal{P}\}, \tag{2.61}$$

in which $\sim_{\mathcal{M}}$ means that the groups are conjugated by an element in $\mathcal{M}$. Next, given a residual subspace $L = c_L + V^L \subseteq V$, we denote by $\mathrm{Stab}(L)$ and $\mathrm{Fix}(L)$ the subgroups of W_0 consisting of those elements $w \in W_0$ for which $w(L) = L$ and those that fixes L pointwise, respectively. Note that if $\Phi_L \subseteq \Phi_0$ denotes the set of roots whose coroots are constant on L and $W(\Phi_L)$ is its Weyl group, we have $\mathrm{Fix}(L) = W(\Phi_L) \cap W_{c_L}$. We define the **Weyl group of** L to be

$$W(L) := \mathrm{Stab}(L)/\,\mathrm{Fix}(L). \tag{2.62}$$

Assuming that $({}^LM, {}^LP) \leftrightarrow L$ in (2.60), we wish to relate the Weyl groups $W(L)$ and $W({}^LM, {}^LP)$.

Lemma 2.5.8. *If $L = c_L + V^L$, let $\mathrm{Stab}(V^L, W(\Phi_L)c_L)$ denote the subgroup of those $w \in W_0$ such that $w(V^L) = V^L$ and $wc_L \in W(\Phi_L)c_L$. Then, we have an isomorphism $W(L) \cong \mathrm{Stab}(V^L, W(\Phi_L)c_L)/W(\Phi_L)$.*

Proof. The injection $\mathrm{Stab}(L) \hookrightarrow \mathrm{Stab}(V^L, W(\Phi_L)c_L)$ induces a well-defined isomorphism between the two quotients. □

Proposition 2.5.9. *The group $W(L)$ is isomorphic to $W({}^LM, {}^LP)$.*

Proof. In the proof, we will omit the superscript ^{L} and write ${}^LG = \mathcal{G}$, ${}^LM = \mathcal{M}$ and so on (we also omit the superscript ^{L} from the respective Lie algebras). Denote by $Z_{\mathcal{M}} \subseteq \mathcal{G}$ the center of $\mathcal{M}$. Note that $\mathrm{Norm}_{\mathcal{G}}(\mathcal{M})/\mathcal{M}$ is equal to $\mathrm{Norm}_{\mathcal{G}}(Z_{\mathcal{M}})/\,\mathrm{Cent}_{\mathcal{G}}(Z_{\mathcal{M}})$ and that

$$\begin{aligned} \mathrm{Norm}_{\mathcal{G}}(\mathcal{M})/\mathcal{M} &\cong \mathrm{Norm}_{\mathcal{G}}(\mathcal{M},\mathcal{T})/\,\mathrm{Norm}_{\mathcal{M}}(\mathcal{T}) \\ &\cong \{w \in W_0 \mid w\mathcal{M}w^{-1} = \mathcal{M}\}/W(\Phi_{\mathcal{M}}), \end{aligned} \tag{2.63}$$

in which $W(\Phi_{\mathcal{M}}) := \mathrm{Norm}_{\mathcal{M}}(\mathcal{T})/\mathcal{T}$. Decompose the Cartan subalgebra $\mathfrak{t} = V_{\mathbb{C}}$ as $V_{\mathcal{M},\mathbb{C}} \oplus V_{\mathbb{C}}^{\mathcal{M}}$, with $V_{\mathbb{C}}^{\mathcal{M}}$ being the Lie algebra of $Z_{\mathcal{M}}$, as mentioned in the text between equations (2.57) and (2.58). Then, the parabolic $\mathcal{P}$ defines a dominant (with respect to the positive subsystem induced by $\mathcal{B} \cap \mathcal{M}$) element $\gamma \in V_{\mathcal{M}}$ with the property that, if $\Phi_{\mathcal{M}}^{\mathcal{P}} := \{\alpha \in \Phi_{\mathcal{M}} \mid \langle \alpha^\vee, \gamma\rangle \geq 0\}$, the Lie algebra of $\mathcal{P}$ can be written as $\mathfrak{p} = V_{\mathcal{M},\mathbb{C}} \oplus \sum_{\alpha \in \Phi_{\mathcal{M}}^{\mathcal{P}}} \mathfrak{g}_{\alpha^\vee}$ (see (2.54)). We claim that:

$$W(\mathcal{M},\mathcal{P}) = \{w \in \mathrm{Norm}_{\mathcal{G}}(\mathcal{M})/\mathcal{M} \mid w\gamma \in W(\Phi_{\mathcal{M}})\gamma\}. \tag{2.64}$$

Indeed, given $w \in \mathrm{Norm}_{\mathcal{G}}(\mathcal{M})/\mathcal{M}$, under (2.63), we can assume that w is represented by an element $\dot{w} \in W_0$ that stabilizes $\Phi_{\mathcal{M}}, V_{\mathcal{M}}$ and $V^{\mathcal{M}}$. One then checks that for $v \in W(\Phi_{\mathcal{M}})$ we have $\dot{w}\Phi_{\mathcal{M}}^{\mathcal{P}} = v\Phi_{\mathcal{M}}^{\mathcal{P}}$ if and only if $\dot{w}\gamma = v\gamma$, yielding the equality of the sets defined in (2.61) and (2.64). Finally, using that $\gamma = 2c_L$ and that $W(\Phi_{\mathcal{M}}) = W(\Phi_L)$, the result follows from Lemma 2.5.8. □

2.6 Global fields

A **number field** is a finite extension of the rational numbers $\mathbb{Q}$ and a **function field in one variable over** k is a field extension of k of transcendence degree one [82, Section 4.4]. By definition, a **global field** is either a number field or a finitely generated function field in one variable over $k = \mathbb{F}_q$, where the latter is the finite field with q elements. In this thesis, more emphasis is given to number fields and the only function field that will be considered is $F = \mathbb{F}_q(t)$, which is viewed as the field of rational functions over the projective line $\mathbb{P}^1$ over $\mathbb{F}_q$.

2.6.1 Zeta function of a number field

Recall that the Dedekind zeta function of a number field F is defined for $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$ by (see, for example, [69, Definition (5.1)])

$$\zeta(s) = \sum_{I \subseteq \mathfrak{o}_F} \frac{1}{(N_{F/\mathbb{Q}}(I))^s},$$

where I ranges over all the non-zero ideals in the ring of integers $\mathfrak{o}_F$ of F. It is well known that this function has meromorphic continuation to the whole complex plane with only a single pole at $s = 1$. The celebrated class number formula states that if $[F : \mathbb{Q}] = r_1 + 2r_2$ with r_1 the number of real embeddings of F, r_2 the number of pairs of complex embeddings, h is the class number, R is the regulator of F, w is the number of roots of unity in F and d is the discriminant of the extension $F/\mathbb{Q}$ then

$$\operatorname{res}_{s=1} \zeta(s) = \frac{2^{r_1}(2\pi)^{r_2} hR}{\sqrt{|d|}w}. \tag{2.65}$$

Let $\Gamma(s)$ denote the usual gamma function and define the gamma factors at infinity by

$$\Gamma_{\mathbb{F}}(s) = \begin{cases} \pi^{-s/2}\Gamma(s/2), & \text{if } \mathbb{F} = \mathbb{R} \\ 2\pi(2\pi)^{-s}\Gamma(s), & \text{if } \mathbb{F} = \mathbb{C}. \end{cases} \tag{2.66}$$

Remark. For the gamma factor at the complex place, the literature is not uniform. We have followed [103, Ch. VII, §6, Lemma 8] and [98, pp.318-319]. This is the reason for the extra factor of 2π. In [69, p.456], the gamma factor is π^{-1} times what we wrote here and in [82, Section 7.1, p.251], although this gamma factor is not actually defined, their computations for the local zeta functions yield $(2\pi)^{-1}\Gamma_{\mathbb{C}}(s)$.

Define the **completed Dedekind zeta function** by

$$\Lambda(s) = |d|^{s/2}\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta(s). \tag{2.67}$$

Then, the functional equation satisfied by the Dedekind zeta function can be expressed as $\Lambda(s) = \Lambda(1 - s)$. The completed Dedekind zeta has meromorphic continuation to the whole complex plane with simple poles at $s = 0$ and $s = 1$. Its residues are given by

$$\operatorname{res}_{s=1} \Lambda(s) = \frac{2^{r_1}(2\pi)^{r_2} hR}{w}, \tag{2.68}$$

and $\operatorname{res}_{s=0} \Lambda(s) = -\operatorname{res}_{s=1} \Lambda(s)$.

Remark. Note that as far as the functional equation $\Lambda(s) = \Lambda(1-s)$ is concerned, we could have multiplied the gamma factor $\Gamma_{\mathbb{F}}(s)$ by any nonzero constant and still the functional equation would hold. Different constants does, on the other hand, change the definition of $\Lambda(s)$ and the value of (2.68). However, for the applications we have in mind in Chapter 4, we will be interested in quotients of $\Lambda(s)$ (see Corollaries 4.1.6 and 4.1.7).

We will also need the regularized version of the completed Dedekind zeta function at $s = 0$ and $s = 1$

$$\rho(s) := s(s-1)\Lambda(s), \tag{2.69}$$

which has the advantage of being holomorphic. The following proposition summarizes the analytic properties of the function $\rho(s)$:

Proposition 2.6.1. *The function $\rho(s)$ satisfies the following properties:*

(a) $\rho(s)$ is an entire function,

(b) $\rho(1) = \operatorname{res}_{s=1} \Lambda(s)$,

(c) $\rho(s)$ has zeros only for $0 < \operatorname{Re}(s) < 1$,

(d) $\rho(s)$ satisfies the functional equation $\rho(s) = \rho(1-s)$,

(e) $\rho(s)$ is at most of polynomial growth as $|t| \to \infty$ on vertical lines $\sigma + it$,

(f) $\rho(s)^{-1}$ is at most of polynomial growth on vertical lines $\sigma + it$ for $\sigma \geq 1$.

Proof. The analytic properties of items (e) and (f) use the rapid decay of the gamma function on vertical lines and the estimates for $\zeta(s)$ and $\zeta(s)^{-1}$ near the line $\sigma = 1$, see [95] and [47, Section 10.6]. □

2.6.2 Adèles

The purpose of this subsection is to recall some important facts about the ring of adèles of a global field and fix some notation. We refer the details to [103] and [82]. Given a global field F, we let $\mathcal{V}$ denote the set of places of F. This set is subdivided into the set of Archimedean and non-Archimedean places which will be denoted by $\mathcal{V}_\infty$ and $\mathcal{V}_f$, respectively. When $F = \mathbb{Q}$, the set of places is characterized by Ostrowski's Theorem as either being a non-Archimedean p-adic place or the usual norm of $\mathbb{Q}$, which is an Archimedean place. When F is a function field, every place is non-Archimedean. To every place v, we will denote by F_v the completion of F as a metric space. For a number field, the Archimedean completions are either $\mathbb{R}$ or $\mathbb{C}$ and the non-Archimedean are $\mathfrak{p}$-adic fields.

In this thesis, unless otherwise stated, we will only consider **normalized absolute values**. For a non-Archimedean v, this means that if $\mathfrak{o}_v$ denote the ring integers of F_v and we let ϖ_v denote a fixed generator of the unique maximal ideal of $\mathfrak{o}_v$, then the absolute value is such that $|\varpi_v|_v = q_v^{-1}$, where $q_v = [\mathfrak{o}_v : \varpi_v\mathfrak{o}_v]$. For an Archimedean place, when v is complex, we have that $|x|_v = x\overline{x}$, the square of the usual absolute value of a complex number and for v real, $|\cdot|_v$ is the usual absolute value of $\mathbb{R}$.

Each local completion F_v of F is a locally compact topological ring and for each finite subset $S \subseteq \mathcal{V}$ with $\mathcal{V}_\infty \subseteq S$, we define $\mathbb{A}_S := \prod_{v\in S} F_v \times \prod_{v\notin S} \mathfrak{o}_v$. For $S \subseteq T$ we get an embedding $\iota_{ST} : \mathbb{A}_S \to \mathbb{A}_T$ and the ring of adèles $\mathbb{A}$ of F is defined as

$$\mathbb{A} := \varinjlim_{S} \mathbb{A}_S$$

the limit of directed system $\{\mathbb{A}_S, (\iota_{ST})\}$ of topological rings, with S ranging in the set of subsets of $\mathcal{V}$ that contain $\mathcal{V}_\infty$. As a set, we have that

$$\mathbb{A} = \left\{ (x_v)_v \in \prod_v F_v \mid x_v \in \mathfrak{o}_v \text{ for all, but finitely many } v \in \mathcal{V} \right\}. \tag{2.70}$$

Endowed with the direct limit topology, we have

Proposition 2.6.2. *The ring $\mathbb{A}$ is locally compact, $F \subseteq \mathbb{A}$ diagonally as a discrete subset and $\mathbb{A}/F$ is compact.*

From the nature of the adèle group there are two relevant ways to normalize the Haar measure. One could prescribe local measures in a certain way and piece them together to give a global measure, or one could normalize the Haar measure so that the volume of the fundamental domain for the action of F is one. Denote the latter by ν. For the former, to each place v of F, fix Haar measures μ_v according to (see [103, Ch.V, §4]):

$$\begin{array}{rcll} \mu_v(\mathfrak{o}_v) &=& 1, & v \text{ non-Archimedean}, \\ d\mu_v &=& dx, & v \text{ real}, \\ d\mu_v &=& |dx \wedge d\overline{x}|, & v \text{ complex}. \end{array} \tag{2.71}$$

Here, the measure on the real place is the usual Lebesgue measure and in the complex places it is twice the usual Lebesgue measure. Let $\mu := \prod_v \mu_v$ be the measure on $\mathbb{A}$ given by the local measures as in (2.71). The comparison between these measure on $\mathbb{A}$ is:

Proposition 2.6.3. *Let d be the discriminant of F. Then $\mu = |d|^{1/2}\nu$.*

Proof. In broad strokes, fix a $\mathbb{Z}$-basis $\{\xi_1, \dots, \xi_n\}$ of $\mathfrak{o}_F$, the ring of integers of F. Then, this is also a basis of the $\mathbb{R}$-vector space $F_\infty = \prod_{v \in \mathcal{V}_\infty} F_v$, where the latter is the product over all Archimedean places of F. Next, consider the identification $\theta : \mathbb{R}^n \to F_\infty$ that sends $(x_1, \dots, x_n) \mapsto \sum_i x_i \xi_i$. Using it, one shows that the set $\theta([0,1]^n) \times \prod_{v \in \mathcal{V}_f} \mathfrak{o}_v \subseteq \mathbb{A}$ is a fundamental domain modulo F. Then, one computes

$$\mu\left(\theta([0,1]^n) \times \prod_{v \in \mathcal{V}_f} \mathfrak{o}_v\right) = |d|^{1/2},$$

completing the proof. The details are in [103, Proposition 7, Ch.V, §4]. □

We stress that the factor 2 present in the definition of the local measure μ_v for v complex is essential. With the standard Lebesgue measure in each factor one computes the volume of the set $\theta([0,1]^n) \subseteq F_\infty$ to be $2^{-r_2}|d|^{1/2}$ (see [98, Lemma 4.1.4]).

Example 2.6.4. Here, $\mu_\infty := \prod_{v \in \mathcal{V}_\infty} \mu_v$. Let $a \in \mathbb{Z}$ be square-free. It is well-known that for quadratic extensions $F = \mathbb{Q}(\sqrt{a})$, one has $d = a$ if $a \equiv 1 \mod 4$ and $d = 4a$ otherwise. Also, the ring of integers has basis $\{1, (1+\sqrt{a})/2\}$ if $a \equiv 1 \mod 4$ and $\{1, \sqrt{a}\}$ otherwise. For $F = \mathbb{Q}(i), a = -1$, we have $d = -4$. The ring of integers has an integral basis given by $\{1, i\}$. It is clear that with the usual Lebesgue measure, the volume of $\theta([0,1]^2) = 1$, thus, $\mu_\infty(\theta([0,1]^2)) = 2 = |d|^{1/2}$. For $F = \mathbb{Q}(\sqrt{2}), a = 2$, we have $d = 8$ and the image of the basis $\{1, \sqrt{2}\}$ in F_∞ is $\{(1,1), (\sqrt{2}, -\sqrt{2})\}$. Hence, $\mu_\infty(\theta([0,1]^2)) = 2\sqrt{2}$.

One can be explicit about the local measures of $\nu = \prod_v \nu_v$. Namely, the local measures ν_v are given by (see [98, Section 2])

$$\begin{aligned} \nu_v(\mathfrak{o}_v) &= (N\mathfrak{d}_v)^{-1/2}, && v \text{ non-Archimedean}, \\ \nu_v &= \mu_v, && v \text{ Archimedean}. \end{aligned} \tag{2.72}$$

Here, $N\mathfrak{d}_v$ is the norm of the different ideal $\mathfrak{d}_v$ of the local place v. We refer the reader to [103, Ch. VIII] for details on the different ideal. An important feature is that $\prod_{v \in \mathcal{V}_f} N\mathfrak{d}_v = |d|$ (see the Corollary to [103, Ch. VII, §4, Proposition 12]).

2.6.3 Idèles

For each place $v \in \mathcal{V}$, consider $F_v^\times \subseteq F_v$, the multiplicative subgroup of F_v. Similar to the additive case, these are locally compact groups so we can assign Haar measures. Fix a Haar measure α_v on the additive group F_v, and define

$$\alpha_v^\times := m_v \frac{\alpha_v}{|x|_v}, \tag{2.73}$$

where $m_v = 1$ for v Archimedean and $m_v := \frac{q_v}{q_v - 1}$ for v non-Archimedean. In this way, we have $\alpha_v^\times(\mathfrak{o}_v^\times) = \alpha_v(\mathfrak{o}_v)$ for all $v \in \mathcal{V}_f$, where $\mathfrak{o}_v^\times$ denotes the units of $\mathfrak{o}_v$.

Similar to the adèles, for each finite subset $S \subseteq \mathcal{V}$ with $\mathcal{V}_\infty \subseteq S$, we define the rings $\mathbb{I}_S := \prod_{v \in S} F_v^\times \times \prod_{v \notin S} \mathfrak{o}_v^\times$ and embeddings $\iota_{ST} : \mathbb{I}_S \to \mathbb{I}_T$, whenever $S \subseteq T$. The ring of idèles $\mathbb{I}$ is then defined as the limit of the directed system $\{\mathbb{I}_S, (\iota_{ST})\}$ of topological rings, with S ranging in the set of subsets of $\mathcal{V}$ that contain $\mathcal{V}_\infty$. Also, it is well-known that there is an algebraic isomorphism $\mathbb{A}^\times \cong \mathbb{I}$ between the units of the ring of adèles and the idèles, but this algebraic isomorphism is not topological [82, Section 5.2]. We will, however, view $\mathbb{I} \subseteq \mathbb{A}$.

Likewise the adèles, $\mathbb{I}$ is a locally compact topological ring in which $F^\times \subseteq \mathbb{I}$ diagonally as a discrete subset, but the quotient is not compact. Nevertheless, there is an important compact subgroup of this quotient which we now describe. First, the quotient $C_F := \mathbb{I}/F^\times$ is called the **idèle class group**. The normalized absolute value in each place yields an absolute value on $\mathbb{A}$ via

$$|x| = \prod_v |x_v|_v, \tag{2.74}$$

for all $x = (x_v)_v \in \mathbb{A}$. When restricted to $\mathbb{I}$, we have:

Proposition 2.6.5 ([82], Theorem 5.14)**.** *The morphism* $|\cdot| : \mathbb{I} \to \mathbb{R}_+$ *satisfies:*

(a) it is surjective if F is a number field and of the form $q^{\mathbb{Z}}$ if F is a function field with q a power of the characteristic of F,

(b) $|x| = 1$ for all $x \in F^\times$.

The kernel of $|\cdot| : \mathbb{I} \to \mathbb{R}_+$ is denoted by $\mathbb{I}^1$, and in view of item (b) of Proposition 2.6.5, which is known in the literature as Artin's product formula, we define the subgroup $C_F^1 := \mathbb{I}^1/F^\times$, called the **norm-one idèle class group** of F. The groups C_F and C_F^1 sit inside the exact sequence

$$1 \to C_F^1 \to C_F \to V(\mathbb{I}) \to 0$$

in which, from Proposition 2.6.5(a), $V(\mathbb{I})$ is either isomorphic to $\mathbb{R}_+$ or to the lattice $q^{\mathbb{Z}}$. Moreover, when $\mathbb{I}$ is equipped with the Haar measure $\nu^\times = \prod_v \nu^\times$, as in (2.73) for $\alpha_v = \nu_v$, we have that C_F^1 satisfies the properties:

Proposition 2.6.6. *The topological group C_F^1 is compact. When F is a number field, its volume is*

$$\mathrm{vol}(C_F^1) = \mathrm{res}_{s=1}\,\zeta(s),$$

where $\zeta(s)$ is the Dedekind zeta function of the number field F.

Proof. See [103, Proposition 9, Ch.V, §4]. In his proof, A. Weil computes with $\mu_v^\times$ (that assigns $\mathfrak{o}_v^\times$ volume one, for $v \in \mathcal{V}_f$) instead of $\nu_v^\times$ so the result there is $|d|^{1/2}$ times the result stated here. $\square$

2.7 Global harmonic analysis

Our interest in this thesis is to study unramified **K**-spherical automorphic forms for split connected groups over a number field F, where $\mathbf{K}$ is a maximal compact of $\underline{G}(\mathbb{A})$. This situation is a particular case of a bigger framework laid down by Langlands and explained in details in [67]. The aim of this section is to summarize this setting and, in special, state the decomposition theorem (6.2) mentioned in the Introduction, as well as the inner product formula for pseudo-Eisenstein series that will be the starting point for the results obtained in Chapter 4.

Let $\underline{G}$ be a connected reductive algebraic group defined over a global field F and let $\mathbb{A}$ be the ring of adèles of F. For each place v of F we denote by F_v the corresponding completion of F at v and $G_v = \underline{G}(F_v)$ its group of F_v-points. For a non-Archimedean v, we let $\mathfrak{o}_v$ denote the ring of integers of F_v and $\mathfrak{p}_v = \varpi_v \mathfrak{o}_v \subseteq \mathfrak{o}_v$ denotes the unique maximal ideal of $\mathfrak{o}_v$ in which ϖ_v is a uniformizer element. It is known that except for a finite set of places S_b (including the Archimedean ones) the group G_v is defined over $\mathfrak{o}_v$ [92, Lemma 4.9]. In any case, at each place v we fix a maximal compact $K_v \subseteq G_v$ and with $K_v = \underline{G}(\mathfrak{o}_v)$ for all $v \notin S_b$. In what follows, let

$$\mathbf{K} := \prod_v K_v \tag{2.75}$$

be the maximal compact subgroup of $\underline{G}(\mathbb{A})$ for the K_v as described above.

Denote by $\underline{Z}_G$ the center of $\underline{G}$. Following [55], fix a closed subgroup $Z_0 \subseteq \underline{Z}_G(\mathbb{A})$ such that $Z_0\underline{Z}(F)$ is closed and $Z_0\underline{Z}(F)\backslash\underline{Z}(\mathbb{A})$ is compact. For example, one could take $Z_0 = \underline{Z}_G(\mathbb{A})$ or, if $\underline{Z}_G^\circ$ is the identity component of $\underline{Z}_G$, one could take $Z_0 = \underline{Z}_G^\circ(\mathbb{A})$, according to [8, Proposition 1.9], which states that for a reductive group $\underline{H}$ over a number field, $\underline{H}^\circ(\mathbb{A})\backslash\underline{H}(\mathbb{A})$ is compact. Fix a unitary character ω of Z_0 trivial on $Z_0 \cap \underline{Z}_G(F)$ and Haar measures on $\underline{G}(F), \underline{G}(\mathbb{A})$ and on the quotient.

Definition 2.7.1. The space $L^2(\underline{G}(F)Z_0\backslash\underline{G}(\mathbb{A}))_\omega$ consists of all measurable functions φ on $\underline{G}(F)\backslash\underline{G}(\mathbb{A})$ satisfying $\varphi(zg) = \omega(z)\varphi(g)$ and

$$\int_{\underline{G}(F)Z_0\backslash\underline{G}(\mathbb{A})} |\varphi(g)|^2 \, dg < \infty.$$

The space just defined is a Hilbert space in which $\underline{G}(\mathbb{A})$ acts by the right regular representation, R. As mentioned in the Introduction, an important problem in the area is to decompose the space $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))$ in terms of irreducible representations of $\underline{G}(\mathbb{A})$. We are therefore faced with the problem of decomposing $L^2(\underline{G}(F)Z_0\backslash\underline{G}(\mathbb{A}))_\omega$ for various ω and Z_0. Our goal in this section is to state a result for the spectral decomposition of $L^2(\underline{G}(F)Z_0\backslash\underline{G}(\mathbb{A}))_\omega$ for a fixed ω and for $Z_0 = \underline{Z}_G(\mathbb{A})$. For semisimple groups, such decomposition was first obtained by Langlands in [55]. The standard reference for this part is the first two chapters of [67]. Before we continue, we state the following useful result about the decomposition of unitary $\underline{G}(\mathbb{A})$-representations, due to L. Clozel. A proof can be found in [22, Theorem A.1].

Proposition 2.7.2. *The group $\underline{G}(\mathbb{A})$ is of type I.*

This result asserts that, for example, a decomposition of $L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\omega$ as a Hilbert integral exists and is unique.

We fix $\underline{P}_0 \subseteq \underline{G}$ a minimal parabolic of $\underline{G}$ and we let $\underline{M}_0\underline{U}_0$ be its Levi decomposition, with all three being defined over F. By a **standard parabolic**, we mean a parabolic $\underline{P} \subseteq \underline{G}$, over F, containing $\underline{P}_0$. Similarly, a **standard Levi** is a Levi subgroup of $\underline{G}$, over F, with $\underline{M}_0 \subseteq \underline{M}$. Up until 2.7.9, we assume that we have fixed an arbitrary standard parabolic subgroup $\underline{P} \subseteq \underline{G}$ with Levi decomposition $\underline{P} = \underline{M}\,\underline{U}$ with $\underline{M}_0 \subseteq \underline{M}$.

2.7.1 Unramified global characters

Recall that $X^*(\underline{M}_0)$ denotes the group of F-rational characters of $\underline{M}_0$. Let $a_0^* = \mathbb{R} \otimes X^*(\underline{M}_0)$. If $\underline{T}_0$ denotes the maximal split torus in the center of $\underline{M}_0$, the restriction from $\underline{M}_0$ to $\underline{T}_0$ induces an isomorphism $a_0^* \cong \mathbb{R} \otimes X^*(\underline{T}_0)$. We let $\Phi_0 = \Phi(\underline{T}_0, \underline{G}) \subseteq a_0^*$ be the roots of $\underline{G}$ relative to $\underline{T}_0$, Φ_0^+ is the set of positive roots determined by $\underline{P}_0$ and Δ_0 is the corresponding set of simple roots. The space spanned by Φ_0 will be denoted $(a_0^G)^*$. Via the restriction from $\underline{G}$ to $\underline{M}_0$, we view $a_G^* = \mathbb{R} \otimes X^*(\underline{G})$ as a subspace of a_0^* and we have a decomposition

$$a_0^* = a_G^* \oplus (a_0^G)^*. \tag{2.76}$$

We let W_0 be the Weyl group of $\underline{T}_0$, which is identified with the Weyl group of the root system $\Phi \subseteq a_0^*$. We fix a W_0-invariant inner product on a_0^* with respect to which (2.76) is orthogonal.

More generally, for the given standard Levi $\underline{M}$, we let $\underline{T}_M$ be the maximal F-split torus in the center of $\underline{M}$ and denote $\Phi_M = \Phi(\underline{T}_M, \underline{G})$ the roots of $\underline{G}$ with respect to the $\underline{T}_M$, which does not form a root system, in general. We let $\Phi_M^+ = \Phi_M \cap \Phi_0^+$ and we denote by ϱ_P half of the sum of all roots in Φ_M^+. We write ϱ_0 instead of ϱ_{P_0}, and when $\underline{M} = \underline{G}$, this vector is trivial. By restriction from $\underline{M}$ to $\underline{T}_M$, we identify Φ_M with a subset of $a_M^* = \mathbb{R} \otimes X^*(\underline{M})$, and the space it generates is denoted by a_M^{G*}. Similarly, from the restriction from $\underline{G}$ to $\underline{M}$, there is an inclusion of $a_G^* \subseteq a_M^*$ and we thus obtain a decomposition

$$a_M^* = a_G^* \oplus a_M^{G*}. \tag{2.77}$$

The inner product of a_0^* restricts to an inner product of a_M^* so that the decomposition (2.77) is orthogonal. We will denote the complexified spaces $\mathbb{C} \otimes a_M^*$ as $a_{M,\mathbb{C}}^*$. We thus have a decomposition $a_{M,\mathbb{C}}^* = a_{G,\mathbb{C}}^* \oplus a_{M,\mathbb{C}}^{G*}$.

For any character $\chi \in X^*(\underline{M})$ let χ_v be the corresponding homomorphism $\underline{M}(F_v) \to F_v^*$ and define $|\chi|$ to be the continuous homomorphism $\underline{M}(\mathbb{A}) \to \mathbb{R}_+$ that, for all $m \in \underline{M}(\mathbb{A})$ assigns

$$m^{|\chi|} = \prod_v |m_v^{\chi_v}|_v, \tag{2.78}$$

where we are using the notation of [67] that the evaluation of a character χ on an element m is written m^χ.

Definition 2.7.3. The **space of unramified characters of** $\underline{M}$, X_M, is defined as the group of all continuous characters $\underline{M}(\mathbb{A}) \to \mathbb{C}^\times$ that are trivial on $\underline{M}(\mathbb{A})^1 := \cap_{\chi \in X^*(\underline{M})} \ker|\chi|$. The group X_M^G is the subgroup of all continuous characters of $\underline{M}(\mathbb{A})^1 \backslash \underline{M}(\mathbb{A})$ which are trivial on the center $\underline{Z}_G(\mathbb{A})$. The groups $\operatorname{Re} X_M$ and $\operatorname{Re} X_M^G$ are the respective subgroups of real-valued characters.

There is a surjective group homomorphism $\kappa : a_{M,\mathbb{C}}^* \to X_M$ given by: if $\lambda = \sum_i s_i \otimes \chi_i \in a_{M,\mathbb{C}}^*$ and $m \in \underline{M}(\mathbb{A})$, set

$$\kappa(\lambda)(m) = \prod_i m^{s_i|\chi_i|}. \tag{2.79}$$

When restricted to $a_{M,\mathbb{C}}^{G*}$, the homomorphism $\kappa : a_{M,\mathbb{C}}^{G*} \to X_M^G$ is bijective in the number field case and, in the function field case, the kernel is a group of the form $(2\pi i/\log q)\Lambda$, with $\Lambda \subseteq a_M^{G*}$ a lattice. In both cases, when restricted to a_M^*, κ yields an isomorphism $a_M^* \cong \operatorname{Re} X_M$ and $\operatorname{Re} X_M = \operatorname{Re} X_G \oplus \operatorname{Re} X_M^G$, similar to (2.77). We also make the definitions

$$\operatorname{Im} X_M := \kappa(ia_M^*) \quad \text{and} \quad \operatorname{Im} X_M^G := \kappa(ia_M^{G*}). \tag{2.80}$$

2.7.2 About Haar measures

Recall that we have fixed a minimal parabolic $\underline{P}_0 = \underline{M}_0\underline{U}_0$ (defined over F). Following [67, I.1.13] we will assume we have Haar measures du, dk and dm on $\underline{U}_0(\mathbb{A})$, $\mathbf{K}$ and on $\underline{M}_0(\mathbb{A})$, such that the measures of $\underline{U}_0(F)\backslash\underline{U}_0(\mathbb{A})$ and of $\mathbf{K}$ is one. We do not specify the measure dm yet. The condition on $\underline{U}_0$ holds for any unipotent $\underline{U}'$. From the Iwasawa decomposition $\underline{G}(\mathbb{A}) = \underline{P}_0(\mathbb{A})\mathbf{K}$, we get a Haar measure dg on $\underline{G}(\mathbb{A})$ via

$$f \mapsto \int_{\underline{U}_0(\mathbb{A})} \int_{\underline{M}_0(\mathbb{A})} \int_{\mathbf{K}} f(umk) m^{-2\varrho_0}\, dk\, dm\, du,$$

for any continuous and compactly supported function on $\underline{G}(\mathbb{A})$. For our fixed standard parabolic $\underline{P} = \underline{M}\,\underline{U}$, we have a similar formula with $\underline{U}_0, \underline{M}_0$ and ϱ_0 replaced by $\underline{U}, \underline{M}$ and ϱ_P.

2.7.3 Automorphic forms

Suppose for a moment that F is a number field. Recall that we can write $\underline{G}(\mathbb{A}) = G_\infty \times G_f$, with G_∞ be the product of G_v over all Archimedean places and G_f the product over all non-Archimedean places (we similarly write H_∞ and H_f, such that $H = H_\infty H_f$, for any subgroup $H \subseteq \underline{G}(\mathbb{A})$). From [9, 4.1], we may view G_∞ as the Lie group $(R_{F/\mathbb{Q}}\underline{G})(\mathbb{R})$, where $R_{F/\mathbb{Q}}\underline{G}$ is the algebraic group over $\mathbb{Q}$ obtained by the restriction of scalars functor. We let $\mathrm{Lie}(G_\infty)$ be the real Lie algebra of G_∞, $\mathfrak{g}_\infty = \mathrm{Lie}(G_\infty) \otimes \mathbb{C}$ and $\mathcal{U}$ is the enveloping algebra of $\mathfrak{g}_\infty$. The group G_∞ and $\mathcal{U}$ acts on smooth functions on $\underline{G}(\mathbb{A})$ by the right regular action, R. Given an element $g \in \underline{G}(\mathbb{A})$, write $g = g_\infty g_f$ with $g_\infty \in G_\infty$ and $g_f \in G_f$.

Back to the situation where F is any global field, let $\phi : \underline{G}(\mathbb{A}) \to \mathbb{C}$ be a function. Recall that, if F is a function field, ϕ is **smooth** if it is locally compact and if F is a number field, ϕ is smooth if it is C^∞ in its Archimedean coordinate and locally constant in the non-Archimedean coordinate. In both cases it is said to be of **moderate growth** if there exists constants $C > 0$ and $r \geq 0$ such that

$$|\phi(g)| \leq C\|g\|^r,$$

where $\|g\|$ denote the height of an element $g \in \underline{G}(\mathbb{A})$ [67, Section I.2.2]. Further, ϕ is said to be **K-finite** if the vector space spanned by $\{R(k)\phi \mid k \in \mathbf{K}\}$ is finite dimensional. In the number field case, we let $\mathcal{Z} \subseteq \mathcal{U}$ be the center of the universal enveloping algebra (see [9] for the details and for the function field case). With the notations above, a function $\phi : \underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}) \to \mathbb{C}$ is called an **automorphic form for** $\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A})$ if (see [67, I.2.17]):

(i) ϕ is smooth,

(ii) ϕ is of moderate growth,

(iii) ϕ is **K**-finite,

(iv) there is an ideal $J \subseteq \mathcal{Z}$ of finite codimension that annihilates ϕ.

We denote by $A(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))$ the space of all automorphic forms for $\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A})$. If ω is a character of $\underline{Z}_M(\mathbb{A})$ which is trivial on $\underline{Z}_M(F)$, we will call it an **automorphic character** of $\underline{Z}_M$. Fix an automorphic character of $\underline{Z}_M$. The subspace of all ϕ in $A(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))$ with the extra property

(v) $\phi(zg) = z^{\varrho_P}\omega(z)\phi(g)$, for all $z \in \underline{Z}_M(\mathbb{A})$ and $g \in \underline{G}(\mathbb{A})$,

is denoted $A(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\omega$. A function for which (v) is satisfied will be said to **have central character** ω.

Remark. Note that in the particular case in which $\underline{M} = \underline{G}$ (so $\underline{U}$ is trivial), we obtain the definition of an automorphic form for $\underline{G}(F)\backslash\underline{G}(\mathbb{A})$. Similarly, we have a definition of automorphic forms for $\underline{M}(F)\backslash\underline{M}(\mathbb{A})$.

If F is a function field, there is an action of $\underline{G}(\mathbb{A})$ by right translations on both $A(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))$ and $A(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\omega$. If F is a number we do not have an action of $\underline{G}(\mathbb{A})$. Let K_∞ and K_f be such that $\mathbf{K} = K_\infty K_f$. In the number field case, we only have an action of $G_f \times (\mathfrak{g}_\infty, K_\infty)$ (we refer to [102] or [14] for the notion of $(\mathfrak{g}, K)$-modules). The problem is that K_v-finiteness is, in general, not preserved in the Archimedean places. To put this in a slightly more precise way (see [23, Lecture 2, Section 3]), let $\phi \in A(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))$ and put $\phi' = R(g')\phi$, with $g' \in \underline{G}(\mathbb{A})$. Then, ϕ is $\mathbf{K}$-finite, while ϕ' is naturally $\mathbf{K}'$-finite, where $\mathbf{K}' = g'\mathbf{K}(g')^{-1}$ is a conjugate of $\mathbf{K}$. Now, at the non-Archimedean places, since K_f and K'_f are both compact and open, the intersection $K_f \cap K'_f$ is of finite index in both. So there is no difference between K_f-finiteness and K'_f-finiteness. On the other hand, at the Archimedean places the intersection of K_∞ and K'_∞ can be very small, even consist only of the identity element. In any case, there is an action of $M_f \times (\mathfrak{m}_\infty, M_\infty \cap \mathbf{K})$ on the spaces $A(\underline{M}(F)\backslash\underline{M}(\mathbb{A}))$ and $A(\underline{M}(F)\backslash\underline{M}(\mathbb{A}))_\omega$, in the number field case, and one of $\underline{M}(\mathbb{A})$ on the function field case.

If we prescribe the data of δ, a finite dimensional representation of K_∞, $C \subseteq K_f$ a compact open, $J \subseteq \mathcal{Z}$ an ideal of finite codimension and ω an automorphic character of $\underline{Z}_M(\mathbb{A})$, let $A(\delta, C, J)_\omega$ be the subspace of all automorphic forms $\phi \in A(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\omega$ subject to this data, that is, $R(k_\infty)\phi = \delta(k_\infty)\phi$, $R(c)\phi = \phi$ for $c \in C$, ϕ is annihilated by J as well as (v), above. We have, in this setting (see [67, p.38]), that the following result of Harish-Chandra [33, Theorem 1] (see also [9]) holds:

Proposition 2.7.4. *The dimension of $A(\delta, C, J)_\omega$ is finite.*

Definition 2.7.5. The isomorphism class, with respect to the relevant action, depending if F is a function field or a number field, of an irreducible subquotient E of $A(\underline{M}(F)\backslash\underline{M}(\mathbb{A}))$ is called an **automorphic representation of $\underline{M}$**. An irreducible subspace $E \subseteq A(\underline{M}(F)\backslash\underline{M}(\mathbb{A}))$ is called an **automorphic subrepresentation of $\underline{M}$**.

2.7.4 Global Hecke algebras

Recall that we have a fixed maximal compact $\mathbf{K} = \prod_v K_v$ in which $K_v = \underline{G}(\mathfrak{o}_v)$ for almost all places $v \in \mathcal{V}$. For a non-Archimedean place, let $\mathcal{H}_v$ denote the convolution algebra $(C_c^\infty(G_v), *)$ of locally constant compactly supported $\mathbb{C}$-valued function on G_v, with respect to a Haar measure dg_v on G_v. Recall that a smooth representation (π_v, E_v) of G_v is one for which the stabilizer of each $f \in E_v$ is an open subgroup of G_v. Given a smooth representation (π_v, E_v) of G_v, we obtain an action of $\mathcal{H}_v$ via, for any $h \in \mathcal{H}_v$ and $f \in E_v$, $\pi_v(h)f = \int_{G_v} h(g_v)\pi_v(g_v)f\, dg_v$. One can show that this assignment yields and equivalence between the category of smooth representations of G_v and the category of nondegenerate modules for $\mathcal{H}_v$ [17, 1.5]. For each compact subgroup $K'_v \subseteq G_v$, let $e_{K'_v}$ be the idempotent $\mathrm{meas}(K'_v)^{-1}\,\mathrm{ch}_{K'_v}$, where ch_A denotes the characteristic function of the subset A. Then, we define the algebra $\mathcal{H}(G_v, K'_v) := e_{K'_v} * \mathcal{H}_v * e_{K'_v}$. It is known that for the K_v as above, the algebra $\mathcal{H}(G_v, K_v)$ is commutative for $v \notin S_b$ (see [17, Corollary 4.1]). In this case, we will call it a **spherical Hecke algebra**. The commutativity is a Corollary to the following important result, a proof of which can be found in [17, Theorem 4.1]. For this result, assume that $\underline{M} = \underline{M}_0$, and that M_v is the local completion of $\underline{M}_0(F)$ at the place $v \in \mathcal{S}_b$.

Proposition 2.7.6 (Satake Isomorphism)**.** *Let du_v be a Haar measure on U_v such that the measure of $U_v \cap K_v$ is one. The linear map, called the Satake transform, $\mathcal{H}(G_v, K_v) \to \mathcal{H}(M_v, M_v \cap K_v)$, given by*

$$S_v f(t) = \delta(t)^{1/2} \int_{U_v} f(tu_v)\, du_v$$

is an algebra isomorphism from $\mathcal{H}(G_v, K_v)$ onto the algebra $\mathbb{C}[M_v/(M_v \cap K_v)]^{W_0}$.

When $\underline{G}$ is split, $\mathcal{H}(G_v, K_v)$ is a spherical Hecke algebra for all $v \in \mathcal{V}_f$. In this case, we will write $\underline{M}_0 = \underline{T}$, a maximal torus that is split over F. It follows that for each place $v \in \mathcal{V}_f$ we have isomorphisms $S_v : \mathcal{H}(G_v, K_v) \to \mathbb{C}[X_*(\underline{T})]^{W_0}$.

Now let v be an Archimedean place. The Hecke algebra $\mathcal{H}_v$ in this case is the Hecke algebra of the pair $(\mathfrak{g}_v, K_v)$ in the sense of [29], where $\mathfrak{g}_v$ is the complexified Lie algebra of G_v. This is the convolution algebra of left and right K_v-finite distributions on G_v supported on K_v. Denote by A_{K_v} the algebra of finite measures on K_v. If δ is a finite dimensional representation of K_v, $d_\delta = \dim \delta$ and χ_δ is the character of δ then the measure

$$\mu_\delta := d_\delta^{-1} \chi_\delta\, dk_v, \tag{2.81}$$

acting on $\phi \in C^\infty(K_v)$ via $\langle \mu_\delta, \phi\rangle = d_\delta^{-1} \int_{K_v} \chi_\delta(k_v)\phi(k_v) dk_v$ is an example of an element of A_{K_v}. When viewed as distributions on G_v, A_{K_v} is contained in $\mathcal{H}_v$ and one has an isomorphism

$$\mathcal{H}_v \cong \mathcal{U}(\mathfrak{g}_v) \otimes_{\mathcal{U}(\mathfrak{k}_v)} A_{K_v}, \tag{2.82}$$

induced by the multiplication map $(X, \mu) \mapsto X * \mu$ from $\mathcal{U}(\mathfrak{g}_v) \times A_{K_v}$ to the space of distributions on G_v. Moreover, together with the set of idempotents of A_{K_v}, that is, the set of finite sum of measures of the type μ_δ as in (2.81), the algebra $\mathcal{H}_v$ is an idempotented algebra ([29] and [14, Section 3.4]). Moreover, there is an equivalence between the category of admissible $(\mathfrak{g}_v, K_v)$ modules and the one of admissible $\mathcal{H}_v$ modules [29, 3.], where we recall that an admissible module, for both algebras, is one for which the dimension of the isotypic component of the equivalence class of a irreducible unitary representation of K_v is finite.

Consider $e_{K_v} \in A_{K_v}$ as the element μ_{triv} corresponding to the trivial representation of K_v in (2.81). The **spherical Hecke algebra** in the present Archimedean context is defined as the algebra $e_{K_v} * \mathcal{H}_v * e_{K_v}$ of bi-K_v-invariant elements of $\mathcal{H}_v$. We will denote this algebra by $\mathcal{H}(G_v, K_v)$ and we remark that this notation is not consistent with [9]. Using the description of (2.82), one can show that $\mathcal{H}(G_v, K_v)$ is isomorphic to $\mathcal{U}(\mathfrak{g}_v)^{K_v}/\mathcal{U}(\mathfrak{g}_v)\mathfrak{k}_v \cap \mathcal{U}(\mathfrak{g}_v)^{K_v}$, where the exponent means $\mathrm{Ad}(K_v)$-invariance. The latter is isomorphic to the algebra of invariant differential operators on G_v/K_v (see [39]). We thus have

Proposition 2.7.7. *Let $a_{T,\mathbb{C}} = \mathbb{C} \otimes X_*(\underline{T})$. The spherical Archimedean Hecke algebra $\mathcal{H}(G_v, K_v)$ is isomorphic to $\mathrm{Sym}(a_{T,\mathbb{C}})^{W_0}$, via the Harish-Chandra's isomorphism.*

In both the number and function field cases, let $\mathcal{H} := \otimes'_v \mathcal{H}_v$, be the restricted tensor product of the family of algebras $\{\mathcal{H}_v \mid c \in \mathcal{V}\}$ with respect to the idempotents $\{e_{K_v} \mid v \notin S_b\}$ (see text above (2.75) for S_b). This means that elements in $\mathcal{H}$ are finite linear combinations of vectors of the type

$$(\otimes_{v\in S} h_v) \otimes (\otimes_{v\notin S} e_{K_v}),$$

where $S \subseteq \mathcal{V}$ is a finite set of places with $\mathcal{V}_\infty \subseteq S_b \subseteq S$ and $h_v \in \mathcal{H}_v$. This algebra is called the **global Hecke algebra**. The upshot of this section is that in both cases of global fields, the various spaces of automorphic forms on $\underline{G}(\mathbb{A})$ are admissible modules for the convolution action of the global Hecke algebra $\mathcal{H}$. We can similarly define the global Hecke algebra for any Levi subgroup $\underline{M}$.

2.7.5 Constant terms and cuspidality

Let ϕ be a function on $\underline{M}(F)\underline{U}(F)\backslash\underline{G}(\mathbb{A})$, measurable and locally L^1. The **constant term of** ϕ **along** $\underline{P}$ is the measurable and locally L^1 function ϕ_P on $\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A})$ defined by (du as in Section 2.7.2)

$$\phi_P(g) := \int_{\underline{U}(F)\backslash\underline{U}(\mathbb{A})} \phi(ug)du. \tag{2.83}$$

If, in the definition above, ϕ is assumed to be smooth and of moderate growth, then its constant term along $\underline{P}$ shares the same properties. An element $\phi \in A(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))$ is called a **cuspidal automorphic form** if, for every parabolic subgroup $\underline{P}'$ such that $\underline{P}_0 \subseteq \underline{P}' \subsetneq \underline{P}$ it holds that $\phi_{P'} = 0$. The space of all cuspidal automorphic forms will be denoted by $A_0(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))$ and $A_0(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\omega$, for ω a character of $Z_M(\mathbb{A})$. These spaces are preserved by the action of the global Hecke algebra on the number and function field cases. By letting $\underline{M} = \underline{G}$ (so that $\underline{U}$ is trivial) we can talk about cuspidal automorphic forms for $\underline{G}(F)\backslash\underline{G}(\mathbb{A})$. Similarly for any levi subgroup $\underline{M}$.

Definition 2.7.8. An irreducible subspace $E \subseteq A_0(\underline{M}(F)\backslash\underline{M}(\mathbb{A}))$ is called a **cuspidal automorphic representation of $\underline{M}$.**

2.7.6 Cuspidal data

Let (π, E_π) be a fixed cuspidal automorphic representation of $\underline{M}(\mathbb{A})$. Denote by $A_0(\underline{M}(F)\backslash\underline{M}(\mathbb{A}))_\pi$ the isotypic component of $A_0(\underline{M}(F)\backslash\underline{M}(\mathbb{A}))$ of type π. By definition, this set is the sum of all invariant subspaces $E' \subseteq A_0(\underline{M}(F)\backslash\underline{M}(\mathbb{A}))$ isomorphic to E_π as an $\underline{M}(\mathbb{A})_f \times (\mathfrak{m}_\infty, \mathbf{K} \cap M_\infty)$-modules.

Define $A_0(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\pi$ as the set of all $\phi \in A_0(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))$ such that, for all $k \in \mathbf{K}$, the function $\phi_k : \underline{M}(F)\backslash\underline{M}(\mathbb{A}) \to \mathbb{C}$ defined by

$$\phi_k(m) = m^{-\varrho_P}\phi(mk) \tag{2.84}$$

is in $A_0(\underline{M}(F)\backslash\underline{M}(\mathbb{A}))_\pi$, with ϱ_P as in 2.7.1.

Lemma 2.7.9. *Let $\phi \in A_0(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\pi$. The assignment $\phi \mapsto f_\phi$, with $f_\phi(k) = \phi_k$, for $k \in \mathbf{K}$ yields an isomorphism between $A_0(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\pi$ and $\mathrm{ind}_{\underline{M}(\mathbb{A})\cap\mathbf{K}}^{\mathbf{K}} A_0(\underline{M}(F)\backslash\underline{M}(\mathbb{A}))_\pi$.*

Proof. The inverse is given by $f \mapsto \phi_f \in A_0(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\pi$ with $\phi_f(g) = m_P(g)^{\varrho_P} f(k_g)(m_g)$, with $g = u_g m_g k_g$. $\square$

If π' is another irreducible automorphic representation of $\underline{M}(\mathbb{A})$, we say that π' is equivalent to π if there is an unramified character $\lambda \in X_M^G$ such that $\pi' \cong \pi \otimes \lambda$. Let $\mathfrak{P}$ be the equivalence class of π. Define $\mathrm{Fix}_{X_M^G}(\mathfrak{P}) := \{\lambda \in X_M^G \mid \lambda \otimes \pi' \cong \pi'\}$, which is independent of the chosen $\pi' \in \mathfrak{P}$. Then, X_M^G acts transitively on $\mathfrak{P}$ and yields a bijection

$$\mathfrak{P} \cong X_M^G / \mathrm{Fix}_{X_M^G}(\mathfrak{P}). \tag{2.85}$$

Remark. In case F is a number field, $\mathrm{Fix}_{X_M^G}(\mathfrak{P})$ is trivial. Up until now we were developing, following [67], the theory in parallel for both number and function field cases. But our main interest in this thesis is on the number field case. From now on, we will state the formulas and results only in the number field case. This will mean that in many formulas one would have to take the group $\mathrm{Fix}_{X_M^G}(\mathfrak{P})$ into consideration.

The spaces $A_0(\underline{M}(F)\backslash\underline{M}(\mathbb{A}))_\pi$ and $A_0(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\pi$ transform under the action of X_M^G by

$$\begin{aligned} A_0(\underline{M}(F)\backslash\underline{M}(\mathbb{A}))_{\pi\otimes\lambda} &= \lambda\, A_0(\underline{M}(F)\backslash\underline{M}(\mathbb{A}))_\pi \\ A_0(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_{\pi\otimes\lambda} &= (\lambda \circ m_P)\, A_0(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\pi, \end{aligned} \tag{2.86}$$

where $m_P : \underline{G}(\mathbb{A}) \to \underline{M}(\mathbb{A})^1\backslash\underline{M}(\mathbb{A})$ is the map defined by the Iwasawa decomposition $\underline{G}(\mathbb{A}) = \underline{M}(\mathbb{A})\underline{U}(\mathbb{A})\mathbf{K}$ (see [67, Section I.1]).

Convention. Given an element $\phi \in A(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\pi$, cuspidal or not, and $\lambda \in X_M^G$, the element $(\lambda \circ m_P)\phi \in A(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_{\pi\otimes\lambda}$ will be denoted by $\lambda\phi$.

Definition 2.7.10. Let ω be an automorphic character of $\underline{Z}_G$. A pair $(\underline{M}, \mathfrak{P})$, in which $\underline{M}$ is a standard Levi of $\underline{G}$ and $\mathfrak{P}$ is the equivalence class of a cuspidal automorphic representation of $\underline{M}$ whose restriction to $\underline{Z}_G(\mathbb{A})$ is ω, is called a **cuspidal datum**. If one needs to specify the character ω, we say it is a cuspidal datum **of central character** ω. Two cuspidal data are said to be equivalent and we write $(\underline{M}, \mathfrak{P}) \sim (\underline{L}, \mathfrak{Q})$ if there exists a $w \in W_0$ such that $w(\underline{M}, \mathfrak{P}) = (\underline{L}, \mathfrak{Q})$, that is, $w\underline{M}w^{-1} = \underline{L}$ and $w\mathfrak{P} = \mathfrak{Q}$. The equivalence class of a cuspidal datum $(\underline{M}, \mathfrak{P})$ is called a **class of cuspidal data**, and will be denoted $[\underline{M}, \mathfrak{P}]$.

2.7.7 Automorphic-valued Paley-Wiener functions

Recall that the space X_M is endowed with a complex analytic structure coming from $a^*_{M,\mathbb{C}}$ by means of the surjective map $\kappa : a^*_{M,\mathbb{C}} \to X_M$ of (2.79). Moreover, the subspace $X_M^G \subseteq X_M$ is a closed analytic set [67, p.79], which in the number field case is identified via κ with the complex vector space $a^{G*}_{M,\mathbb{C}}$. Denote by $PW(X_M^G)$ the space of all complex-valued Paley-Wiener functions on X_M^G.

Given a cuspidal datum $(\underline{M}, \mathfrak{P})$, fix a base point $\pi \in \mathfrak{P}$. Let ϕ be a function on $\mathfrak{P}$ such that $\phi(\pi') \in A_0(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_{\pi'}$ and that the image of the associated $\tilde{\phi} : X_M^G \to A_0(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\pi$, given by $\tilde{\phi}(\lambda) = \lambda^{-1}\phi(\pi \otimes \lambda)$, generates a finite dimensional vector subspace of $A_0(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\pi$.

Definition 2.7.11. A function ϕ as above is called an **automorphic-valued function on** $\mathfrak{P}$. We will say that an automorphic-valued function ϕ is **Paley-Wiener** if the associated function $\tilde{\phi}$ can be identified with an element of $PW(X_M^G) \otimes A_0(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\pi$. We denote by $P_{(\underline{M},\mathfrak{P})}$ the space of all automorphic-valued Paley-Wiener functions on $\mathfrak{P}$.

We remark at once that, in the number field case, there is an identification $PW(X_M^G) \otimes A_0(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\pi \xrightarrow{\sim} P_{(\underline{M},\mathfrak{P})}$, which is explicitly given by

$$\iota(\varphi \otimes \psi)(\pi \otimes \lambda)(g) = m_P(g)^\lambda \varphi(\lambda)\psi(g), \tag{2.87}$$

for $\varphi \in PW(X_M^G)$ and $\psi \in A_0(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\pi$. In the function field case, one has to take the finite group $\mathrm{Fix}_{X_M^G}(\mathfrak{P})$ into consideration [67, II.1(5)].

If $\mathbf{K}' \subseteq \mathbf{K}$ is a compact subgroup, we denote by $P^{\mathbf{K}'}_{(\underline{M},\mathfrak{P})}$ the space of all $\mathbf{K}'$-spherical automorphic-valued Paley-Wiener functions on $\mathfrak{P}$ in the sense that any $\phi \in P^{\mathbf{K}'}_{(\underline{M},\mathfrak{P})}$ is such that the associated $\tilde{\phi}$ can be identified with an element of $A_0(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}'}_\pi \otimes PW(X_M^G)$. In the number field case we have

$$P^{\mathbf{K}'}_{(\underline{M},\mathfrak{P})} \cong A_0(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}'}_\pi \otimes PW(X_M^G). \tag{2.88}$$

Here, the exponent $\mathbf{K}'$ means that the automorphic forms considered are $R(\mathbf{K}')$-invariant.

2.7.8 Eisenstein series and intertwining operators

Let π be an irreducible subrepresentation of $\underline{M}(\mathbb{A})$ (Definition 2.7.5). We obtain a character $\omega_\pi : \underline{Z}_M(\mathbb{A}) \to \mathbb{C}^\times$. If π is an automorphic representation, from condition (v) of Section 2.7.3 (note that we do not have the factor z^{ϱ_P} in the present case), it follows that ω_π is automorphic. We then obtain an element $|\omega_\pi| \in \operatorname{Re} X_M \cong a_M^*$ as in (2.78) (recall that $\mathbb{R} \otimes X^*(\underline{M}) \cong \mathbb{R} \otimes X^*(\underline{Z}_M)$). Define $\operatorname{Re}\pi := |\omega_\pi|$. The imaginary part is defined by the equation

$$\operatorname{Im}\pi = \pi \otimes (-\operatorname{Re}\pi). \tag{2.89}$$

We let also $-\pi$ be the contragredient of π and $-\overline{\pi}$ be the conjugate of the contragredient.

Fix a unitary automorphic character ω of $\underline{Z}_G$ throughout this subsection. Let $(\underline{M}, \mathfrak{P})$ be a cuspidal datum of central character ω and pick a base point $\pi \in \mathfrak{P}$. Let $\psi \in A(\underline{M}(F)\underline{U}(\mathbb{A})\backslash \underline{G}(\mathbb{A}))_\pi$. The **Eisenstein series** is defined as

$$E(\psi, \pi)(g) := \sum_{\gamma \in \underline{P}(F)\backslash \underline{G}(F)} \psi(\gamma g). \tag{2.90}$$

Whenever convergent, it defines a function on $\underline{G}(F)\backslash \underline{G}(\mathbb{A})$.

Proposition 2.7.12 ([67], Proposition II.1.5). *Let $\mathcal{C}$ be the open cone on $\operatorname{Re} X_M^G$ given by $\{\lambda \in \operatorname{Re} X_M^G \mid \langle \lambda, \alpha^\vee \rangle > \langle \varrho_P, \alpha^\vee \rangle$ for all $\alpha \in \Phi_M^+\}$. Then, the series defining $E(\lambda\psi, \pi \otimes \lambda)(g)$ converges normally for g in a compact set and λ in a neighborhood of 0 for all π such that $\operatorname{Re}\pi \in \mathcal{C}$.*

More generally, for $\phi \in P_{(\underline{M}, \mathfrak{P})}$, we define an Eisenstein series via

$$E(\phi, \pi)(g) := \sum_{\gamma \in \underline{P}(F)\backslash \underline{G}(F)} \phi(\pi)(\gamma g). \tag{2.91}$$

Convention. Whenever a character $\lambda_0 \in \operatorname{Re} X_M^G$ is in $\mathcal{C}$, so that it is in the region of convergence of the Eisenstein series, we will write $\lambda_0 \gg 0$.

Let $\underline{L}$ be another standard Levi of $\underline{G}$ corresponding to the standard parabolic $\underline{Q} = \underline{L}\,\underline{N}$. Suppose $w \in \underline{G}(F)$ is such that $w\underline{M}w^{-1} = \underline{L}$. Then, the element w sends $\mathfrak{P} \mapsto w\mathfrak{P} = \mathfrak{Q}$, an equivalence class of a cuspidal automorphic subrepresentation of $\underline{L}$. Let $\pi \in \mathfrak{P}$ and $\psi \in A(\underline{M}(F)\underline{U}(\mathbb{A})\backslash \underline{G}(\mathbb{A}))_\pi$. We define the **intertwining operator** as

$$(M(w, \pi)\psi)(g) = \int_{\underline{N}(\mathbb{A}) \cap w\underline{U}(\mathbb{A})w^{-1} \backslash \underline{N}(\mathbb{A})} \psi(w^{-1}ug)\, du, \tag{2.92}$$

whenever convergent. The measure du is as in Section 2.7.2. Because of the assumption that $\mathfrak{P}$ consists of cuspidal representations, this is the case for $\operatorname{Re}\pi \in \mathcal{C}$ and it defines an element of $A_0(\underline{L}(F)\underline{N}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_{w\pi}$. Still assuming π in the domain of absolute convergence of the Eisenstein series, if $\overline{\pi}$ denote its conjugate representation, with respect to the Hermitian product on $A(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_{-\overline{\pi}} \times A(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_{\pi}$ given by

$$(\psi,\phi) \mapsto \int_{(\underline{M}(F)\underline{Z}_M(\mathbb{A})\backslash\underline{M}(\mathbb{A}))\times\mathbf{K}} m^{-2\varrho_P}\,\overline{\psi}(mk)\phi(mk)\,dm\,dk, \tag{2.93}$$

for all $\psi \in A(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_{-\overline{\pi}}$ and $\phi \in A(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_{\pi}$.

2.7.9 Pseudo-Eisenstein and spectral decomposition

In this section, we assume F to be a number field. The function field case is very similar but one has to take the finite group $\operatorname{Fix}_{X_M^G}\mathfrak{P}$ into consideration [67, II.1.3]. Let us fix a Haar measure on $\operatorname{Re} X_M^G$. We fix one via the isomorphism $\operatorname{Re} X_M^G \cong a_T^{G*}$. We obtain also a measure on $\operatorname{Im} X_M^G$ and on the sets of the type $\{\lambda \in X_M^G \mid \operatorname{Re}\lambda = \lambda_0 \in \operatorname{Re} X_M^G\}$, denoted $d\lambda$, by means of (2.80) ([67, Section I.1.13]). We let $\mathcal{F} : PW(X_M^G) \to C_c^\infty(\underline{M}(\mathbb{A})^1\underline{Z}_G(\mathbb{A})\backslash\underline{M}(\mathbb{A}))$ be the Fourier transform defined by, for $\lambda_0 \in \operatorname{Re} X_M^G$ deep in the positive chamber

$$\mathcal{F}(\varphi)(m) = \int_{\operatorname{Re}\lambda=\lambda_0} \varphi(\lambda)m^{\lambda}d\lambda, \tag{2.94}$$

for all $\varphi \in P(X_M^G)$ with inverse

$$\mathcal{F}^{-1}(f)(\lambda) = \int_{\underline{M}(\mathbb{A})^1\underline{Z}_G(\mathbb{A})\backslash\underline{M}(\mathbb{A})} f(m)m^{-\lambda}dm, \tag{2.95}$$

for all $f \in C_c^\infty(\underline{M}(\mathbb{A})^1\underline{Z}_G(\mathbb{A})\backslash\underline{M}(\mathbb{A}))$. We require the measure dm to be dual to $d\lambda$. The next step is to define the Fourier transform of an element in $P_{(\underline{M},\mathfrak{P})}$. For the fixed base point $\pi \in \mathfrak{P}$, let

$$\tilde{A}(\underline{M}(F)\backslash\underline{M}(\mathbb{A}))_\pi := R(\underline{M}(\mathbb{A}))(A(\underline{M}(F)\backslash\underline{M}(\mathbb{A}))_\pi)$$

be the space spanned by all right translates of the right regular representation of $\underline{M}(\mathbb{A})$ on $A(\underline{M}(F)\backslash\underline{M}(\mathbb{A}))_\pi$. In the function field case, we have that $\tilde{A}(\underline{M}(F)\backslash\underline{M}(\mathbb{A}))_\pi = A(\underline{M}(F)\backslash\underline{M}(\mathbb{A}))_\pi$, but in the number field case these spaces do not coincide as the right regular action does not preserve $\mathbf{K}\cap\underline{M}(\mathbb{A})$-finiteness in the Archimedean places. Given $\phi \in P_{(\underline{M},\mathfrak{P})}$, the Fourier transform

$$\mathcal{F}_\pi(\phi) : \underline{G}(\mathbb{A}) \to \tilde{A}(\underline{M}(F)\backslash\underline{M}(\mathbb{A}))_\pi$$

is given by, for $g = u_g m_g k_g$ according to the Iwasawa decomposition

$$\mathcal{F}_\pi(\phi)(g)(m) = \int_{\operatorname{Re}\lambda=\lambda_0} \phi(\pi\otimes\lambda)(mm_gk_g)m^{-(\lambda+\varrho_P)}d\lambda. \tag{2.96}$$

Denote by $\widehat{P}_{(\underline{M},\mathfrak{P})}$ the image of the Fourier transform $\mathcal{F}_\pi$ and consider the map υ form the space $C_c^\infty(\underline{M}(\mathbb{A})^1\underline{Z}_G(\mathbb{A})\backslash\underline{M}(\mathbb{A}))\otimes A(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\pi$ into the set of functions on $\underline{G}(\mathbb{A})$ with values on $\tilde{A}(\underline{M}(F)\backslash\underline{M}(\mathbb{A}))_\pi$ defined by

$$\upsilon(f\otimes\psi)(g)(m)=f(m_P(g))\psi(mg)m^{-\varrho_P},$$

for $f\in C_c^\infty(\underline{M}(\mathbb{A})^1\underline{Z}_G(\mathbb{A})\backslash\underline{M}(\mathbb{A}))$ and $\psi\in A(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\pi$.

Lemma 2.7.13 ([67], Section II.1(5))**.** *In the situation described above, the following diagram is commutative:*

$$\begin{tikzcd}
P_{(\underline{M},\mathfrak{P})} \arrow[d, "\mathcal{F}_\pi"] & PW(X_M^G)\otimes A(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\pi \arrow[l, "\sim"', "\iota"] \arrow[d, "\mathcal{F}\otimes id"] \\
\widehat{P}_{(\underline{M},\mathfrak{P})} & C_c^\infty(\underline{M}(\mathbb{A})^1\underline{Z}_G(\mathbb{A})\backslash\underline{M}(\mathbb{A}))\otimes A(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\pi. \arrow[l, "\sim"', "\upsilon"]
\end{tikzcd} \tag{2.97}$$

Further, let $\epsilon:\tilde{A}(\underline{M}(F)\backslash\underline{M}(\mathbb{A}))_\pi\to\mathbb{C}$ denote the evaluation at the unit element. One obtains a map $\widehat{P}_{(\underline{M},\mathfrak{P})}\to\epsilon\widehat{P}_{(\underline{M},\mathfrak{P})}$, and this last space is characterized in [67] as the set of all **K**-finite functions $\Phi:\underline{U}(\mathbb{A})\underline{M}(F)\backslash\underline{G}(\mathbb{A})\to\mathbb{C}$ that can be written as, for all $g\in\underline{G}(\mathbb{A})$,

$$\Phi(g)=\sum_i f_i(m_P(g))\psi_i(g), \tag{2.98}$$

for $\{f_i\}\subseteq C_c^\infty(\underline{M}(\mathbb{A})^1\underline{Z}_G(\mathbb{A})\backslash\underline{M}(\mathbb{A}))$ and $\{\psi_i\}\subseteq A(\underline{M}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\pi$.

Next, let $\phi\in P_{(\underline{M},\mathfrak{P})}$ and fix a base point $\pi\in\mathfrak{P}$. The **pseudo-Eisenstein series** are defined by

$$\theta_\phi(g)=\sum_{\gamma\in\underline{P}(F)\backslash\underline{G}(F)}\epsilon\mathcal{F}_\pi(\phi)(\gamma g), \tag{2.99}$$

whenever this series converges. It turns out that such a series is absolutely convergent, uniform when g remains in a compact set and defines an element $L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\omega$ [67, Proposition II.1.10]. Moreover, for $\lambda_0\in\operatorname{Re}X_M^G$ in the region of convergence of the Eisenstein $E(\phi,\pi)$, it can be written as

$$\theta_\phi(g):=\int_{\operatorname{Re}\pi=\operatorname{Re}\lambda_0}E(\phi,\pi)(g)\,d\pi, \tag{2.100}$$

where $d\pi$ is obtained from the measure on $\operatorname{Im}X_M^G$ (see text above (2.94) and [67, Remark II.1.11]). An important property of the pseudo-Eisenstein series is the following density result [67, Theorem II.1.12]:

Proposition 2.7.14. *The space $L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\omega$ is topologically generated by the linear span of all pseudo-Eisenstein series θ_ϕ when ϕ ranges over all $P_{(\underline{M},\mathfrak{P})}$ with $(\underline{M},\mathfrak{P})$ a cuspidal datum of central character ω.*

As the pseudo-Eisenstein series generate the space we wish to decompose, it is important to have a formula for their L^2-inner product. Whenever we have $(\underline{M},\mathfrak{P}) \sim (\underline{L},\mathfrak{Q})$, let $W((\underline{M},\mathfrak{P}),(\underline{L},\mathfrak{Q})) := \{w \in W_0 \mid w(\underline{M},\mathfrak{P}) = (\underline{L},\mathfrak{Q})\}$.

Proposition 2.7.15 ([67], Theorem II.2.1)**.** *Let $(\underline{L},\mathfrak{Q})$ and $(\underline{M},\mathfrak{P})$ be two cuspidal data of central character ω and fix $\lambda_0 \gg 0$. Let $\phi \in P_{(\underline{L},\mathfrak{Q})}$ and $\psi \in P_{(\underline{M},\mathfrak{P})}$. Then*

$$(\theta_\phi, \theta_\psi) = \begin{cases} \int\limits_{\operatorname{Re}\pi=\lambda_0} \sum\limits_{w} (\phi(-w\overline{\pi}), M(w,\pi)\psi(\pi))\, d\pi, & \text{if } (\underline{M},\mathfrak{P}) \sim (\underline{L},\mathfrak{Q}) \\ 0, & \text{otherwise,} \end{cases}$$

with the sum ranging over all $w \in W((\underline{M},\mathfrak{P}),(\underline{L},\mathfrak{Q}))$ and the Hermitian pairing is the one described in (2.93).

We are now ready to state the promised result regarding the decomposition of the space $L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\omega$ along cuspidal data. It will follow from Propositions 2.7.14 and 2.7.15. But before that, let $\mathfrak{C}$ denote the set of all classes of cuspidal data $\mathfrak{X} := [\underline{M},\mathfrak{P}]$ with central character ω.

Definition 2.7.16. For $\mathfrak{X} \in \mathfrak{C}$, we define $L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}$ as the closure of the subspace of $L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\omega$ generated by all pseudo-Eisenstein θ_ϕ with $\phi \in P_{(\underline{M},\mathfrak{P})}$ ranging over the cuspidal data $(\underline{M},\mathfrak{P})$ in the class $\mathfrak{X}$.

Corollary 2.7.17. *We have the following completed orthogonal decomposition*

$$L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_\omega = \widehat{\bigoplus_{\mathfrak{X}\in\mathfrak{C}}} L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}.$$

2.7.10 A particular, yet general case

Assume that F is a number field and that $\underline{G}$ is connected, reductive and split over F. In this case, the Levi factor $\underline{M}_0$ of the minimal parabolic $\underline{P}_0$ is a maximal split torus. We will use the notation $\underline{M}_0 = \underline{T}$, $\underline{P}_0 = \underline{B}$, $\underline{U}_0 = \underline{U}$ and $\varrho_0 = \varrho$ for the Weyl vector. For every place v we have a maximal compact K_v, in which $K_v = \underline{G}(\mathfrak{o}_v)$ for v non-Archimedean. Note that, because $\underline{T}$ is a torus, there is no proper parabolic $\underline{P}$ to verify the condition for cuspidality. Hence, every element in $A(\underline{T}(F)\backslash\underline{T}(\mathbb{A}))$ is automatically cuspidal. Thus, the equivalence class of the trivial representation of $\underline{T}(\mathbb{A})$ on $A(\underline{T}(F)\backslash\underline{T}(\mathbb{A}))$ is a cuspidal automorphic representation and yields a class of cuspidal data $[\underline{T},\imath]$, with

$$\imath := \{\text{triv} \otimes \lambda \mid \lambda \in X_T^G\}. \tag{2.101}$$

In this case, for any $w \in W_0$ it holds that $w(\underline{T},\imath) = (\underline{T},\imath)$ so the set indexing the sum in the description of the scalar product in Proposition 2.7.15 is the Weyl group W_0. In the split case, there will always be a summand $L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_{[\underline{T},\imath]}$ for any reductive group $\underline{G}$, so this case can be treated in a general setting. Our interest is in $L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}}_{[\underline{T},\imath]}$, the space of $\mathbf{K}$-invariant elements of $L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_{[\underline{T},\imath]}$.

In this particular setting, much of the discussion of sections 2.7.7 until 2.7.9 can be simplified. Firstly, the isotypic component of the trivial representation $A(\underline{T}(F)\backslash\underline{T}(\mathbb{A}))_{\mathrm{triv}}$ is isomorphic to $\mathbb{C}$. Thus, the space of $\mathbf{K}$-spherical Paley-Wiener automorphic-valued functions on $\imath$ is identified with $PW(a^{G*}_{T,\mathbb{C}})$ (we use here the identification of X^G_T and $a^{G*}_{T,\mathbb{C}}$ given by κ in the number field case (2.79)), because using (2.88) and Lemma 2.7.9, we obtain

$$\begin{aligned} P^{\mathbf{K}}_{(\underline{T},\imath)} &\cong A(\underline{T}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}}_{\mathrm{triv}} \otimes PW(a^{G*}_{T,\mathbb{C}}) \\ &\cong (\mathrm{ind}^{\mathbf{K}}_{\underline{T}(\mathbb{A})\cap\mathbf{K}}\mathbb{C})^{\mathbf{K}} \otimes PW(a^{G*}_{T,\mathbb{C}}) \\ &\cong PW(a^{G*}_{T,\mathbb{C}}). \end{aligned} \tag{2.102}$$

Proposition 2.7.18. *The identification $PW(a^{G*}_{T,\mathbb{C}}) \cong P^{\mathbf{K}}_{(\underline{T},\imath)}$ of (2.102) maps φ in $PW(a^{G*}_{T,\mathbb{C}})$ onto $j\varphi \in P^{\mathbf{K}}_{(\underline{T},\imath)}$, with $j\varphi(\mathrm{triv}\otimes\lambda)(g) = \varphi(\lambda)m_B(g)^{\lambda+\varrho}$.*

Proof. Because $A(\underline{T}(F)\backslash\underline{T}(\mathbb{A}))_{\mathrm{triv}} \cong \mathbb{C}$, we have identifications

$$\mathbb{C} \cong (\mathrm{ind}^{\mathbf{K}}_{\underline{T}(\mathbb{A})\cap\mathbf{K}} A(\underline{T}(F)\backslash\underline{T}(\mathbb{A}))_{\mathrm{triv}})^{\mathbf{K}} \cong A(\underline{T}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}}_{\mathrm{triv}},$$

that maps $1 \mapsto f \mapsto \nu$. Here, f is such that $f(k)(t) = f(1)(1) = 1$ for all $k \in \mathbf{K}$ and $t \in \underline{T}(\mathbb{A})$ and $\nu = \nu_f$ is the inverse of the isomorphism of Lemma 2.7.9. Therefore, $\nu(g) = m_B(g)^{\varrho}$ (note that ν is in $A(\underline{T}(F)\underline{U}(\mathbb{A})\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}}_{\mathrm{triv}}$ and corresponds to the constant function 1 because of (2.84)). We tensor the above map with the identity on $PW(a^{G*}_{T,\mathbb{C}})$ and compose with the identification ι of (2.87) to obtain

$$j\varphi(\mathrm{triv}\otimes\lambda)(g) = \iota(\varphi\otimes\nu)(\mathrm{triv}\otimes\lambda)(g) = \varphi(\lambda)m_B(g)^{\lambda+\varrho},$$

as required. ☐

The identification of Proposition 2.7.18 above allow us to define Eisenstein and pseudo-Eisenstein series for all $\varphi \in PW(a^{G*}_{T,\mathbb{C}})$. Specifically, we put $E(\varphi,\lambda) := E(j\varphi,\lambda)(g) = \sum_{\gamma\in\underline{B}(F)\backslash\underline{G}(F)} \varphi(\lambda)m_B(\gamma g)^{\lambda+\varrho}$ and $\theta_\varphi := \theta_{j\varphi}$. Define the **unramified Eisenstein series** by

$$\mathcal{E}(\lambda, g) = \sum_{\gamma\in\underline{B}(F)\backslash\underline{G}(F)} m_B(\gamma g)^{\lambda+\varrho}. \tag{2.103}$$

From (2.100) and (2.91) and the definition of $j\varphi$ in Proposition 2.7.18 we obtain

$$\theta_\varphi(g) = \int_{\mathrm{Re}\,\lambda=\lambda_0} \varphi(\lambda)\mathcal{E}(\lambda, g)\, d\lambda. \tag{2.104}$$

The conclusion is that $L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}}_{[\underline{T},\imath]}$ is topologically generated by the span of all pseudo-Eisenstein series θ_φ for $\varphi \in PW(a^{G*}_{T,\mathbb{C}})$ as in (2.104). In the next subsection we will make the inner product formula explicit in this situation.

2.7.11 Inner product in the unramified case

In the situation of the previous subsection, (we emphasize that $\underline{G}$ is split) we can write the inner product formula of Proposition 2.7.15 as follows. Recall that we have a Haar measure that gives $\mathbf{K}$ measure one. Fix a measure on a_T^{G*} and a measure $d\lambda$ on $\lambda_0 + ia_T^{G*}$ (we will specify the precise constants to be used in Section 2.7.12), with $\lambda_0 \gg 0$ an element such that $\lambda - \varrho$ is in the (open) fundamental chamber for W_0. For $\varphi, \psi \in PW(a_{T,\mathbb{C}}^{G*})$, we have (we will write $\mathrm{triv} \otimes \lambda$ simply as λ)

$$\begin{aligned}(\theta_\varphi, \theta_\psi) &= \int_{\mathrm{Re}\,\lambda=\lambda_0} \sum_{w\in W_0} (j\varphi(-w\overline{\lambda}), M(w,\lambda)(j\psi)(\lambda))\, d\lambda \\ &= \int_{\mathrm{Re}\,\lambda=\lambda_0} \sum_{w\in W_0} \overline{\varphi(-w\overline{\lambda})}\psi(\lambda) \int_{\mathbf{K}} (M(w,\lambda)(m_B(\cdot)^{\lambda+\varrho})(k)\, dk\, d\lambda.\end{aligned} \tag{2.105}$$

We must compute $(M(w,\lambda)m_B(\cdot)^{\lambda+\varrho})(1)$. We will peel the global spherical vector (see (2.109), below) and compute the actions of the local intertwiners. This is a classical computation that dates back to Gindikin and Karpelevich [31] in the Archimedean place. The non-Archimedean case was obtained by Langlands (we will refer to [18], for Casselman's proof, below). One must check that the various choices of measures are coherent, and that we will do below. A survey of this computation can be found in [15].

For $\underline{G}, \underline{B}, \underline{T}$ and $\underline{U}$ as in Section 2.7.10. We write Φ_0 for the root system of $(\underline{G}, \underline{T})$. Let $a_T^* = \mathbb{R} \otimes X^*(\underline{T})$ and $a_T = \mathbb{R} \otimes X_*(\underline{T})$ be as before and $\langle \cdot, \cdot \rangle$ be the bilinear pairing $a_{T,\mathbb{C}}^* \times a_{T,\mathbb{C}} \to \mathbb{C}$. Recall that a character $\chi \in X^*(\underline{T})$ defines a character $\chi_v : T_v \to F_v^\times$ for each local $v \in \mathcal{V}$. Further, let $H_v : T_v \to a_T$ be defined by the equation below that holds for all $\chi \in X^*(\underline{T})$ and $t_v \in T_v$

$$|\chi_v(t_v)|_v = q_v^{\langle \chi, H(t_v)\rangle}, \tag{2.106}$$

where $|\cdot|_v$ is the normalized absolute value on each local place and

$$q_v = \begin{cases} \#(\mathfrak{o}_v/\varpi_v\mathfrak{o}_v), & v \text{ non-Archimedean} \\ e, & v \text{ Archimedean.} \end{cases}$$

Similarly, for $\chi \in X^*(\underline{T})$ and $t = (t_v)_v \in \underline{T}(\mathbb{A})$, let $H : \underline{T}(\mathbb{A}) \to a_T$ be given by $t^{|\chi|} = e^{\langle \chi, H(t)\rangle}$. For any $\lambda \in a_{T,\mathbb{C}}^{G*}$ it holds

$$m_B(t)^\lambda = \prod_v q_v^{\langle \lambda, H_v(t)\rangle} = e^{\langle \chi, H(t)\rangle}. \tag{2.107}$$

Given $\lambda \in a_{T,\mathbb{C}}^{G*}$, define an unramified local character of T_v via

$$\chi_{v,\lambda}(t_v) := \begin{cases} q_v^{\langle \lambda, H_v(t_v)\rangle}, & v \text{ non-Archimedean,} \\ e^{\langle \lambda, H_v(t_v)\rangle}, & v \text{ real,} \\ e^{\langle 2\lambda, H_v(t_v)\rangle}, & v \text{ complex.} \end{cases} \tag{2.108}$$

Define the **unramified principal series** $(\pi_v(\lambda), I_v(\lambda))$ at the parameter $\lambda \in a_{T,\mathbb{C}}^{G*}$ as the representation of G_v (of $(\mathfrak{g}_v, K_v)$ for v Archimedean) whose representation space is given by the smooth (and K_v-finite) functions $f : G_v \to \mathbb{C}$ satisfying

$$f(b_v g_v) = \chi_{v,\lambda+\varrho}(b_v) f(g_v).^{8}$$

Such G_v-representation contains a spherical vector $f_{v,\lambda}^{\circ}$ normalized so that $f_{v,\lambda}^{\circ}(1) = 1$. Namely,

$$f_{v,\lambda}^{\circ}(n_v t_v k_v) = \chi_{v,\lambda+\varrho}(t_v).$$

for all $g_v = n_v t_v k_v$ given by its Iwasawa decomposition $G_v = B_v K_v$. Denote by $(\pi(\lambda), I(\lambda))$ the global representation of $(\mathfrak{g}_\infty, K_\infty) \times G_f$. It decomposes as $I(\lambda) = \otimes_v I_v(\lambda)$. The unique normalized global spherical vector defined as $f_\lambda^{\circ} := \otimes_v f_{v,\lambda}^{\circ} \in I(\lambda)$, satisfies, by (2.107) and (2.108):

$$f_\lambda^{\circ}(g) = m_B(g)^{\lambda+\varrho}. \tag{2.109}$$

It is known that for each place $v \in \mathcal{V}$ and for each $\lambda \in a_{T,\mathbb{C}}^{G*}$, there is a unique irreducible spherical subquotient of $I_v(\lambda)$. For later usage, let us make the following:

Definition 2.7.19. For $\lambda \in a_{T,\mathbb{C}}^{G*}$ and v any place of F, we will denote by $(\pi_{v,\lambda}, E_{v,\lambda})$ the unique irreducible spherical subquotient of $(\pi_v(\lambda), I_v(\lambda))$.

Let $\Gamma_v(s)$ be defined by the gamma factors at infinity of the Dedekind zeta function (2.66) for v Archimedean, and for v non-Archimedean,

$$\Gamma_v(s) = \frac{1}{1 - q_v^{-s}}. \tag{2.110}$$

Fix the Haar measure μ_v of F_v as in (2.71). Using a fixed pinning of the Lie algebra of $\underline{G}$, define measures on each unipotent subgroup $\underline{U}_\alpha(F_v)$ for each root $\alpha \in \Phi_0$. For $w \in W$, let du_v be the measure on $U_v \cap wU_v w^{-1} \backslash U_v$ induced by the measure μ_v. Note that this group is topologically (not algebraically) the product of the groups

$$\prod_{\alpha \in \Phi(w)} \underline{U}_\alpha(F_v)$$

with $\Phi(w) = \Phi^+ \cap w^{-1}\Phi^-$ and hence it is homeomorphic to $\ell(w)$ copies of F_v, each of which is endowed with the measure μ_v. Let the local intertwining operator $M(w, \lambda) : I_v(\lambda) \to I_v(w\lambda)$ (whenever convergent) be given by

$$M_v(w, \lambda) f(g_v) = \int_{U_v \cap wU_v w^{-1} \backslash U_v} f(w^{-1} u_v g_v)\, du_v.$$

[8] This is equivalent to saying that $f_v(bg) = \chi_{v,\lambda} \delta^{1/2}(b) f_v(g)$, for δ the modular character.

Proposition 2.7.20. *For every $v \in \mathcal{V}$ and λ in the domain of convergence of $M_v(w,\lambda)$, the action of the intertwining operator on the spherical vector evaluated at one satisfies*

$$M_v(w,\lambda)f^\circ_{v,\lambda}(1) = \prod_{\alpha\in\Phi(w)} \frac{\Gamma_v(\langle\lambda,\alpha^\vee\rangle)}{\Gamma_v(\langle\lambda,\alpha^\vee\rangle+1)}.$$

Here, $\Phi(w) = \{\alpha \in \Phi_0^+ \mid w\alpha \in \Phi_0^-\}$.

Proof. We refer to [18, Theorem 3.1], for the result in the non-Archimedean places. For v Archimedean, this is the original formula of Gindikin and Karpelevich. We refer to [39, Chapter IV, Theorem 6.14]. □

Example 2.7.21. In this example we discuss the Gindikin-Karpelevich formula for $\underline{G} = \underline{SL_2}$ and v Archimedean. By this example it should be clearer the need for the factors 2 used in (2.108) and (2.71) in the complex places. We start by discussing the general formula for Harish-Chandra's c-function, following [39]. Let G be a connected semisimple Lie group, K a maximal compact subgroup so that $X = G/K$ is a symmetric space of noncompact type. The group K is the set of fixed points of the Cartan involution θ. Denote this involution on $\mathfrak{g}$ also by θ and we let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition into eigenspaces of θ. Fix a maximal abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{p}$, let $\Phi_0 = \Phi(\mathfrak{g},\mathfrak{a}) \subseteq \mathfrak{a}^*$ be the root system with W_0 the Weyl group. We fix $(\cdot,\cdot)$ a W_0-invariant inner product and identify $\mathfrak{a}$ with $\mathfrak{a}^*$. We will write as before $\langle\cdot,\cdot\rangle$ for the pairing between $\mathfrak{a}^*$ and $\mathfrak{a}$. Let also Φ_0^+ be a system of positive roots corresponding to a choice of a chamber $\mathfrak{a}^*_+$. Given a root $\alpha \in \Phi_0$, let $m_\alpha := \dim_{\mathbb{R}} \mathfrak{g}_\alpha$ denote the multiplicity of the root α in the decomposition $\mathfrak{g}$ into root spaces. In this general setting, the Weyl vector is given by

$$\varrho = \frac{1}{2}\sum_{\alpha\in\Phi_0^+} m_\alpha \alpha.$$

Further, let $\mathfrak{n} = \oplus_{\alpha\in\Phi_0^+}\mathfrak{g}_\alpha$. We obtain the Iwasawa decompositions $\mathfrak{g} = \mathfrak{k}+\mathfrak{a}+\mathfrak{n}$ and $G = KAN$, where A and N are the analytic subgroups of G whose Lie algebras are $\mathfrak{a}$ and $\mathfrak{n}$, respectively. Given $g \in G$, let $A(g)$ be the A factor of its Iwasawa decomposition and set $\tilde{H} : G \to \mathfrak{a}$ by $\tilde{H}(g) := \log A(g)$. We note that, above, we use the map H for the opposite Iwasawa decomposition $G_v = B_vK_v$. Let also $\overline{N} := \theta(N)$ be the opposite unipotent. Let $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ be such that $\mathrm{Re}(i\lambda) \in \mathfrak{a}^*_+$. The Harish-Chandra c-function is given by the formula

$$c(\lambda) = \int_{\overline{N}} \exp(-\langle i\lambda+\varrho, \tilde{H}(\overline{n})\rangle)\, d\overline{n}, \tag{2.111}$$

with the Haar measure $d\overline{n}$ normalized so that $c(-i\varrho) = 1$ ([39, Chapter IV, §6.1]). We note that by changing coordinates using $\mathrm{Ad}(w_0)$ in (2.111), in the real case we have $c(-i\lambda) = (M(w_0,\lambda)f^\circ_{v,\lambda})(1)$, while in the complex case, $c(-2i\lambda) = (M(w_0,\lambda)f^\circ_{v,\lambda})(1)$. We have to compare our choice of Haar measure μ_v with $d\overline{n}$ above. With respect to $d\overline{n}$, if $\Phi_{0,\mathrm{ind}} \subseteq \Phi_0$ denotes the set of indivisible roots and $\alpha_0 := \alpha(\alpha,\alpha)^{-1}$, the c-function satisfies the general formula [39, Chapter IV, Theorem 6.14]

$$c(\lambda) = \prod_{\alpha\in\Phi^+_{0,\mathrm{ind}}} \frac{2^{(1/2)m_\alpha+m_{2\alpha}-(i\lambda,\alpha_0)}\Gamma(\frac{1}{2}(m_\alpha+m_{2\alpha}+1))\Gamma((i\lambda,\alpha_0))}{\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha+1+(i\lambda,\alpha_0)))\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha+m_{2\alpha}+(i\lambda,\alpha_0)))}. \tag{2.112}$$

This formula simplifies in the cases we are interested in, namely, when G is split over $\mathbb{R}$ and when G is complex.

(a) if G is split over $\mathbb{R}$, then $m_\alpha = 1$ and $m_{2\alpha} = 0$ for all α and

$$c(\lambda) = \prod_{\alpha\in\Phi_0^+} \frac{\Gamma((i\lambda,\alpha_0))}{\sqrt{\pi}\Gamma(\frac{1}{2}+(i\lambda,\alpha_0))},$$

where we used the duplication formula $\Gamma(z+\frac{1}{2})\Gamma(z) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$.

(b) if G is complex we have $m_\alpha = 2$ and $m_{2\alpha} = 0$. Then (see [39, Theorem 5.7])

$$c(\lambda) = \prod_{\alpha\in\Phi_0^+} \frac{(\varrho,\alpha)}{(i\lambda,\alpha)}.$$

Let us finally compare the normalization $d\overline{n}$ with μ_v for $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$. For $SL_2(\mathbb{R})$, denote by $\tilde{c}(\lambda)$ the c-function using the measure dx on the unipotent group

$$\overline{N} = \left\{ n_x = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}.$$

Since the Haar measures satisfies $dx = \kappa d\overline{n}$, we have $\tilde{c}(\lambda) = \kappa c(\lambda)$ with

$$\kappa = \tilde{c}(-i\varrho) = \int_{\mathbb{R}} e^{-2\langle\varrho,\tilde{H}(n_x)\rangle}\, dx. \tag{2.113}$$

So we must compute κ. One computes the Iwasawa decomposition of n_x

$$n_x = \begin{pmatrix} \xi^{-1} & -x\xi^{-1} \\ x\xi^{-1} & \xi^{-1} \end{pmatrix} \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \begin{pmatrix} 1 & x\xi^{-2} \\ 0 & 1 \end{pmatrix},$$

with $\xi = \sqrt{1+x^2}$. So it follows that $-2\langle\varrho,\tilde{H}(n_x)\rangle = -2\log\xi$, and hence

$$\kappa = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi. \tag{2.114}$$

Using $\tilde{c}(-i\lambda) = \kappa c(-i\lambda)$, (2.114), $(\lambda, 2\alpha_0) = \langle\lambda,\alpha^\vee\rangle$, (2.66) and (a) above, we get:

$$\tilde{c}(-i\lambda) = \frac{\sqrt{\pi}\Gamma(\frac{1}{2}\langle\lambda,\alpha^\vee\rangle)}{\Gamma(\frac{1}{2}(\langle\lambda,\alpha^\vee\rangle+1))} = \frac{\Gamma_{\mathbb{R}}(\langle\lambda,\alpha^\vee\rangle)}{\Gamma_{\mathbb{R}}(\langle\lambda,\alpha^\vee\rangle+1)}.$$

This formula is in accordance with Proposition 2.7.20 in this case. We do the same for $SL_2(\mathbb{C})$ with the Haar measure $dx = |dx \wedge d\overline{x}| = 2rdrdt$, in polar coordinates. We must compute κ as in (2.113) so that $\tilde{c}(\lambda) = \kappa c(\lambda)$, now with $\mathbb{C}$ instead of $\mathbb{R}$. Given $n_x \in \overline{N}$, the Iwasawa decomposition of n_x is

$$n_x = \begin{pmatrix} \xi^{-1} & -\overline{x}\xi^{-1} \\ x\xi^{-1} & \xi^{-1} \end{pmatrix} \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \begin{pmatrix} 1 & \overline{x}\xi^{-2} \\ 0 & 1 \end{pmatrix},$$

now with $\xi = \sqrt{1 + x\overline{x}}$. In the complex case (see [39, Ch. IV, §5, no. 2]), we have that $\varrho = \alpha$ so that $-2\langle \varrho, \tilde{H}(n_x)\rangle = -4\log\xi$, and hence

$$\kappa = 2\int_0^{2\pi} dt \int_0^{\infty} \frac{rdr}{(1+r^2)^2} = 2\pi. \tag{2.115}$$

Using (2.66), (2.115), $(\lambda, 2\alpha_0) = \langle \lambda, \alpha^\vee \rangle$, item (b) above and $\Gamma(x+1) = x\Gamma(x)$, we get:

$$\tilde{c}(-i\lambda) = \frac{2\pi}{(\lambda, \alpha_0)} = \frac{2\pi\Gamma((\lambda, \alpha_0))}{\Gamma(\langle \lambda, \alpha_0\rangle + 1)} = \frac{\Gamma_{\mathbb{C}}(\langle \lambda/2, \alpha^\vee\rangle)}{\Gamma_{\mathbb{C}}(\langle \lambda/2, \alpha^\vee\rangle + 1)},$$

or equivalently, that

$$\tilde{c}(-2i\lambda) = \frac{\Gamma_{\mathbb{C}}(\langle \lambda, \alpha^\vee\rangle)}{\Gamma_{\mathbb{C}}(\langle \lambda, \alpha^\vee\rangle + 1)},$$

which is in accordance to Proposition 2.7.20. This ends the example.

We now finish the computation of the inner product. Recall that the measures du_v used to compute the local intertwining operators were induced from the measure μ_v of the local field. However, in the global intertwiner operator (2.92), the measure used was the one that assigned volume one to each $\underline{U}_\alpha(F)\backslash\underline{U}_\alpha(\mathbb{A})$, for $\alpha \in \Phi_0$. We should, therefore, use the measures ν_v of (2.72) instead of μ_v. But, using the formula of Proposition 2.6.3, we have $\nu = |d|^{-1/2}\mu$, where d is the discriminant of F. Also, from (2.67), we have $\prod_v \Gamma_v(s) = |d|^{-s/2}\Lambda(s)$. Noting further that $(\underline{U}(\mathbb{A}) \cap w\underline{U}(\mathbb{A})w^{-1}\backslash\underline{U}(\mathbb{A}))$ is homeomorphic to $\ell(w)$ copies of $\mathbb{A}$, we obtain

$$\begin{aligned}(M(w,\lambda)f_\lambda^\circ)(1) &= |d|^{-\ell(w)/2} \prod_{\alpha\in\Phi(w)} \left(\prod_v \frac{\Gamma_v(\langle \lambda, \alpha^\vee\rangle)}{\Gamma_v(\langle \lambda, \alpha^\vee\rangle + 1)} \right) \\ &= \prod_{\alpha\in\Phi(w)} \frac{\Lambda(\langle \lambda, \alpha^\vee\rangle)}{\Lambda(\langle \lambda, \alpha^\vee\rangle + 1)}.\end{aligned}$$

We conclude from this and (2.105) that for $\varphi, \psi \in PW(a_{T,\mathbb{C}}^{G*})$ and λ_0 such that $\lambda - \varrho$ is in the interior of the fundamental chamber, we have

$$(\theta_\varphi, \theta_\psi) = \int_{\mathrm{Re}\,\lambda=\lambda_0} \sum_{w\in W_0} \prod_{\alpha\in\Phi(w)} \frac{\Lambda(\langle \lambda, \alpha^\vee\rangle)}{\Lambda(\langle \lambda, \alpha^\vee\rangle + 1)} \overline{\varphi(-w\overline{\lambda})}\psi(\lambda)\, d\lambda. \tag{2.116}$$

2.7.12 Measures

For the situation of Section 2.7.10, which we are concerned with in this thesis, we fixed Haar measures on the unipotent subgroups and on $\mathbf{K}$ in Section 2.7.2, but we left unspecified the measure $d\lambda$ of the space $\lambda_0 + ia_T^{G*}$ used in the inner product formula (2.116) and the measure on $\underline{T}(\mathbb{A})$. These choices are not essential for the residue calculus that lies ahead of us. But, it is quite important in order to obtain the isometry of Theorem A, and, specially in the semisimple case, to obtain some nontrivial identities using harmonic analysis. We fix a measure on a_T^* in the following way. Fix a basis $\{y_1, \ldots, y_R\} \subseteq X_*(\underline{T})$ and define $(2\pi)^{-R} dy_1 \wedge \ldots \wedge dy_R$ as a measure on a_T^*. Then, let $\{y_1, \ldots, y_r\}$ be a basis of the lattice Y, given as the image under the orthogonal projection $a_T \to a_T^G$ (see Section 4.2) of the cocharacter lattice $X_*(\underline{T})$, with r the semisimple rank of $\underline{G}$. Define a measure on the real vector space a_T^{G*} by $d\mu^Y := dy_1 \wedge \cdots \wedge dy_r$. The measure on ia_T^{G*} used in (2.116) is

$$d\lambda := (2\pi)^{-r} d\mu^Y(\operatorname{Im} \lambda). \tag{2.117}$$

We fix the measure on $\underline{T}(\mathbb{A})$ by letting

$$\operatorname{vol}(\underline{T}(F)\backslash\underline{T}(\mathbb{A})^1) = 1 \tag{2.118}$$

(we know from Proposition 2.6.6 that $\underline{T}(F)\backslash\underline{T}(\mathbb{A})^1$ is compact) and we define a measure on $\underline{T}(\mathbb{A})^1\backslash\underline{T}(\mathbb{A}) \cong a_T$ by duality with the measure $(2\pi)^{-r} d\mu^Y$ of a_T^*.

Chapter 3

Transition between spherical spectra

The aim of this chapter is to compute the precise rational constants of (2.50) occurring the in Y-functional of the spectral problem of the Yang system. In its turn, the Plancherel measure of the Yang system ν_P (see Definition 2.4.8) will be an important ingredient in the formulation of the spectral measure for the unramified spherical automorphic spectrum.

The computation of these constants is a special case of the proof of the HII conjecture [40] given in [76] in the context of affine Hecke algebras. The new contribution to subject that will be presented in this chapter is a limit transition between the spherical Plancherel measure of the relevant affine Hecke algebra and the measure of the Yang system.

3.1 Initial considerations

Let $\underline{G}$ be a Chevalley (semisimple) group and F a number field. Let also $\underline{T} \subseteq \underline{B} \subseteq \underline{G}$ be a maximal split torus and a Borel subgroup. The choice of $\underline{B}$ defines a set of simple roots Δ_0. By exchanging the roles of $X^*(\underline{T})$ and $X_*(\underline{T})$, we have $\mathcal{R}^\vee = (X_*(\underline{T}), \Delta_0^\vee, X^*(\underline{T}), \Delta_0)$, a based root datum dual to $\mathcal{R}(\underline{G}, \underline{B}, \underline{T})$. We let $({}^L G, {}^L B, {}^L T)$ be the triple consisting of an algebraic group, a Borel subgroup and a maximal torus, all defined over $\mathbb{C}$, associated to $\mathcal{R}^\vee$, whose existence up to isomorphism was discussed in Proposition 2.2.3. Let

$$\begin{aligned} a_T &= \mathbb{R} \otimes X_*(\underline{T}) \cong \mathbb{R} \otimes X^*({}^L T) \supseteq \Phi_0^\vee \\ a_T^* &= \mathbb{R} \otimes X_*({}^L T) \cong \mathbb{R} \otimes X^*(\underline{T}) \supseteq \Phi_0. \end{aligned}$$

Let also W_0 denote the finite Weyl group of the root system Φ_0 and r be the rank of $\underline{G}$. We also fix, once and for all, a W_0-invariant inner product on a_T turning the pair $(a_T, (\cdot,\cdot))$ into a Euclidean space. We thus have $V^*, (\cdot,\cdot), V, \langle\cdot,\cdot\rangle, \mathcal{R}^\vee$, and W_0 as in Section 2.3.1, where $V^* := a_T$ and $V := a_T^*$. Due to the dualities involved, the notations for roots and coroots will be swapped from what was used in Chapter 2.

Convention. In this Chapter, let us write $X := X^*(\underline{T})$ and $Y := X_*(\underline{T})$. We will denote elements in $V_\mathbb{C}^* \supseteq Y$ by ξ or y and elements of $V_\mathbb{C} \supseteq X$ by λ or x.

3.2 Approximating functions

We let $q \in \mathbb{C}^\times$. We wish to vary the parameters q in order to obtain useful information. But our main concern is for $q = q_v \in \mathbb{R}_{>1}$, the cardinality of the residue field of F_v, for a finite place $v \in \mathcal{V}_f$. In fact, it will be more relevant the reciprocal q_v^{-1}. So consider the affine Hecke algebra $\mathcal{H}(\mathcal{R}^\vee, q^{-1})$ with q arbitrary and let $k \in \mathbb{C}$ with $q = e^k$. Here, the spherical algebra $\mathcal{H}^+$ is given by $T_0^+\mathcal{H}(\mathcal{R}^\vee, q^{-1})T_0^+$. Let V_+ be the fundamental chamber with respect to Δ_0. Given $\lambda \in V_\mathbb{C} = \mathbb{C} \otimes X$, consider the element $s := q^\lambda = e^{k\lambda} \in {}^LT$, viewed as the complex homomorphism $y \mapsto q^{\langle y,\lambda\rangle} = e^{k\langle y,\lambda\rangle}$ on the lattice Y. The latter expression extends to an entire function of $k \in \mathbb{C}$, $\lambda \in V_\mathbb{C}$ and $\xi \in V_\mathbb{C}^*$. Using this and Proposition 2.3.8(b), we lift and extend the definition of the spherical function φ_s to any $\xi \in V_\mathbb{C}^*$. For $k \in \mathbb{C}$ and $\lambda \in V_\mathbb{C}$ we define:

$$\begin{aligned} \psi(k,\lambda;\xi) &:= P_{W_0}(q)^{-1} \textstyle\sum_{w\in W} c(q^{w\lambda}, q^{-1}) e^{\langle \xi, w\lambda\rangle} \\ &= P_{W_0}(q)^{-1} \textstyle\sum_{w\in W} c(q^{w\lambda}, q^{-1}) q^{\langle \xi/k, w\lambda\rangle}. \end{aligned} \tag{3.1}$$

By the first line it is clear that ψ extends to a meromorphic function in k, λ, and ξ with poles at the union $\cup_{n\in\mathbb{Z}\setminus\{0\},\alpha\in\Phi^+} H_{n,\alpha} \times V_\mathbb{C}^*$, where $H_{n,\alpha} \subseteq \mathbb{C}\times V_\mathbb{C}$ denotes the complex hypersurface defined by $H_{n,\alpha} := \{(k,\lambda) \mid k\langle\alpha^\vee,\lambda\rangle + n2\pi i = 0\}$ (recall the expression (2.18) for the c-function). Informally, for $k > 0$, we can think of (3.1) as $\psi(k,\lambda;\xi) = \varphi_{e^{k\lambda}}(\theta^+_{\xi/k})$ if the right hand side makes sense. In particular, if $y \in Y^+$ we have:

$$\varphi_{q^\lambda}(\theta_y^+) = \psi(k,\lambda;ky). \tag{3.2}$$

Consider the entire function R in two complex variables k and z given by $R(k,z) := (kz)^{-1}(e^{kz} - 1)$. Note that R is holomorphic on both parameters and has zeroes along $H_{n,\alpha}$, for $n \in \mathbb{Z}\setminus\{0\}$ and $\alpha \in \Phi_0^+$. For each $\alpha \in \Phi_0$ define

$$G_\alpha(k,\lambda) := \frac{R(k, \langle\alpha^\vee,\lambda\rangle - 1)}{R(k,\langle\alpha^\vee,\lambda\rangle)}, \tag{3.3}$$

and define $G(k,\lambda) := \prod_{\alpha\in\Phi_0^+} G_\alpha(k,\lambda)$.

Lemma 3.2.1. *For all $\alpha \in \Phi_0$, G_α is meromorphic on $\mathbb{C}\times V_\mathbb{C}$ with poles inside the complex hypersurface $\cup_{n\in\mathbb{Z}\setminus\{0\},\alpha\in\Phi_0^+} H_{n,\alpha}$, and $G_\alpha(0,\lambda) = 1$. Moreover,*

$$c(q^\lambda, q^{-1}) = q^{|\Phi_0^+|} G(k,\lambda) c_\mathbb{H}(\lambda, -1).$$

Proof. For each $\alpha \in \Phi_0^+$, note that

$$\begin{aligned} c_\alpha(q^\lambda, q^{-1}) &= \frac{1 - qq^{\langle -\alpha^\vee, \lambda \rangle}}{1 - q^{\langle -\alpha^\vee, \lambda \rangle}} \\ &= q\frac{q^{\langle \alpha^\vee, \lambda \rangle - 1} - 1}{q^{\langle \alpha^\vee, \lambda \rangle} - 1} \\ &= qG_\alpha(k, \lambda)\gamma_\alpha(\lambda, -1), \end{aligned}$$

proving that the equation in the statement is valid. The meromorphicity, location of the poles and the evaluation at $k = 0$ are obvious (using $R(0, z) = 1$). $\square$

The next Corollary summarizes the analytic properties of the function $\psi(k, \lambda; \xi)$ that we need. To state it, let Z_{W_0} denote the set of zeroes of $P_{W_0}(q^{-1})$, viewed as a function of k and ${}_-V \subseteq V$ denote the antidominant chamber $\sum_{\alpha \in \Delta_0} \mathbb{R}_- \alpha$. We also recall that for a distinguished unipotent orbit $\mathbf{o}$, $2\lambda(\mathbf{o}) \in V$ is the associated (dominant) weighted Dynkin diagram (see (2.56) and (2.57)).

Corollary 3.2.2. *(1) If $\lambda \in V_\mathbb{C}$ is regular, then for all $\xi \in V^*$ we have*

$$\psi(k, \lambda; \xi) = \frac{1}{P_{W_0}(q^{-1})} \sum_{w \in W_0} G(k, w\lambda) c_\mathbb{H}(w\lambda, -1) e^{\langle \xi, w\lambda \rangle}.$$

(2) The function ψ extends to a meromorphic function in k, λ, and ξ, with poles along the set

$$\left(\cup_{n \in \mathbb{Z} \setminus \{0\}, \alpha \in \Phi_0^+} H_{n,\alpha} \times V_\mathbb{C}^* \right) \cup Z_{W_0} \times V_\mathbb{C} \times V_\mathbb{C}^*.$$

Given $k \in \mathbb{R}$, ψ is holomorphic on $\cap_{\alpha \in \Phi_0^+} \{(y, \lambda) \mid |\operatorname{Im}\langle \alpha^\vee, \lambda \rangle| < 2\pi/|k|\}$, and the domain of holomorphicity of $\psi(k, \lambda; y)$ contains the set $\{0\} \times V \times V_\mathbb{C}^*$.

(3) For each distinguished unipotent orbit $\mathbf{o}$ one has an expansion of the form

$$\psi(k, \lambda(\mathbf{o}); \xi) = \sum_{\lambda \in W_0 \lambda(\mathbf{o})} a(k, \lambda; \xi) e^{\langle \xi, \lambda \rangle}, \tag{3.4}$$

where $a(k, \lambda; \xi)$ is a harmonic polynomial in $\xi \in V^*$ with respect to the reflection group W_λ, with coefficients which are holomorphic in k in a neighborhood of $k = 0$. Moreover, $a(k, \lambda; \xi) = 0$ unless $\lambda \in {}_-V$.

(4) With respect to λ, $\psi(k, \lambda; \xi)$ is W_0-invariant.

(5) When evaluated at $k = 0$, $\psi(0, \lambda; \xi) = \phi(\lambda, -1; \xi)$.

(6) For $k_0 \in \mathbb{R}_{>0}$ sufficiently small, there exists a $\mu_\mathbf{o}$ in the interior of the antidominant chamber ${}_-V$ and $C > 0$ such that for all $k \in [0, k_0]$ and $\xi \in V_+^*$ we have $|\psi(k, \lambda(\mathbf{o}); \xi)| \leq Ce^{\langle \xi, \mu_\mathbf{o} \rangle}$.

Remark. The harmonic polynomials $a(k,\lambda;\xi)$ introduced in Corollary 3.2.2(3) coincides, when $k=0$, with the harmonic polynomials $a(\lambda,-1;\xi)$ of [36, Corollary 3.8] associated to $\phi(\lambda,-1;\xi)$, because of item (5). We refer to [39, Chapter III] for the generalities of harmonic polynomials.

Proof. Using the definition (3.1) and the fact that $P_{W_0}(q)q^{-|\Phi_0^+|} = P_{W_0}(q^{-1})$, item (1) follows immediately from Lemma 3.2.1. For item (2), note that for λ generic we can write, by (1):

$$\psi(k,\lambda;\xi) = \frac{\sum_{w\in W_0}\epsilon(w)\prod_{\alpha\in\Phi_0^+}(\langle w\alpha^\vee,\lambda\rangle+1)G(k,w\lambda)e^{\langle w\xi,\lambda\rangle}}{P_{W_0}(q^{-1})\prod_{\alpha\in\Phi_0^+}\langle\alpha^\vee,\lambda\rangle}, \tag{3.5}$$

where $\epsilon(w)=(-1)^{\ell(w)}$. The numerator divided by the factor $P_{W_0}(q^{-1})$ of the denominator is visibly skew invariant in $\lambda\in V_{\mathbb{C}}$ for the action of W_0, and meromorphic in k, λ and ξ as claimed by the properties of $G(k,\lambda)$. The skew invariance implies that the numerator is divisible by the remaining factor $\prod_{\alpha\in\Phi_0^+}\langle\alpha^\vee,\lambda\rangle$ of the denominator, and the result follows.

As to (3): let $\lambda(\mathbf{o})\in\overline{V_+}$. Its stabilizer $W_{\lambda(\mathbf{o})}$ is generated by reflections along hyperplanes orthogonal (with respect to $\langle\cdot,\cdot\rangle$) to some coroots in $\Phi_0^\vee$ and denote by $W^{\lambda(\mathbf{o})}$ the set of representatives of minimal length of $W_0/W_{\lambda(\mathbf{o})}$. Let $\Phi_{\lambda(\mathbf{o})}$ denote the set of roots whose associated reflection is in $W_{\lambda(\mathbf{o})}$. Let also $V_{\lambda(\mathbf{o})}$ denote the real subspace of V spanned by $\Phi_{\lambda(\mathbf{o})}$. Write $\lambda_t=\lambda(\mathbf{o})+t\mu$ for $t\in\mathbb{R}_+$, where $\mu\in V_{\lambda(\mathbf{o})}$ is regular and dominant (with respect to $\Phi^+_{\lambda(\mathbf{o})}:=\Phi_0^+\cap\Phi_{\lambda(\mathbf{o})}$). Clearly, for sufficiently small positive t, λ_t is regular and dominant. By (2) we have $\psi(k,\lambda(\mathbf{o});\xi)=\lim_{t\to 0}\psi(k,\lambda_t;\xi)$. From (3.5) it follows that $\psi(k,\lambda_t;\xi)$ admits an expansion of the form

$$\begin{aligned}\psi(k,\lambda_t;\xi) &= \sum_{u\in W^{\lambda(\mathbf{o})}}\prod_{\alpha\in\Phi^+_{\lambda(\mathbf{o})}}\frac{1}{\langle\alpha^\vee,\lambda_t\rangle}\sum_{v\in W_{\lambda(\mathbf{o})}}\epsilon(v)H_u(k,v\lambda_t)e^{\langle\xi,uv\lambda_t\rangle}\\ &= \sum_{u\in W^{\lambda(\mathbf{o})}}\left(\sum_{v\in W_{\lambda(\mathbf{o})}}\frac{\epsilon(v)H_u(k,\lambda(\mathbf{o})+tv\mu)e^{\langle\xi,uv(t\mu)\rangle}}{\prod_{\alpha\in\Phi^+_{\lambda(\mathbf{o})}}\langle\alpha^\vee,t\mu\rangle}\right)e^{\langle\xi,u\lambda(\mathbf{o})\rangle},\end{aligned}$$

where for all u, $H_u(k,\lambda)$ is holomorphic in a neighborhood of $(k,\lambda)=(0,\lambda(\mathbf{o}))$. It is standard to see [83, Section 3] that in the limit for $t\to 0$ the expression between brackets yields a harmonic polynomial $a(k,u\lambda(\mathbf{o});\xi)$ as claimed in (3.4). What remains to show is that $a(k,\lambda;\xi)$ vanished identically for all λ outside the interior of the anti-dual cone. By the holomorphicity of $a(k,\lambda;\xi)$ as a function of k in a small neighborhood of 0, it is enough to show such vanishing for all k small and positive. So let $k>0$ be sufficiently small, and consider (3.2) for $\lambda=\lambda(\mathbf{o})$. It follows that the restriction of $|\psi(k,\lambda(\mathbf{o});y)|^2$ to $y\in kY^+$ is summable. By Casselman's criterion [19, Section 4.4] this implies the result.

Item (4) is straight-forward, and for (5) we observe that it suffices (by (2)) to show this for regular λ only. But then the result is clear from (1), the identity $P_{W_0}(1)=|W_0|$ and Lemma 3.2.1.

For (6), take $\mu_{\mathbf{o}} \in {}_{-}V$ small enough such that for all $\lambda \in W\lambda(\mathbf{o})$ satisfying $a(k, \lambda(\mathbf{o}); \xi)$ is not identically zero, we have $\lambda \in \mu_{\mathbf{o}} + {}_{-}V$. This is possible because all such λ belong to ${}_{-}V$, which is an open cone.

Since the degree of harmonic polynomials is bounded by the number of positive roots [39, Theorem 3.6], and since the coefficients $a(k, \lambda(\mathbf{o}); \xi)$ are holomorphic for $k \in [0, k_0]$ (provided that k_0 is chosen small enough, by (3)), the task at hand clearly reduces to proving that for any monomial $m = \alpha_1^{p_1} \ldots \alpha_n^{p_n}$ (with α_i running through the basis of simple roots) and any $\kappa = \lambda - \mu_{\mathbf{o}} \in {}_{-}V$, there exists a constant $C_m > 0$ such that $m(\xi) \leq C_m e^{-\langle \xi, \kappa \rangle}$ for all $\xi \in V_+^*$. This is elementary. Since there are only finitely many nonzero terms to consider, we can take the same constant $C > 0$ for all m and $\lambda \in W\lambda(\mathbf{o})$ which appear in the expansion of $\psi(k, \lambda(\mathbf{o}); \xi)$. This finishes the proof. □

3.3 Fourier transform

Given a distinguished orbit $\mathbf{o}$, we let $s_{\mathbf{o}} := q^{\lambda(\mathbf{o})} \in {}^L T$. We obtain a spherical representation $\pi_{\mathbf{o}} := \pi_{s_{\mathbf{o}}}$ of $\mathcal{H}(\mathcal{R}, q^{-1})$ (see Section 2.3.3(d)) and an elementary spherical function $\varphi_{\mathbf{o}} := \varphi_{s_{\mathbf{o}}}$ as in (2.19). We will explain below that $\pi_{\mathbf{o}}$ is a spherical discrete series representation of $\mathcal{H}(\mathcal{R}, q^{-1})$ and $\varphi_{\mathbf{o}}$ is a matrix coefficient of $\pi_{\mathbf{o}}$. The aim of this section is to relate the L^2-norm of $\varphi_{\mathbf{o}}$ and the Plancherel measure of $\pi_{\mathbf{o}}$. Let us start with the latter.

Recall that the restriction of the trace τ to the spherical subalgebra $\mathcal{H}(q)^+ = T_0^+ \mathcal{H}(\mathcal{R}^\vee, q) T_0^+$ can be decomposed as (see [72, Section 3.4], [73], [35])

$$\tau|_{\mathcal{H}(q)^+} = \int_{\widehat{\mathcal{H}(q)^+} \subseteq W_0 \backslash {}^L T} \chi_\pi \, d\nu^+(\pi, q), \tag{3.6}$$

where $d\nu^+(\cdot, q)$ is the Plancherel measure of $\widehat{\mathcal{H}(q)^+}$. For $q = q_v \in \mathbb{R}_{>1}$, the cardinality of the residue field of F_v at a finite place $v \in \mathcal{V}_f$, it is well-known (see [59] and [72, Section 3.4]) that $\mathcal{H}(q)^+$ only has continuous spectrum. But, as mentioned before, the relevant algebra we will consider in our context is $\mathcal{H}^+ := \mathcal{H}(q^{-1})^+$, which does have discrete spectrum (see [35], [36], [72], [73]).

Similar to Section 2.3.4 for the affine Hecke algebra, we consider the Hilbert space $L^2(\mathcal{H}^+)$ and $(\mathcal{H}^+)^*$. Recall that the linear dual $(\mathcal{H}^+)^*$ of $\mathcal{H}^+$ (with its weak-star topology) is identified with the direct product $\prod_{y \in Y^+} \mathbb{C}\theta_y^+$ (with its product topology). Further, the Schwartz completion $S(\mathcal{H}^+)$ of $\mathcal{H}^+$ is the subalgebra $T_0^+ S(\mathcal{H}(\mathcal{R}, q^{-1})) T_0^+$ of the Schwartz completion of the affine Hecke algebra. It is known that $T_0^+ S(\mathcal{H}(\mathcal{R}, q^{-1})) T_0^+$ is a unital Fréchet algebra, which has the commutative algebra $\mathcal{H}^+$ as a dense subalgebra. Hence, $S(\mathcal{H}^+)$ is itself commutative. We also have an inclusions $\mathcal{H}^+ \subseteq S(\mathcal{H}^+) \subseteq L^2(\mathcal{H}^+) \subseteq (\mathcal{H}^+)^*$, as dense subspaces.

Recall (see [73], [25]) that the support $\widehat{\mathcal{H}^+}$ of $\nu^+(\pi, q^{-1})$, the Plancherel measure of the trace τ^+ of $\mathcal{H}^+$, is the set of irreducible tempered characters of $\mathcal{H}^+$, that is, those irreducible characters of $\mathcal{H}^+$ which extend continuously to $S(\mathcal{H}^+)$. In its turn, this set is in natural bijection with the set of tempered spherical irreducible representations of $\mathcal{H}(\mathcal{R}^\vee, q^{-1})$. This bijection assigns to a tempered spherical irreducible representation of $\mathcal{H}(\mathcal{R}^\vee, q^{-1})$ the tempered character of $\mathcal{H}^+$

$$(\pi, E) \mapsto (\pi^+, E^+), \tag{3.7}$$

where π^+ is the restriction of π to $\mathcal{H}^+$, acting on the line $E^+ \subseteq E$ of spherical vectors in E. An irreducible spherical tempered representation of $\mathcal{H}(\mathcal{R}^\vee, q^{-1})$ is a discrete series representation if it extends continuously to $L^2(\mathcal{H}(\mathcal{R}^\vee, q^{-1}))$. The corresponding character (π^+, E^+) is called a discrete series character. It is known [73, Lemma 2.22] that all matrix elements of a discrete series representation (π, E) of $\mathcal{H}(\mathcal{R}^\vee, q^{-1})$ belong to $S(\mathcal{H}(\mathcal{R}^\vee, q^{-1}))$. Let us try to explain (3.7) for some discrete series representations that will important to us.

For $\mathbf{o}$ a distinguished orbit, let $(\pi_{\mathbf{o}}, E_{\mathbf{o}})$ be the unique spherical subquotient of the minimal principal series at $s_{\mathbf{o}} = q^{\lambda(\mathbf{o})}$. Since $s_{\mathbf{o}}$ is a real residual point of ${}^LT$ it satisfies $s_{\mathbf{o}}^{-1} \in W_0 s_{\mathbf{o}}$ (see Proposition 2.4.4 and [73, Theorem A.14]). Also, $\pi_{\mathbf{o}}$ is isomorphic to the unique spherical subquotient of any of the minimal principal series representations $\pi_{ws_{\mathbf{o}}}$ with $w \in W_0$. It follows that $\pi_{\mathbf{o}}$ is a self-dual representation. The fact that $s_{\mathbf{o}}$ is residual also implies that $\pi_{\mathbf{o}}$ is a discrete series representation (see [35, Theorem 3.1] and the references [48], [58], therein). Now equip $E_{\mathbf{o}}$ with a (unique up to scaling by a positive factor) inner product such that $\pi_{\mathbf{o}}$ is a $*$-representation. Let $v_{\mathbf{o}} \in E_{\mathbf{o}}$ denote the unit spherical vector. Being a discrete series and a self-dual $*$-representation implies the following:

(a) the spherical function $\varphi_{\mathbf{o}}(h) = (hv_{\mathbf{o}}, v_{\mathbf{o}})$, is an element of $L^2(\mathcal{H}^+)$,

(b) $\varphi_{\mathbf{o}}(h^*) = \overline{\varphi_{\mathbf{o}}(h)}$ for all $h \in L^2(\mathcal{H}^+)$, or $\varphi_{\mathbf{o}}^* = \varphi_{\mathbf{o}}$,

(c) the representation $\pi_{\mathbf{o}} : \mathcal{H}(\mathcal{R}^\vee, q^{-1}) \to \mathrm{End}(E_{\mathbf{o}})$ has a unique extension to $L^2(\mathcal{H}(\mathcal{R}^\vee, q^{-1}))$,

(d) elements of $L^2(\mathcal{H}^+)$ map $E_{\mathbf{o}}$ to the line $E_{\mathbf{o}}^+$, spanned by $v_{\mathbf{o}} \in E_{\mathbf{o}}$.

Hence, $\pi_{\mathbf{o}}(h)$ is a multiple of the orthogonal projection operator $v_{\mathbf{o}}^* \otimes v_{\mathbf{o}}$ for all $h \in L^2(\mathcal{H}^+)$. The restriction of $\pi_{\mathbf{o}}(h)$ to $E_{\mathbf{o}}^+$ defines a discrete series character $\pi_{\mathbf{o}}^+$ of $\mathcal{H}^+$.

Now, consider the spherical Fourier transform $\mathcal{F}$ (see [73, Definition 4.41], [35, (2.6)]) evaluated at $\pi_{\mathbf{o}}$:

$$\mathcal{F}_{\mathbf{o}} := \mathcal{F}_{\pi_{\mathbf{o}}} : L^2(\mathcal{H}^+) \to \mathrm{End}(E_{\mathbf{o}}^+) = \mathbb{C}. \tag{3.8}$$

For $h \in L^2(\mathcal{H}^+)$ we have $\pi_{\mathbf{o}}(h)v_{\mathbf{o}} = (hv_{\mathbf{o}}, v_{\mathbf{o}})v_{\mathbf{o}} = \varphi_{\mathbf{o}}(h)v_{\mathbf{o}}$, hence $\mathcal{F}_{\mathbf{o}}(h) = \varphi_{\mathbf{o}}(h)$. Otherwise phrased, we have for all $h \in \mathcal{H}^+$ that

$$\mathcal{F}_{\mathbf{o}}(h) = \varphi_{\mathbf{o}}(h) = \mathrm{Tr}_{E_{\mathbf{o}}}((v_{\mathbf{o}}^* \otimes v_{\mathbf{o}})\pi_{\mathbf{o}}(h)) = \mathrm{Tr}_{E_{\mathbf{o}}}(\pi_{\mathbf{o}}(h)) = \pi_{\mathbf{o}}^+(h). \tag{3.9}$$

(The third equality follows because we know a priori that $\pi_{\mathbf{o}}(h)$ is a multiple of the rank one idempotent $v_{\mathbf{o}}^* \otimes v_{\mathbf{o}}$, as explained above).

Now suppose that (E, π) is any tempered irreducible spherical representation of $\mathcal{H}$. We fix an inner product on E such that π becomes a $*$-representation and let v_π be the spherical vector. Since $\varphi_{\mathbf{o}} \in S(\mathcal{H}^+)$ and since the Fourier transform $\mathcal{F}$ on $\mathcal{H}^+$ extends uniquely and continuously to $S(\mathcal{H}^+)$, we can apply $\varphi_{\mathbf{o}}$ to the spherical Fourier transform evaluated at π. As before, we have:

$$\mathcal{F}_\pi(\varphi_{\mathbf{o}}) = (\pi(\varphi_{\mathbf{o}})v_\pi, v_\pi) = \varphi_\pi(\varphi_{\mathbf{o}}).$$

Now $\mathcal{H}^+$ is commutative, and the algebra homomorphisms $\mathcal{H}^+ \to \mathbb{C}$ are in bijection with $W_0 \backslash {}^L T$. Let $W_0 s \in W_0 \backslash {}^L T$ be the algebraic character associated to π^+, and $W_0 s_{\mathbf{o}}$ the algebraic character associated to $\pi_{\mathbf{o}}^+$. Assume that π^+ is not equivalent to $\pi_{\mathbf{o}}^+$, or equivalently $W_0 s_{\mathbf{o}} \neq W_0 s$. Let $h \in \mathcal{H}^+$. Then, on the one hand, we have $\mathcal{F}_\pi(h\varphi_{\mathbf{o}}) = \mathcal{F}_\pi(W_0 s_{\mathbf{o}}(h)\varphi_{\mathbf{o}}) = W_0 s_{\mathbf{o}}(h)\mathcal{F}_\pi(\varphi_{\mathbf{o}})$ while on the other, we have $\mathcal{F}_\pi(h\varphi_{\mathbf{o}}) = \mathcal{F}_\pi(h)\mathcal{F}_\pi(\varphi_{\mathbf{o}}) = W_0 s(h)\mathcal{F}_\pi(\varphi_{\mathbf{o}})$. Since this holds for all $h \in \mathcal{H}^+$ we conclude that $\mathcal{F}_\pi(\varphi_{\mathbf{o}}) = 0$. It follows from this and from (3.9) that

$$\mathcal{F}(\varphi_{\mathbf{o}}) = \{\pi \mapsto \mathcal{F}_\pi(\varphi_{\mathbf{o}})\} = (\varphi_{\mathbf{o}}, \varphi_{\mathbf{o}})\delta_{\cdot,\pi_{\mathbf{o}}},$$

where, $\delta_{\pi,\pi_{\mathbf{o}}}$ is the Kronecker delta. By this and the fact that the Fourier transform $\mathcal{F}$ is an isometry [73, Theorem 4.43] we obtain the identity:

$$(\varphi_{\mathbf{o}}, \varphi_{\mathbf{o}}) = \nu^+(\pi_{\mathbf{o}}, q^{-1})^{-1}. \tag{3.10}$$

3.4 Normalizations

In this section, we retain the notation $q > 1$, $s_{\mathbf{o}} = q^{\lambda(\mathbf{o})}$, $\mathcal{H}^+ = \mathcal{H}(q^{-1})^+$ and $\varphi_{\mathbf{o}} = \varphi_{s_{\mathbf{o}}}$ for $\mathbf{o}$ a distinguished unipotent orbit. By manipulating equation (3.10), we will arrive at an expression that will allow us to obtain the precise constants of discrete part of the Plancherel measure of the Yang system. We will start by normalizing the trace τ of the Hecke algebra $\mathcal{H}(\mathcal{R}^\vee, q^{-1})$ in the sense of [75, Section 3]. Let $d := (q^{1/2} - q^{-1/2})^{-r}$ (cf. [76, Section 4.6]), with r the rank of $\underline{G}$. The normalization to be used is

$$\tau_{\mathrm{n}} := d\tau. \tag{3.11}$$

With this normalization we obtain a new Hermitian pairing on the Hecke algebra $\mathcal{H}(\mathcal{R}^\vee, q^{-1})$ (cf. (2.15) and Proposition 2.3.6) which will be denoted $(\cdot,\cdot)_{\mathrm{n}}$.

Lemma 3.4.1. *With respect to the normalized pairing we have*

$$(\varphi_{\mathbf{o}}, \varphi_{\mathbf{o}})_{\mathrm{n}} = d \sum_{y \in Y^+} |\psi(k, \lambda(\mathbf{o}); ky)|^2 \frac{P_{W_0}(q^{-1})^2}{q^{-\ell(w^y)} P_{W_y}(q^{-1})}.$$

Proof. Using Proposition 2.3.9 and $(\cdot,\cdot)_{\mathrm{n}} = d(\cdot,\cdot)$, we have

$$\begin{aligned}(\varphi_{\mathbf{o}},\varphi_{\mathbf{o}})_{\mathrm{n}} &= d\sum_{y,z\in Y^+}\frac{(\varphi_{\mathbf{o}},\theta_y^+)}{(\theta_y^+,\theta_y^+)}\frac{\overline{(\varphi_{\mathbf{o}},\theta_z^+)}}{(\theta_z^+,\theta_z^+)}(\theta_y^+,\theta_z^+)\\ &= d\sum_{y\in Y^+}|\varphi_{\mathbf{o}}(\theta_y^+)|^2\frac{P_{W_0}(q^{-1})^2}{q^{-\ell(w^y)}P_{W_y}(q^{-1})}\end{aligned}$$

where use was made of Proposition 2.3.6. The proof ends by using (3.2). □

With the normalization (3.11) of the trace, from (3.6), we obtain a normalized version of the Plancherel measure given by $\nu_{\mathrm{n}}^+(\pi,q^{-1}) = d.\nu^+(\pi,q^{-1})$ (the dot means scalar multiplication). Now, there is a formula for $\nu^+(\pi_{\mathbf{o}},q^{-1})$ (see [84, (0.4, 0.5)], [35, (2.19)], [40, 3.4] and [76, (37, 38)]). We thus obtain an expression for $\nu_{\mathrm{n}}^+(\pi,q^{-1})$. Namely, if $A_{\mathbf{o}}$ denote the component group of the centralizer in ${}^LG$ of an element of $\mathbf{o}$, then

$$\nu_{\mathrm{n}}^+(\pi_{\mathbf{o}},q^{-1}) = \pm d\frac{q^{|\Phi_0^+|}}{|A_{\mathbf{o}}|}\frac{\prod'_{\alpha\in\Phi_0}(q^{\langle-\alpha^\vee,\lambda(\mathbf{o})\rangle}-1)}{\prod'_{\alpha\in\Phi_0}(q^{\langle-\alpha^\vee,\lambda(\mathbf{o})\rangle+1}-1)}, \tag{3.12}$$

where the notation $\prod'_{\alpha\in\Phi_0}$ denotes the product over all roots α for which the factor is nonzero. Also, the sign $\pm$ is there to ensure that the right-hand side is positive. Combining Lemma 3.4.1, (3.12) and (3.10), we obtain

$$\pm\frac{1}{d}\frac{|A_{\mathbf{o}}|}{q^{|\Phi_0^+|}}\frac{\prod'_{\alpha\in\Phi_0}(q^{\langle-\alpha^\vee,\lambda(\mathbf{o})\rangle+1}-1)}{\prod'_{\alpha\in\Phi_0}(q^{\langle-\alpha^\vee,\lambda(\mathbf{o})\rangle}-1)} = \frac{1}{d}\sum_{y\in Y^+}|\psi(k,\lambda(\mathbf{o});ky)|^2\frac{P_{W_0}(q^{-1})^2}{q^{-\ell(w^y)}P_{W_y}(q^{-1})}. \tag{3.13}$$

3.5 A limit argument

Recall that $k=\log q$. Define $a(q) := d^{-1} = (q^{1/2}-q^{-1/2})^r = (e^{k/2}-e^{-k/2})^r$. We will write $a(k)$ instead of $a(q)$ whenever it is convenient. After the statement of Proposition 3.5.2, below, it should become clearer the reason why we made this definition. In any case, our next task is to take the limit $k\to 0$ (or equivalently $q\to 1$) on both sides of equation (3.13). Firstly:

Lemma 3.5.1. *We have that $a(q)\sim(\log q)^r$, that is,* $\lim_{q\to 1}\frac{a(q)}{(\log q)^r} = 1$.

Proof. We have $a(q) = (q^{1/2}-q^{-1/2})^r = q^{-r/2}(e^{\log q}-1)^r$. We expand

$$a(q) = \frac{(\log q)^r}{q^{r/2}}\left(1+\frac{\log q}{2}+\frac{(\log q)^2}{3!}+\frac{(\log q)^3}{4!}+\cdots\right)^r, \tag{3.14}$$

and thus the result. □

Next, as $\lambda(\mathbf{o})\in V$ is a $(V,\Phi_0^\vee,-1)$-residual point (see Definition 2.4.3 and the Remark that follows), we have that $|\{\alpha\in\Phi_0 \mid \langle\alpha^\vee,\lambda(\mathbf{o})\rangle = 1\}|$ is equal to

$|\{\alpha \in \Phi_0 \mid \langle \alpha^\vee, \lambda(\mathbf{o})\rangle = 0\}| + r$. When $q \to 1$, both the numerator and the denominator of the fractional expression with $\prod'_{\alpha\in\Phi_0}$ on the left hand side of (3.13) tends to zero, but the residual condition just mentioned above implies that this fractional expression has a pole of order r in the limit $q \to 1$ whereas, from (3.14), $a(q)$ has a zero of order r. Hence, the limit exists and moreover

$$\lim_{q\to 1} \frac{|A_{\mathbf{o}}|a(q)}{q^{|\Phi_0^+|}} \frac{\prod'_{\alpha\in\Phi_0}(q^{\langle -\alpha^\vee,\lambda(\mathbf{o})\rangle+1} - 1)}{\prod'_{\alpha\in\Phi_0}(q^{\langle -\alpha^\vee,\lambda(\mathbf{o})\rangle} - 1)} = |A_{\mathbf{o}}| \frac{\prod'_{\alpha\in\Phi_0}\langle -\alpha^\vee, \lambda(\mathbf{o})\rangle + 1}{\prod'_{\alpha\in\Phi_0}\langle -\alpha^\vee, \lambda(\mathbf{o})\rangle}. \tag{3.15}$$

The right hand side of (3.13) is slightly more complicated. Pick a positively oriented basis $\{x_1, \dots, x_r\}$ of the lattice X, which is dual to the lattice Y. Let

$$d\mu^X := dx_1 \wedge \cdots \wedge dx_r. \tag{3.16}$$

This measure satisfy $d\mu^X = \mathrm{vol}(X) d\mu^E$, where $\mathrm{vol}(X)$ is the (Euclidean) volume of the the parallelepiped in V spanned by the chosen basis. It is known that this volume is independent of the choice made (see, for example, [69, p.26]). Then:

Proposition 3.5.2. *In the limit $k \to 0$, the series in (3.13) satisfy*

$$\sum_{y\in Y^+} d^{-1}|\psi(k, \lambda(\mathbf{o}); ky)|^2 \to \int_{V_+^*} |\phi(\lambda(\mathbf{o}), -1; \xi)|^2 \, d\mu^X(\xi).$$

Note that the normalizing factor d^{-1} functions as an *area increment* used to define the Riemann integral above as $k \to 0$. Hopefully, this is a good justification for writing $a(q) = d^{-1} = a(k)$. To prove this proposition we will need some preliminary results. But before that, recall from Corollary 3.2.2(6) that $|\psi(k, \lambda(\mathbf{o}); \xi)|$ is of exponential decay on $\xi \in V_+^*$, and so an L^2-function. The next proposition implies that the established pointwise convergence of $\psi(k, \lambda(\mathbf{o}); y)$ to $\phi(\lambda(\mathbf{o}), -1; y)$ (Corollary 3.2.2(5)) when $k \to 0$ is also convergence in the L^2-sense, for $k \in \mathbb{R}_+$. We will not actually need this fact, but since we are here, it is worth to mention it:

Proposition 3.5.3. *Let $\mu_{\mathbf{o}}$ be as in Corollary 3.2.2(6). Given $\epsilon > 0$, there is a $\delta > 0$ such that $k < \delta$ implies, for all $\xi \in V^*$*

$$|\psi(k, \lambda(\mathbf{o}); \xi) - \phi(\lambda(\mathbf{o}), -1; \xi)| < \epsilon e^{\langle \xi, \mu_{\mathbf{o}}\rangle}. \tag{3.17}$$

Proof. The proof is similar to the one of Corollary 3.2.2(6), once one remarks that

$$\psi(k, \lambda(\mathbf{o}); \xi) - \phi(\lambda(\mathbf{o}), -1; \xi) = \sum_{\lambda\in W\lambda(\mathbf{o})} b(k, \lambda; \xi) e^{\langle \xi, \lambda\rangle},$$

similar to the expansion of ψ, with harmonic polynomials $b(k, \lambda; \xi)$ which are holomorphic in k for small k such that $b(0, \lambda; \xi) = 0$ (see the Remark after Corollary 3.2.2). Consider the expansion $b(k, \lambda; \xi) = \sum_m b_m(k, \lambda) m(\xi)$ in monomials $m = \alpha_1^{p_1} \dots \alpha_n^{p_n}$ as in the proof of Corollary 3.2.2(6). For each $b(k, \lambda; \xi)$ this is a finite sum, because the degree of the $b(k, \lambda; \xi)$ as polynomial in ξ is bounded by the number of positive roots $|\Phi^+|$. Let us say that the total number of monomial terms with nonzero coefficients we obtain in this way in the expansion of $\psi(k, \lambda(\mathbf{o}); \xi) - \phi(\lambda(\mathbf{o}), -1, \xi)$ equals N. All coefficients $b_m(k, \lambda)$ are continuous, and $b_m(0, \lambda) = 0$ for all m and λ. Using the notation of the proof of Corollary 3.2.2(6), it thus suffices to choose $\delta > 0$ such that $|b_m(k, \lambda)| < \epsilon/NC$ for all $k \in [0, \delta)$. $\square$

Let us now focus on proving Proposition 3.5.2. We start with an easy Lemma, in which for $\eta \in V_+^*$, we let $D_\eta := \{\xi \in V_+^* \mid 0 \leq \langle \xi, \alpha \rangle \leq \langle \eta, \alpha \rangle \text{ for } \alpha \in \Delta_0\}$.

Lemma 3.5.4. *Let $f \in L^2(V_+^*, d\mu^X)$. For all $\epsilon > 0$, there is an $\eta \in V_+^*$ such that*

$$\int_{V_+^* \setminus D_\eta} |f(\xi)|^2 \, d\mu^X(\xi) < \epsilon. \tag{3.18}$$

This applies in particular to the function $\xi \mapsto e^{\langle \xi, \mu_\mathbf{o} \rangle}$ of Corollary 3.2.2(6).

Proof. Fix a point y in the interior of V_+^* and consider the sequence $\{y_n\} \subseteq V_+^*$ in which $y_n = ny$ with $n \in \mathbb{Z}_{\geq 0}$. For each n, let $D_n := D_{y_n}$ and define

$$a_n := \int_{D_n \setminus D_{n-1}} |f(\xi)|^2 \, d\mu^X(\xi).$$

We can thus write

$$\int_{V_+^*} |f(\xi)|^2 \, d\mu^X(\xi) = \sum_{n=1}^{\infty} a_n.$$

Since $f \in L^2(V_+^*, d\mu^X)$, the series $\sum_{n=1}^{\infty} a_n$ converges and the terms are positive, so there exists an $N \in \mathbb{Z}_{\geq 0}$ such that $\sum_{n=N}^{\infty} a_n < \epsilon$, and the lemma is proved for $\eta = y_N$. $\square$

Lemma 3.5.5. *For all $\epsilon > 0$ there exists a $\eta \in V_+^*$ and $\delta > 0$ such that for all $k < \delta$,*

$$\sum_{\{y \in Y^+ \mid ky \notin D_\eta\}} a(k) e^{2\langle ky, \mu_\mathbf{o} \rangle} < \epsilon. \tag{3.19}$$

Proof. Given $\epsilon > 0$ pick η as in Lemma 3.5.4. Let $\{y_1, \ldots, y_r\}$ be a basis of Y, dual to the basis of X used to define $d\mu^X$ with all $y_i \in Y^+$. For $k > 0$, let $\{N_{k,\eta,i} \in \mathbb{Z}_{\geq 0} \mid i = 1, \ldots, r\}$ be such that $\eta(k) := \sum_i k N_{k,\eta,i} y_i$ is the element of $kY^+ \cap D_\eta$ closest to η in the Euclidean distance. In particular, $\lim_{k \to 0} k N_{k,\eta,i} = \langle \eta, x_i \rangle$. For each $J \subseteq \{1, \ldots, r\}$, define:

$$Y_{J,\eta,k}^+ := \left\{ y = \sum_i n_i y_i \in Y^+ \mid n_i > N_{k,\eta,i} \text{ if } i \in J \text{ and } n_i \leq N_{k,\eta,i} \text{ if } \; i \notin J \right\}.$$

Note that $Y_{\emptyset,\eta,k} = D_{\eta(\mathbf{k})} \subseteq D_\eta$. For each i, put $\mu_i := \langle y_i, \mu_\mathbf{o} \rangle$, so that $\mu_i < 0$ as $\mu_\mathbf{o} \in {}_-V$. Then clearly $\{y \in Y^+ \mid ky \notin D_\eta\} = \coprod_{J \neq \emptyset} Y_{J,\eta,k}^+$. Manipulating the geometric series $\sum_{n_i > N_{n,\eta,i}} (e^{2k\mu_i})^{n_i}$ for $i \notin J$ and the finite sum

$\sum_{n_i \le N_{n,\eta,i}} (e^{2k\mu_i})^{n_i}$ for $i \in J$, one checks that

$$\sum_{y \in Y^+_{J,\eta,k}} a(k) e^{2\langle ky, \mu_{\mathbf{o}}\rangle} = \prod_{i \in J} \frac{(e^{k/2} - e^{-k/2}) e^{2k\mu_i (N_{k,\eta,i}+1)}}{(1 - e^{2k\mu_i})} \times \prod_{i \notin J} \frac{(e^{k/2} - e^{-k/2})(1 - e^{2k\mu_i(N_{k,\eta,i}+1)})}{(1 - e^{2k\mu_i})}.$$

Clearly, as $k \to 0$ the sum tends to zero for each $J \neq \emptyset$, and thus the result. □

Proof of Proposition 3.5.2. Fix $\epsilon > 0$ and let $\eta \in V^*_+$ for which (3.18) holds for the function $\xi \mapsto e^{\langle 2\xi, \mu_{\mathbf{o}}\rangle}$. Since all the functions involved are continuous and D_η is compact, the convergence as $k \to 0$ of Corollary 3.2.2(5)

$$|\psi(k, \lambda(\mathbf{o}); \xi)|^2 \to |\phi(\lambda(\mathbf{o}), -1, \xi)|^2 \tag{3.20}$$

is uniform on D_η. It follows from (3.20) and (3.14) that, for the chosen $\epsilon > 0$, we can find a $\delta > 0$ such that $0 < |k| < \delta$ implies

$$\sum_{kY^+ \cap D_\eta} |\phi(\lambda(\mathbf{o}), -1; \xi)|^2 k^r - \epsilon < \sum_{kY^+ \cap D_\eta} |\psi(k, \lambda(\mathbf{o}); \xi)|^2 a(k) < \sum_{kY^+ \cap D_\eta} |\phi(\lambda(\mathbf{o}), -1; \xi)|^2 k^r + \epsilon,$$

from which we conclude by taking $k \to 0$, then $\epsilon \to 0$ and the squeeze theorem

$$\sum_{\xi \in kY^+ \cap D_\eta} |\psi(k, \lambda(\mathbf{o}); \xi)|^2 a(k) \to \int_{D_\eta} |\phi(\lambda(\mathbf{o}), -1, \xi)|^2 \, d\mu^X(\xi).$$

Finally, using Corollary 3.2.2(6) and Lemma 3.5.5, we can control the sum

$$\sum_{\{y \in Y^+ | ky \notin D_\eta\}} |\psi(k, \lambda(\mathbf{o}); \xi)|^2 a(k),$$

and thus the result. □

3.6 Constants for the discrete part

We are now ready to compute the rational constants of the Plancherel measure of the Yang system $\nu_P = \sum_{L \in \mathcal{L}} \nu_L$ when L is a residual point.

Theorem 3.6.1. *For each distinguished orbit* $\mathbf{o}$*, the Plancherel measure of the Yang system satisfies*

$$\nu_P(\{\lambda(\mathbf{o})\}) = \pm \frac{1}{\mathrm{vol}(Y)} \frac{|W_{\lambda(\mathbf{o})}|}{|A_{\mathbf{o}}|} \frac{\prod'_{\alpha \in \Phi_0} \langle \alpha^\vee, \lambda(\mathbf{o})\rangle}{\prod'_{\alpha \in \Phi_0} \langle \alpha^\vee, \lambda(\mathbf{o})\rangle + 1}. \tag{3.21}$$

Proof. From Proposition 3.5.2 and (3.15), after the limit $k \to 0$, (3.13) becomes

$$\pm|A_{\mathbf{o}}|\frac{\prod'_{\alpha\in\Phi_0}\langle -\alpha^\vee, \lambda(\mathbf{o})\rangle + 1}{\prod'_{\alpha\in\Phi_0}\langle -\alpha^\vee, \lambda(\mathbf{o})\rangle} = |W_0|^2 \int_{V_+^*} |\phi(\lambda(\mathbf{o}), -1; \xi)|^2 d\mu^X(\xi),$$

from which we conclude (note that in $\prod'_{\alpha\in\Phi_0}$, the sign of $\alpha^\vee$ is immaterial)

$$\int_{V^*} |\phi(\lambda(\mathbf{o}), -1; \xi)|^2 d\mu^X(\xi) = \pm\frac{|A_{\mathbf{o}}|}{|W_0|}\frac{\prod'_{\alpha\in\Phi_0}\langle \alpha^\vee, \lambda(\mathbf{o})\rangle + 1}{\prod'_{\alpha\in\Phi_0}\langle \alpha^\vee, \lambda(\mathbf{o})\rangle}. \tag{3.22}$$

Now, recall that $d\mu^X = \mathrm{vol}(X)d\mu^E$. Using the identity 2.4.9(3), and noting that all the computations in the Yang system used $d\mu^E$, we get that

$$\int_{V^*} |\phi(\lambda(\mathbf{o}), -1; \xi)|^2 d\mu^X(\xi) = \pm\,\mathrm{vol}(X)\frac{|W_{\lambda(\mathbf{o})}|}{|W_0|}\frac{1}{\nu_P(\{\lambda(\mathbf{o})\})}, \tag{3.23}$$

where $\frac{|W_0|}{|W_{\lambda(\mathbf{o})}|}$ is the cardinality of $W_0\lambda(\mathbf{o})$. Using (3.22), we obtain

$$\frac{1}{\nu_P(\{\lambda(\mathbf{o})\})} = \pm\,\mathrm{vol}(X)^{-1}\frac{|A_{\mathbf{o}}|}{|W_{\lambda(\mathbf{o})}|}\frac{\prod'_{\alpha\in\Phi_0}\langle \alpha^\vee, \lambda(\mathbf{o})\rangle + 1}{\prod'_{\alpha\in\Phi_0}\langle \alpha^\vee, \lambda(\mathbf{o})\rangle}, \tag{3.24}$$

and thus the result, as $\mathrm{vol}(X)^{-1} = \mathrm{vol}(Y)$. □

Remark. Let $\{y_1, \ldots, y_r\}$ be a positively oriented basis of Y and let $d\mu^Y$ the measure on V obtained as in (3.16). In view of Theorem 3.6.1, if we modify slightly the Y-functional (2.43) of the Yang system (its unfortunate, but there are two different meanings for Y in this remark) by considering $\eta'(\lambda) = d\mu^Y(\mathrm{Im}\,\lambda)/(c_{\mathbb{H}}(-\lambda,k)c_{\mathbb{H}}(\lambda,k))$ and defining Y'_{V,p_V} with respect to η', from $d\mu^Y = \mathrm{vol}(Y)d\mu^E$, we would get that $Y'_{V,p_V} = \mathrm{vol}(Y)Y_{V,p_V}$ so that $\nu'_P = \mathrm{vol}(Y)\nu_P$. We would have obtained the more natural expression

$$\nu'_P(\{\lambda(\mathbf{o})\}) = \frac{|W_{\lambda(\mathbf{o})}|}{|A_{\mathbf{o}}|}\frac{\prod'_{\alpha\in\Phi_0}\langle \alpha^\vee, \lambda(\mathbf{o})\rangle}{\prod'_{\alpha\in\Phi_0}\langle \alpha^\vee, \lambda(\mathbf{o})\rangle + 1}.$$

3.7 A comparative example

This section is independent of the rest of the chapter. The reason we include it is to compare the results obtained so far with what is known in the literature. The idea is to compute the L^2-norm of $\phi(\lambda(\mathbf{o}), -1; \cdot)$ when $\lambda(\mathbf{o}) = \varrho$, the Weyl vector, and compare it with the right-hand side of (3.22). In this case, the associated distinguished orbit $\mathbf{o}$ is the regular orbit $\mathbf{o}_{\mathrm{reg}}$.

As mentioned in Section 2.4, the Euclidean measure $d\mu^E$ was used to make the computations in [36]. Here, the measure in the space V^* used to compute $\|\phi(\lambda(\mathbf{o}), -1; \cdot)\|^2$ was the measure $d\mu^X = \mathrm{vol}(X)d\mu^E$. Equation (3.22) reads, for $\lambda(\mathbf{o}) = \varrho$ as follows. Let $\{d_1, \ldots, d_r\}$ denote the primitive degrees and

$\{m_1, \ldots, m_r\}$ denote the exponents of W_0 (see [42, Chapter 3] for details). These quantities are related by $m_i = d_i - 1$. We have

$$\prod_{\alpha \in \Phi_0^+} \frac{\mathrm{ht}(\alpha^\vee) + 1}{\mathrm{ht}(\alpha^\vee)} = \prod_i d_i = |W_0|, \tag{3.25}$$

$$\prod_{\alpha \in \Phi_0^+ \setminus \Delta_0} \frac{\mathrm{ht}(\alpha^\vee)}{\mathrm{ht}(\alpha^\vee) - 1} = \prod_i m_i, \tag{3.26}$$

where $\mathrm{ht}(\alpha^\vee)$ denotes the height of the coroot $\alpha^\vee$. A proof of (3.25) can be found in [42, Theorem 3.9]. For (3.26), recall that $m_r \geq \cdots \geq m_1 = 1$ is a partition of $|\Phi_0^+|$ whose dual partition $n_1 \geq \cdots \geq n_{h-1} = 1$ encodes the numbers $n_i := \{\alpha \in \Phi_0 \mid \mathrm{ht}(\alpha^\vee) = i\}$ [42, Theorem 3.19]. Here, h is the Coxeter number and $h-1$ is the height of the longest coroot. Using this, one checks the desired equation. Then, right-hand side of (3.22) is

$$\begin{aligned} \frac{|A_{\mathbf{o}_{\mathrm{reg}}}|}{|W_0|} \frac{\prod'_{\alpha \in \Phi_0} \langle \alpha^\vee, \varrho \rangle + 1}{\prod_{\alpha \in \Phi_0} \langle \alpha^\vee, \varrho \rangle} &= (-1)^r \frac{|A_{\mathbf{o}_{\mathrm{reg}}}|}{|W_0|} \prod_{\Phi_0^+} \frac{\langle \alpha^\vee, \varrho \rangle + 1}{\langle \alpha^\vee, \varrho \rangle} \prod_{\Phi_0^- \setminus (-\Delta_0)} \frac{\langle \alpha^\vee, \varrho \rangle + 1}{\langle \alpha^\vee, \varrho \rangle} \\ &= (-1)^r \frac{[Y : Q^\vee]}{\prod_i m_i}, \end{aligned} \tag{3.27}$$

where we have used $|A_{\mathbf{o}_{\mathrm{reg}}}| = |Z_G| = [Y : Q^\vee]$ (see [91, Theorem 4.11]). For the left-hand side, enumerate the simple roots $\Delta_0 = \{\alpha_1, \ldots, \alpha_r\}$. Write $2\varrho = \sum_i l_i \alpha_i$, with $l_i \geq 0$. Then note that $\phi(\varrho, -1; \xi) = e^{\langle \xi, \varrho \rangle}$. Therefore $|\phi(\varrho, -1; \xi)|^2 = e^{\langle \xi, 2\varrho \rangle} = \prod_i e^{l_i \langle \xi, \alpha_i \rangle}$, so consider the coordinates $z_i := \langle \xi, \alpha_i \rangle$. From $d\alpha_1 \wedge \ldots \wedge d\alpha_r = [X : Q] dx_1 \wedge \ldots \wedge dx_r$, we obtain that

$$\begin{aligned} \int_{V^*} |\phi(\varrho, -1; \xi)|^2 \, d\mu^X(\xi) &= \frac{1}{[X : Q]} \int_{\xi \in V^*} \prod_i e^{l_i \langle \alpha_i, \xi \rangle} \, d\alpha_1 \wedge \ldots \wedge d\alpha_r \\ &= \frac{|W_0|}{[X : Q]} \prod_i \int_{\mathbb{R}_{<0}} e^{l_i z_i} dz_i \\ &= \frac{|W_0|}{[X : Q] \prod_i l_i}. \end{aligned} \tag{3.28}$$

Using (3.22), (3.27) and (3.28), the conclusion is that

$$\frac{|W_0|}{[X : Q] \prod_i l_i} = \frac{[Y : Q^\vee]}{\prod_i m_i}.$$

In other words, the following equality (cf. [10, Ch. 6, Exercise 6, Section 4])

$$|W_0| \prod_i m_i = [P : Q] \prod_i l_i$$

holds, where use was made of $[Y : Q^\vee] = [P : X]$ (see Lemma 2.2.2). This observation was originally made in [36, Example 3.22]. Note that this equation depends only on the root system, not in the extra data coming from the lattices of the root datum.

3.8 Higher spectral series

In this section we compute the precise constants of $d\nu_L$ (see (2.50)). Recall the bijection (2.59) relating nilpotent orbits, pairs as in the Bala-Carter correspondence and residual subspaces. Let $L_{\mathbf{o}} = \lambda_{M_{\mathbf{o}}}(\mathbf{o}) + V^{L_{\mathbf{o}}}$ be a residual subspace with corresponding pair $({}^L M_{\mathbf{o}}, {}^L P_{\mathbf{o}})$. Consider $L^t_{\mathbf{o}} = \lambda_{M_{\mathbf{o}}}(\mathbf{o}) + iV^{L_{\mathbf{o}}}$. In the statement below, $\Phi_{M_{\mathbf{o}}} = \Phi_{L_{\mathbf{o}}}$ denotes the root system of the Levi $\underline{M}_{\mathbf{o}} \subseteq \underline{G}$ corresponding to ${}^L M_{\mathbf{o}}$, which is equal to the root subsystem of Φ_0 whose coroots are constant on $L_{\mathbf{o}}$. Also, the subspace $L_{\mathbf{o}}$ induces a decomposition of V given by $V_{L_{\mathbf{o}}} \oplus V^{L_{\mathbf{o}}}$. Denote by $\pi : V^* \to (V^{L_{\mathbf{o}}})^{\perp}$ (the $\perp$ is with respect to $\langle \cdot, \cdot \rangle$) the orthogonal projection. Next, write $r = s + t$ with $s := \dim V_{L_{\mathbf{o}}}$ and $t := \dim V^{L_{\mathbf{o}}}$. Fix

$$\begin{array}{rcl} \{y_1, \ldots, y_r\} & \subseteq & Y^+ \\ \{x_1, \ldots, x_r\} & \subseteq & X^+ \\ \{\overline{y}_1, \ldots, \overline{y}_s\} & \subseteq & \overline{Y} := \pi(Y) \\ \{y^1, \ldots, y^t\} & \subseteq & Y \cap (V_{L_{\mathbf{o}}})^{\perp} \end{array} \tag{3.29}$$

basis of the lattices $Y, X, \overline{Y}$ and $Y \cap (V_{L_{\mathbf{o}}})^{\perp}$, respectively, such that the basis of X and Y are dual to each other. One checks that, up to a sign (which can be taken care of by interchanging the order) we have

$$dy_1 \wedge \ldots \wedge dy_r = (d\overline{y}_1 \wedge \ldots \wedge d\overline{y}_s) \wedge (dy^1 \wedge \ldots \wedge dy^t). \tag{3.30}$$

Theorem 3.8.1. *For a nilpotent orbit* $\mathbf{o}$*, let* $\lambda = \lambda_{M_{\mathbf{o}}}(\mathbf{o}) + i\lambda^{L_{\mathbf{o}}} \in L^t_{\mathbf{o}}$*. The summands* $\nu_{L_{\mathbf{o}}}$ *of the Plancherel measure of the Yang system satisfy*

$$d\nu_{L_{\mathbf{o}}}(\lambda) = \left(\pm \frac{|W(\Phi_{M_{\mathbf{o}}})_{\lambda_{M_{\mathbf{o}}}(\mathbf{o})}|}{\mathrm{vol}(\overline{Y})|A_{(\mathbf{o}\,\cap\,{}^L M_{\mathbf{o}})}|} \frac{\prod'_{\alpha \in \Phi_{M_{\mathbf{o}}}} \langle \alpha^\vee, \lambda_{M_{\mathbf{o}}}(\mathbf{o}) \rangle}{\prod'_{\alpha \in \Phi_{M_{\mathbf{o}}}} \langle \alpha^\vee, \lambda_{M_{\mathbf{o}}}(\mathbf{o}) \rangle + 1} \right) \frac{d\mu^{L_{\mathbf{o}}}(\lambda^{L_{\mathbf{o}}})}{(2\pi)^t}, \tag{3.31}$$

with $d\mu^{L_{\mathbf{o}}}$ *as in (2.48).*

Recall from Proposition 2.4.6 that the local contributions of the Y-functional at $c \in \mathcal{C}$ decomposes as a sum of the functionals Y_L with L such that $c_L = c$ and

$$Y_L(g) = Y_{\Psi_L, c_L}(\{c_L\}) \int_{\lambda^L \in iV^L} g(\lambda^L) d\mu^L(\mathrm{Im}\, \lambda^L) \tag{3.32}$$

for all $g \in PW(V_{\mathbb{C}})$. Our point here is that the lower rank contribution $Y_{\Phi_L, c_L}(\{c_L\})$ can be computed by (3.21).

Remark. For the equation (3.21) and (3.31) we see that it matters if the orbit $\mathbf{o}$ is viewed as an orbit in ${}^L G$ or in ${}^L M_{\mathbf{o}}$. In fact, one can show that $A_{(\mathbf{o}\,\cap\,{}^L M_{\mathbf{o}})} \subseteq A_{\mathbf{o}}$ and the quotient is well understood, namely it is an R-group (see [50, Section 9]).

Proof of Theorem 3.8.1. Let $d\mu^Y := dy_1 \wedge \ldots \wedge dy_r$. In view of $d\mu^Y = \mathrm{vol}(Y) d\mu^E$, the Y-functional of (2.43) can be rewritten as

$$Y_{V,p_V}(f) = \frac{1}{\mathrm{vol}(Y)} \int_{p_V + iV} f(\lambda) \frac{d\mu^Y(\mathrm{Im}\,\lambda)}{c_{\mathbb{H}}(\lambda,-1)c_{\mathbb{H}}(-\lambda,-1)}. \tag{3.33}$$

for all $f \in PW(V_{\mathbb{C}})$. Recall that the expression for $c_{\mathbb{H}}(\lambda,-1)$ (see (2.111)) is a product over all positive roots. For any affine subspace $L \subseteq V$, let $c_{\mathbb{H},L}(\lambda,-1)$ denote the product with factors indexed by Φ_L^+ and let $c_{\mathbb{H}}^L(\lambda,-1)$ be such that $c_{\mathbb{H}}(\lambda,-1) = c_{\mathbb{H},L}(\lambda,-1)c_{\mathbb{H}}^L(\lambda,-1)$. Let $p_{L_{\mathbf{o}}} := \pi(p_V) \in V_{L_{\mathbf{o}}}$. The local contribution at the $(V_{L_{\mathbf{o}}}, \Phi_{L_{\mathbf{o}}}^{\vee}, -1)$-residual point $\lambda_{M_{\mathbf{o}}}(\mathbf{o})$ of the lower rank functional $Y_{V_{L_{\mathbf{o}}},p_{L_{\mathbf{o}}}}$ (defined mutatis mutandis as (3.33)) can be computed by Theorem 3.6.1. Let $L_{\mathbf{o}}^t := \lambda_{M_{\mathbf{o}}}(\mathbf{o}) + iV^{L_{\mathbf{o}}}$. Then, we split the integral (3.33) using (3.30) and compute the contribution along $L_{\mathbf{o}}$ as

$$\begin{aligned} Y_{L_{\mathbf{o}}}(f(\lambda_{M_{\mathbf{o}}}(\mathbf{o})+\cdot)) = {} & \left(\frac{(-2\pi)^s |W(\Phi_{M_{\mathbf{o}}})_{\lambda_{M_{\mathbf{o}}}(\mathbf{o})}|}{\mathrm{vol}(\overline{Y})|A_{(\mathbf{o}\,\cap\,{}^L M_{\mathbf{o}})}|} \frac{\prod'_{\alpha\in\Phi_{M_{\mathbf{o}}}} \langle \alpha^\vee, \lambda_{M_{\mathbf{o}}}(\mathbf{o})\rangle}{\prod'_{\alpha\in\Phi_{M_{\mathbf{o}}}} \langle \alpha^\vee, \lambda_{M_{\mathbf{o}}}(\mathbf{o})\rangle + 1} \right) \times \\ & \int_{L_{\mathbf{o}}^t} f(\lambda)\, \frac{d\mu^E(\mathrm{Im}\,\lambda)}{c_{\mathbb{H}}^{L_{\mathbf{o}}}(\lambda,-1)c_{\mathbb{H}}^{L_{\mathbf{o}}}(-\lambda,-1)}, \end{aligned}$$

for all $f \in PW(V_{\mathbb{C}})$ (the factor $(-2\pi)^s$ comes from taking residues while shifting from a point in the negative chamber towards 0. See also [36, Example 3.16]). Then, from the characterization of $d\nu_L$ given in Definition 2.4.8 and the definition of $d\mu^L$ in (2.48) (see Remark before Definition 2.4.8), the result follows. □

3.9 Final remarks

We end this chapter with a few remarks about the decomposition of the Yang system, recalled in Proposition 2.4.9(2), namely that the Fourier transform yields a unitary injection of Hilbert spaces

$$\mathcal{F}(k) : L^2(V^*, k)^{W_0} \to L^2(V_{\mathbb{C}}, \nu_P)^{W_0} = \left(\bigoplus_{L\in\mathcal{L}} L^2(L^t, \nu_L) \right)^{W_0}.$$

For each $L \in \mathcal{L}$, the measure $\nu_{W_0 L} = \sum_{L' \subseteq W_0 L} \nu_{L'}$ is W_0-invariant. Pick a representative of the orbit $W_0 L$ (say, L) and define

$$\tilde{\nu}_L := \frac{|W_0|}{|\,\mathrm{Stab}(L)|} \nu_L. \tag{3.34}$$

Note that the number of distinct subspaces in the orbit $W_0 L$ is exactly $\frac{|W_0|}{|\,\mathrm{Stab}(L)|}$. Denote by $W_0 \backslash \mathcal{L}$ the set of W_0-orbits of residual subspaces. We can write

$$\left(\bigoplus_{L\in\mathcal{L}} L^2(L^t, \nu_L) \right)^{W_0} = \bigoplus_{L\in W_0\backslash\mathcal{L}} L^2(L^t, \tilde{\nu}_L)^{W(L)}, \tag{3.35}$$

with $W(L)$ the Weyl group of L as defined in (2.62). We can use the representatives of $W_0\backslash\mathcal{L}$ that are in the fixed bijection with the nilpotent orbits $\mathbf{o}$, as in (2.60). Further, we can concentrate the measure on a fundamental domain for the action of $W(L)$ on L^t and define $\tilde{\nu}_{W(L)\backslash L} := |W(L)|\tilde{\nu}_L$. From this and (3.35) we get

$$\begin{aligned} L^2(V_{\mathbb{C}},\nu_P)^{W_0} &= \left(\bigoplus_{L\in\mathcal{L}} L^2\left(L^t,\nu_L\right)\right)^{W_0} \\ &\cong \bigoplus_{\mathbf{o}} L^2\left(L^t_{\mathbf{o}},\tilde{\nu}_{L_{\mathbf{o}}}\right)^{W(L_{\mathbf{o}})} \\ &\cong \bigoplus_{\mathbf{o}} L^2\left(W(L_{\mathbf{o}})\backslash L^t_{\mathbf{o}},\tilde{\nu}_{W(L_{\mathbf{o}})\backslash L_{\mathbf{o}}}\right). \end{aligned}$$

Let $c_{\mathbf{o}} := \lambda_{M_{\mathbf{o}}}(\mathbf{o})$. Using that, for any $L\in\mathcal{L}$, $|W(L)||\operatorname{Fix}(L)| = |\operatorname{Stab}(L)|$ and also that $W(\Phi_L)_{c_L} = W_{c_L}\cap W(\Phi_L) = \operatorname{Fix}(L)$, we obtain from (3.31)

$$|W(L_{\mathbf{o}})|\frac{|W_0|}{|\operatorname{Stab}(L_{\mathbf{o}})|}\frac{|W(\Phi_{M_{\mathbf{o}}})_{c_{\mathbf{o}}}|}{|A_{\mathbf{o}}|} \quad = \quad \frac{|W_0|}{|A_{\mathbf{o}}|},$$

so it follows from Theorem 3.8.1 and the definition of $d\tilde{\nu}_{W(L_{\mathbf{o}})\backslash L_{\mathbf{o}}}$ that for $\lambda = \lambda_{M_{\mathbf{o}}}(\mathbf{o}) + i\lambda^{L_{\mathbf{o}}}$ in (a fundamental domain for) $W(L_{\mathbf{o}})\backslash L^t_{\mathbf{o}}$

$$\begin{aligned} d\tilde{\nu}_{W(L_{\mathbf{o}})\backslash L_{\mathbf{o}}}(\lambda) = \frac{1}{\operatorname{vol}(\overline{Y})}&\left(\frac{|W_0|}{|A_{\mathbf{o}}|}\frac{\prod'_{\alpha\in\Phi_{M_{\mathbf{o}}}}\langle\alpha^\vee,\lambda_{M_{\mathbf{o}}}(\mathbf{o})\rangle}{\prod'_{\alpha\in\Phi_{M_{\mathbf{o}}}}\langle\alpha^\vee,\lambda_{M_{\mathbf{o}}}(\mathbf{o})\rangle+1}\right) \\ \times&\left(\prod_{\Phi_+\backslash\Phi_{M_{\mathbf{o}}}}\frac{\langle\alpha^\vee,\lambda_{M_{\mathbf{o}}}(\mathbf{o})\rangle^2+\langle\alpha^\vee,\lambda^{L_{\mathbf{o}}}\rangle^2}{(\langle\alpha^\vee,\lambda_{M_{\mathbf{o}}}(\mathbf{o})\rangle-1)^2+\langle\alpha^\vee,\lambda^{L_{\mathbf{o}}}\rangle^2}\right)\frac{d\mu^E(\lambda^{L_{\mathbf{o}}})}{(2\pi)^t}. \end{aligned} \tag{3.36}$$

Chapter 4

On the spherical automorphic spectrum

In this chapter, we formulate and prove our main results concerning the problem (a) mentioned in the Introduction. We will state all the results in the first section and the remaining sections will be on how to prove them.

4.1 Statements

The setup is the one of Section 2.7.10: we have a connected reductive $\underline{G}$ which we assume split over a number field F. We let $\mathbf{K} = \prod K_v$ with K_v a maximal compact for each v with $K_v = \underline{G}(\mathfrak{o}_v)$ for v non-Archimedean. Denote by $\underline{Z}_G$ the center of $\underline{G}$ and consider the $\underline{G}(\mathbb{A})$-representation $L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))$, (the space defined in Definition 2.7.1, for $Z_0 = \underline{Z}_G(\mathbb{A})$ and ω the trivial character). In this chapter, we are interested in spherical automorphic forms, (that is, the subrepresentation of $L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))$ generated by their $\mathbf{K}$-invariant functions) which are supported by the class of cuspidal datum $\mathfrak{X} = [T, \imath]$, described in Section 2.7.10. Let $L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_{[T,\imath]}$ denote this automorphic space. Recall from the end of Section 2.7.10, that the space of $\mathbf{K}$-fixed vectors is topologically generated by the unramified pseudo-Eisenstein series θ_ϕ, for $\phi \in PW(a_{T,\mathbb{C}}^{G*})$ described in (2.104) (we recall that $a_T^{G*} \subseteq a_T^* = \mathbb{R} \otimes X^*(\underline{T})$ is the real vector space spanned by the root lattice Q). Moreover, we have the precise formula (2.116) for the inner product $(\theta_\phi, \theta_\psi)$ for all $\phi, \psi \in PW(a_{T,\mathbb{C}}^{G*})$. We can state the main result of this chapter in terms of the bijection (2.59):

Theorem 4.1.1. *For each unipotent orbit $\mathbf{o}$, let $\mu_{\mathbf{o}}$ be the measure on $W_0 L_{\mathbf{o}}^t$ defined below in (4.50). For each $\phi, \psi \in PW(a_{T,\mathbb{C}}^{G*})$ let $\langle \phi, \psi \rangle_{\mathbf{o}}$ denote the positive* *semidefinite Hermitian pairing defined in (4.49). Then, we have*

$$(\theta_\phi, \theta_\psi) = \sum_{\mathbf{o}} \langle \phi, \psi \rangle_{\mathbf{o}},$$

for all $\phi, \psi \in PW(a^{G*}_{T,\mathbb{C}})$. This induces an isometry of Hilbert spaces

$$L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}}_{[T,\mathbf{1}]} \cong \left(\oplus_{\mathbf{o}} L^2(W_0 L^t_{\mathbf{o}}, \mu_{\mathbf{o}})\right)^{W_0}. \tag{4.1}$$

For each unipotent orbit $\mathbf{o}$, let $W(^L M_{\mathbf{o}}, {}^L P_{\mathbf{o}})$ be the Weyl group of the corresponding pair $(^L M_{\mathbf{o}}, {}^L P_{\mathbf{o}})$ (in the Bala-Carter correspondence (2.53)) which is isomorphic to Weyl group $W(L_{\mathbf{o}})$ of the subspace $L_{\mathbf{o}}$ (see 2.5.9). We now proceed in a similar way as in Section 3.9. In (4.47), we defined measures μ_L on L^t, for each residual subspace $L \in \mathcal{L}$. We let $m_{\mathbf{o}}$ denote the number of distinct affine subspaces in the W_0-orbit of $L_{\mathbf{o}}$. We first concentrate the measure $\mu_{\mathbf{o}}$ on the representative $L^t_{\mathbf{o}}$ and then concentrate it further on a fundamental domain for $W(L_{\mathbf{o}})\backslash L^t_{\mathbf{o}}$ in $L^t_{\mathbf{o}}$. We define:

$$\begin{array}{rcl} \tilde{\mu}_{\mathbf{o}} & := & m_{\mathbf{o}} \mu_{L_{\mathbf{o}}} \\ \tilde{\mu}_{W(L_{\mathbf{o}})\backslash L_{\mathbf{o}}} & := & |W(L_{\mathbf{o}})| \tilde{\mu}_{\mathbf{o}} \\ \mu_{\mathrm{aut}} & := & \sum_{\mathbf{o}} \tilde{\mu}_{W(L_{\mathbf{o}})\backslash L_{\mathbf{o}}}. \end{array} \tag{4.2}$$

From Theorem 4.1.1, we obtain (see Proposition 2.5.6):

Corollary 4.1.2. *The automorphic space can be expressed as:*

$$\begin{array}{rcl} L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}}_{[T,\mathbf{1}]} & \cong & \oplus_{\mathbf{o}} L^2(L^t_{\mathbf{o}}, \tilde{\mu}_{\mathbf{o}})^{W(^L M_{\mathbf{o}}, {}^L P_{\mathbf{o}})}. \\ & \cong & L^2(P^{\mathrm{su}}, \mu_{\mathrm{aut}}) \end{array}$$

Let $\mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K}) = \otimes'_v \mathcal{H}(G_v, K_v)$ denote the global spherical Hecke algebra, in which $\mathcal{H}(G_v, K_v)$ denotes the corresponding spherical Hecke algebra of each local place v, as discussed in Section 2.7.4. This algebra is equipped with a $*$-structure coming from each local factor. It acts naturally by convolution on the space of $\mathbf{K}$-invariant functions on $\underline{G}(F)\underline{Z}_G\backslash\underline{G}(\mathbb{A})$ and diagonally on $\left(\oplus_{\mathbf{o}} L^2(W_0 L^t_{\mathbf{o}}, d\mu_{\mathbf{o}})\right)^{W_0}$. In Section 4.9.2 we introduce a normalization of the unramified Eisenstein series, denoted $\mathcal{E}_0(\lambda, g)$ (see (4.54)). These form a smooth family of automorphic forms in λ, for λ in a neighborhood of the support $\cup_{\mathbf{o}} W_0 L^t_{\mathbf{o}}$ of the spectral measure μ_{aut}. Then, Theorem 4.1.1 can be stated as:

Theorem 4.1.3. *Let $\mathcal{F}_{\mathrm{aut}} : L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}}_{[T,\mathbf{1}]} \to \left(\oplus_{\mathbf{o}} L^2(W_0 L^t_{\mathbf{o}}, \mu_{\mathbf{o}})\right)^{W_0}$ be the integral transform given by*

$$\mathcal{F}_{\mathrm{aut}}(f)(\lambda) = \int_{\underline{G}(F)\underline{Z}_G\backslash\underline{G}(\mathbb{A})} f(g)\mathcal{E}_0(-\lambda, g)\, dg \tag{4.3}$$

defined for $\lambda \in \cup_{\mathbf{o}} W_0 L^t_{\mathbf{o}}$. It is an isometry of $*$-unitary $\mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K})$-modules.

Assuming that $\mathbf{o}$ is distinguished, for any Archimedean or non-Archimedean place v, let $\pi_{v,\lambda(\mathbf{o})}$ be the unique irreducible spherical subquotient of the (unramified) principal series representation of the local group G_v induced by the character of T_v corresponding to $\lambda(\mathbf{o})$ (Definition 2.7.19), and denote by $\pi_{\lambda(\mathbf{o})}$ the irreducible representation of $\underline{G}(\mathbb{A})$ given by

$$\pi_{\lambda(\mathbf{o})} = \otimes_v' \pi_{v,\lambda(\mathbf{o})}, \tag{4.4}$$

which has a $\mathbf{K}$-invariant vector. Let $L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_{[T,\mathbf{1}],d}$ be the discrete part of $L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_{[T,\mathbf{1}]}$, that is, the closure of the span of all topologically irreducible $\underline{G}(\mathbb{A})$-subrepresentations. Let also $L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))^{\mathrm{sph}}_{[T,\mathbf{1}],d}$ be the smallest closed $\underline{G}(\mathbb{A})$-invariant subspace of $L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))_{[T,\mathbf{1}],d}$ containing $L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}}_{[T,\mathbf{1}],d}$. We obtain the following representation-theoretic corollaries:

Corollary 4.1.4. *The space $L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))^{\mathrm{sph}}_{[T,\mathbf{1}],d}$ is multiplicity-free and decomposes as*

$$L^2(\underline{G}(F)\underline{Z}_G\backslash\underline{G}(\mathbb{A}))^{\mathrm{sph}}_{[T,\mathbf{1}],d} \cong \bigoplus_{\mathbf{o}} \pi_{\lambda(\mathbf{o})},$$

with the sum indexed by the finite set of distinguished unipotent orbits.

Corollary 4.1.5. *For each distinguished $\mathbf{o}$, the local factors $\pi_{v,\lambda(\mathbf{o})}$ of $\pi_{\lambda(\mathbf{o})}$ are unitary representations for all places v.*

Finally, it follows from Theorem 4.1.3 that for every distinguished unipotent orbit $\mathbf{o}$, the corresponding normalized Eisenstein $\mathcal{E}_0(\lambda(\mathbf{o}),\cdot)$ is a square integrable function which is a multiple of the residue of the unramified Borel Eisenstein series at the parameter $\lambda(\mathbf{o})$. Assuming $\underline{G}$ semisimple, and using the isometry of the above theorems to compare norms of vectors in both sides of (4.3), we obtain some non-trivial identities. For example, we recover the classical results of Siegel-Langlands regarding the volume of the $\underline{G}(F)\backslash\underline{G}(\mathbb{A})$ (cf. [54], [89]). To state the next results, recall the function $\rho(s)$ of (2.69), the regularized version of the completed Dedekind zeta function $\Lambda(s)$. We will also need the entire function $r(\lambda)$ defined for $\lambda \in a^{G*}_{T,\mathbb{C}}$ as the product $r(\lambda) := \prod_{\alpha\in\Phi_0^+} \rho(\langle\alpha^\vee,\lambda\rangle)$. We fix Haar measures on $\underline{U}(\mathbb{A})$, $\mathbf{K}$, $\underline{T}(\mathbb{A})$ and on $\underline{G}(\mathbb{A})$ as in Section 2.7.12.

Corollary 4.1.6. *Consider $\tilde{\mathcal{E}}_0(\lambda,g) := r(\lambda)^{-1}\mathcal{E}_0(\lambda,g)$. Then, this function is well defined for $\lambda = \lambda(\mathbf{o})$ when $\mathbf{o}$ is a distinguished unipotent orbit. Moreover, we have*

$$\|\tilde{\mathcal{E}}_0(\lambda(\mathbf{o}),\cdot)\|^2 = \frac{|A_{\mathbf{o}}|}{|W_0|^2}\left|\frac{\prod'_{\alpha\in\Phi_0}(\langle\alpha^\vee,\lambda(\mathbf{o})\rangle+1)}{\prod'_{\alpha\in\Phi_0}\langle\alpha^\vee,\lambda(\mathbf{o})\rangle}\right|\frac{\prod_{\alpha\in\Phi_0^+}\rho(\langle\alpha^\vee,\lambda(\mathbf{o})\rangle+1)}{\prod_{\alpha\in\Phi_0^+}\rho(\langle\alpha^\vee,\lambda(\mathbf{o})\rangle)},$$

where $|A_{\mathbf{o}}|$ is the cardinality of the component group of the centralizer of an element of $\mathbf{o}$ and $\prod'$ denotes the product over all nonzero factors.

Corollary 4.1.7. *Let $d_1, \dots, d_r$ denote the primitive degrees of W_0, $Q^\vee$ denote the coroot lattice. Then,*

$$\mathrm{vol}(\underline{G}(F)\backslash\underline{G}(\mathbb{A})) = \frac{[X_*(T) : Q^\vee]}{\rho(1)^r} \prod_{i=1}^{r} \Lambda(d_i),$$

where $\rho(1)$ is the residue at $s = 1$ of the completed Dedekind zeta function $\Lambda(s)$ and r is the rank of $\underline{G}$.

4.2 Remarks for the reductive case

In Chapter 3 we computed some rational constants that will be an ingredient in the spectral measure for our automorphic space. But the group there was semisimple, while here it is reductive. Now, it is not entirely clear which semisimplification[1] of $\underline{G}$ should be considered in order to use the computations of Chapter 3. The approach we will take is to follow the procedure sketched in (3.29) on how to use the residue calculus to compute higher spectral series. From $(\underline{G}, \underline{B}, \underline{T})$, we obtain the triple $({}^L G, {}^L B, {}^L T)$ and the associated based root datum $R^\vee = (X_*(\underline{T}), \Delta_0^\vee, X^*(\underline{T}), \Delta_0)$. We write $a_T^* := \mathbb{R} \otimes X^*(\underline{T})$ and $a_T := \mathbb{R} \otimes X_*(\underline{T})$ and use the usual notation for their complexifications.

Now, from section 2.7.10 the relevant space for our harmonic analysis is $a_T^{G*} = \mathbb{R} \otimes Q$. Let also a_G^*, a_T^G and a_T^G be defined by the tensor product over $\mathbb{Z}$ of $\mathbb{R}$ with the lattices $X^*(\underline{G}), Q^\vee$ and $X_*(\underline{G})$, respectively. Since $Q^\vee \cap Q^\perp = \{0\}$ and $Q^\vee + Q^\perp$ has finite index on $X_*(\underline{T})$ (see [92, Lemma 1.2]), we have an orthogonal direct sum $a_T = a_G \oplus a_T^G$. Let $Y \subseteq a_T^G$ be the image under the orthogonal projection $\pi : a_T \to a_T^G$. It follows that $Y \cong X_*(\underline{T})/Q^\perp$. We similarly have $a_T^* = a_G^* \oplus a_T^{G*}$. Let $X \subseteq a_T^{G*}$ be the lattice dual to Y, which is identified with $(Q^\perp)^\perp$. We obtain a based root datum

$$(\mathcal{R}^\vee)_{\mathrm{ss}} = (Y, \Delta_0^\vee, X, \Delta_0), \tag{4.5}$$

which is the root datum of $({}^L G', {}^L T_1, {}^L B')$, the derived group of ${}^L G$, with ${}^L T_1 \subseteq {}^L T$ (see equation (2.2) and the text below it) and ${}^L B' = {}^L B \cap {}^L G'$. We write $V^* := a_T^G$ and $V = a_T^{G*}$ and we have objects $V^*, (\cdot,\cdot), V, \langle\cdot,\cdot\rangle, (\mathcal{R}^\vee)_{\mathrm{ss}}$ and W_0 with $\Phi_0^\vee \subseteq V^*$ and $\Phi_0 \subseteq V$.

Now, let $(\underline{G}_{\mathrm{ss}}, \underline{B}_{\mathrm{ss}}, \underline{T}_{\mathrm{ss}})$ be the triple corresponding to $((\mathcal{R}^\vee)_{\mathrm{ss}})^\vee$, with $\underline{T}_{\mathrm{ss}} \subseteq \underline{G}_{\mathrm{ss}}$ a maximal F-split torus. It is clear that $\underline{G}_{\mathrm{ss}}$ is a semisimple split group. The character lattice of the maximal torus $\underline{T}_{\mathrm{ss}}$ of $\underline{G}_{\mathrm{ss}}$ is by definition $X_{\mathrm{ss}} = (Q^\perp)^\perp$. This is the maximal sublattice of $X^*(\underline{T})$ in which Q has finite index, and in particular $X^*(\underline{T})/X_{\mathrm{ss}}$ is itself a lattice. This quotient lattice canonically corresponds to a subtorus $\underline{Z}_G^\circ \subseteq \underline{T}$ where $\underline{Z}_G^\circ$ is the identity component of center of $\underline{G}$, and $\underline{T}_{\mathrm{ss}} = \underline{T}/\underline{Z}_G^\circ$. Because of connectedness, we have $\underline{G}_{\mathrm{ss}}(F) = \underline{G}(F)/\underline{Z}_G^\circ(F)$,

[1] We could, for example, take the derived group or quotient by the full center.

$\underline{G}_{\mathrm{ss}}(F_v) = \underline{G}(F_v)/\underline{Z}_G^\circ(F_v)$ for all local place v and $\underline{G}_{\mathrm{ss}}(\mathbb{A}) = \underline{G}(\mathbb{A})/\underline{Z}_G^\circ(\mathbb{A})$. From A. Borel's result [8, Proposition 1.9], we have $\underline{Z}_G^\circ(\mathbb{A})\backslash\underline{Z}_G(\mathbb{A}) \subseteq \mathbf{K}$. Therefore, the double cosets $\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A})/\mathbf{K} = \underline{G}(F)\underline{Z}_G^\circ(\mathbb{A})\backslash\underline{G}(\mathbb{A})/\mathbf{K}$ coincide, so for the decomposition that lies ahead of us, this is immaterial. If $\underline{G}$ is semisimple, with the construction above, we recover the group we started.

4.3 Rewriting the scalar product

Our goal is to decompose the space $L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}}_{[T,\mathbf{1}]}$, which is generated by the pseudo-Eisenstein. The path we will take is the same as Langlands [55], namely, from the expression of the inner product (2.116), we will shift the contour from $\lambda_0 \in a_T^{G*} = V$ towards the origin and pick poles along the way that will yield the spectral decomposition. But the systematic way to analyze the residues will be in the optics of [36] (that we recalled in Section 2.4). So, let us now rewrite (2.116) in such a way that the Residue Lemma (see Section 2.4.1) becomes applicable. The ingredients we will need are the graded Hecke algebra c-functions $c_{\mathbb{H}}(\lambda, k)$, for $k = \pm 1$ and the functions $\Lambda(s), \rho(s)$ (see Section 2.6.1, equations (2.67) and (2.69)) associated to the field F. In addition, define the global γ-factor

$$\gamma(s) := \frac{1}{s^2\Lambda(-s)} \tag{4.6}$$

and let $c(\lambda) := \prod_{\alpha\in\Phi_0^+}\gamma(\langle\alpha^\vee,\lambda\rangle)$ be the global c-function.

Convention. In Chapter 2, we defined many objects that depended on the parameter k. From here onwards, by default, we will assume $k = 1$ and omit it from the notation.

Before we continue, we highlight a simple elementary result that will be useful throughout. We recall that, for each $w \in W_0$, we denote

$$\Phi(w) := \Phi_0^+ \cap w^{-1}\Phi_0^-.$$

Lemma 4.3.1. *Let f be a function of one complex variable, and define a function φ on $V_{\mathbb{C}}$ by $\varphi(\lambda) = \prod_{\alpha\in\Phi_0^+} f(\langle\alpha^\vee,\lambda\rangle)$. For all $w \in W_0$ we have*

$$\frac{\varphi(w\lambda)}{\varphi(\lambda)} = \prod_{\alpha\in\Phi(w)} \frac{f(-\langle\alpha^\vee,\lambda\rangle)}{f(\langle\alpha^\vee,\lambda\rangle)}$$

Recall that we have defined the entire function $r(\lambda)$ on $V_{\mathbb{C}}$ used in the statement of Corollary 4.1.6

$$r(\lambda) = \prod_{\alpha\in\Phi_0^+} \rho(\langle\alpha^\vee,\lambda\rangle).$$

As a consequence of Lemma 4.3.1, the W_0-invariance of the functions $c_{\mathbb{H}}(-\lambda)c_{\mathbb{H}}(\lambda)$ and $r(-\lambda)r(\lambda)$ and the functional equations of $\Lambda(s)$, $\rho(s)$ (see Section 2.6), we obtain various identities, like:

$$\begin{aligned}\frac{c(-w\lambda)}{c(-\lambda)} &= \prod_{\alpha\in\Phi(w)} \frac{\Lambda(\langle\alpha^\vee,\lambda\rangle)}{\Lambda(\langle\alpha^\vee,\lambda\rangle+1)} \\ &= \frac{c_{\mathbb{H}}(-w\lambda)}{c_{\mathbb{H}}(-\lambda)}\frac{r(\lambda)}{r(w\lambda)} = \frac{c_{\mathbb{H}}(\lambda)}{c_{\mathbb{H}}(w\lambda)}\frac{r(\lambda)}{r(w\lambda)} = \frac{c(\lambda)}{c(w\lambda)}.\end{aligned} \tag{4.7}$$

Similarly, we see that

$$\frac{r(\lambda)}{r(w\lambda)} = \prod_{\alpha\in\Phi(w)} \frac{\rho(\langle\alpha^\vee,\lambda\rangle)}{\rho(\langle\alpha^\vee,\lambda\rangle+1)}. \tag{4.8}$$

It follows that the pole set of the meromorphic function $\frac{r(\lambda)}{r(w\lambda)}$ is a union of hyperplanes of the form $\langle\alpha^\vee,\lambda\rangle = z$ where $-1 < \mathrm{Re}(z) < 0$ and $\alpha \in \Phi(w)$.

Next, denote $\phi^-(\lambda) := \overline{\phi(\overline{\lambda})}$. We define the meromorphic function R_ϕ by

$$R_\phi(\lambda) := \sum_{w\in W_0} c_{\mathbb{H}}(-w\lambda)\phi^-(-w\lambda)\frac{r(\lambda)}{r(w\lambda)}. \tag{4.9}$$

We note in passing that the meromorphic function $r^{-1}R_\phi$ is W_0-invariant. Due to its importance, we define the summand $R_{\phi,w}$ as the rational function of $V_{\mathbb{C}}$

$$R_{\phi,w}(\lambda) := c_{\mathbb{H}}(-w\lambda)\phi^-(-w\lambda)\frac{r(\lambda)}{r(w\lambda)}, \tag{4.10}$$

so that we can write $R_\phi = \sum_w R_{\phi,w}$. Inserting (4.7) in (2.116) we obtain

$$(\theta_\phi, \theta_\psi) = \int_{\mathrm{Re}(\lambda)=\lambda_0\gg 0} R_\phi(\lambda)\psi(\lambda)\frac{d\lambda}{c_{\mathbb{H}}(-\lambda)}, \tag{4.11}$$

with $d\lambda$ as in (2.117). For a point $p_V \in V$ outside the set of poles of $c_{\mathbb{H}}(-\lambda)^{-1}$ we define a linear functional similar to (2.43) (but here, we recall $k = 1$) $\bar{X}_{V,p_V}$ on $PW(V_{\mathbb{C}})$, the space of Paley-Wiener functions on $V_{\mathbb{C}}$, by:

$$\bar{X}_{V,p_V}(f) := \int_{p_V+iV} f(\lambda)\frac{d\lambda}{c_{\mathbb{H}}(-\lambda)}. \tag{4.12}$$

For $p_V = \lambda_0$, we use (4.12) to rewrite (4.11) as

$$(\theta_\phi, \theta_\psi) = \bar{X}_{V,p_V}(\psi R_\phi). \tag{4.13}$$

4.4 The strategy for the proofs

We would like to apply the support results recalled in Proposition 2.4.5 to the expression (4.13). In order to do so we need to show that the additional poles of R_ϕ coming from the zeroes of $\rho(s)$ contained in the factors $\frac{r(\lambda)}{r(w\lambda)}$ with $w \in W_0$ (see (4.8)) do not interfere with the process of iteratively taking residues as in [36]. Before we sketch the argument we will use, let us describe in simple

2-dimensional examples what are the difficulties involved. Much of the computations we will now show were already present in the appendix to Langland's seminal work [55] (see also [67, Appendix III]).

Recall that our goal is to evaluate the integral

$$\bar{X}_{V,p_V}(\psi R_\phi) := \int_{p_V+iV} \psi(\lambda) R_\phi(\lambda) \frac{d\lambda}{c_{\mathbb{H}}(-\lambda)},$$

in which $\psi \in PW(V_{\mathbb{C}})$, $R_\phi = \sum_w R_{\phi,w}$ with $R_{\phi,w}$ as in (4.10). The functional $\bar{X}_{V,p_V} \in PW(V_{\mathbb{C}})^*$ is well-understood. It decomposes *canonically* as a finite sum of tempered residue distributions (called local contributions) indexed by the centers of residual subspaces (see the Residue Lemma in Section 2.4.1 and Corollary 2.4.7). One computes these local contributions via the decomposition

$$\bar{X}_{V,p_V}(f) = \sum_L \bar{X}_L(f)$$

running over the residual subspaces L and where the $\bar{X}_L$ are distributions on $V_{\mathbb{C}}$ with support contained in $L^t = c_L + iV^L$. The latter were studied in details in [73] and [74]. The strategy will be to devise *an* algorithm to compute these $\bar{X}_L$ (they are *not* canonically defined) by means of a contour-shift argument. The iterated contour-shifts to be used will essentially be along line segments connecting a given point on a residual subspace towards its center, starting with the point p_V in the residual subspace V whose center is the origin. As an illustration to what lies ahead of us, let us consider the summand R_{ϕ,w_0} with w_0 the longest element of the Weyl group W_0, which we will assume, for now, to be of type A_2. Its critical part is:

$$\frac{r(\lambda)}{r(w_0\lambda)} = \frac{\rho(\langle\alpha^\vee,\lambda\rangle)\rho(\langle\beta^\vee,\lambda\rangle)\rho(\langle\lambda,\gamma^\vee\rangle)}{\rho(\langle\alpha^\vee,\lambda\rangle+1)\rho(\langle\beta^\vee,\lambda\rangle+1)\rho(\langle\lambda,\gamma^\vee\rangle+1)}, \tag{4.14}$$

in which $\{\alpha,\beta,\gamma=\alpha+\beta\}$ are the positive roots. In figure 4.1, we describe the first step of our algorithm. We shift the contour from p_V to the origin linearly, crossing the residual subspaces $L_\delta := \{\lambda \mid \langle\lambda,\delta^\vee\rangle - 1 = 0\}$, for $\delta \in \Phi_0^+$ in the points p_δ. We will also use the notation $[\delta^\vee = 1]$ to denote L_δ, whenever this is convenient to us. We write c_δ for the centers of these subspaces and we shade the area corresponding to the union of the critical strips $\mathrm{Re}(\langle\lambda,\delta^\vee\rangle) \in (-1,0)$ of the factors occurring in the denominator of (4.14).

Note that, for example, the center c_α of the subspace L_α (or $[\alpha^\vee = 1]$) is right in the middle of the critical strip, where, presumably, all the poles in question are concentrated. However, restricting (4.14) to this subspace, some cancellations occur, and we obtain

$$\left.\frac{r(\lambda)}{r(w_0\lambda)}\right|_{(L_\alpha)_{\mathbb{C}}} = \frac{\rho(1)\rho(\langle\overline{\beta}^\vee,\lambda\rangle)\rho(\langle\overline{\beta}^\vee,\lambda\rangle+1)}{\rho(2)\rho(\langle\overline{\beta}^\vee,\lambda\rangle+1)\rho(\langle\overline{\beta}^\vee,\lambda\rangle+2)} = \frac{\rho(1)\rho(\langle\overline{\beta}^\vee,\lambda\rangle)}{\rho(2)\rho(\langle\overline{\beta}^\vee,\lambda\rangle+2)}, \tag{4.15}$$

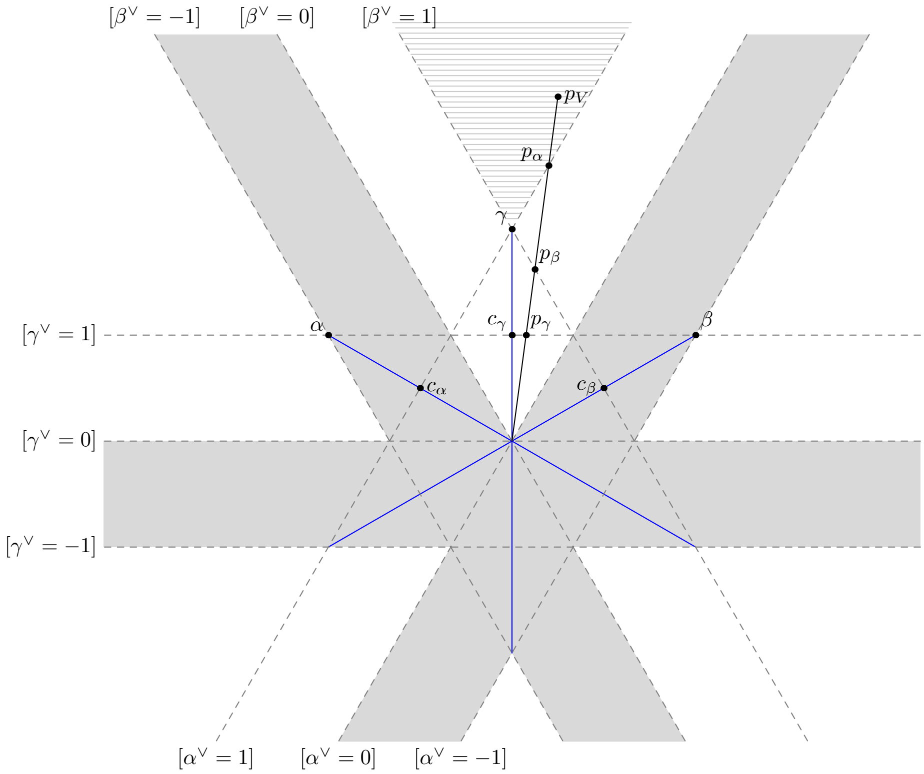

Figure 4.1: The pole regions in the subspace V, in type A_2.

in which $\overline{\beta}^\vee = \beta^\vee|_{(L_\alpha)_\mathbb{C}}$. The critical strip is now shifted to $\mathrm{Re}(\langle\overline{\beta}^\vee, \lambda\rangle) \in (-2, -1)$. The schematics for the pole region along the subspace L_α is depicted in figure 4.2. It is visible that we can now shift the contour from p_α linearly all the way towards c_α without ever crossing the pole region of $\frac{r(\lambda)}{r(w_0\lambda)}\big|_{(L_\alpha)_\mathbb{C}}$.

Consider next a group whose root system is of type G_2. Again, we analyze the summand R_{ϕ,w_0}, that contains all factors. It is given by

$$\frac{r(\lambda)}{r(w_0\lambda)} = \frac{\rho(\alpha_1^\vee)\rho(\alpha_2^\vee)\rho(\alpha_3^\vee)\rho(\alpha_4^\vee)\rho(\alpha_5^\vee)\rho(\alpha_6^\vee)}{\rho(\alpha_1^\vee+1)\rho(\alpha_2^\vee+1)\rho(\alpha_3^\vee+1)\rho(\alpha_4^\vee+1)\rho(\alpha_5^\vee+1)\rho(\alpha_6^\vee+1)}. \tag{4.16}$$

Here, $\rho(\alpha_i^\vee)$ actually means $\rho(\langle\alpha^\vee, \lambda_i\rangle)$ and similar for $\rho(\alpha_i^\vee + 1)$, but for lack of space we write it that way and $\{\alpha_i \mid i = 1, \ldots 6\}$ are the positive roots, in which α_1 and α_6 are the simple roots and

$$\begin{array}{lccc} \text{long roots:} & \alpha_1 = \alpha_1, & \alpha_3 = 2\alpha_1 + 3\alpha_6, & \alpha_5 = \alpha_1 + 3\alpha_6, \\ \text{short roots:} & \alpha_2 = \alpha_1 + \alpha_6, & \alpha_4 = \alpha_1 + 2\alpha_6, & \alpha_6 = \alpha_6. \end{array} \tag{4.17}$$

In figure 4.3, we describe the first step of our algorithm. We shift the contour from p_V to the origin linearly, crossing the residual subspaces $L_i = [\alpha_i^\vee = 1]$,

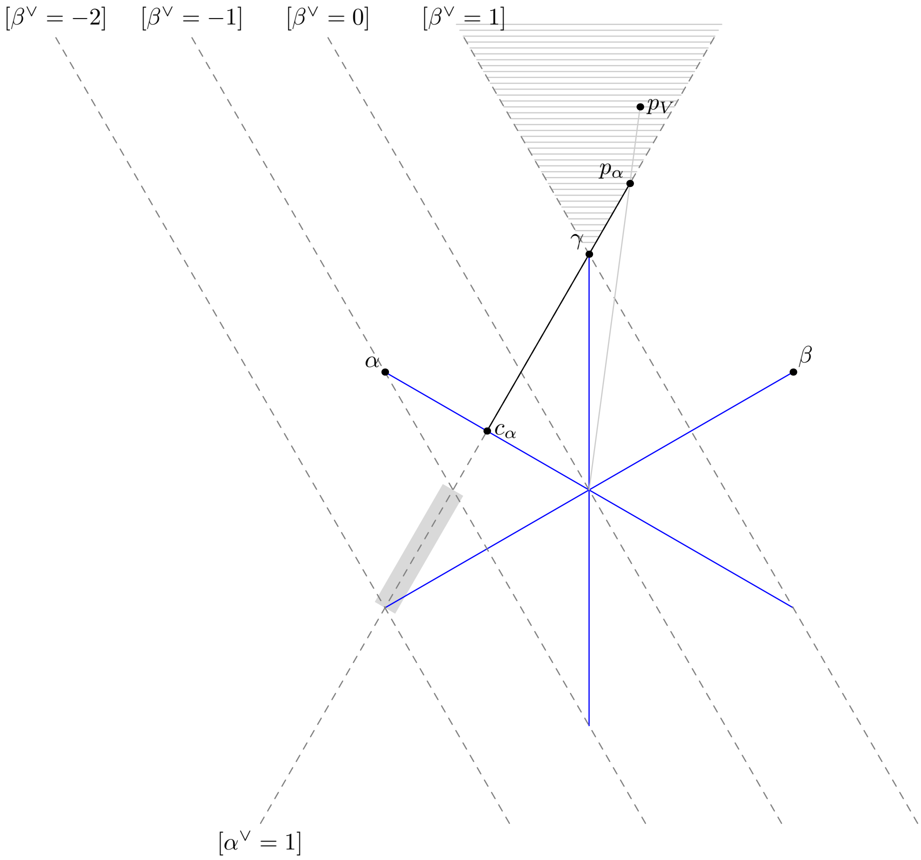

Figure 4.2: The pole region in the subspace L_α.

for $\alpha_i \in \Phi_0^+$ in the points p_i. We write c_i for the centers of these subspaces and we shade the area corresponding to the union of the critical strips of the various factors occurring in the denominator of (4.14). From the figure, it is visible that the segments $[p_i, c_i]$ which will be used in the second step of the algorithm crosses various critical strips for $i = 1, 2, 5, 6$ and also, for example, the center c_6 of the residual subspace $[\alpha_6^\vee = 1]$ is right in the middle of the critical strips $\mathrm{Re}(\langle \lambda, \alpha_i^\vee \rangle + 1) \in (0, 1)$, for $i = 1, 2$. The message we wish to reinforce is that when the expression (4.16) is restricted to these subspaces, some cancellations occur. To describe them, recall that the dual roots are given by

$$\begin{array}{llll} \text{short dual roots:} & \alpha_1^\vee = \alpha_1^\vee, & \alpha_3^\vee = 2\alpha_1^\vee + \alpha_6^\vee, & \alpha_5^\vee = \alpha_1^\vee + \alpha_6^\vee, \\ \text{long dual roots:} & \alpha_2^\vee = 3\alpha_1^\vee + \alpha_6^\vee, & \alpha_4^\vee = 3\alpha_1^\vee + 2\alpha_6^\vee, & \alpha_6^\vee = \alpha_6^\vee. \end{array} \tag{4.18}$$

Restricting the expressions of (4.18) to L_1 we obtain (write $\overline{\alpha_i}^\vee = \alpha_i^\vee|_{(L_i)_\mathbb{C}}$)

$$\begin{array}{llll} \text{short dual roots:} & \overline{\alpha}_1^\vee = 1, & \overline{\alpha}_3^\vee = 2 + \overline{\alpha}_6^\vee, & \overline{\alpha}_5^\vee = 1 + \overline{\alpha}_6^\vee, \\ \text{long dual roots:} & \overline{\alpha}_2^\vee = 3 + \overline{\alpha}_6^\vee, & \overline{\alpha}_4^\vee = 3 + 2\overline{\alpha}_6^\vee, & \overline{\alpha}_6^\vee = \overline{\alpha}_6^\vee. \end{array} \tag{4.19}$$

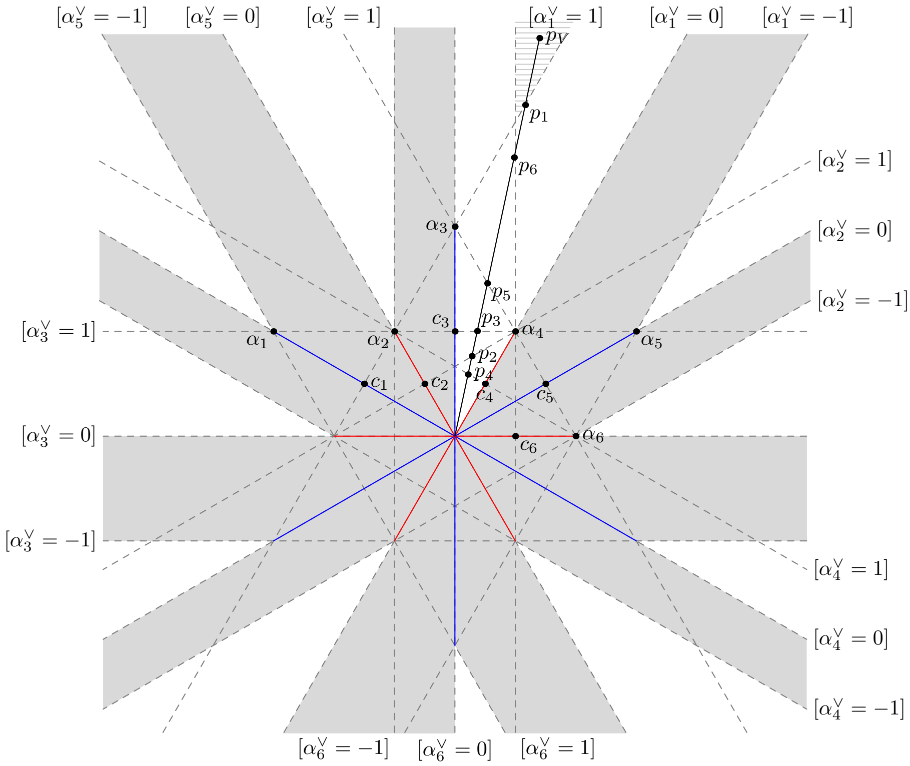

Figure 4.3: The pole regions in the subspace V, in type G_2.

We substitute (4.19) in the expression (4.16) to obtain

$$\begin{aligned} \left.\frac{r(\lambda)}{r(w_0\lambda)}\right|_{(L_1)_\mathbb{C}} &= \frac{\rho(1)\rho(3+\overline{\alpha}_6^\vee)\rho(2+\overline{\alpha}_6^\vee)\rho(3+2\overline{\alpha}_6^\vee)\rho(1+\overline{\alpha}_6^\vee)\rho(\overline{\alpha}_6^\vee)}{\rho(2)\rho(4+\overline{\alpha}_6^\vee)\rho(3+\overline{\alpha}_6^\vee)\rho(4+2\overline{\alpha}_6^\vee)\rho(2+\overline{\alpha}_6^\vee)\rho(\overline{\alpha}_6^\vee+1)} \\ &= \frac{\rho(1)\rho(3+2\overline{\alpha}_6^\vee)\rho(\overline{\alpha}_6^\vee)}{\rho(2)\rho(4+\overline{\alpha}_6^\vee)\rho(4+2\overline{\alpha}_6^\vee)}. \end{aligned} \tag{4.20}$$

The pole set of $\left.\frac{r(\lambda)}{r(w_0\lambda)}\right|_{(L_1)_\mathbb{C}}$ are the zeroes of the denominator of (4.20) located in the region $\mathrm{Re}(\overline{\alpha_6}^\vee) \in (-4,-3) \cup (-2,-\frac{3}{2})$. This is depicted in figure 4.4. Again, after the restriction to $(L_1)_\mathbb{C}$ we now can shift the contour from p_1 linearly all the way towards c_1 without crossing the pole region. But note that this time, different than what happened in the A_2-case, the center c_1 is in the *boundary* of a critical strip. With our methods, this is as far as we can go without assuming the Riemann Hypothesis. Similar phenomena happen for the subspaces L_2, L_5 and L_6.

It will turn out that we never need to enter a critical strip, so we can use the analysis of the Yang system to conclude our results in the automorphic world. The heart of the proof is based on an $\mathfrak{sl}_2$-argument given in Lemma 4.7.1, which will take care of the combinatorics involved to guarantee that enough

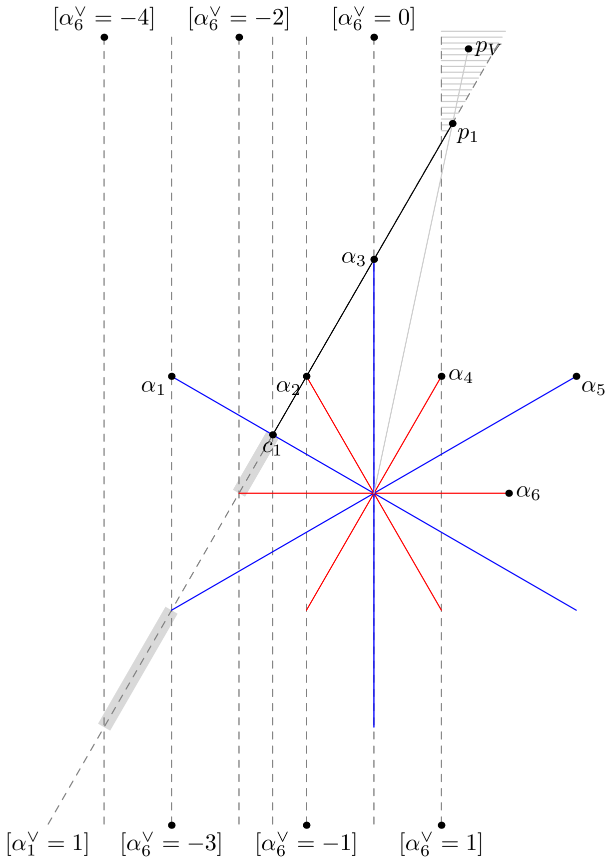

Figure 4.4: The segment $[p_1, c_1]$ does not cross the pole region in L_1.

cancellations take place in the restriction of the kernel to lower dimensional residual subspaces L, creating a window of holomorphy wide enough to perform the necessary contour shifts without ever seeing the critical poles (see Lemma 4.7.2).

4.5 Residual distributions

Recall that the notion of residual subspaces was introduced in Section 2.4.2. Write $\omega(\lambda) := \frac{d\lambda}{c_{\mathbb{H}}(-\lambda)}$ for the rational n-form of the functional $\bar{X}_{V,p_V}$ defined above in (4.12), and denote by $\mathcal{P}(\omega)$ the collection of hyperplanes it defines. Let $\mathcal{L}(\omega)$ be the lattice of intersection of elements of $\mathcal{P}(\omega)$ and $\mathcal{C}(\omega)$ be the set of their centers. Using the Residue Lemma, it follows that there is a unique collection of tempered distributions $\{\bar{X}_c \mid c \in \mathcal{C}(\omega)\}$ such that

(a) $\operatorname{supp}(\bar{X}_c) \subseteq \cup\{iV^L \mid L \in \mathcal{L}(\omega) \text{ with } c_L = c\}$

(b) $\bar{X}_{V,p_V}(f) = \sum_{c\in\mathcal{C}(\omega)} \bar{X}_c\left(f(c+\cdot)\right)$, for all $f \in PW(V_{\mathbb{C}})$.

We know that among the local contributions $\{\bar{X}_c \mid c \in \mathcal{C}(\omega)\}$, many are identically zero. A first step towards determining which $\bar{X}_c$ is nonzero, and thus the support of $\bar{X}_{V,p_V}$, is to introduce the functional $\bar{Y}_{V,p_V} \in PW(V_{\mathbb{C}})^*$ given by

$$\bar{Y}_{V,p_V}(f) := \int_{\mathrm{Re}(\lambda)=p_V} f(\lambda)\eta(\lambda), \tag{4.21}$$

for all $f \in PW(V_{\mathbb{C}})$ and with $\eta(\lambda) := \frac{d\lambda}{c_{\mathbb{H}}(-\lambda)c_{\mathbb{H}}(\lambda)}$. We similarly define the sets $\mathcal{H}(\eta)$, $\mathcal{L}(\eta)$ and $\mathcal{C}(\eta)$. We then apply the Residue Lemma to $\bar{Y}_{V,p_V}$ to obtain a collection of local distributions $\{\bar{Y}_c \mid c \in \mathcal{C}(\eta)\}$ whose properties we summarize in the following proposition. In it, for each residual subspace $L \in \mathcal{L}$, from $V = V_L \oplus V^L$, write $\dim V = r = s + t$ with $s := \dim V_L$ and $t := \dim V^L$. We let p_L denote the projection of p_V onto V_L and $\{y^1, \ldots, y^t\}$ be a basis of the lattice $Y \cap (V_L)^\perp$ (see 3.29). We have a lower rank functional $\bar{Y}_{V_L,p_L}$ and we denote by $\bar{Y}_{\Phi_L,c_L}$ its local contribution at the $(V_L, \Phi_L, 1)$-distinguished point $c_L \in V_L$ (cf. [36, Theorem 3.13] and Proposition 2.4.6).

Proposition 4.5.1. *For $c \in \mathcal{C}(\eta) \cap \overline{V_+}$, the local contribution $\bar{Y}_c$ of $\bar{Y}_{V,p_V}$ admits a decomposition*

$$\bar{Y}_c(f(c+\cdot)) = \sum_{\{L\in\mathcal{L}(\eta)\ \mid\ c_L=c\}} \bar{Y}_L(f(c+\cdot)), \tag{4.22}$$

where the functional $\bar{Y}_L$ is a positive measure on iV with support on iV^L that satisfies $\bar{Y}_L = 0$ unless L is residual. More precisely, $\bar{Y}_L$ is defined by

$$\bar{Y}_L(g) := \bar{Y}_{\Phi_L,c_L}(\{c_L\}) \int_{iV^L} g(\lambda^L)d\mu^L(\mathrm{Im}\,\lambda^L),$$

for any $g \in PW(V_{\mathbb{C}})$, in which, $\bar{Y}_{\Phi_L,c_L}(\{c_L\})$ denotes the total mass of $\bar{Y}_{\Phi_L,c_L}$ and for $\lambda^L \in V^L$,

$$d\mu^L(\lambda^L) := \prod_{\alpha\in\Phi_0^+\setminus\Phi_L} \frac{\langle\alpha^\vee, c_L\rangle^2 + \langle\alpha^\vee, \lambda^L\rangle^2}{(\langle\alpha^\vee, c_L\rangle - 1)^2 + \langle\alpha^\vee, \lambda^L\rangle^2} d\lambda^L \tag{4.23}$$

with $d\lambda^L$ a measure on V^L given by

$$d\lambda^L := (2\pi)^{-t}(dy^1 \wedge \cdots \wedge dy^t)(\lambda^L). \tag{4.24}$$

Moreover, in each W_0-orbit of residual subspaces there exists at least one element L for which $\bar{Y}_{\Phi_L,c_L}(\{c_L\}) > 0$.

Proof. Given a function f on $V_{\mathbb{C}}$, let $\check{f}(\lambda) = f(-\lambda)$. If we let $\bar{Y}^{k=-1}_{V,\gamma}$ denote the functional whose corresponding n-form η have parameter $k = -1$ instead of $k = 1$, then, $\bar{Y}_{V,p_V}(f) = \bar{Y}^{k=-1}_{V,-p_V}(\check{f})$ (up to a positive constant, this is the functional $Y_{V,-p_V}$ of the Yang system (2.43)), and we can use Proposition 2.4.6. □

Next, recall the notion of averaging operators given in (2.45). Similarly to Proposition 2.4.5, we have:

Proposition 4.5.2. *Let $c \in \mathcal{C}(\eta) \cap \overline{V_+}$. Then for all $w \in W_0$, it holds*

$$\bar{X}_{wc} = \bar{Y}_c \circ w^{-1} \circ A_{wc}. \tag{4.25}$$

Corollary 4.5.3. *A local contribution $\bar{X}_c$ of $\bar{X}_{V,p_V}$ satisfies $\bar{X}_c = 0$, unless $c = c_L$ with L residual. In particular,*

$$\bar{X}_{V,p_V}(f) = \sum_{c \in \mathcal{C}} \bar{X}_c(f(c + \cdot)), \tag{4.26}$$

for all $f \in PW(V_{\mathbb{C}})$.

Proof. Note that $\mathcal{C}(\omega) \cap \overline{V_+} = \mathcal{C}(\eta) \cap \overline{V_+}$ and that $\mathcal{C}(\omega) \subseteq W_0(\mathcal{C}(\omega) \cap \overline{V_+})$. The result now follows from Proposition 4.5.2 and Proposition 4.5.1. □

Looking closer to the existence part of the Residue Lemma one obtains, similarly to (2.47), that there is a decomposition of each local contribution of $\bar{X}_{V,p_V}$ as

$$\bar{X}_c(f(c + \cdot)) = \sum_{\{L \in \mathcal{L} \mid c_L = c\}} \bar{X}_L(f(c + \cdot)), \tag{4.27}$$

for some distributions $\bar{X}_L$ on $c + iV$ with support on L^t. Such a discussion was carried out in [74, Section 3.1, (3.28)] using results of [73]. The splitting of $\bar{X}_c$ in terms of a collection of distributions $\{\bar{X}_L \mid L \text{ such that } c_L = c\}$ as in (4.27) is not canonical, as it depends on how one deforms the contours of integration. But, from the Residue Lemma, the sum $\bar{X}_c$ is canonical.

Remark. A consequence of the definition of $\bar{X}_L$ as in [74] is that, if nonzero its support is always L^t. In particular, $\bar{X}_c$ is nonzero if and only if there is an L with $c_L = c$ and $\bar{X}_L$ is nonzero.

Remark. The distributions $\bar{X}_L$, which a priori are distributions on $PW(V_{\mathbb{C}})$, are extended to functions involving factors $\rho(\langle \alpha^\vee, \lambda \rangle)$ or $\rho(\langle \alpha^\vee, \lambda \rangle + 1)^{-1}$, for $\lambda \in L^t = c_L + iV^L$, as long as $\langle \alpha^\vee, c_L \rangle \geq 0$, because of the polynomial growth of these factors in this region (see Proposition 2.6.1).

4.6 Non-vanishing conditions.

The aim of this subsection is to give a sharp criterion to determine when the local contributions $\{\bar{X}_c \mid c \in \mathcal{C}\}$ of (4.26) are nonzero. We recall that the root datum relevant to our situation is $(\mathcal{R}^\vee)_{\mathrm{ss}}$ (see (4.5)). We let $q \in \mathbb{R}_{>1}$ be the cardinality of the residue field of the completion of F at a non-Archimedean place[2]. Consider $\mathcal{H}((\mathcal{R}^\vee)_{\mathrm{ss}},q)$ and $\mathbb{H}((\mathcal{R}^\vee)_{\mathrm{ss}},1)$ the corresponding affine and graded affine Hecke algebras, respectively, with the indicated parameters. The idea is to link the distributions $\bar{X}_{wc}$, with the representation theory of these algebras. Recall that a residual subspace L defines a parabolic root system Φ_L with Weyl group $W(\Phi_L)$. Recall also that $c_{\mathbb{H},L}(\lambda) := \prod_{\alpha\in\Phi_L^+}\gamma_\alpha(\lambda)$ and $c_{\mathbb{H}}^L$ is defined by the equation $c_{\mathbb{H}}(\lambda) = c_{\mathbb{H}}^L(\lambda)c_{\mathbb{H},L}(\lambda)$. Let $\mathcal{M}^L$ denote the localization of the ring of holomorphic functions on $V_{\mathbb{C}}$ with respect to the complement of the prime ideal corresponding to $L_{\mathbb{C}}$. Let also $\mathcal{M}(L_{\mathbb{C}})$ denote the ring of meromorphic functions on $L_{\mathbb{C}}$. There is a well-defined evaluation map $e_L : \mathcal{M}^L \to \mathcal{M}(L_{\mathbb{C}})$ restricting an element of $\mathcal{M}^L$ to $L_{\mathbb{C}}$. Observe that $c_{\mathbb{H}}^L \in \mathcal{M}^L$ according to the above condition. We define an operator $A^L(k) \in \mathrm{End}(\mathcal{M}^L)$ by

$$A^L(k)(f)(\lambda) := \frac{1}{|\operatorname{Fix}(L)|}\sum_{w\in\operatorname{Fix}(L)} c_{\mathbb{H}}(w\lambda,k)f(w\lambda). \tag{4.28}$$

We extend our convention of Section 4.3 to the operators $A^L(k)$ and we omit the parameter when $k=1$.

Remark. Observe that A^L does not preserve holomorphic functions because of the factors $c_{\mathbb{H}}^L \in \mathcal{M}^L$. Note also that $A^L(f) = A_{\Phi_L,c_L}(c_{\mathbb{H}}^L f)$, in which A_{Φ_L,c_L} denotes the averaging operator of the lower rank root system Φ_L. Specifically, this means that in (2.45), the sum is over $W(\Phi_L)_c$. If $L=\{c\}$ is a residual point, we have $A^L = A_c$. More generally, if $c=c_L$, we also have:

$$A_c(f)(\lambda) = \frac{|\operatorname{Fix}(L)|}{|W_c|}\sum_{w\in W_c/\operatorname{Fix}(L)} A^{wL}(f)(w\lambda). \tag{4.29}$$

Finally, these operators satisfy (we recall $\check{f}(\lambda) = f(-\lambda)$):

$$A^L(f)(\lambda) = A^L(-1)(\check{f})(-\lambda). \tag{4.30}$$

Lemma 4.6.1. *Let L be a residual subspace with $c_L = c \in \mathcal{C}\cap\overline{V_+}$ and $w\in W_0$. Then, for every $f\in PW(V_{\mathbb{C}})$, it holds that*

$$\bar{X}_{wc}(f(wc+\cdot)) = \bar{Y}_{\Phi_L,c}(\{c\})\sum_{\{M\in\mathcal{L}\ |\ c_M=c\}}\int_{M^t} A^{wM}(f)(w\lambda)d\mu^L(\mathrm{Im}(\lambda)).$$

Proof. We use (4.25), (4.29) and Proposition 4.5.1 to obtain that, for all Paley-Wiener function f, it holds that

$$\begin{aligned}\bar{X}_{wc}(f(c+\cdot)) &= \sum_{\{L\ |\ c_L=c\}}\bar{Y}_L\left(A_{wc}(f)(w(c+\cdot))\right)\\ &= \bar{Y}_{\Phi_L,c}(\{c\})\sum_{\{M\ |\ c_M=c\}}\int_{M^t}A^{wM}(f)(w\lambda)d\mu^L(\mathrm{Im}(\lambda)),\end{aligned}$$

[2]Which finite place is being considered is not relevant. The nature of our method is that we are interested in the limit $q\to 1^+$, with $q>1$.

where, for the last equality, we used that $\bar{Y}_{\Phi_L,c}(\{c\}) = \bar{Y}_{\Phi_M,c}(\{c\})$, for all L, M such that $c_L = c = c_M$ (see [73, Theorem 3.27(i)]). $\square$

The next Proposition is essential to our understanding of the support of the functional $\bar{X}_{V,p_V}$. For its proof, we remind the reader of Proposition 2.3.3, in which M. Reeder gave a condition for the nonvanishing of the weight spaces of the unique anti-spherical irreducible constituent of the unramified minimal principal series for the Hecke algebra $\mathcal{H}((\mathcal{R}^\vee)_{\mathrm{ss}}, q)$, with central character $W_0\tau$, for $\tau \in {}^LT_1 \subseteq {}^LT$. Let $\mathfrak{q} := \{x \in {}^L\mathfrak{g} \mid \mathrm{ad}(L)x = x\}$ and $C := C_{^LG}(L)^\circ$. Also, let $\mathfrak{q}_0$ denote the unique dense orbit of C on $\mathfrak{q}$.

Proposition 4.6.2. *Let L be a residual coset and let $c = c_L$ be its center. Assume that the residual subspace L is chosen in its W_0-orbit such that $c = c_L$ is dominant. Let $\mathbf{o}_{W_0L} \subseteq {}^L\mathfrak{g}$ be the nilpotent orbit associated with W_0L as in Proposition 2.5.4 and let $w \in W_0$. The following are equivalent:*

- *(a) There exists a homomorphism $\varphi : \mathfrak{sl}_2 \to {}^L\mathfrak{g}$ such that $\varphi(h) = 2c$ and $\varphi(e) \in \mathfrak{q} \cap \mathbf{o}_{W_0L} \cap \mathrm{Ad}(w^{-1})({}^L\mathfrak{n})$.*
- *(b) The restriction to wL of $A^{wL}(f)$ is not identically equal to 0 for some Paley-Wiener functions f on $V_\mathbb{C}$.*
- *(c) The distribution $\bar{X}_{wc}$ is nonzero.*

Proof. It will be proven (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c). For the first, note that $\mathfrak{q}_0 = \mathbf{o}_{W_0L} \cap \mathfrak{q}$. Pick $\lambda \in V^L$ such that $c + i\lambda \in L_\mathbb{C}$ is a generic element and let $t := q^{i\lambda}$. We are then in condition to apply Proposition 2.3.3 and conclude that (a) is equivalent to the nonvanishing of the $w\tau$ weight space of the unique antispherical module for $\mathcal{H}((\mathcal{R}^\vee)_{\mathrm{ss}}, q)$ with central character $W_0\tau$, for $\tau = q^{-c}t^{-1}$. Using the involution $\iota : \mathcal{H}((\mathcal{R}^\vee)_{\mathrm{ss}}, q^{-1}) \to \mathcal{H}((\mathcal{R}^\vee)_{\mathrm{ss}}, q)$ that sends $T_i \mapsto -q^{-1}T_i$ (see Lemma 2.3.2) and Lusztig's reduction theorems [57] (see Proposition 2.3.14) we see that the last assertion is equivalent to the nonvanishing of the $-w(c+i\lambda)$-weight space of the unique spherical module for $\mathbb{H}((\mathcal{R}^\vee)_{\mathrm{ss}}, -1)$ with central character $-W_0(c+i\lambda)$. To prove that this last condition is equivalent to (b), let $\mu := c+i\lambda$. We know from Proposition 2.3.13 that the unique spherical module for $\mathbb{H}((\mathcal{R}^\vee)_{\mathrm{ss}}, -1)$ with central character $W_0(-\mu)$ is generated by the elementary spherical function $\phi(-\mu, -1; \xi)$ which can be written as (recall $A(-1)_0$ means the averaging operator $A(k)_c$ of (2.45) with $k = -1$ and $c = 0$)

$$A(-1)_0(e^\xi)(-\mu) = \frac{|\operatorname{Fix}(L)|}{|W_0|} \sum_{w \in W_0/\operatorname{Fix}(L)} A^{wL}(-1)(e^\xi)(-w\mu),$$

for all $\xi \in V_\mathbb{C}^*$. Thus, using (4.30), the nonvanishing of the weight space is equivalent to saying that $A^{wL}(e^{-\xi})(w\mu)$ is not constantly zero as a function of ξ. Let us now show that, if $A^{wL}(e^{-\xi})(w\mu) = 0$ for all $\mu \in L_\mathbb{C}$, then it holds that $A^{wL}f|_{wL} \equiv 0$ for all $f \in PW(V_{\mathbb{C}})$. From the Euclidean Paley-Wiener Theorem, given $f \in PW(V_{\mathbb{C}})$, there is a compactly supported smooth function $\varphi \in C_c^\infty(V^*)$ such that $f(\lambda) = \int_{\xi\in V^*} \varphi(\xi)e^{-\xi(\lambda)}d\mu^E(\xi)$ ($d\mu^E$ is the Euclidean measure). Thus, for all $\lambda \in V_{\mathbb{C}}$

$$A^{wL}(f)(w\lambda) = \int_{\xi\in V^*} \varphi(\xi)A^{wL}(e^{-\xi})(w\lambda)d\mu^E(\xi),$$

in which the exchange of the summation symbol and the integration symbol is justified because of the compact support of φ. It follows from this that the vanishing of $A^{wL}(e^{-\xi})(w\mu)$, as a function of ξ, for $\mu \in L_{\mathbb{C}}$, implies $A^{wL}f|_{wL_{\mathbb{C}}} = 0$, for all Paley-Wiener functions in $V_{\mathbb{C}}$. Conversely, suppose $A^{wL}(e^{-\xi})(w\mu)$ is nonvanishing. Let f_1 be a W_{wc}-invariant Paley-Wiener function on $V_{\mathbb{C}}$ such that $f_1(w\mu) \neq 0$. Put $f_2 = f_1e^{-\xi} \in PW(V_{\mathbb{C}})$. Then,

$$A^{wL}(f_2)(w\mu) = f_1(w\mu)A^{wL}(e^{-\xi})(w\mu) \neq 0.$$

We thus proved (a) is equivalent to (b).

In view of Lemma 4.6.1 and the positivity of $\bar{Y}_L$, (c) implies (b). Conversely, note that each residual subspace $L_{\mathbb{C}}$ is determined by a finite set of affine hyperplanes $\langle \alpha^\vee, \lambda\rangle = 1$. For each such hyperplane, let $g_\alpha(\lambda) := (\langle \alpha^\vee, \lambda\rangle - 1)$ be the corresponding defining function. Then,

$$g(\lambda) = \prod g_\alpha, \tag{4.31}$$

with the product over all $\alpha \in \Phi_0$ such that $\langle \alpha^\vee, L\rangle$ is not constantly 0, is a holomorphic Fix(L)-invariant function on $V_{\mathbb{C}}$ such that $g|_M$ is constantly 0 for all $M \neq L$. Assuming (b) and applying Lemma 4.6.1 for the function ${}^wgf \in PW(V_{\mathbb{C}})$, (with ${}^wg(\lambda) = g(w^{-1}\lambda)$), one has

$$\bar{X}_{wc}({}^wgf(wc+\cdot)) = \bar{Y}_{\Phi_L,c}(\{c\}) \int_{L^t} g(\lambda)A^{wL}(f)(w\lambda)d\mu^L(\mathrm{Im}(\lambda)).$$

and the result will follow if $\bar{Y}_{\Phi_L,c}(\{c\})$ is nonzero. If it were the case that $\bar{Y}_{\Phi_L,c}(\{c\})$ is zero, then the unique spherical representation with central character $W(\Phi_L)c$ of the lower rank graded Hecke algebra would not be a discrete series. But, assumption (b) (because equivalent to (a)) implies that this module is a discrete series, using Reeder's result Proposition 2.3.3. The argument is similar to the one sketched in Proposition 2.5.5. □

At this point, we have enough information about the functional $\bar{X}_{V,p_V}$ at our disposal, and we remind the reader that our goal is to evaluate (4.13). Our next task will be to determine when the summand $R_{\phi,w}|_{L_{\mathbb{C}}}$ of (4.10) is nonvanishing, for $L \in \mathcal{L}$.

Lemma 4.6.3. *For any $w \in W_0$ and any residual L we have $r(-w\lambda)^{-1} \in \mathcal{M}^L$.*

Proof. The set of poles of $r(-w\lambda)^{-1}$ is a union of hyperplanes of the form $\langle\alpha^\vee,\lambda\rangle = z$ with $\mathrm{Re}(z) \in (-1,0)$. On the other hand, if $\alpha^\vee$ is constant on L then $\alpha \in \Phi_L$, and by Proposition 2.5.4(d) this implies that $\langle\alpha^\vee, L\rangle \in \mathbb{Z}$. $\square$

Recall the notations that, for any function f on $V_{\mathbb{C}}$, $\check{f}(\lambda) = f(-\lambda)$ and also that $\phi^-(\lambda) = \overline{\phi(\overline{\lambda})}$. By the previous lemma we conclude that it makes sense to write

$$R_\phi(-\lambda) = |W_0| r(-\lambda) A_0\left(\frac{\phi^-}{\check{r}}\right)(\lambda)$$

and that the argument of the operator A_0 belongs to $\mathcal{M}^L$ for every residual subspace L. Let L be a residual subspace, with pointwise fixator $\mathrm{Fix}(L) \subseteq W_0$. Let $W^L \subseteq W_0$ denote a complete set of representatives for $W_0/\mathrm{Fix}(L)$. We can write the last equation, using (4.29), in the form

$$R_\phi(-\lambda) = |\,\mathrm{Fix}(L)| r(-\lambda) \sum_{w \in W^L} \left(A^{wL}\left(\frac{\phi^-}{\check{r}}\right) \circ w\right)(\lambda).$$

Using (4.30), we can write

$$R_\phi(\lambda) = |\,\mathrm{Fix}(L)| r(\lambda) \sum_{w \in W^L} \left(A^{wL}(-1)\left(\frac{\check{\phi}^-}{r}\right) \circ w\right)(\lambda). \tag{4.32}$$

Note further, that for each $u \in W^L$, we have

$$\left(r\left[A^{wL}(-1)\left(\frac{\check{\phi}^-}{r}\right) \circ w\right]\right)(\lambda) = \frac{1}{|\,\mathrm{Fix}(L)|} \sum_{u \in w\,\mathrm{Fix}(L)} R_{\phi,u}(\lambda). \tag{4.33}$$

In any case, the restriction of R_ϕ to $L_{\mathbb{C}}$ is well defined as a meromorphic function. We end this subsection with the promised condition about the nonvanishing of $R_{\phi,w}|_{L_{\mathbb{C}}}$. For the next corollary, recall that ${}^L\overline{\mathfrak{n}}$ is the nilpotent algebra opposite to ${}^L\mathfrak{n}$ (see 2.51).

Corollary 4.6.4. *Let L be as in Proposition 4.6.2 and let $v, w \in W_0$. The only summands $R_{\phi,w}$ of (4.10) which have a nonzero restriction to $vL_{\mathbb{C}}$ are those such that there is an $\mathfrak{sl}_2$-homomorphism with $\varphi(h) = 2c$ and $\varphi(e) \in \mathfrak{q}_0 \cap$* $\mathrm{Ad}(v^{-1}w^{-1})({}^L\overline{\mathfrak{n}})$. *This set of Weyl group elements is a union of left cosets for* $\mathrm{Fix}(vL)$.

Proof. If $u \in \mathrm{Fix}(L)$ then $\mathrm{Ad}(u)$ comes from conjugation by an element of $N_C({}^L\mathfrak{t})$. Therefore $\mathrm{Ad}(u)(\mathfrak{q}_0) = \mathfrak{q}_0$, showing that the condition on w is preserved by multiplication on the right with elements of $\mathrm{Fix}(vL)$.

Next, we analyze the restriction to vL of the summands $R_{\phi,w}$ of R_ϕ. Typically, the restriction of a single summand $R_{\phi,w}$ to $vL_{\mathbb{C}}$ is not defined, as $R_{\phi,w}$ may be singular is $vL_{\mathbb{C}}$. But the restriction to $vL_{\mathbb{C}}$ of a sum over a $\mathrm{Fix}(vL)$-coset, as in (4.33), is a well-defined meromorphic function of $vL_{\mathbb{C}}$. Let us call a sum over a $\mathrm{Fix}(vL)$-coset as in (4.33) by $R_{\phi,wvL}$. Now, since for every residual subspace L one has $-L \in W_0L$ (see Proposition 2.4.4, or [74] for an intrinsic argument), it follows that $-w_0\lambda \in L$, for all $\lambda \in L$, where w_0 is the longest element of W_0. Using (4.10), and (4.28) we get

$$R_{\phi,wvL}(v\lambda) = \left(r\left[A^{wvw_0L}\left(\frac{\phi^-}{\check{r}}\right) \circ (-w)\right]\right)(v\lambda),$$

thus, it holds that $R_{\phi,wvL}|_{vL_{\mathbb{C}}}$ is nonzero if and only if $A^{wvw_0L}\left(\frac{\phi^-}{\check{r}}\right)$ restricted to $wvw_0L_{\mathbb{C}}$ is nonzero, which implies that there exists a Paley-Wiener function f such that $A^{wvw_0L}(f)$ is not constantly zero. From Proposition 4.6.2, we can find φ such that $\varphi(e) \in \mathfrak{q}_0 \cap \mathrm{Ad}(w_0v^{-1}w^{-1})({}^L\mathfrak{n})$. Replacing φ by $\varphi' := \theta \circ \mathrm{Ad}(w_0) \circ \varphi$ where θ is the Cartan involution of ${}^L\mathfrak{g}$, one has that φ' satisfies the criterion, because $\theta\,\mathrm{Ad}(w_0)$ preserves $\mathfrak{q}_0$ (we have $-w_0c_L = c_L$). The converse is similar. $\square$

4.7 The contour shift argument.

In this section we will carry out the algorithm to compute the distribution $\{\bar{X}_L \mid L \in \mathcal{L}\}$ of (4.27), in such a way that we never need to enter a critical strip. Let $L \in \mathcal{L}$ be such that c_L is dominant, $v, w \in W_0$ and assume that both $\bar{X}_{vc_L} \neq 0$ and that $R_{\phi,w}|_{vL_{\mathbb{C}}} \neq 0$. Using Proposition 4.6.2 and Corollary 4.6.4, there exist Lie algebra homomorphisms $\varphi', \varphi'' : \mathfrak{sl}_2 \to {}^L\mathfrak{g}$ such that $\varphi'(h) = \varphi''(h) = 2c_L$ and

$$\begin{array}{rcl} \varphi'(e) & \in & \mathfrak{q}_0 \cap \mathrm{Ad}(v^{-1})({}^L\mathfrak{n}) \\ \varphi''(e) & \in & \mathfrak{q}_0 \cap \mathrm{Ad}(v^{-1}w^{-1})({}^L\overline{\mathfrak{n}}) \end{array} \tag{4.34}$$

As both $\varphi'(e), \varphi''(e) \in \mathfrak{q}_0$, we can in fact take $\varphi' = \varphi''$ in (4.34). Therefore in this situation there exists a homomorphism $\varphi = \mathrm{Ad}(v)\varphi'$ such that

$$\begin{array}{rcl} \varphi(h) & = & 2vc_L \\ \varphi(e) & \in & {}^L\mathfrak{n} \cap \mathrm{Ad}(w^{-1})({}^L\overline{\mathfrak{n}}). \end{array} \tag{4.35}$$

In view of equation (4.35), the roots occurring in ${}^L\mathfrak{n} \cap \mathrm{Ad}(w^{-1})({}^L\overline{\mathfrak{n}})$ are those in $\Phi(w) = \Phi_0^+ \cap w^{-1}\Phi_0^-$. The next lemma is essential to guarantee that the contour shift we will describe below never enters the critical strips of the factors $\rho(\langle\alpha^\vee, \lambda\rangle + 1)$ (recall that $\rho(s)$ is the regularization of the completed Dedekind zeta function, see (2.69)).

Lemma 4.7.1. *Let $L \in \mathcal{L}$ such that $c_L \in \overline{V_+}$ and $v, w \in W_0$ such that $R_{\phi,w}|_{vL_{\mathbb{C}}}$ is nonzero. Consider the restriction of the product $\frac{r(\lambda)}{r(w\lambda)}$ to vL. The roots $\alpha \in \Phi(w)$ such that the restriction of $\rho(\langle\alpha^\vee, \lambda\rangle + 1)$ to vL is nonconstant and appears in the denominator of the restriction of (4.8) to vL satisfy $\langle\alpha^\vee, vc_L\rangle \geq 0$.*

Proof. When we restrict $\frac{r(\lambda)}{r(w\lambda)}$ to vL certain cancellations will occur in (4.8) because of the additional affine integral relations between the restricted roots of the set $\Phi(w)$ due to root strings of coroots which have a constant value on vL. By (4.35) there exists a homomorphism $\varphi : \mathfrak{sl}_2 \to {}^L\mathfrak{g}$ such that $\varphi(h) = 2vc_L$ and $\varphi(e) \in {}^L\mathfrak{n} \cap \mathrm{Ad}(w^{-1})({}^L\bar{\mathfrak{n}})$. Recall that φ was given by $\varphi = \mathrm{Ad}(v)\varphi'$ with φ' satisfying $\varphi'(h) = 2c_L$ and (4.34). In particular, one has that

$$\varphi(e) \in \sum {}^L\mathfrak{g}_{v\alpha^\vee}, \tag{4.36}$$

with the sum over the roots $\{\alpha \in \Phi_L \mid \langle \alpha^\vee, c_L\rangle = 1\}$. That said, recall from $\mathfrak{sl}_2$-representation theory that elements in the kernel of $\mathrm{ad}(\varphi(e))$ are linear combination of highest weight vectors. Thus, $\mathrm{ad}(\phi(e)) \in \mathrm{End}({}^L\mathfrak{n} \cap \mathrm{Ad}(w^{-1})({}^L\bar{\mathfrak{n}}))$ will be *injective* if restricted to a direct sum of root spaces ${}^L\mathfrak{g}_{\alpha^\vee}$ with $\alpha \in \Phi(w)$ and $\langle \alpha^\vee, vc_L\rangle < 0$. Now, given an $\alpha \in \Phi(w)$, let

$$\Phi(\alpha, v) := \{\beta \in \Phi(w) \mid \beta^\vee|_{vL} = \alpha^\vee|_{vL}\},$$

and similarly define $\Phi(\alpha+1, v)$. Note that if $\gamma \in \Phi_L$ with $\langle \gamma^\vee, c_L\rangle = 1$ and $\beta \in \Phi(\alpha, v)$, then $(\beta^\vee + v\gamma^\vee)|_{vL} = \alpha^\vee|_{vL} + 1$. Hence, from (4.36), it follows that

$$\mathrm{ad}(\varphi(e))\colon \left(\sum_{\beta \in \Phi(\alpha,v)} {}^L\mathfrak{g}_{\beta^\vee}\right) \to \left(\sum_{\beta \in \Phi(\alpha+1,v)} {}^L\mathfrak{g}_{\beta^\vee}\right). \tag{4.37}$$

From what we said above, if $\langle \alpha^\vee, vc_L\rangle < 0$, we have that the map in (4.37) is injective and hence $|\Phi(\alpha, v)| \le |\Phi(\alpha+1, v)|$. Writing $\Phi^{\mathrm{comp}} = \Phi(w) \setminus (\Phi(\alpha, v) \cup \Phi(\alpha+1, v))$, one rewrites (4.8) as

$$\frac{r(\lambda)}{r(w\lambda)} = \prod_{\Phi(\alpha,v)} \frac{\rho(\langle \beta^\vee, \lambda\rangle)}{\rho(\langle \beta^\vee, \lambda\rangle + 1)} \times \prod_{\Phi(\alpha+1,v)} \frac{\rho(\langle \beta^\vee, \lambda\rangle)}{\rho(\langle \beta^\vee, \lambda\rangle + 1)} \times \prod_{\Phi^{\mathrm{comp}}} \frac{\rho(\langle \beta^\vee, \lambda\rangle)}{\rho(\langle \beta^\vee, \lambda\rangle + 1)}$$

and concludes that $\rho(\alpha^\vee|_{vL} + 1)$ does not occur in the denominator of (4.8). It follows that all factors $\rho(\alpha^\vee|_{vL} + 1)$ which have a negative multiplicity (i.e. appear in the denominator of (4.8)) satisfy $\langle \alpha^\vee, c_{vL}\rangle \ge 0$. □

With Lemma 4.7.1 at our disposal we are ready to describe an algorithm that consists of linear shifts of the contours of integration to compute the local residue contributions of $\bar{X}_{V,p_V}$ in a way that we will not meet the poles of R_ϕ.

We start by remarking that given a subspace $L \in \mathcal{L}(\omega)$ and a point $p_L \in L$, we will denote by $\bar{X}_{L,p_L}(f)$ the integral of a function f with Schwartz decay in the imaginary direction with contour $p_L + iV^L$:

$$\bar{X}_{L,p_L}(f) := \int_{p_L + iV^L} f(\lambda)\omega^L(\lambda), \tag{4.38}$$

in which $\omega^L(\lambda)$ is a meromorphic form on $L_{\mathbb{C}}$, regular outside the union of all codimension 1 ω-residual (see Definition 2.4.1) subspaces of L. Such a form ω^L is produced by an iterative contour shift, as the one that we will describe.

Next, note that because the collections $\mathcal{H}(\omega)$ and $\mathcal{L}(\omega)$ are finite, there exists an $\epsilon > 0$ such that, for all $L \in \mathcal{L}(\omega)$, the ball $B_{\mathbb{R}}(c_L, \epsilon) \subseteq V$, avoids all hyperplanes of $\mathcal{H}(\omega)$ except those that contain c_L and we are thus in the situation of [74, Proposition 3.6].

Now, a linear shift of contour from p_L to c_L should be interpreted in the following sense: in the straight line $[p_L, c_L]$, choose $\epsilon_L \in [p_L, c_L] \cap B_{\mathbb{R}}(c_L, \epsilon)$. If necessary, we deform ϵ_L in a small neighborhood to ensure that the path $[p_L, \epsilon_L]$ intersects the codimension 1 subspaces $\mathcal{L}(\omega) \ni M \subseteq L$ only in generic points p_M. Then, the linear shift from p_L to c_L is actually the shift from p_L to ϵ_L, along the line segment $[p_L, \epsilon_L]$, but we remark that the choice of ϵ_L can be made so that if $\langle \alpha^\vee, p_L \rangle > 0$ and $\langle \alpha^\vee, c_L \rangle \geq 0$ for a root $\alpha \in \Phi_0^+$, then, it still holds $\langle \alpha^\vee, \epsilon_L \rangle > 0$ as if ϵ_L were in the interior of the segment $[p_L, c_L]$. In this situation for any meromorphic function f in an open neighborhood of the form $U_L + iV$ of $c_L + iV$ which has Schwartz decay in the imaginary direction and whose poles do not meet the contours of integration in $L_{\mathbb{C}}$ of the form $p + iV^L$ with $p \in [c_L, \epsilon_L]$ we have that $\bar{X}_{L,\epsilon_L}(f) = X_L^{\epsilon_L}(f(c_L + \cdot))$ for a unique tempered distribution $X_L^{\epsilon_L}$ on iV with support on iV^L. Namely it is a boundary value distribution in the sense of Hormander[41, Theorem 3.1.15]. These distributions $\{X_L^{\epsilon_L} \mid L \in \mathcal{L}(\omega)\}$, are the distributions $\bar{X}_L$ of (4.27) (see [74], (3.28) and Section 3.1), and the superscript emphasizes that their definition depended on the choices of ϵ_L made.

Let us now describe the promised algorithm to compute the $\bar{X}_L$. Fix $w \in W_0$. When we shift the contour, it could be the case that $R_{\phi,w}$ is constantly a pole for an $L \in \mathcal{L}(\omega)$ (we can only guarantee that $R_{\phi,w}$ is not constantly a pole if L is residual, see Lemma 4.6.3) so we may not, a priori, restrict ourselves only to the residual subspaces when performing the algorithm. For any $L \in \mathcal{L}(\omega)$ let

$$S(w, L) := \left\{ \alpha \in \Phi(w) \;\middle|\; \begin{array}{c} \rho(\alpha^\vee|_{L_{\mathbb{C}}} + 1) \text{ is not constant and occurs} \\ \text{in the denominator of } \left.\frac{r(\lambda)}{r(w\lambda)}\right|_{L_{\mathbb{C}}}. \end{array} \right\}. \tag{4.39}$$

Inductively on j, the codimension of an affine subspace, we construct sets

$$\mathcal{S}(j) := \{L \in \mathcal{L}(\omega) \mid \operatorname{codim}(L) = j \text{ and that } L \text{ satisfies (a) - (d) below}\} \tag{4.40}$$

(a) there is a finite set $P(L) = \{p_{L,i}\} \subseteq L$ obtained by previous iterations of the algorithm,

(b) $\langle \alpha^\vee, p_{L,i} \rangle > 0$ for all $p_{L,i} \in P(L)$ and $\alpha \in S(w, L)$,

(c) $R_{\phi,w}|_{L_{\mathbb{C}}}$ is not constantly zero,

(d) $\bar{X}_{V,p_V}(R_{\phi,w}) = \sum_N \bar{X}_N(R_{\phi,w}) + \sum_{L,i} C(L,i)\bar{X}_{L,p_{L,i}}(R_{\phi,w})$, in which, in (d), the first sum is indexed by $\{N \in \mathcal{L} \mid \operatorname{codim} N < j$ and $\bar{X}_N \neq 0\}$, the second sum is indexed by $L \in \mathcal{S}(j)$ and the points $\{p_{L,i}\} \subseteq L$, and $C(L,i)$ are constants that depend on $p_{L,i}$. The construction goes as follows. Starting with $j = 0$, let

$$\mathcal{S}(0) = \{V\}, \quad P(V) = \{p_V\},$$

so that (a) - (d) are trivially satisfied. Assuming that $\mathcal{S}(j)$ is defined, perform the following steps:

(1) For each $L \in \mathcal{S}(j)$, choose a $p_{L,0} \in P(L)$,

(2) shift the contours of $C(L,i)\bar{X}_{L,p_{L,i}}$ to $p_{L,0}$, linearly,

(3) if $\bar{X}_L \neq 0$, shift $p_{L,0}$ to c_L linearly.

Define $\mathcal{S}(j+1)$ to be the collection of $M \in \mathcal{L}(\omega)$ of codimension 1 in each $L \in \mathcal{S}(j)$ that were crossed on points p_M during the linear shifts of contour described in steps (2) and (3) and such that $R_{\phi,w}|_M \not\equiv 0$.

Lemma 4.7.2. *In both shifts of contour described in (2) and (3), the pole-set of* $\left.\frac{r(\lambda)}{r(w\lambda)}\right|_{L_{\mathbb{C}}}$ *is not crossed.*

Proof. Recall that the pole-set of $\frac{r(\lambda)}{r(w\lambda)}$ is a union of hyperplanes $\langle \alpha^\vee, \lambda \rangle = z$, with $\operatorname{Re}(z) \in (-1, 0)$. Thus, to prove the Lemma it suffices to show that for every point p involved in the linear shifts described in steps (2) and (3) we have $\langle \alpha^\vee, p \rangle \geq 0$ for all $\alpha \in S(w, L)$.

In step (2), because of property (b) of the set $\mathcal{S}(j)$ defined in (4.40), any point $p \in [p_{L,i}, p_{L,0}]$ satisfy $\langle \alpha^\vee, p \rangle > 0$ for all $\alpha \in S(w, L)$. Further, the condition that $\bar{X}_L \neq 0$ to perform step (3) implies that L is residual (see Corollary 4.5.3 and the first remark after it), so if we write $L = vN$ with N residual in dominant position and $v \in W_0$ we have $c_L = vc_N$ and we are in condition to use Lemma 4.7.1 to conclude that for all p in the interior of the segment $[p_{L,0}, c_L]$ it holds that $\langle \alpha^\vee, p \rangle > 0$ for every $\alpha \in S(w, L)$. $\square$

Proposition 4.7.3. *The set $\mathcal{S}(j+1)$ satisfies (a) - (d), so that the induction is well-defined.*

Proof. When we perform the linear shifts as in (2) and (3), we cross subspaces $M \in \mathcal{L}(\omega)$ of codimension 1 in the $L \in \mathcal{S}(j)$. Notice that a same codimension 1 subspace $M \in \mathcal{L}(\omega)$ may be crossed several times during the shift of different L, but as all the collection of hyperplanes $\mathcal{L}(\omega)$ is finite, we end up with a finite set $P(M)$ as in (a). These possible multiple crossings of subspace M is what creates the constants $C(M,i) \in \mathbb{Z}$. Moreover, in the proof of Lemma 4.7.2, we saw that all points p_M produced satisfy $\langle \alpha^\vee, p_M \rangle > 0$, for all $\alpha \in S(w, L)$. Now, note that if $\beta \in S(M, w)$, and $M \subseteq L$, then there exist $\alpha \in S(L, w)$ such that

$\alpha^\vee|_M = \beta^\vee|_M$. Hence, (b) is also satisfied and (c) is satisfied by construction. We are left to show that condition (d) holds. By the inductive hypothesis, it holds for $\mathcal{S}(j)$. Next, note that when we perform step (2), we combine all the constants $C(L,i)$ to obtain a unique integration with contour $p_{L,0} + iV^L$ and we get

$$\sum_{L\in\mathcal{S}(L,j)} \sum_i C(i)\bar{X}_{L,p_{L,i}}(R_{\phi,w}) = \sum_{L\in\mathcal{S}(j)} \bar{X}_{L,p_{L,0}}(R_{\phi,w}) + \{\text{codimension 1 integrals}\}.$$

Because of property (b), we have that $R_{\phi,w}|_{L_\mathbb{C}}$ is not constantly a pole along L. Thus, if it is the case that $\bar{X}_L = 0$, from [41, Theorem 3.1.15], one has that actually the sum of the coefficient functions used to define $\bar{X}_L$ (see [74, (3.28)]) are already zero, and hence, the integration $\bar{X}_{L,p_{L,0}}(R_{\phi,w}) = 0$. Hence, we can disconsider every $L \in \mathcal{S}(j)$ such that $\bar{X}_L = 0$ and only the $\bar{X}_L$ with L residual remain. Finally, when we shift the contour using step (3), we obtain a contribution $\bar{X}_L(R_{\phi,w})$ plus some codimension 1 integrals for $M \in \mathcal{S}(j+1)$, say $\bar{X}_{M,p_M}(R_{\phi,w})$. We thus obtain

$$\bar{X}_{V,p_V}(R_{\phi,w}) = \sum_{\{N\in\mathcal{L} \mid \operatorname{codim} N \le j\}} \bar{X}_N(R_{\phi,w}) + \sum_{M,i} C(M,i)\bar{X}_{M,p_{M,i}}(R_{\phi,w}),$$

with the last sum ranging over all the $M \in \mathcal{S}(j+1)$ and the $p_{M,i}$ just constructed, so (d) is satisfied. □

Corollary 4.7.4. *The inner product $(\theta_\phi, \theta_\psi)$ decomposes as*

$$(\theta_\phi, \theta_\psi) = \sum_{c\in\mathcal{C}} \bar{X}_c\left(\psi R_\phi(c+\cdot)\right). \tag{4.41}$$

Proof. After the induction procedure described above we obtain $\bar{X}_{V,p_V}(\psi R_\phi) = \sum_L \bar{X}_L(\psi R_\phi)$, with the sum over all the L such that $\bar{X}_L \neq 0$ and such that there is a $w \in W_0$ for which $R_{\phi,w}|_{L_\mathbb{C}} \not\equiv 0$. Grouping all the terms and using (4.26), (4.27) and (4.13), the result follows. □

4.8 The expression for the scalar product.

The last task is to rewrite (4.41) in a symmetric form to obtain a useful expression. Using (4.25) and a computation similar to [36, Theorem 3.18], we obtain:

$$(\theta_\phi, \theta_\psi) = \frac{1}{|W_0|}\sum_{L\in\mathcal{L}} \int_{L^t} \sum_{v\in W_0} \left\{ \sum_{w\in W_0} \phi^-(-wv\lambda)c_\mathbb{H}(-wv\lambda)\frac{r(v\lambda)}{r(wv\lambda)} \right\} \times \psi(v\lambda)c_\mathbb{H}(v\lambda)\, d\overline{\nu}_L(\lambda), \tag{4.42}$$

in which, similarly to the measure ν_L of the Yang system, $\overline{\nu}_L$ is the unique positive measure supported on $L^t = c_L + iV^L \subseteq V_{\mathbb{C}}$ characterized by the requirement that, for all $f \in PW(V_{\mathbb{C}})$, we have $\int_{L^t} f d\overline{\nu}_L = \bar{Y}_L(f(c_L + \cdot))$ if c_L is dominant, and such that

$$\overline{\nu}_{W_0 L} := \sum_{L' \subseteq W_0 L} \overline{\nu}_{L'} \tag{4.43}$$

is a W_0-invariant measure. In fact, since the Y-functional in the automorphic setting is closely related to the Y-functional of the Yang system (compare equations (2.42) and (2.43) with (2.117) and (4.21)), we can use the computations done in Chapter 3, namely Theorems 3.6.1 and 3.8.1, and in view of the Remark of Section 3.6, we conclude that $\overline{\nu}_L$ is the push forward of a smooth measure on $c_L + iV^L$ of the form: for $\lambda = c_L + i\lambda^L \in L^t$,

$$d\overline{\nu}_L(\lambda) = \frac{|W(\Phi_L)_{c_L}|}{|A_{\mathbf{o}\cap {}^L M_{\mathbf{o}}}|} \frac{\prod'_{\alpha\in\Phi_L} \langle \alpha^\vee, c_L\rangle}{\prod'_{\alpha\in\Phi_L} (\langle \alpha^\vee, c_L\rangle + 1)} \times \prod_{\alpha\in\Phi_0^+\setminus\Phi_L} \frac{\langle \alpha^\vee, c_L\rangle^2 + \langle \alpha^\vee, \lambda^L\rangle^2}{(\langle \alpha^\vee, c_L\rangle - 1)^2 + \langle \alpha^\vee, \lambda^L\rangle^2} \frac{d\lambda^L}{(2\pi)^t}, \tag{4.44}$$

where $d\lambda^L$ is given in (4.24), $|A_{\mathbf{o}\cap {}^L M_{\mathbf{o}}}|$ is the cardinality of the component group of the centralizer of the image of the $\mathfrak{sl}_2$-homomorphism associated to $W_0 L$, $\mathbf{o} = \mathbf{o}_{W_0 L}$ and ${}^L M_{\mathbf{o}}$ is the associated Levi (see Proposition 2.5.4, (2.60) and (3.31)) and $\prod'$ denotes the product over the nonzero factors.

4.9 Proof of the theorems

4.9.1 Proof of Theorem 4.1.1

Let us rewrite (4.42) in order to exhibit its most important properties. Namely that in this formula every summand corresponding to a W_0-orbit of residual subspaces contributes a positive semidefinite Hermitian form on the space of Paley-Wiener functions. For that, let $W_0\backslash\mathcal{L}$ denote a complete set of representatives for W_0-orbits of residual subspaces. Note that if $\lambda = c_L + i\mu \in L^t$ and if $w_L \in W(\Phi_L)$ denotes the longest element in the Weyl group $W(\Phi_L)$ then $w_L(c_L) = -c_L$ and $w_L\mu = \mu$ (see Proposition 2.4.4). In other words, for $\lambda \in L^t$ we have:

$$-w_L\lambda = \overline{\lambda}. \tag{4.45}$$

Lemma 4.9.1. *The expression (4.42) for the inner product can be written in the following equivalent way:*

$$\sum_{L\in W_0\backslash\mathcal{L}} |W_0| \int_{W_0 L^t} \overline{A_0(r\phi)(\lambda)} A_0(r\psi)(\lambda) \frac{d\overline{\nu}_{W_0 L}(\lambda)}{r(-\lambda) r(\lambda)}. \tag{4.46}$$

Proof. First observe that $r(-\lambda)r(\lambda)$ does not vanish identically on L^t. Indeed, $\alpha^\vee \in \Phi_0$ is constant on L^t if and only if $\alpha^\vee \in \Phi_L$. By Proposition 2.5.4(d), the constant value of $r(-\lambda)r(\lambda)$ on L^t is in $\mathbb{Z}$, hence outside of the critical strip. Thus it makes sense to multiply and divide the integrand of (4.42) by $r(-w\lambda)$, for each $w \in W_0$, to obtain

$$(r(-\lambda)r(\lambda))^{-1} \sum_{v,w\in W_0} \phi^-(-w\lambda)c_{\mathbb{H}}(-w\lambda)r(-w\lambda)\psi(v\lambda)c_{\mathbb{H}}(v\lambda)r(v\lambda).$$

Then, using (4.45) and the definition of A_0 in (2.45), one rewrites this as

$$|W_0|^2(r(-\lambda)r(\lambda))^{-1}\overline{A_0(r\phi)(\lambda)}A_0(r\psi)(\lambda),$$

and the result follows from the equality $A_0(r\phi^-)(-\lambda) = \overline{A_0(r\phi)(\lambda)}$, on L^t. $\square$

Lemma 4.9.2. *The integrand of each factor of (4.46) is Hermitian, regular and nonnegative on L^t.*

Proof. As $\rho(s)$ is positive for $s \in \mathbb{Z}$, the constant factors of $r(\lambda)$ (see Proposition 2.5.4) on L^t are all positive. This implies that $r(-\lambda)r(\lambda)$ is a nonnegative function on L^t. Indeed, using that $-w_L$ is a permutation of $\Phi_0 \setminus \Phi_L$ we have

$$r(-\lambda)r(\lambda) = R \prod_{\alpha\in\Phi_0^+\setminus\Phi_L} \rho(\langle\alpha^\vee,\lambda\rangle)\overline{\rho(\langle\alpha^\vee,\lambda\rangle)},$$

in which use was made of (4.45) and $R > 0$ is the product of the constant values of ρ. Thus, the factor $\overline{A_0(r\phi)(\lambda)}A_0(r\psi)(\lambda)$ is entire on $V_\mathbb{C}$, W_0-invariant, and nonnegative on L^t if $\phi = \psi$. Interchanging the role of ϕ and ψ clearly results in complex conjugation of the restriction to L^t. On the other hand, $(r(-\lambda)r(\lambda))^{-1}$ is meromorphic on $V_\mathbb{C}$, W_0-invariant, and nonnegative on L^t. However, *it may possibly have singularities on L^t corresponding to critical zeros of ρ.* But the product

$$\overline{A_0(r\phi)(-\lambda)}A_0(r\psi)(\lambda)(r(-\lambda)r(\lambda))^{-1},$$

which can be written as (4.42) by Lemma 4.9.1, is the restriction to L^t of a meromorphic function which has no singularities on L^t or is identically zero. $\square$

Let us now consider the growth behavior on L^t of each factor of (4.46). Lemma 4.7.1 implies also that each summand of the integrand written in the form (4.42) on L^t is given by the product of certain Paley-Wiener functions on L^t (we can, because of the double summation over W_0, absorb the rational factors $c_\mathbb{H}$ by these Paley-Wiener functions) times factors of the form $\frac{\rho(\langle\alpha^\vee,\lambda\rangle+a)}{\rho(\langle\alpha^\vee,\lambda\rangle+b)}$ where a and b are in $\mathbb{R}$ such that the argument is *not* in the critical strip. Such factors are of moderate growth on L^t by the analytical properties of $\rho(s)$ and discussed in Proposition 2.6.1. In view of (4.46) and (4.43), we define, for any $L \in \mathcal{L}$

$$d\mu_L(\lambda) = \frac{|W_0|}{r(-\lambda)r(\lambda)}\, d\overline{\nu}_L(\lambda). \tag{4.47}$$

We have, therefore, shown that:

Proposition 4.9.3. *For each $L \in W_0\backslash\mathcal{L}$, let μ_L be as in (4.47) and μ_{W_0L} be the positive measure on W_0L^t defined by*

$$d\mu_{W_0L}(\lambda) := \sum_{L' \subseteq W_0L} d\mu_{L'}. \tag{4.48}$$

Then, the corresponding summand of (4.46)

$$\langle\phi,\psi\rangle_{W_0L} := \int_{W_0L^t} \overline{A_0(r\phi)(\lambda)} A_0(r\psi)(\lambda) d\mu_{W_0L}(\lambda) \tag{4.49}$$

defines a positive semidefinite Hermitian form on the space of Paley-Wiener functions on $V_{\mathbb{C}}$. The radical of this pairing consists of Paley-Wiener functions ϕ for which we have $A_0(r\phi)|_{W_0L^t}$ is zero. We have thus a continuous map $A_{W_0L} : PW(V_{\mathbb{C}}) \to L^2(W_0L^t, \mu_{W_0L})^{W_0}$, isometric with respect to $\langle\phi,\psi\rangle_{W_0L}$ and with dense image and given by $\phi \mapsto A_0(r\phi)|_{W_0L^t}$. Finally we have

$$(\theta_\phi, \theta_\psi) = \sum_{L \in W_0\backslash\mathcal{L}} \langle\phi,\psi\rangle_{W_0L}.$$

In view of Proposition 2.5.4 and the bijections (2.60), we set (see (4.48)):

$$\mu_{\mathbf{o}} := \mu_{W_0L_{\mathbf{o}}} \tag{4.50}$$

and $\langle\cdot,\cdot\rangle_{\mathbf{o}} := \langle\cdot,\cdot\rangle_{W_0L_{\mathbf{o}}}$ and we obtain Theorem 4.1.1 from Proposition 4.9.3. Corollary 4.1.2 follows from Propositions 2.5.6 and 2.5.9, in view of the definitions of the measures in (4.2) and what was done in Section 3.9.

4.9.2 Proof of Theorem 4.1.3

Recall that the unramified Borel Eisenstein series $\mathcal{E}(\lambda, g)$, discussed in (2.103), is given by

$$\mathcal{E}(\lambda, g) = \sum_{\gamma \in B(F)\backslash G(F)} m_B(\gamma g)^{\lambda+\varrho}, \tag{4.51}$$

where m_B is the map coming from the Iwasawa decomposition and $\varrho \in a_T^*$ is the Weyl vector. It is well known that this series is absolutely convergent for $\mathrm{Re}(\lambda) = \lambda_0$ and $\lambda_0 - \varrho$ is in the interior of the fundamental chamber. Its constant term along $\underline{B}$ is given by (see, for example [88, §6.2])

$$c_B\mathcal{E}(\lambda, g) = \sum_{w \in W_0} \frac{c(\lambda)}{c(w\lambda)} m_B(g)^{w\lambda+\varrho}, \tag{4.52}$$

it has meromorphic continuation for all $\lambda \in V_{\mathbb{C}}$ and satisfies functional equations

$$\mathcal{E}(w\lambda, g) = \frac{c(w\lambda)}{c(\lambda)} \mathcal{E}(\lambda, g), \tag{4.53}$$

for all $w \in W_0$. For $g \in \underline{G}(\mathbb{A})$, as a function of λ, let us write as before $\check{\mathcal{E}}(\lambda, g) = \mathcal{E}(-\lambda, g)$. We define the **normalized Eisenstein series** by

$$\mathcal{E}_0(\lambda, g) := \frac{1}{|W_0|} A_0(r\check{\mathcal{E}}(\cdot\ , g))(-\lambda). \tag{4.54}$$

Lemma 4.9.4. *The normalized Eisenstein $\mathcal{E}_0(\lambda, g)$ is, as a function of λ, holomorphic, W_0-invariant and satisfies*

$$\mathcal{E}_0(\lambda, g) = \frac{1}{|W_0|} c_{\mathbb{H}}(-\lambda) r(-\lambda) \mathcal{E}(\lambda, g).$$

Proof. For the formula, we have:

$$\begin{aligned}
\mathcal{E}_0(\lambda, g) &= \frac{1}{|W_0|^2} \sum_{w \in W_0} c_{\mathbb{H}}(-w\lambda) \check{\mathcal{E}}(-w\lambda, g) r(-w\lambda) \\
&= \frac{1}{|W_0|^2} \sum_{w \in W_0} c_{\mathbb{H}}(-w\lambda) \frac{c(w\lambda)}{c(\lambda)} \mathcal{E}(\lambda, g) r(-w\lambda) \\
&= \frac{1}{|W_0|^2} \sum_{w \in W_0} c_{\mathbb{H}}(-w\lambda) \frac{c_{\mathbb{H}}(-\lambda) r(-\lambda)}{c_{\mathbb{H}}(-w\lambda) r(-w\lambda)} \mathcal{E}(\lambda, g) r(-w\lambda) \\
&= \frac{1}{|W_0|} c_{\mathbb{H}}(-\lambda) r(-\lambda) \mathcal{E}(\lambda, g),
\end{aligned}$$

where we applied the definition in the first line, the functional equation (4.53) in the second line and (4.7) in the third line. Together with the fact implied by the functional equations (4.53) that $\mathcal{E}(\lambda, g)$ has zeroes along the hyperplanes $\langle \alpha^\vee, \lambda \rangle = 0$ for α simple, it follows that $\mathcal{E}_0(\lambda, g)$ is holomorphic in the closure of the fundamental chamber. Indeed, if $s = s_\alpha$ for α a simple root then (4.53) implies that, for λ outside the hyperplane $[\alpha^\vee = 0]$, we have (see (4.7))

$$\mathcal{E}(s\lambda, g) - \mathcal{E}(\lambda, g) \frac{\Lambda(-\langle \alpha^\vee, \lambda \rangle)}{\Lambda(\langle \alpha^\vee, \lambda \rangle)} = 0.$$

Taking a limit towards a λ' such that $\langle \alpha^\vee, \lambda' \rangle = 0$ (thus $s\lambda' = \lambda'$) we obtain that $2\mathcal{E}(\lambda', g) = 0$. Invoking the W_0-invariance implied by the averaging operator, it is holomorphic everywhere. ☐

It follows from Lemma 4.9.4 and (4.52) that the constant term of the normalized Eisenstein series $\mathcal{E}_0(\lambda, g)$ along $\underline{B}$ is given by

$$c_B \mathcal{E}_0(\lambda, g) = \frac{1}{|W_0|} \sum_{w \in W_0} c_{\mathbb{H}}(-w\lambda) r(-w\lambda) m_B(g)^{w\lambda + \varrho},$$

which is an expression similar to the spherical function $\phi(\lambda, -1; \cdot)$, although the sign of the exponential term is not quite the same of the c-function, and of course, the present expression has the factor of the r-function that is not present in Definition 2.3.12.

Lemma 4.9.5. *Given $\phi \in PW(V_{\mathbb{C}})$ define*

$$u_\phi(\lambda) := \int_{\underline{G}(F)\underline{Z}_G\backslash\underline{G}(\mathbb{A})} \overline{\theta_\phi(g)}\mathcal{E}(\lambda, g)\, dg$$

This integral converges for $\mathrm{Re}(\lambda) = p_V \gg 0$ *and defines a holomorphic function in its domain of convergence that satisfies*

$$u_\phi(\lambda) = c_{\mathbb{H}}(-\lambda)^{-1}R_\phi(\lambda).$$

Proof. Given $\psi \in PW(V_{\mathbb{C}})$, one has

$$\int_{\mathrm{Re}(\lambda)=p_V} \psi(\lambda) \int_{\underline{G}(F)\underline{Z}_G\backslash\underline{G}(\mathbb{A})} \overline{\theta_\phi(g)}\mathcal{E}(\lambda, g)\, dg\, d\lambda = \int_{\mathrm{Re}(\lambda)=p_V} R_\phi(\lambda)\psi(\lambda)\frac{d\lambda}{c_{\mathbb{H}}(-\lambda)},$$

in which the exchange of integrals is allowed by the estimates used in the proof of [67, Proposition II.1.10] and use was made of equations (2.104) and (4.11). Since this holds for all $\psi \in PW(V_{\mathbb{C}})$ the result follows. □

Lemma 4.9.6. *Given $\phi \in PW(V_{\mathbb{C}})$, it holds that*

$$\int_{\underline{G}(F)\underline{Z}_G\backslash\underline{G}(\mathbb{A})} \overline{\theta_\phi(g)}\mathcal{E}_0(\lambda, g)\, dg = A_0(r\phi^-)(-\lambda).$$

Proof. Using Lemmas 4.9.4, 4.9.5 and (4.9), we get

$$\begin{aligned}\int_{\underline{G}(F)\underline{Z}_G\backslash\underline{G}(\mathbb{A})} \overline{\theta_\phi(g)}\mathcal{E}_0(\lambda, g)\, dg &= \frac{1}{|W_0|}c_{\mathbb{H}}(-\lambda)r(-\lambda)u_\phi(\lambda) \\ &= \frac{1}{|W_0|}\sum_{w\in W_0} c_{\mathbb{H}}(-w\lambda)r(-w\lambda)\phi^-(-w\lambda),\end{aligned}$$

proving the Lemma. □

Proposition 4.9.7. *The transform $\mathcal{F}_{\mathrm{aut}}$ of Theorem 4.1.3 is an isometric isomorphism.*

Proof. Using that $-\overline{\lambda} \in W_0\lambda$ (see (4.45)) so that $A_0(r\phi^-)(-\lambda) = \overline{A_0(r\phi)(\lambda)}$ and $\overline{\mathcal{E}_0(\lambda, g)} = \mathcal{E}_0(-\lambda, g)$ for all λ in the support $\cup_{\mathbf{o}}\ W_0L^t_{\mathbf{o}}$, we obtain from Lemma 4.9.6 and by inserting $f = \theta_\phi$ in (4.3), that $\mathcal{F}_{\mathrm{aut}}(\theta_\phi)(\lambda) = A_0(r\phi)(\lambda)$ for all $\phi \in PW(V_{\mathbb{C}})$. Hence the isometry part of Theorem 4.1.3 follows from Proposition 4.9.3. □

The last part we need to check is the equivariance of the transform $\mathcal{F}_{\mathrm{aut}}$. Recall that $\mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K}) = \otimes'_v\mathcal{H}(G_v, K_v)$ is the global spherical Hecke algebra, where for v non-Archimedean, $\mathcal{H}(G_v, K_v)$ is the algebra of compactly supported functions on $K_v\backslash G_v/K_v$ and For v Archimedean $\mathcal{H}(G_v, K_v)$ is the subalgebra of left and right invariant elements of the corresponding Archimedean Hecke algebra (the Archimedean Hecke algebra is described in [29, Paragraph 3] or [9, 1.1] although our notation differs from the one in [9]).

For each $v \in \mathcal{V}$, let $\mathfrak{o}_v^\times = \{x \in F_v^\times \mid |x|_v = 1\} \subseteq F_v^\times$, that is:

$$\mathfrak{o}_v^\times = \begin{cases} \{x \in F_v^\times \mid v(x) \geq 0\}, & v \text{ non-Archimedean} \\ \{-1, 1\}, & v \text{ real} \\ S^1, & v \text{ complex.} \end{cases}$$

Define $Y_v = X_*(\underline{T}) \otimes_{\mathbb{Z}} (F_v^\times / \mathfrak{o}_v^\times)$, that is:

$$Y_v = \begin{cases} X_*(\underline{T}) \otimes_{\mathbb{Z}} \mathbb{Z}, & v \text{ non-Archimedean} \\ X_*(\underline{T}) \otimes_{\mathbb{Z}} \mathbb{R}, & v \text{ Archimedean,} \end{cases}$$

where for the Archimedean places we used the isomorphism $\mathbb{R} \to \mathbb{R}_+$ given by the exponential map. Let $\widehat{Y_v}$ be the Pontryagin dual of Y_v. Its group of complex points then satisfy:

$$\widehat{Y_v}(\mathbb{C}) = \begin{cases} {}^L T & v \text{ non-Archimedean} \\ \mathrm{Lie}({}^L T), & v \text{ Archimedean.} \end{cases}$$

Of course, $\mathrm{Lie}({}^L T) = {}^L\mathfrak{t} = \mathbb{C} \otimes X^*(\underline{T})$ contains $V_{\mathbb{C}}$. if $v \in \mathcal{V}_f$ let q_v denote the cardinality of residue field of F_v. Then, for $\lambda \in V_{\mathbb{C}}$ we obtain an element $q_v^{-\lambda} \in {}^L T \cong \mathrm{Hom}(X_*(\underline{T}), \mathbb{C})$ given by $y \mapsto q_v^{-\langle \lambda, y \rangle}$. Let $\beta_v : V_{\mathbb{C}} \to \widehat{Y_v}(\mathbb{C})$ be the group homomorphism given by

$$\beta_v(\lambda) = \begin{cases} q_v^{-\lambda} \in {}^L T, & v \text{ non-Archimedean} \\ \lambda \in {}^L\mathfrak{t}, & v \text{ real} \\ 2\lambda \in {}^L\mathfrak{t}, & v \text{ complex.} \end{cases}$$

Let $S_v : \mathcal{H}(G_v, K_v) \to \mathbb{C}[\widehat{Y_v}(\mathbb{C})]^{W_0}$. For v non-Archimedean, $\mathbb{C}[\widehat{Y_v}(\mathbb{C})]^{W_0} \cong \mathbb{C}[X_*(\underline{T})]^{W_0}$ and S_v is the Satake isomorphism (see Proposition 2.7.6). Whenever v is an Archimedean place, $\mathbb{C}[\widehat{Y_v}(\mathbb{C})]^{W_0} \cong \mathrm{Sym}(X_*(\underline{T}) \otimes \mathbb{C})^{W_0}$ and S_v is the Harish-Chandra isomorphism (see Proposition 2.7.7). We thus have that $\mathcal{H}(G(\mathbb{A}), \mathbf{K}) \cong \otimes_v' \mathbb{C}[\widehat{Y_v}(\mathbb{C})]^{W_0}$ under the isomorphism $S = \otimes_v' S_v$. The $*$-structure of $\mathcal{H}(G_v, K_v)$, for v non-Archimedean, can be described via

$$S_v(h^*)(t) = \overline{S_v(h)(\overline{t^{-1}})}, \tag{4.55}$$

while for v Archimedean it is described by

$$S_v(h^*)(\lambda) = \overline{S_v(h)(-\overline{\lambda})}. \tag{4.56}$$

Each $\lambda \in V_{\mathbb{C}}$ defines a character $\chi_{v,\lambda}$ of $\mathcal{H}(G_v, K_v)$ by means of S_v and thus a character χ_λ of $\mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K})$, given by, for $h = \otimes_v h_v \in \mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K})$,

$$\chi_\lambda(\otimes_v h_v) = \prod_v S_v(h_v)(\beta_v(\lambda)).$$

Lemma 4.9.8. *If λ is such that $-\overline{\lambda} \in W_0 \lambda$ (see 4.45), then*

$$\chi_\lambda(h^*) = \overline{\chi_\lambda(h)}, \tag{4.57}$$

Proof. It suffices to show for $h = \otimes_v h_v \in \mathcal{H}(G(\mathbb{A}), \mathbf{K})$. But,

$$\begin{aligned}
\chi_\lambda(h^*) &= \prod_v S_v(h_v^*)(\beta_v(\lambda)) \\
&= \prod_v \overline{S_v(h_v)(\beta_v(-\overline{\lambda}))} \\
&= \prod_v \overline{S_v(h_v)(\beta_v(w\lambda))} \\
&= \overline{\chi_\lambda(h)},
\end{aligned}$$

as required, where use was made of (4.55) and (4.56) on the second equality, the assumption on λ on the third equality (note that for all $\lambda \in \cup_{\mathbf{o}} W_0 L_{\mathbf{o}}^t$ this condition is satisfied, because of (4.45)) and W_0- invariance on the last. □

Proposition 4.9.9. *The transform $\mathcal{F}_{\mathrm{aut}}$ is $\mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K})$-equivariant.*

Proof. It is known that the unramified Borel Eisenstein series $\mathcal{E}(\lambda, g)$ is an eigenfunction for the convolution action of $\mathcal{H}(G_v, K_v)$ with eigenvalue $\chi_{v,\lambda}$ for each v. For $h_v \in \mathcal{H}(G_v, K_v)$ and $f \in L^2(\underline{G}(F)\underline{Z}_G(\mathbb{A})\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}}$ denote by $h_v \cdot f$ the convolution action

$$h_v \cdot f(g) = \int_{G_v} h_v(g_v) f(gg_v)\, dg_v,$$

with dg_v normalized to $\mathrm{meas}(K_v) = 1$. For $\lambda \in \cup_{\mathbf{o}} W_0 L_{\mathbf{o}}^t$, we have

$$\begin{aligned}
\mathcal{F}_{\mathrm{aut}}(h_v \cdot f)(\lambda) &= \int_{\underline{G}(F)\underline{Z}_G\backslash\underline{G}(\mathbb{A})} \mathcal{E}_0(-\lambda, g)(h_v \cdot f)(g)\, dg \\
&= \int_{\underline{G}(F)\underline{Z}_G\backslash\underline{G}(\mathbb{A})} \mathcal{E}_0(-\lambda, g) \left(\int_{G_v} h_v(g_v) f(gg_v)\, dg_v \right) dg \\
&= \int_{\underline{G}(F)\underline{Z}_G\backslash\underline{G}(\mathbb{A})} \left(\int_{G_v} \mathcal{E}_0(-\lambda, gg_v^{-1}) h_v(g_v)\, dg_v \right) f(g)\, dg \\
&= \int_{\underline{G}(F)\underline{Z}_G\backslash\underline{G}(\mathbb{A})} \overline{(h_v^* \cdot \mathcal{E}_0(\lambda, g))} f(g)\, dg \\
&= \chi_{v,\lambda}(h_v) \mathcal{F}_{\mathrm{aut}}(f)(\lambda),
\end{aligned}$$

proving the equivariance of $\mathcal{F}_{\mathrm{aut}}$. □

4.10 Proof of the corollaries

Corollary 4.1.5 is immediate once we prove Corollary 4.1.4, so let us show that this indeed follows from Theorem 4.1.3 (or Theorem 4.1.1). In this section, Let $\mathcal{A} := L^2(\underline{G}(F)\underline{Z}_G\backslash\underline{G}(\mathbb{A}))_{[T,1]}$. Recall the definitions:

(a) $\mathcal{A}_d := \overline{\mathrm{span}\{H \subseteq \mathcal{A} \mid H \text{ is closed, } \underline{G}(\mathbb{A})\text{-invariant, irreducible}\}}$

(b) $(\mathcal{A}^{\mathbf{K}})_d := \overline{\mathrm{span}\{U \subseteq \mathcal{A}^{\mathbf{K}} \mid U \text{ is closed, } \mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K})\text{-invariant, irreducible}\}}$.

Remark. Note that the span of the invariant closed subspaces of $\mathcal{A}^{\mathbf{K}}$ is already closed by Theorem 4.1.3 (it is a finite dimensional vector space) so one does not need to take the closure in (b).

Lemma 4.10.1. *Suppose H is an irreducible unitary $\underline{G}(\mathbb{A})$ representation. Then,* $\dim(H^{\mathbf{K}}) \leq 1$. *In particular, it is an irreducible $\mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K})$-module or* 0.

Proof. The representation H has a factorization $H = \widehat{\otimes}_v H_v$ (see [29, Theorem 4]) and each of the representations H_v of G_v satisfy $\dim H_v^{K_v} \leq 1$. Thus, $\dim H^{\mathbf{K}} \leq 1$, which implies the result. $\square$

Lemma 4.10.2. *Let H be a unitary $\underline{G}(\mathbb{A})$ representation and $0 \neq \phi \in H$ a $\mathbf{K}$-invariant vector. Define $H_\phi := \overline{\mathcal{H} * \phi}$ (where $\mathcal{H}$ is the global Hecke algebra). Then H_ϕ is an admissible $\underline{G}(\mathbb{A})$-representation, and $H_\phi^{\mathbf{K}} = \mathbb{C}\phi$.*

Proof. It is a Theorem of Harish-Chandra ([33] see also the survey [23, Theorem 3.5]) that H_ϕ is an admissible $\underline{G}(\mathbb{A})$-representation. The idempotent $e_{\mathbf{K}}$ is a bounded operator (a self-adjoint projection), hence

$$H_\phi^{\mathbf{K}} = e_{\mathbf{K}}(\overline{\mathcal{H} * \phi}) \subseteq \overline{e_{\mathbf{K}}(\mathcal{H} * \phi)} = \overline{\mathbb{C}\phi} = \mathbb{C}\phi,$$

and thus the result. $\square$

Lemma 4.10.3. *Let H be a unitary $\underline{G}(\mathbb{A})$ representation and $\mathbb{C}\phi_1, \mathbb{C}\phi_2 \subseteq H$ be inequivalent $\mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K})$-modules. Then, H_{ϕ_1} and H_{ϕ_2} are inequivalent irreducible $\underline{G}(\mathbb{A})$ representations.*

Proof. Let $0 \neq X \subseteq H_{\phi_i}$ be a closed, $\underline{G}(\mathbb{A})$-invariant and irreducible subspace. We claim X contains ϕ_i. If not, from Lemma 4.10.1 and 4.10.2, $X^{\mathbf{K}} = 0$ and hence, since $H_{\phi_i} = X \oplus (X^\perp \cap H_{\phi_i})$, we get $\phi_i \in X^\perp \cap H_{\phi_i} \subseteq H_{\phi_i}$, a closed and $\underline{G}(\mathbb{A})$-invariant subspace. Thus $0 \neq X \subseteq H_{\phi_i} \subseteq X^\perp$, contradiction. This shows that H_{ϕ_i} is irreducible. Now, given any nonzero $\underline{G}(\mathbb{A})$-homomorphism $H_{\phi_1} \to H_{\phi_2}$ we obtain a nonzero $\mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K})$-homomorphism $H_{\phi_1}^{\mathbf{K}} \to H_{\phi_2}^{\mathbf{K}}$, which is necessarily an isomorphism. This finishes the proof. $\square$

Lemma 4.10.4. *We have* $(\mathcal{A}^{\mathbf{K}})_d = (\mathcal{A}_d)^{\mathbf{K}}$.

Proof. We first prove $\subseteq$: by Theorem 4.1.3 we know that $(\mathcal{A}^{\mathbf{K}})_d = \oplus_{\mathbf{o}} \mathbb{C}\phi_{\mathbf{o}}$, a finite direct sum indexed by the distinguished orbits (with $\phi_{\mathbf{o}}$ being $\mathcal{E}_0(\lambda(\mathbf{o}), \cdot)$, the normalized Eisenstein with parameter $\lambda(\mathbf{o})$). Therefore, each $H_{\phi_{\mathbf{o}}} \subseteq \mathcal{A}$ is closed, invariant and irreducible (Lemma 4.10.3), so $\oplus_{\mathbf{o}} H_{\phi_{\mathbf{o}}} \subseteq \mathcal{A}_d$. Hence

$$(\oplus_{\mathbf{o}} H_{\phi_{\mathbf{o}}})^{\mathbf{K}} = \oplus_{\mathbf{o}} H_{\phi_{\mathbf{o}}}^{\mathbf{K}} = (\mathcal{A}^{\mathbf{K}})_d \subseteq (\mathcal{A}_d)^{\mathbf{K}},$$

where we used Lemma 4.10.2 in the second equality.

Now we prove $\supseteq$: write S for span of all H as in (a), so $\mathcal{A}_d = \overline{S}$. The map

$$\bigoplus_{H \text{ as in (a)}} H^{\mathbf{K}} \longrightarrow\!\!\!\!\rightarrow e_{\mathbf{K}}(S)$$

is surjective since it is the functorial image of the canonical surjection

$$\bigoplus_{H \text{ as in (a)}} H \longrightarrow\!\!\!\!\rightarrow S,$$

under the exact functor of *taking* $\mathbf{K}$*-invariants* from the category of $\mathcal{H}$-modules to the category of $\mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K})$-modules. It then follows that $e_{\mathbf{K}}(S) \subseteq (\mathcal{A}^{\mathbf{K}})_d$, by (b). Further, (recall $e_{\mathbf{K}}$ is continuous)

$$\begin{aligned}(\mathcal{A}_d)^{\mathbf{K}} &= e_{\mathbf{K}}(\overline{S}) \subseteq \overline{e_{\mathbf{K}}(S)} \\ &\subseteq \overline{(\mathcal{A}^{\mathbf{K}})_d} = (\mathcal{A}^{\mathbf{K}})_d,\end{aligned}$$

as the latter is already closed, as noted above. □

4.10.1 Proof of Corollary 4.1.4

By L. Clozel's result (see Proposition 2.7.2), there is a unique decomposition of $(\mathcal{A}^{\mathbf{K}})_d$ which, by Theorem 4.1.3 and Lemma 4.10.3 is given by

$$(\mathcal{A}^{\mathbf{K}})_d \cong \bigoplus_{\mathbf{o}} \pi^{\mathbf{K}}_{\lambda(\mathbf{o})}, \tag{4.58}$$

a finite multiplicity-free orthogonal direct sum of $\mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K})$-modules. On the other hand, $\mathcal{A}_d$ decomposes uniquely (since $\underline{G}(\mathbb{A})$ is a type I group, see Proposition 2.7.2) as

$$\mathcal{A}_d = \bigoplus_{\pi} m(\pi)\pi,$$

an orthogonal direct sum of irreducible $\underline{G}(\mathbb{A})$-invariant subspaces. Using Lemma 4.10.4 and comparing with (4.58), the result follows.

4.10.2 Proof of Corollary 4.1.6

We recall that we assume from now on that $\underline{G}$ is semisimple and that $\tilde{\mathcal{E}}_0(\lambda, g) = r(\lambda)^{-1}\mathcal{E}_0(\lambda, g)$. In the present case, we write $Y = X_*(\underline{T})$ and $X = X^*(\underline{T})$. We can Theorem 4.1.3 above to obtain some nontrivial identities. First, note that for every distinguished orbit $\mathbf{o}$, the corresponding $\{\lambda(\mathbf{o})\}$ is a residual subspace. Thus, by Proposition 2.5.4(d), we have that $\langle \alpha^\vee, \lambda(\mathbf{o})\rangle \in \mathbb{Z}$, for all roots $\alpha \in \Phi_0$. It follows that $\tilde{\mathcal{E}}_0(\lambda(\mathbf{o}), g)$ is well-defined, as the zeros of $\rho(s)$ satisfy $\mathrm{Re}(s) \in (0, 1)$. Now, for all $\lambda \in \cup_{\mathbf{o}} L^t_{\mathbf{o}}$,

$$\begin{aligned}\mathcal{F}_{\mathrm{aut}}(\mathcal{E}_0(\lambda(\mathbf{o}), \cdot))(\lambda) &= \int_{\underline{G}(F)\backslash \underline{G}(\mathbb{A})} \mathcal{E}_0(\lambda(\mathbf{o}), g)\mathcal{E}_0(-\lambda, g)\, dg \\ &= \delta_{W_0\lambda(\mathbf{o})}(\lambda)\|\mathcal{E}_0(\lambda(\mathbf{o}), \cdot)\|^2\end{aligned}$$

where $\delta_{W_0\lambda(\mathbf{o})}(\lambda)$ is the characteristic function of the set $W_0\{\lambda(\mathbf{o})\}$. Thus, computing the norm of $\mathcal{F}_{\mathrm{aut}}(\mathcal{E}_0(\lambda(\mathbf{o}),\cdot))$ with the measure $\sum_{\mathbf{o}}\mu_{\mathbf{o}}$ of Theorem 4.1.1 and using the fact that $\mathcal{F}_{\mathrm{aut}}$ is an isometry, we obtain

$$\|\mathcal{E}_0(\lambda(\mathbf{o}),\cdot)\|^2\mu_{W_0\{\lambda(\mathbf{o})\}}(\{W_0\lambda(\mathbf{o})\}) = 1. \tag{4.59}$$

We use (4.48) and the precise expression (4.44) of the measure $\overline{\nu_L}$ (even better, we use Theorem 3.6.1 and the Remark of Section 3.6) to compute

$$\mu_{W_0\{\lambda(\mathbf{o})\}}(\{W_0\lambda(\mathbf{o})\}) = \frac{|W_0|}{|W_{\lambda(\mathbf{o})}|}\left(\frac{|W_0|}{r(\lambda(\mathbf{o}))r(-\lambda(\mathbf{o}))}\right)\times \left|\frac{|W_{\lambda(\mathbf{o})}|}{|A_{\mathbf{o}}|}\frac{\prod'_{\alpha\in\Phi_0}\langle\alpha^\vee,\lambda(\mathbf{o})\rangle}{\prod'_{\alpha\in\Phi_0}\langle\alpha^\vee,\lambda(\mathbf{o})\rangle+1}\right|. \tag{4.60}$$

Hence, setting $\tilde{\mathcal{E}}_0(\lambda(\mathbf{o}),g) = r(\lambda(\mathbf{o})^{-1}\mathcal{E}_0(\lambda(\mathbf{o}),g)$ and using (4.59), we get the reciprocal of (4.60) divided by $|r(\lambda(\mathbf{o}))|^2$, that is we obtain the desired expression

$$\|\tilde{\mathcal{E}}_0(\lambda(\mathbf{o}),\cdot)\|^2 = \frac{|A_{\mathbf{o}}|}{|W_0|^2}\left|\frac{\prod'_{\alpha\in\Phi_0}\langle\alpha^\vee,\lambda(\mathbf{o})\rangle+1}{\prod'_{\alpha\in\Phi_0}\langle\alpha^\vee,\lambda(\mathbf{o})\rangle}\right|\left(\prod_{\alpha\in\Phi_0^+}\frac{\rho(\langle\alpha^\vee,\lambda(\mathbf{o})\rangle+1)}{\rho(\langle\alpha^\vee,\lambda(\mathbf{o})\rangle)}\right), \tag{4.61}$$

where the rightmost bracket is the working out of the expression $\frac{r(-\lambda(\mathbf{o}))}{r(\lambda(\mathbf{o}))}$, using the functional equation $\rho(-s) = \rho(1+s)$.

4.10.3 Proof of Corollary 4.1.7

Using the harmonic analysis laid down by Theorems 4.1.1 and 4.1.3 we will compute $\mathrm{vol}(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))$, for $\underline{G}$ semisimple, and recover Langlands' result (see [54] for the case $F=\mathbb{Q}$). From $\|\mathcal{F}_{\mathrm{aut}}(f)\|^2 = \|f\|^2$ for all $f\in L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}}_{[T,1]}$, we substitute $f=\mathcal{E}_0(\varrho,g)$ where ϱ is the Weyl vector. On the one hand its constant term along $\underline{B}$ is

$$\begin{aligned} c_B\mathcal{E}_0(\varrho,g) &= \frac{1}{|W_0|}\sum_{w\in W_0} c_{\mathbb{H}}(-w\varrho)r(-w\varrho)m_B(g)^{\varrho+w\varrho} \\ &= \frac{1}{|W_0|}\sum_{w\in W_0}\prod_{\alpha\in\Phi_0^+}\frac{\langle\alpha^\vee,w\varrho\rangle-1}{\langle\alpha^\vee,w\varrho\rangle}\rho(1+\langle\alpha^\vee,w\varrho\rangle)m_B(g)^{\varrho+w\varrho} \\ &= \frac{1}{|W_0|}\prod_{\alpha\in\Phi_0^+}\frac{\mathrm{ht}(\alpha^\vee)+1}{\mathrm{ht}(\alpha^\vee)}\rho(\langle\alpha^\vee,\varrho\rangle) \\ &= r(\varrho), \end{aligned}$$

where we used the definitions in the first lines, that $\prod_{\alpha\in\Phi_0^+}(\langle\alpha^\vee,w\varrho\rangle-1)$ is only nonzero when $w=w_0$, the longest element in the third line and (3.25) in view of $\langle\alpha^\vee,\varrho\rangle = \mathrm{ht}(\alpha^\vee)$, in the last line. It follows that $\mathcal{E}_0(\varrho,g) = r(\varrho)$ (is a constant function of g), and thus

$$\|\mathcal{E}_0(\varrho,\cdot)\|^2 = r(\varrho)^2\,\mathrm{vol}(G(F)\backslash G(\mathbb{A})). \tag{4.62}$$

Hence, it follows from (4.62) that $\mathrm{vol}(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))$ is the L^2-norm of the function $\tilde{\mathcal{E}}_0(\lambda(\mathbf{o}),\cdot)$ introduced in Corollary 4.1.6 for $\lambda(\mathbf{o})=\varrho$. Thus, the desired volume will follow by setting $\lambda(\mathbf{o})=\varrho$ on equation (4.61). First, we open the first expression between brackets in (4.61) to obtain

$$(-1)^r c_{\mathbb{H}}(\varrho)\times\prod_{\alpha\in\Phi_0^+\backslash\Delta_0}\frac{\langle\alpha^\vee,\varrho\rangle-1}{\langle\alpha^\vee,\varrho\rangle}. \tag{4.63}$$

Further, we open the second expression between brackets in (4.61). For that, we remind the reader that $r(\lambda)=\prod_{\alpha\in\Phi_0^+}\rho(\langle\alpha^\vee,\lambda\rangle)$ and $\rho(s)=s(s-1)\Lambda(s)$ (see (2.69)). We thus obtain

$$c_{\mathbb{H}}(\varrho)\times\prod_{\Phi_0^+\backslash\Delta_0}\frac{\langle\alpha^\vee,\varrho\rangle}{\langle\alpha^\vee,\varrho\rangle-1}\times\prod_{\Delta_0}\frac{\Lambda(\langle\alpha^\vee,\varrho\rangle+1)}{(\langle\alpha^\vee,\varrho\rangle-1)\Lambda(\langle\alpha^\vee,\varrho\rangle)}\times\prod_{\Phi_0^+\backslash\Delta_0}\frac{\Lambda(\langle\alpha^\vee,\varrho\rangle+1)}{\Lambda(\langle\alpha^\vee,\varrho\rangle)} \tag{4.64}$$

Finally, using that the orbit associated to ϱ is the regular orbit $\mathbf{o}_{\mathrm{reg}}$, so that $|A_{\mathbf{o}_{\mathrm{reg}}}|=[Y:Q^\vee]$ and also that $|W_0|=c_{\mathbb{H}}(\varrho)$ (see (3.25)), we substitute these and (4.63), (4.64) in (4.61) to obtain

$$\|\tilde{\mathcal{E}}_0(\varrho,\cdot)\|=\frac{[Y:Q^\vee]\prod_i\Lambda(d_i)}{\rho(1)^r},$$

as desired.

We recall from Proposition 2.6.1, that $\rho(1)=\mathrm{res}_{s=1}\Lambda(s)$ which was computed in (2.68). It is the volume of the norm-one idèle class group of F (see Section 2.6.3) times $|d|^{1/2}\Gamma_{\mathbb{R}}(1)^{t_1}\Gamma_{\mathbb{C}}(1)^{t_2}$, where d is the discriminant and t_1, $2t_2$ is the number of real and complex embeddings of F. When $F=\mathbb{Q}$, $\rho(1)=1$.

Chapter 5

Contraction of the Bruhat-Tits building

In this chapter, following an idea of M. Bestvina and G. Savin (unpublished manuscript, [6]) for the building of SL_n, we propose to study Euclidean buildings, $\mathscr{B}$, by means of piecewise linear (PL, in short) Morse theory. The novelty that we introduce is a construction of a Morse function that will allow us to define the direction of steepest descent towards a fixed central vertex around any vertex x of $\mathscr{B}$. With this notion at hand, we can write down an explicit contraction of the complex which we conjecture (see Conjecture 5.4.4) that has optimal bounds in its coefficients. Should this conjecture be true, we are able to use similar arguments than the ones of E. Opdam and M. Solleveld ([70] and [71]) to study extensions of tempered representations of a reductive p-adic group. The Morse function we construct will satisfy the properties of a *nice Morse function* as was defined by M. Bestvina and G. Savin in another unpublished manuscript [7]. We remark that we used as a starting point the ideas of [6] for SL_n, and arrived at a similar outcome in this more general situation as M. Bestvina and G. Savin in [7] by independent methods. We make a comparison with their results below, in Section 5.5.

Apart from sharing the geometry of root systems, this chapter is independent of the previous ones, so it will be a mixture between new results, definitions and known facts.

5.1 Overview of the chapter

We now sketch our constructions. We first assume the root system to be irreducible. In 5.3.1, we endow the Coxeter complex with a (finite) Weyl group invariant PL Morse function. This function is obtained by choosing a generic vector on a Weyl chamber so that we have a height function on a cone. The properties of this function, discussed in Theorem 5.3.8 allow us to define the direction of steepest descent around any vertex in the Coxeter complex, so that a contraction towards the origin (the cone point) can be explicitly written in the level of chain groups.

Granted the existence of the Morse function on the Coxeter complex, in 5.3.2 we construct a Morse function on a locally finite and discrete Euclidean building by using the following principle: fix a special vertex p_0. Given another point p, pick an apartment containing both of them, identify the apartment with a Coxeter complex so that p_0 corresponds to the origin and use the already defined Morse function in the apartment to define a map $p \mapsto \varphi(p)$ on the building. Once we prove that this map is well-defined, it is the content of Theorem 5.3.16 that this function shares many properties of its sister function on the Coxeter complex. In particular, we have sufficient conditions to write an explicit contraction towards a central (special) vertex by following the directions of steepest descent. The construction of this contraction is discussed in Section 5.4, by first dealing with the simplicial case, in 5.4.1, and then passing to the general polysimplicial case in 5.4.2, where we assume that each irreducible factor is endowed with a PL Morse function with the desired properties. In Section 5.6, pass our constructions to the Bruhat-Tits building.

5.2 PL Morse theory

5.2.1 A class of complexes

The aim of this section is to introduce our main tool, namely, piecewise linear Morse theory, following [5]. Also, we will recall some notions of algebraic topology to set the notations and conventions. We follow [34] and [56]. Most of the definitions of this paragraph can be found in [56, Chapter 5]. Our interest is on finite-dimensional, locally finite and regular cell complexes. For us, an **r-cell** is a topological space homeomorphic to the open unit ball in $\mathbb{R}^r$ and a **cell** is an r-cell for some r. A **closed cell** is a topological space homeomorphic to a closed ball. A point is a 0-cell and also a closed cell. Given a closed cell F, the image of the interior of the closed ball used to define F is a cell, called the interior of F and denoted $\mathrm{int}(F)$. A point is a 0-cell and also a closed cell. Given a nonempty topological space X, a **cell decomposition** is a partition of X into subsets that are cells of various dimensions. We denote this partition by $\mathcal{E} = \mathcal{E}(X)$ and demand that the following condition is satisfied: for each r-cell e with $r > 0$, there exists a closed r-cell F_e and a continuous map $\chi_e : F_e \to X$ (called a **characteristic map** for e) that restricts to a homeomorphism of the interior of F_e onto e and maps the boundary ∂F_e into the union of all cells in $\mathcal{E}$ of dimensions strictly less than r. By a **cell complex** we mean a pair $(X, \mathcal{E})$

with X a nonempty Hausdorff topological space together with a cell decomposition. A cell complex is a **CW-complex** if, moreover, it satisfies the properties (C), the closure of each cell is contained in the union of finitely many cells, and (W), a subset $A \subseteq X$ is closed if and only if $A \cap \overline{e}$ is closed for all $e \in \mathcal{E}$ [104, §5]. A cell complex is **locally finite** if the partition $\mathcal{E}$ is locally finite. Locally finite cell complexes are always CW-complexes [56, Proposition 5.3]. A CW-complex is called **regular** if the characteristic map of each cell $e \in \mathcal{E}$ is a homeomorphism onto $\overline{e}$.

The cells we have in mind are polytopal cells. Recall that a **polyhedron** is a nonempty intersection of finitely many closed half-spaces of $\mathbb{R}^m$. It may or may not be a bounded subset of $\mathbb{R}^m$. A bounded polyhedron is called a **polytope**. Among them, the most relevant types used here will be simplices and polysimplices, which we now recall. An r-simplex is the convex hull of $r+1$ affine independent points $\{x_0, \dots, x_r\} \subseteq \mathbb{R}^m$, called vertices, with $m \geq r$. A **simplex** is then an r-simplex for some r. When referring to simplices, Greek letters such as σ or τ will be used. We will also use the notation $[x_0, \dots, x_r]$ for the simplex spanned by the indicated vertices. Omission of one or more vertices spans another simplex, say τ, of necessarily lower dimension. Such a simplex will be called a **face** of σ. The face relation will be denoted by $\tau \preceq \sigma$. Codimension-1 faces of a given simplex will be called **facets**. The union of the facets of a simplex forms the **boundary** of σ, denoted here as $\partial\sigma$. Note that our notation here differs from the one in [13] and [99] inasmuch as the term facet used there (in [13] the French counterpart "facette" is used) refers to what we call face here.

Given n simplices $\sigma_{(1)}, \dots, \sigma_{(n)}$[1], their product $\sigma_{(1)} \times \dots \times \sigma_{(n)}$ will be called a **polysimplex** and will be denoted $\vec{\sigma}$. A facet of $\vec{\sigma}$ will be a set of the type

$$\sigma_{(1)} \times \dots \times \tau_{(j)} \times \dots \times \sigma_{(n)},$$

in which $\tau_{(j)} \preceq \sigma_{(j)}$ is facet, and a face will be a product of faces of each $\sigma_{(j)}$. Just as in the simplicial case, the notations $\partial\vec{\sigma}$ and $\vec{\tau} \preceq \vec{\sigma}$ will also be used without further ado. In fact, observe that we can define the notion of face, facet and boundary for any polytope. Note also that the face of a polytope is again a polytope and it is always a closed cell.

A **polyhedral cell** is then a point or the relative interior of a polyhedron of positive dimension. They are always convex cells. We will denote such a cell by C. We similarly define polytopal, simplicial and polysimplicial cells. The next definition can be found in [5]:

Definition 5.2.1. An **affine polytope complex** (APC, in short) is a pair $(X, \mathcal{P})$ consisting of a nonempty Hausdorff topological space X and a collection

[1]Here, (i) is an index and the parenthesis has no extra meaning. Below, we will have to index, for example, root systems $\Psi_{0,(i)}$.

$\mathcal{P}$ of closed cells, called polytopes, which are subsets of X. We demand that the set $\mathcal{E} = \{\mathrm{int}(\pi) \mid \pi \in \mathcal{P}\}$ together with X, form a CW-complex in which for each cell $e \in \mathcal{E}$ the characteristic map $\chi_e : \overline{C}_e \to X$ satisfying:

(i) Each C_e is a polytopal cell in a Euclidean space $\mathbb{R}^m$, for a fixed m,

(ii) χ_e is a homeomorphism onto $\overline{e}$,

(iii) for all e, if F is a face of $\overline{C}_e$, and $e' = \mathrm{int}(\chi_e(F))$, then the restriction of χ_e to F agrees with $\chi_{e'} \circ \varphi$, for an affine homeomorphism φ of $\mathbb{R}^m$.

The **dimension of** $(X, \mathcal{P})$ is the maximal dimension of a polytope $\pi \in \mathcal{P}$.

Remark. Condition (ii) of the above definition means that $(X, \mathcal{E})$ is a regular CW-complex. For all e and F (and e') as in condition (iii), we have that F and $\overline{C}_{e'}$ are congruent polytopes. One could allow APCs to be infinite dimensional by not insisting on all cells C_e being contained in $\mathbb{R}^m$ for a fixed m, but to allow m to depend on C_e. In this case, in (iii), one should include $\mathbb{R}^n$ in $\mathbb{R}^m$ in the standard way for $n < m$.

Remark. If, for each e, one replaces χ_e by $\chi'_e = \chi_e \circ \varphi_e$ for affine homeomorphisms φ_e of $\mathbb{R}^m$, the APC $(X, \mathcal{P}')$ obtained this way will be regarded as *identical* to the initial $(X, \mathcal{P})$.

Example 5.2.2. Every finite-dimensional simplicial complex is an APC. In this case, the $\overline{C}_e$ can be taken to be standard simplices. Also every polysimplicial complex ([13, 1.1.6]) are APCs. Note that an APC is not necessarily a topological manifold. A tree of finite valence is an APC which is not a topological manifold.

Example 5.2.3. Let X be an r-simplex, with $r > 0$. If $\mathcal{P} = \{X\}$, then, $(X, \mathcal{P})$ is not an APC as the set $\mathcal{E} = \{\mathrm{int}(X)\}$ is not a partition of X. If $\mathcal{P} = \{F \mid F$ is a face of $X\}$, then we do have an APC.

We will occasionally refer to an APC only by X or $\mathcal{P}$, instead of $(X, \mathcal{P})$ if convenient. If we are given a subset $\mathcal{P}' \subseteq \mathcal{P}$, we will write $|\mathcal{P}'| \subseteq X$ for the topological subspace (induced topology) obtained as the union of all $\pi' \in \mathcal{P}'$. We will use the same notation $|\mathcal{E}'|$ for a subset of $\mathcal{E}$ in a CW-complex $(X, \mathcal{E})$. We will denote by $\mathcal{P}^{(r)} \subseteq \mathcal{P}$ the set of all polytopes of dimension r. The set $X^{(r)} := |\mathcal{P}^{(r)}|$ is called the r**-skeleton** of $\mathcal{P}$. Because of the bijection between $\mathcal{P}$ and $\mathcal{E}$, we may, and occasionally will, index the characteristic maps by $\pi \in \mathcal{P}$ instead of $e \in \mathcal{E}$.

Example 5.2.4 (Coxeter complex)**.** This is a standard object and we refer to [42] for the details. We discuss it to fix the notations. In Section 2.3.1 we discussed the objects $V^*, (\cdot,\cdot), V, \langle\cdot,\cdot\rangle, \mathcal{R}$ and W_0. To avoid more unnecessary notations, let us also assume $\mathcal{X} = Q$ and let us, at once, define $E := V^* \supseteq \Psi_0$ with $\dim E = r$. From the list above we will need $E, (\cdot,\cdot), \Psi_0, \Pi_0, Q$ and W_0. We require Ψ_0 to be reduced. In the section just cited we constructed the Coxeter systems (W_0, S_0) and (W_Q, S) as well as we described the set of affine roots Ψ and its set of simple affine roots Π. Note that with our assumptions, $W = W_Q$.

Suppose n is the number of irreducible components of Ψ_0. Thus, E is the orthogonal direct sum $E = E_{(1)} \oplus \cdots \oplus E_{(n)}$ and Ψ_0 is the disjoint union $\Psi_0 = \Psi_{0,(1)} \cup \cdots \cup \Psi_{0,(n)}$, with $\Psi_{0,(j)}$ an irreducible root system on $E_{(j)}$. For $j = 1, \ldots, n$, let $\theta^\vee_{(j)}$ be the highest coroot of the corresponding irreducible root system. Then, the set of simple affine roots is explicitly given as $\Pi = \{(\alpha^\vee, 0) \mid \alpha \in \Pi_0\} \cup \{(-\theta^\vee_{(j)}, 1) \mid \text{ for } 1 \le j \le n\}$. For each affine root $a = (\alpha^\vee, m) \in \Psi$, as usual, we denote by H_a the affine hyperplane given by the $x \in E$ such that $a(x) = \alpha^\vee(x) + m = 0$. To each $a = (\alpha^\vee, m) \in \Pi$, we let s_a be the associated reflection $x \mapsto x - a(x)\alpha$ with respect to H_a and let $S = \{s_a \mid a \in \Pi\}$. As we mentioned in Section 2.3.1, the pair (W, S) is a Coxeter system of rank[2] $r+n$. It is an affine reflection group acting on E by affine homeomorphisms that permutes the set of connected components of $E \setminus (\cup_{a\in\Psi} H_a)$ in a simply transitive way and has a decomposition $W = W_0 \ltimes t(Q)$. Given the Coxeter system (W, S) as above, for a proper $J \subsetneq \Pi$, let

$$C_J := \{x \in E \mid a(x) = 0, \text{ for } a \in J \text{ and } a(x) > 0 \text{ for } a \in \Pi \setminus J\}. \tag{5.1}$$

Whenever $C_J \neq \emptyset$, this gives a polysimplicial cell in E. The stabilizer of C_J is W_J, the subgroup of W generated by $\{s_a \mid a \in J\}$. Coupled with the action of the affine Weyl group W, one obtains a cell decomposition of E by

$$\mathcal{E}(E) := \{wC_J \mid w \in W/W_J, J \subsetneq \Pi \text{ and } C_J \neq \emptyset\}. \tag{5.2}$$

Note that, if one writes $\Pi = \Pi_{(1)} \cup \cdots \cup \Pi_{(n)}$ (with $\Pi_{(j)} = \Pi_{0,(j)} \cup \{\theta^\vee_{(j)}\}$) and $J = J_{(1)} \cup \cdots \cup J_{(n)}$, then, $C_J \neq \emptyset$ only if $J_{(i)} \subsetneq \Pi_{(i)}$ for each i, because $\cap_{a\in\Pi_{(i)}} H_a = \emptyset$, for all $1 \le i \le n$. We define the set of polysimplices by

$$\Sigma := \{\overline{e} \mid e = wC_J \in \mathcal{E}(E)\}. \tag{5.3}$$

To each non-empty cell $e = wC_J \in \mathcal{E}(E)$, with $w \in W$, the corresponding characteristic map is $\chi_e : \overline{C_J} \to E$, in which $\chi_e = w \circ \iota$, with ι the inclusion. Note that each $\overline{C_J}$ is a face of $\overline{C_\emptyset}$, and, more generally, any polysimplex is the face of a maximal one of the type $w\overline{C_\emptyset}$.

Thus, the pair (E, Σ) just constructed above is a *locally finite* APC of dimension r. It is called the **Coxeter complex**. One sometimes write $\Sigma = \Sigma(W, S)$, if one wants to emphasize the dependence on the Coxeter system (W, S). The maximal cells $wC_\emptyset$, $w \in W$, are called **chambers** or **affine chambers** and $C_\emptyset$

[2]In [42, p.90], the rank of a Coxeter system is defined as the cardinality of S.

is called the **fundamental chamber**, for $\overline{C_\emptyset}$ is a fundamental domain for the action of W on E. Also, with respect to the finite Weyl group W_0, one identifies S_0 with Π_0 and defines, for $J \subsetneq \Pi_0$

$$C_J^v := \{x \in E \mid (\alpha, x) = 0, \text{ for } \alpha \in J \text{ and } (\alpha, x) > 0 \text{ for } \alpha \in \Pi_0 - J\}. \quad (5.4)$$

Then, the polyhedral cells $wC_\emptyset^v$, $w \in W_0$, are called **vectorial chambers** and $C_\emptyset^v$ is called the **fundamental vectorial chamber**, for, likewise, $\overline{C_\emptyset^v}$ is a fundamental domain for the action of W_0 on E [42, Theorem 1.12]. Note that $C_\emptyset^v$ is contained in the cone spanned by the positive roots. Chambers will be denoted by $C \in \Sigma$ and vectorial chambers by C^v. Note the fortunate coincidence that C is also the notation we use for a polyhedral cell.

Remark. As mentioned before, this chapter is independent of the previous chapters, and the notations are slightly different. For example, here we denoted the fundamental vectorial chamber by $C_\emptyset^v$, while in the previous chapters it was denoted by V_+ or V_+^*.

5.2.2 PL Morse functions

Definition 5.2.5. Let $(X, \mathcal{P})$ be an APC. A function $\varphi : X \to \mathbb{R}$ is said to be a **Morse function** if the following holds:

(i) For each $\pi \in \mathcal{P}$, the composition $\varphi\chi_\pi : \overline{C}_\pi \to \mathbb{R}$ is the restriction of an affine linear function $\mathbb{R}^m \to \mathbb{R}$,

(ii) if $\varphi\chi_\pi$ is constant, then $\dim \pi = 0$,

(iii) the image $\varphi(X^{(0)})$ of the 0-skeleton is a discrete subset of $\mathbb{R}$.

Remark. This definition is a good PL analogue of the classical Morse theory on manifolds. For example, in [5, Proposition 2.4] it was proven a PL version of the classical Morse theory statements that critical points are isolated ([64], Corollary 2.3) and that the homotopy type of a manifold does not change in regions without critical points ([64], Theorem 3.1). In fact, we refer to [5] for a number of other nice results that can be obtained with PL Morse theory.

5.3 Existence of Morse functions

5.3.1 The Coxeter complex case

As a warm-up, let us endow the Coxeter complex discussed in Example 5.2.4 with a Morse function and extract some useful properties. These will turn out to be sufficient conditions to write down a contraction map of the complex with a good bound in the coefficients. In this section, we will assume that the root system involved is *irreducible*, so that we have a simplicial complex rather than a polysimplicial complex.

Given a CW-complex $(X, \mathcal{E})$, recall that a **subcomplex** is a pair $(X', \mathcal{E}')$ with $X' = |\mathcal{E}'|$ and $\mathcal{E}' \subseteq \mathcal{E}$ with the property that whenever $e' \in \mathcal{E}'$, then $\overline{e'} \subseteq X'$. Equipped with the subspace topology, a subcomplex is necessarily a closed subspace of X [56, Theorem 5.6]. For example, the r-skeleton $X^{(r)}$ is a subcomplex, for all $r \geq 0$. A subcomplex that will be important for us is the **star** of a cell e. Given $e \in \mathcal{E}(X)$, let $(|\operatorname{St}(e)|, \operatorname{St}(e))$ be the subcomplex

$$\operatorname{St}(e) := \{f \in \mathcal{E}(X) \mid \text{exists } f' \in \mathcal{E}(X) \text{ with } \overline{e} \preceq \overline{f'} \text{ and } \overline{f} \preceq \overline{f'}\} \tag{5.5}$$

(recall $|\mathcal{E}'| = \cup_{e' \in \mathcal{E}'} e$). We will also need $\operatorname{Lk}(e)$, the **link of** e, whose cells are

$$\operatorname{Lk}(e) := \{f \in \operatorname{St}(e) \mid \overline{f} \cap \overline{e} = \emptyset\}.$$

In the case of an APC, for $\pi \in \mathcal{P}$, we write $\operatorname{St}(\pi)$[3] (respectively $\operatorname{Lk}(\pi)$) and consider the set of the closures of all cells of $\operatorname{St}(e)$ (respectively $\operatorname{Lk}(\pi)$), for $e = \operatorname{int}(\pi)$. Sometimes, it may be necessary to specify the star of a cell with respect to a subcomplex $X' \subseteq X$. We write $\operatorname{St}(e; X')$, (respectively $\operatorname{Lk}(\pi; X')$), accordingly.

Example 5.3.1. The star and link of a cell are simple geometric notions. In the figure below, we depict a simplicial complex. The star of the central vertex v is the whole complex, while the link is the boundary.

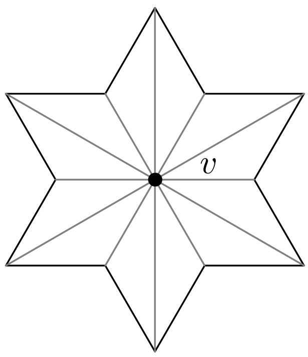

Figure 5.1: The link and the star of the vertex v.

Proposition 5.3.2. *Let $\Sigma(W, S)$ be the Coxeter complex. Then, $|\operatorname{St}(\sigma)|$ is a convex and compact subset of E, for every face $\sigma = \overline{C_J}$ of the fundamental chamber $\overline{C_\emptyset}$.*

[3] Strictly speaking, in Theorem E of the Introduction, we should have written $|\operatorname{St}(\sigma)|$, but we opted not to, for brevity.

Proof. If $\sigma = \overline{C_J}$, the stabilizer of C_J is the standard parabolic W_J, which is known to be finite ([30], section 13.4). Given $\tau \in \mathrm{St}(\sigma)$, from (5.5) there is a chamber $wC_\emptyset$ whose closure contains σ and τ. Now, $C_J \subseteq w\overline{C_\emptyset}$ implies $w \in W_J$, because $\overline{C_\emptyset}$ is a fundamental domain for the action of W. Thus, $|\,\mathrm{St}(\sigma)| = \cup_{w\in W_J} w\overline{C_\emptyset}$, so the star is a compact set.

The convexity is [1, Proposition 3.93]. See also [1, Example 3.133(b)]. □

Corollary 5.3.3. *The star of every simplicial cell of the Coxeter complex is convex and compact.*

Proof. Follows from $\mathrm{St}(w\sigma) = w\,\mathrm{St}(\sigma)$, and because convexity is preserved by affine homeomorphisms. □

Suppose that $\overline{C} \subseteq E$ is a polytope, such as a simplex or a polysimplex. A linear function $\xi : E \to \mathbb{R}$ is said to be **generic with respect to** $\overline{C}$, if ξ is not constant on any of its 1-cells (see [3, Section 1]). We omit the proof of the next lemma.

Lemma 5.3.4. *Let $\overline{C} \subseteq E$ be a (bounded, convex) polytope in a Euclidean space E and $\xi : E \to \mathbb{R}$ a linear function such that ξ is generic with respect to $\overline{C}$. Then, the only local extrema of ξ on $\overline{C}$ are the unique global maximum and the unique global minimum.* □

Proposition 5.3.5. *There exists a vector $\xi \in C^v_\emptyset \subseteq E$ such that the linear function $E \to \mathbb{R}$, $\xi(x) := (\xi, x)$ satisfies $\xi(x) \neq \xi(y)$, whenever x and y are vertices of a* 1*-cell of* Σ.

Proof. We claim it suffices to show the existence of such ξ whenever x and y are the vertices of a 1-simplex of $\mathrm{St}(0)$. Indeed, we saw in Example 5.2.4 that the chambers of Σ are of the type $wC_\emptyset$, with $w \in W$. From the proof of Proposition 5.3.2, we have $|\,\mathrm{St}(0)| = \cup_{v\in W_0} v\overline{C}_\emptyset$. It follows that $\mathrm{St}(0)$ contains a fundamental domain for the action of $t(Q)$ (group of translations on the root lattice) on E. Hence, if x and y are vertices of a 1-simplex of Σ, there exists a $\beta \in Q$ such that such that $t_\beta(x)$ and $t_\beta(y)$ is a 1-simplex of $\mathrm{St}(0)$. The claim then follows from $\xi(t_\beta(x)) - \xi(t_\beta(y)) = \xi(x) - \xi(y)$. But in this case the proposition is certainly true, as we only need to avoid the finite union of hyperplanes indexed by the 1-cells of $\mathrm{St}(0)$: $\cup_e \ \{\xi \in E \mid \xi(x') = \xi(y') \text{ and } x', y' \text{ are vertices of } \overline{e}\}$ (see [3, Proposition 1]). □

Definition 5.3.6. Fix ξ as in Proposition 5.3.5. For any $x \in E$, let $x_\emptyset$ be the unique element in E satisfying the equation

$$W_0 x \cap \overline{C^v_\emptyset} = \{x_\emptyset\}.$$

Define $f : E \to \mathbb{R}$ by $f(x) = (\xi, x_\emptyset)$. It is clearly a W_0-invariant function on E.

Lemma 5.3.7. *The function f is continuous and proper.*

Proof. Because f coincides with a linear function on each $w(\overline{C^v_\emptyset}), w \in W_0$, it is continuous on the closure of each vectorial chamber. Also, if x is on the boundary, say, $x \in w(\overline{C^v_\emptyset}) \cap sw(\overline{C^v_\emptyset})$ for a simple reflection s, then $s(x) = x$, so f agrees on the intersection and hence is continuous.

Given $b \in \mathbb{R}_{>0}$, note that the set $\tau_b := \{x \in \overline{C^v_\emptyset} \mid f(x) \leq b\}$ is an r-simplex. Indeed, let $\{\lambda_1, \dots, \lambda_r\}$ be the fundamental weights of Ψ_0. Then, τ_b is the simplex spanned by the $r+1$ affine independent points $x_0, \dots, x_r$ with $x_0 = 0$ and $x_i = b/f(\lambda_i)$. Note that the latter is well-defined because $\xi \in C^v_\emptyset$. Hence, τ_b is compact. Because of W_0-invariance of f and the fact that $\overline{C^v_\emptyset}$ is a fundamental domain for the action of W_0 on E, it follows that $T_b := f^{-1}([0,b]) = \cup_{w \in W_0} w\tau_b$, and hence compact as well, since W_0 is finite. Clearly, $f^{-1}([a,b]) \subseteq T_b$ and closed, hence compact. □

Theorem 5.3.8. *The function f is a W_0-invariant Morse function on (E, Σ) such that:*

(a) f is non-negative,

(b) $0 \in E$ is the unique minimum of f on E and $f(0) = 0$,

(c) for every $\sigma \in \Sigma$, there exists a unique global minimum of the restriction of f to $|\operatorname{St}(\sigma)|$. Moreover, any local minimum of f on $|\operatorname{St}(\sigma)|$ is the global minimum of f on $|\operatorname{St}(\sigma)|$.

Proof. From its definition, $f\chi_\sigma$ is the restriction of an affine linear function for each σ. From Proposition 5.3.5, f is not constant on 1-cells, so, in order to prove that f is a Morse function, all that is left is to show that the image under f of the 0-skeleton is discrete. From the properness of f proven in Lemma 5.3.7, given $b \geq 0$, the set $f(E^{(0)}) \cap [0,b] = f(E^{(0)} \cap f^{-1}([0,b]))$ is finite. Hence, the image of $E^{(0)}$ is discrete and f is a Morse function.

Now, items (a) and (b) are immediate because $\xi \in C^v_\emptyset$. As for (c), assume, without loss of generality, that $\sigma \subseteq \overline{C^v_\emptyset}$. Given $J \subsetneq \Pi_0$, let $E_J := \cap_{\alpha \in J} H_\alpha$. Note that for any $\sigma \subseteq \overline{C^v_J}$ and x in $|St(\sigma)| \cap \overline{C^v_\emptyset}$, we have a decomposition according to $E = E_J \oplus E^J$:

$$x = x_J + x^J,$$

with $x_J \in \operatorname{St}(\sigma) \cap \overline{C^v_J}$ and with $\alpha(x^J) \geq 0$ for all $\alpha \in J$. Hence, x^J is a non-negative linear combination of $\alpha \in J$, and moreover, we see that $x \in \overline{C^v_J}$ if and only if $x^J = 0$, or equivalently, $x = x_J$. Since $f(x^J) \geq 0$ with $f(x^J) = 0$ if and only if $x^J = 0$, it is clear that f has a maximum on $[x, x_J] \subseteq |St(\sigma)|$ at x . Using the W_J-invariance of f, and (the proof of) Proposition 5.3.2, we see that the same holds for any $x \in |St(\sigma)|$. In other words, a local minimum of f can only be attained in $|\operatorname{St}(\sigma)| \cap \overline{C^v_J}$. The result now follows from Lemma 5.3.4. □

If $a = (\alpha^\vee, m)$ is an affine root, by a **root half-space of** (E, Σ), we mean a set of the type $\{x \in E \mid a(x) \geq 0 \text{ with } a \in \Psi\}$. We denote them by H_a^+. Note that we could have defined also H_a^-, but as one can immediately check, we have $H_a^- = H_{-a}^+$. Given a set of simplices $\{\sigma_1, \ldots, \sigma_s\}$ of Σ, we define their **combinatorial convex hull**, and denote it by $\mathrm{hull}(\sigma_1 \cup \cdots \cup \sigma_s) \subseteq E$ to be the intersection of all root half-spaces of (E, Σ) containing all the simplices. This definition does not agree with the usual concept of convex hull used in Euclidean geometry, thus the usage of the adjective *combinatorial*. On the other hand, in the case of two simplices σ and τ, this notion does agrees with the more usual definition of the combinatorial convex hull as the union of the closure of all chambers occurring in a minimal gallery connecting σ and τ (see [1], Proposition 3.137 and Example 3.139(c)). An equivalent formulation is that the combinatorial convex hull of $\{\sigma_1, \ldots, \sigma_s\}$ is the smallest convex *subcomplex* of (E, Σ) that contains all the simplices σ_i. We refer to [1, Section 3.6] for the details.

Proposition 5.3.9. *Let f be the Morse function on (E, Σ) as above. Then, for any $\sigma \in \Sigma$, we have that $y_\sigma \in \mathrm{hull}(\sigma \cup \{0\})$, where y_σ is the unique minimum of $f|_{|\mathrm{St}(\sigma)|}$.*

Proof. Without loss of generality, we can assume that $\sigma \subseteq \overline{C_\emptyset^v}$, so it holds that $y_\sigma \in \overline{C_\emptyset^v}$. Let also H_a^+ be a half-space containing $(\{0\} \cup \sigma)$, with $a = (\alpha^\vee, m)$ an affine root. It must be shown that y_σ is in H_a^+. Let us consider the cases:

(1) If $\sigma \not\subseteq H_a$, then $|\mathrm{St}(\sigma)| \subseteq H_a^+$, hence $y_\sigma \in H_a^+$ as well.

(2) If $\sigma \subseteq H_a$ but $0 \notin H_a$ (that is, a not a linear root). By definition of $\mathrm{St}(\sigma)$, there is a chamber $C \subseteq \overline{C_\emptyset^v}$ such that σ and y_σ are contained in $\overline{C}$. The definition of f makes it now clear that $\overline{C}$ has to be in H_a^+, since otherwise the restriction of f on $s_a(\overline{C}) \subseteq |\mathrm{St}(\sigma)|$ would take smaller values than $f(y_\sigma)$, in contradiction to the definition of y_σ.

(3) If σ and 0 are both in H_a, then, a is a linear root. As $\sigma \subseteq H_a$, if it were the case that $y_\sigma \notin H_a$, we would have both y_σ and $s_\alpha(y_\sigma)$ as minima of $f|_{|\mathrm{St}(\sigma)|}$, contradicting the uniqueness of y_σ. Thus, $y_\sigma \in H_a \subseteq H_a^+$.

This concludes the proof. □

5.3.2 The Euclidean building case

Our aim is to work with buildings that arise from a reductive group $\underline{G}$ over a non-archimedean local field F. Write $G = \underline{G}(F)$. If we fix a maximal F-split torus $\underline{T}$ of $\underline{G}$, and put $E := \mathbb{R} \otimes X_*(\underline{T})$, (where $X_*(\underline{T})$ is the group of F-rational cocharacters of $\underline{T}$) a construction of these objects can be done roughly in the following way. First, assume that $\underline{G}$ splits. Then by a choice of a basis of the

Lie algebra (a pinning, or "épinglage") one associates a valuation of the root datum[4] and then a building $\mathscr{B}(\underline{G}, F)$ as the quotient of $G \times E$ by a certain equivalence relation ([13], Definition 7.4.2. See also [85, Theorem 3.20]). The general case is even more delicate and one has to take an extension of F, say F_1, over which $\underline{G}$ splits and consider the subset of invariants for the Galois group (the so-called descent theory). For our purposes, we will use a more axiomatic definition of buildings, which can be found in [77, Section II.1.1] (see also [100, Paragraph 4] and [52]). For this definition, let W_0 be a finite reflection group of E. By a **vectorial chamber of** E, we mean any connected component of the complement of the (finite) union of the reflection hyperplanes. If we can write $E = E_1 \oplus E_2$ with E_1 orthogonal to E_2 and $W_0 = W_{0,1} \times W_{0,2}$ with $W_{0,i}$ a finite reflection group of E_i, we say that the pair (W_0, E) is **reducible**. Otherwise, it is called **irreducible**. Consider $\mathcal{T}$ a translation subgroup $\mathcal{T} \subseteq t(E) \subseteq \text{Aff}(E)$ of maximal rank. Here, $\mathcal{T}$ can be dense in E. Define $W' := W_0 \ltimes \mathcal{T}$, with W_0 not necessarily irreducible.

Definition 5.3.10. A **Euclidean building over** (W', E) is a pair $(\mathscr{B}, \mathcal{A})$ in which $\mathscr{B}$ is a set and $\mathcal{A}$ is a family of injections $\phi_A : E \to \mathscr{B}$, called **charts**, whose images $A := \phi_A(E)$ are called **apartments**. This pair is subject to the axioms:

(A1) For any $\phi_A \in \mathcal{A}$ and $w \in W'$, $\phi_{A'} = \phi_A \circ w$ is also in $\mathcal{A}$.

(A2) If $\phi_A, \phi_{A'} \in \mathcal{A}$, then the set $I := \phi_{A'}^{-1}\phi_A(E)$ is a closed and convex subset of E. Moreover, the change of apartments $\phi_A^{-1} \circ \phi_{A'}$ restricted to I coincides with the restriction to I of an element $w \in W'$:

$$\phi_A^{-1} \circ \phi_{A'}|_I = w|_I.$$

(A3) Given $p, q \in \mathscr{B}$ there exists an apartment A containing p and q.

The axioms (A1) - (A3) allow us to map geometric notions of (E, Σ) into $\mathscr{B}$. In particular, a vectorial chamber of $\mathscr{B}$ is the image under $\phi_A \in \mathcal{A}$ of a vectorial chamber of E. Also, there is a well-defined (because of (A2) and (A3)) function $d : \mathscr{B} \times \mathscr{B} \to \mathbb{R}_{\geq 0}$

$$d(p, q) = \|\phi_A^{-1}(p) - \phi_A^{-1}(q)\|, \tag{5.6}$$

for any $A \supseteq \{p, q\}$.

(A4) Given two vectorial chambers $C^v, D^v \subseteq \mathscr{B}$, there exists an apartment A such that $A \cap C^v$ and $A \cap D^v$ contain vectorial chambers.

(A5) For every point $p \in \mathscr{B}$ and apartment $A \ni p$, there is a d-decreasing retraction $\rho : \mathscr{B} \to A$ such that $\rho^{-1}(p) = \{p\}$.

[4] In French, "donée radicielle valuée". In the survey [99, 1.5], J. Tits used this English term. In [85], it was used valuation of the root group datum.

Remark. Using axiom (A5), it follows that the distance function d of (5.6) is actually a metric on $\mathscr{B}$ (that is, it satisfies the triangle inequality, [77, Proposition II.1.3]) so $\mathscr{B}$ is a metric space.

Remark. This is a much more general axiomatic approach than what we will really need. But it turns out that these axioms are well suited for the purpose of passing our Morse construction from the Coxeter complex to the Bruhat-Tits building. We will have to add two different types of *discreteness* to be in the case where we can apply our construction to $\mathscr{B}(\underline{G}, F)$. First, our interest is in the case that $W' = W$ is an affine Coxeter group generated by affine reflections S. In this case, the group $\mathcal{T}$ is *discrete*[5]. We shall also say that $(\mathscr{B}, \mathcal{A})$ is a Euclidean building over (W, S)[6]. Assuming, moreover, that (W_0, E) is irreducible, the pair $(\mathscr{B}, \mathcal{A})$ over (W, S) is the geometric realization of a combinatorial Euclidean building, in the sense of [100, Paragraph 1] (see also [12, §IV.1]). These are simplicial complexes. Although Tits only considered the irreducible case in [100], the definition above works in the polysimplicial case as, in [77, Corollary II.2.4], it was shown that a Euclidean building decomposes as the product of Euclidean buildings corresponding to the irreducible factors of (W_0, E). The second type of discreteness is to demand that the pair $(\mathscr{B}, \mathcal{A})$ is such that the (poly)simplicial complex obtained is *locally finite*.

Remark. In the context of [100] (irreducible case), it was shown (see [100, Theorem 3]) that any valuation of root datum determines a Euclidean building $(\mathscr{B}, \mathcal{A})$. Moreover, for rank greater than or equal to 3 and in the discrete, locally finite case, *every* such pair $(\mathscr{B}, \mathcal{A})$ over (W, S), arises as the building of a *discrete* valuation of root datum associated to an (almost simple) algebraic group $\underline{G}$ over a locally compact and totally disconnected field F ([100], Corollary to Paragraph 15). This justifies the usage of this definition.

From now on, and until the end of this section, we assume that (W_0, E) irreducible, and that we have $(\mathscr{B}, \mathcal{A})$ over an affine Coxeter system (W, S), so $\mathcal{T}$ is discrete. With these assumptions, using the family of injections $\mathcal{A}$, and the simplicial complex structure discussed in Example 5.2.4, it follows that $\mathscr{B}$ is a simplicial complex, and thus an APC. We will write $(\mathscr{B}, \mathcal{P}(\mathscr{B}))$. Before we endow this APC with a Morse function in a similar way as we did with the Coxeter complex $(E, \Sigma(W, S))$, we will need slight enhancements of the axioms (A2) and (A3) in this simplicial case.

Proposition 5.3.11 (A2'). *In the discrete case, the intersection $I := \phi_{A'}^{-1}\phi_A(E)$ is a convex subcomplex of E.*

[5] Note that $\mathcal{T}$ discrete does not imply that W' is a Coxeter group. For example, if $\mathcal{T}$ is the weight lattice, this is not the case and we have (2.4).

[6] J. Tits call such a building as "immeuble de type affine" in [100, p.161].

Proof. In this case, this intersection is a union of simplices. From (A2) it is closed and convex, thus the result. □

Proposition 5.3.12 (A3'). *Given two simplices $\sigma, \tau \in \mathcal{P}(\mathscr{B})$ there exists an apartment $A \supseteq \tau \cup \sigma$.*

This statement appears as axiom (B1) of the definition of a (combinatorial) Euclidean building in [12]. A. Parreau[7] [77, Proposition II.1.17] argues, using the notion of germs of faces, that this condition holds with the set of axioms (A1) - (A5). But one can give a direct proof.

Proof of Proposition 5.3.12. If $p \in \mathscr{B}$ is in a chamber C, then any apartment containing p also contains C. This uses Proposition 5.3.11. Given σ and τ, there are chambers C, D such that $\sigma \preceq \overline{C}$ and $\tau \preceq \overline{D}$. Now pick points p, q in C and D, respectively, and use axiom (A3). □

Corollary 5.3.13. *Given a simplex $\sigma \in \mathcal{P}(\mathscr{B})$, a point $p_0 \in \mathscr{B}$ and $p \in |\operatorname{St}(\sigma)|$, there is an apartment $A \supseteq (\{p_0, p\} \cup \sigma)$.*

Proof. As $p \in |\operatorname{St}(\sigma)|$, there is a chamber C such that $\{p\} \cup \sigma \subseteq \overline{C}$. Then, it follows from Lemma 5.3.12 that there is an apartment containing C and p_0. □

Recall that a point $p_0 \in \mathscr{B}$ is said to be **special** if for any apartment A containing p_0, the image of the stabilizer of p_0 in W under the canonical surjection $W \to W/t(Q) \cong W_0$ is isomorphic to W_0. Suppose further that $(\mathscr{B}, \mathcal{P}(\mathscr{B}))$ is a *locally finite* Euclidean building over (W, S) and that the Coxeter complex $(E, \Sigma(W, S))$ is endowed with the Morse function f of Definition 5.3.6. Fix a special point p_0 of $\mathscr{B}$. Given p in $\mathscr{B}$, pick an apartment $A \supseteq \{p, p_0\}$, which exists by axiom (A3), and a chart ϕ_A such that $p_0 = \phi_A(0)$, which is possible by (A1). Define

$$\varphi(p) := f\phi_A^{-1}(p). \tag{5.7}$$

Proposition 5.3.14. *The function $\varphi : \mathscr{B} \to \mathbb{R}$ discussed in (5.7) is well-defined.*

Proof. Given another apartment $A' \supseteq \{p_0, p\}$ with $p_0 = \phi_{A'}(0)$, axiom (A2) tells us that $\phi_{A'}^{-1}(p) = w^{-1}\phi_A^{-1}(p)$, for a suitable $w \in W$. As $0 = \phi_A^{-1}\phi_{A'}(0) = w(0)$, we get $w \in W_0$. The W_0-invariance of f implies the well-definedness of φ. □

Proposition 5.3.15. *The function φ is continuous and proper.*

[7] A. Parreau even refers to this proposition as property (A3') and remarks that, in the simplicial setting, (A3') is equivalent to axiom (B1) of [12], although there is a small typo in the reference give there.

Proof. In the locally finite case, every point $p \in \mathscr{B}$ has a neighborhood that meets only a finite number of chambers. The continuity then follows from the Pasting Lemma [68, Theorem 18.3] in view of the well-definedness of φ.

Because $\mathscr{B}$ is a metric space and φ is continuous, all we need to prove is that the inverse image under φ of a compact set $K \subseteq \mathbb{R}$ is bounded. From the properness of f, there is a constant $M > 0$ such that $f^{-1}(K) \subseteq B(0, M) \subseteq E$. We claim that $\varphi^{-1}(K) \subseteq B(p_0, M) \subseteq \mathscr{B}$. Indeed, given $p \in \varphi^{-1}(K)$, let $A \supseteq \{p_0, p\}$. As $f\phi_A^{-1}(p) = \varphi(p) \in K$, it follows that $\phi_A^{-1}(p) \in B(0, M)$. From (5.6), it holds that $p \in B(p_0, M) \subseteq \mathscr{B}$. □

Theorem 5.3.16. *The function φ is a Morse function on $\mathscr{B}$ such that:*

(a) φ is non-negative,

(b) $p_0 \in \mathscr{B}$ is the unique minimum of φ on $\mathscr{B}$ and $\varphi(p_0) = 0$,

(c) for every $\sigma \in \mathcal{P}(\mathscr{B})$, there exists a unique global minimum of the restriction of φ to $|\operatorname{St}(\sigma)|$. Moreover, any local minimum of φ on $|\operatorname{St}(\sigma)|$ is the global minimum of φ on $|\operatorname{St}(\sigma)|$.

Proof. Given a 1-simplex $[p, q] \in \mathcal{P}(\mathscr{B})$, there is an apartment $A \supseteq \{p_0, p, q\}$, so property (ii) of Definition 5.2.5 for φ follows from the similar property of f, as well as property (i). As $\varphi(\mathscr{B}) = f(E)$, items (a) and (b) are satisfied and as $f(\Sigma^{(0)})$ is discrete in $\mathbb{R}$, φ is a Morse function.

We are left to prove (c). Given $\sigma \in \mathcal{P}(\mathscr{B})$, for any apartment $A \supseteq \{p_0\} \cup \sigma$, let

$$y_A := \min f|_{|\operatorname{St}(\phi_A^{-1}(\sigma);\Sigma)|} \tag{5.8}$$

which is well-defined (the minimum exists and is unique, by Corollary 5.3.3 and Lemma 5.3.4). Fix such an apartment A with the extra property that $\phi_A(0) = p_0$.

Lemma. *The element $p := \phi_A(y_A) \in A'$, for any apartment $A' \supseteq \{p_0\} \cup \sigma$.*

Proof. Indeed, if $A' \supseteq \{p_0\} \cup \sigma$ is another apartment, using axiom (A1), if necessary, assume $\phi_{A'}(0) = p_0$. Let $\sigma_A, \sigma_{A'}$ be the inverse images of σ under ϕ_A and $\phi_{A'}$, respectively. As $\phi_A(\sigma_A) = \sigma = \phi_{A'}(\sigma_{A'})$, we use axiom (A2) to obtain that

$$\sigma_A = \phi_A^{-1}\phi_{A'}(\sigma_{A'}) = w\sigma_{A'},$$

and $y_A = wy_{A'}$, for a suitable $w \in W$ which is necessarily in W_0 (w fixes $0 \in E$). Now, the intersection $I = \phi_A^{-1}\phi_{A'}(E)$ is a convex subcomplex of (E, Σ) containing 0 and σ_A, so it contains $\operatorname{hull}(\{0\} \cup \sigma_A)$ which, from Proposition 5.3.9, implies that $y_A \in I$. Thus,

$$\phi_A^{-1}(p) = y_A = wy_{A'} = \phi_A^{-1}\phi_{A'}(y_{A'}),$$

from which we conclude $p = \phi_{A'}(y_{A'}) \in A \cap A'$, proving the lemma. □

This element p, just constructed, is easily seen to be the unique global minimum for φ on $|\operatorname{St}(\sigma)|$. Now, if q is a local minimum of φ. Pick an apartment $A \supseteq \{p_0, q\} \cup \sigma$. Then, $\phi_A^{-1}(q)$ is also a local minimum for f on $\phi_A^{-1}(|\operatorname{St}(\sigma)|) = |\operatorname{St}(\sigma_A; \Sigma)|$, so that $\phi_A^{-1}(q) = y_A$, that is, $q = p$. $\square$

5.4 Morse contraction

5.4.1 The simplicial case

Given a Morse function $\varphi : X \to \mathbb{R}$ on an APC $(X, \mathcal{P}(X))$, from properties (i) and (ii) of Definition 5.2.5, it follows that $\varphi\chi_\pi$ coincides with the restriction of a generic linear function on $\overline{C}_\pi$, thus, from Lemma 5.3.4, φ has a unique minimum on π. This unique minimum will be denoted by $\min \varphi|_\pi$. One defines a strict partial order $<$ on X (that is, an irreflexive, transitive and asymmetric binary relation on X) by

$$\pi < \varpi \qquad \text{if, and only if,} \qquad \varphi(\min \varphi|_\pi) < \varphi(\min \varphi|_\varpi). \tag{5.9}$$

This defines a poset structure $(X, \leq)$ by taking the reflexive closure of $<$ (that is, $\pi \leq \varpi$ if, and only if, $\pi < \varpi$ or $\pi = \varpi$). The partial order just defined will be called **Morse ordering**.

Let $(X, \mathcal{P}(X))$ be a simplicial complex. Given an r-simplex σ, an orientation is the choice of a generator of the relative homology group $\tilde{H}_r(\sigma, \partial\sigma) \cong \mathbb{Z}$. For a facet $\tau \preceq \sigma$, the long exact sequence for the relative homology and excision (see [34], Theorems 2.16 and 2.20. See also [11, Theorem 6.5]) imply that there are natural isomorphisms

$$\tilde{H}_r(\sigma, \partial\sigma) \cong \tilde{H}_{r-1}(\partial\sigma) \cong \tilde{H}_{r-1}(\tau, \partial\tau),$$

thus, an orientation on σ induces an orientation on τ and vice-versa. More concretely, put an orientation on σ, by defining an ordering of the vertices $\sigma = [x_0, \ldots, x_r]$. Permutation of the vertices may yield an opposite oriented simplex depending on the product of the sign of the permutations on each simplex. For $r \geq 0$, let $C_r(X)$ be the free $\mathbb{Z}$-module generated by the oriented simplices in dimension r modulo the relation $\sigma^+ = -\sigma^-$, in which σ^- denotes an orientation opposite to σ^+. Let also $C_{-1}(X) = \mathbb{Z}$ and $C_r(X) = 0$ for $r < -1$.

The boundary operator $\partial = \{\partial_r : C_r(X) \to C_{r-1}(X)\}$ is given by the well-known alternating sum expression for $r > 0$ and we let $\partial_0[x] = [\emptyset]$, for all vertices x, in which $[\emptyset]$ is a fixed generator of $C_{-1}(X) = \mathbb{Z}$. The map ∂_0 is usually called an augmentation map (cf. [34], p. 110) and is traditionally denoted by ϵ, instead of ∂_0. We obtain in this way an augmented complex

$$\mathbb{Z} \leftarrow C_0(X) \leftarrow C_1(X) \leftarrow \cdots .$$

Elements of $C_r(X)$ are called **chains**. Given a chain $c = \sum_\tau n_\tau \tau \in C_r(X)$, the **support of** c, denoted $\operatorname{supp}(c)$, is defined as the finite set of τ such that $n_\tau \neq 0$.

From now on, assume that X is simplicial and that $\varphi : X \to \mathbb{R}$ is a Morse function on X, with the associated Morse ordering $\leq$ introduced above. Given an r-simplex $\sigma \in \mathcal{P}(X)$ and a vertex $x \in \sigma$, by σ_x it will be meant the **facet of** σ **opposite to** x. Also, if $y \in |\operatorname{Lk}(\sigma)|$, define the **cone on** σ **with vertex** y, denoted by $y*\sigma$, as the $(r+1)$-dimensional simplex generated by σ and y. Thus, if $\sigma = [x_0, \ldots, x_r]$, then $\sigma_{x_j} = [x_0, \ldots, \widehat{x_j}, \ldots, x_r]$ and $y * \sigma = [y, x_0, \ldots, x_r]$.

Also, for *any* vertex $y \in X$, the operator $y\, * : C_{-1}(X) \to C_0(X)$ sends $[\emptyset] \mapsto [y]$. More generally, given $c = \sum_\tau n_\tau \tau \in C_r(X)$ and $[y] \in \cap_{\operatorname{supp}(c)} \operatorname{Lk}(\tau)$ (whenever this intersection is nonempty), define $y\, * : C_r(X) \to C_{r+1}(X)$ by

$$y * (c) = \sum_\tau n_\tau (y * \tau). \tag{5.10}$$

Lemma 5.4.1. *Let $c = \sum n_\sigma \sigma \in C_r(X)$ be a chain and suppose that there exists a $[y] \in \bigcap_{\operatorname{supp}(c)} \operatorname{Lk}(\sigma)$. Then,*

$$\partial(y * c) = c - y * (\partial c).$$

Proof. For each σ, write $\sigma = [x_0, \ldots, x_r]$, so that $y * \sigma = [y, x_0, \ldots, x_r]$. Then,

$$\begin{aligned} \partial(y * \sigma) &= \partial([y, x_0, \ldots, x_r]) \\ &= \sigma - \sum_{j=0}^{d} (-1)^j [y, x_0, \ldots, \widehat{x_j}, \ldots, x_r] \\ &= \sigma - y * \left(\sum_{j=0}^{d} (-1)^j [x_0, \ldots, \widehat{x_j}, \ldots, x_r] \right) \\ &= \sigma - y * (\partial \sigma). \end{aligned}$$

As $y\, *$ was defined so that it is linear, the result follows. $\square$

Suppose that the simplicial complex X is endowed with a Morse function that satisfies the following properties:

(a) For all $x \in X^{(0)}$, one has $\varphi(x) \geq 0$,

(b) φ has a unique minimum at $y_0 \in X$ and $\varphi(y_0) = 0$,

(c) for every $\sigma \in \mathcal{P}(X)$, there exists a unique global minimum of the restriction of φ to $|\operatorname{St}(\sigma)|$. Moreover, any local minimum of φ on $|\operatorname{St}(\sigma)|$ is the global minimum of φ on $|\operatorname{St}(\sigma)|$.

For example, the Morse functions on the Coxeter complex and on the Euclidean building discussed in Definition 5.3.6 and in (5.7), respectively, fulfill these properties, as was showed in Theorems 5.3.8 and 5.3.16. From these assumptions, note that if τ_0 is a simplex such that $y_0 \in \tau_0$, then τ_0 is a minimal element in the Morse ordering. Also, one has that:

Proposition 5.4.2. *Property (c) implies that* $y = \min \varphi|_{|\operatorname{St}(\sigma)|}$ *is in* σ *if and only if* $y = \min \varphi|_{|\operatorname{St}(\sigma_y)|}$.

Proof. Suppose that $\min \varphi|_{|\operatorname{St}(\sigma)|} = y \in \sigma$. If y is not equal to $\min \varphi|_{|\operatorname{St}(\sigma_y)|}$ then, by Property (c), y is not a local minimum of φ on $|\operatorname{St}(\sigma_y)|$. Hence, there exists a 1-simplex $[y, z]$ in $\operatorname{St}(\sigma_y)$ such that $\varphi(z) < \varphi(y)$. But, by definition of $\operatorname{St}(\sigma_y)$ there exists a simplex τ containing σ_y and $[y, z]$ in that case. In particular, τ is in $\operatorname{St}(\sigma)$, contradicting the assumption $\min(\varphi|_{|\operatorname{St}(\sigma)|}) = y \in \sigma$. The other implication is straightforward. $\square$

Recall that a contraction on a chain complex $C_*(X)$ is a family of linear maps $\gamma = \{\gamma_r : C_r(X) \to C_{r+1}(X)\}$ such that $\gamma\partial + \partial\gamma = id$. It is a chain homotopy between the identity and the zero chain maps ([34], Section 2.1). If X is endowed with a Morse function satisfying the properties (a), (b) and (c) above, define $\gamma : C_r(X) \to C_{r+1}(X)$, in the following recursive way with respect to the Morse ordering: first, set $\gamma([\emptyset]) = [y_0]$. Then, for minimal τ (that is, simplices such that $y_0 \in \tau$) put $\gamma(\tau) = 0$. Given $\sigma \in C_r(X)$, let $y_\sigma := \min \varphi|_{|\operatorname{St}(\sigma)|}$, and suppose γ is defined for $\tau < \sigma$. Define

$$\gamma(\sigma) = \begin{cases} 0, & \text{if } y_\sigma \in \sigma \\ (y_\sigma * \sigma) + \gamma(y_\sigma * (\partial\sigma)), & \text{if } y_\sigma \notin \sigma. \end{cases} \tag{5.11}$$

Remark. The simplices that form the chain $\gamma(\sigma)$ are all smaller than σ in the Morse ordering, so the definition above is indeed recursive. Also, as $\varphi(X^{(0)})$ is discrete, $\varphi(X^{(0)}) \cap [0, \varphi(\min \varphi|_\sigma)]$ is finite, so $\operatorname{supp}(\gamma(\sigma))$ is finite and γ is well-defined.

Theorem 5.4.3. *The linear map* γ *defined above satisfies* $\partial\gamma + \gamma\partial = id$.

Proof. Let $\Delta(\sigma) := (\partial\gamma + \gamma\partial)(\sigma) - \sigma$. Note that if c is a chain, $\Delta(c)$ also makes sense, by linearity of the operators involved. Now, the equation $\partial\gamma + \gamma\partial = id$ is equivalent $\Delta(\sigma) = 0$, for all simplex σ. First, suppose $y_\sigma \in \sigma$. Because $\gamma(\sigma) = 0$, one has to argue that $\gamma\partial(\sigma) = \sigma$. Let $\sigma = [x_0, \ldots, x_r]$, with $x_0 = \min \varphi|_{|\operatorname{St}(\sigma)|}$. From Proposition 5.4.2, $x_0 = \min \varphi|_{|\operatorname{St}(\sigma_{x_0})|}$, so that

$$\begin{aligned} \gamma(\sigma_{x_0}) &= \gamma([x_1, \ldots, x_r]) \\ &= [x_0, \ldots, x_r] + \gamma\left(\sum_{j=1}^{r} (-1)^{j-1} x_0 * [x_1, \ldots, \hat{x}_j, \ldots, x_r]\right) \\ &= \sigma - \sum_{j=1}^{r} \gamma(\sigma_{x_j}), \end{aligned}$$

and hence $\gamma\partial(\sigma) = \gamma(\sigma_{x_0}) + \sum_{j=1}^{r} \gamma(\sigma_{x_j}) = \sigma$, so $\Delta(\sigma) = 0$ in this case. Note that $\gamma(\sigma) = 0$ for any minimal (with respect to the Morse ordering) σ. Next, suppose $y_\sigma \notin \sigma$. Using Lemma 5.4.1,

$$\begin{aligned} \partial\gamma(\sigma) &= \partial\left(y_\sigma * \sigma + \gamma(y_\sigma * (\partial\sigma))\right) \\ &= \sigma - y_\sigma * (\partial\sigma) + \partial\gamma(y_\sigma * (\partial\sigma)), \end{aligned}$$

and because of $\partial\partial = 0$ and Lemma 5.4.1, one has

$$\begin{aligned} \gamma\partial(\sigma) &= \gamma(\partial\sigma - y_\sigma * \partial(\partial\sigma)) \\ &= \gamma\partial(y_\sigma * (\partial\sigma)), \end{aligned}$$

so that

$$\Delta(\sigma) = \Delta(y_\sigma * (\partial\sigma)).$$

As $y_\sigma * (\partial\sigma) < \sigma$, by induction on the Morse ordering, the result follows. □

In an earlier version of this manuscript, the author believed that the contraction constructed in (5.11), with conditions (a), (b) and (c), satisfied the condition that, if $\gamma(\sigma) = \sum n_{\sigma\tau}\tau$, then $n_{\sigma\tau} \in \mathbb{Z}$ with $|n_{\sigma\tau}| \leq 1$. This property is very important for the applications we have in mind. We thank G. Savin for pointing out a mistake on our proof. However, with the Morse function f (of Definition 5.3.6) defined on the Coxeter complex (E, Σ), the Morse contraction γ of (5.11) seems to have the expected property, by computing in lower dimensional examples. We make the following:

Conjecture 5.4.4. *With the Morse function f of Definition 5.3.6, the contraction γ of $C_*(\Sigma)$, defined in (5.11), satisfies the property that, if $\gamma(\sigma) = \sum n_{\sigma\tau}\tau$, then $n_{\sigma\tau} \in \mathbb{Z}$ with $|n_{\sigma\tau}| \leq 1$.*

Our point here is that there is a missing sufficient condition on the Morse function in order for it to yield a contraction as in (5.11) with the nice bound $|n_{\sigma\tau}| \leq 1$ in the coefficients. This missing property, hopefully, can also be transported to a locally finite, discrete Euclidean building, using what was done in Section 5.3.2.

Remark. In [70, Proposition 2.1], a contraction of the Coxeter complex was given (and later, on [71] a contraction of the Bruhat-Tits building). The contraction obtained there had the property that $|n_{\sigma\tau}| \leq M_\gamma$ for a fixed, unspecified constant, that depended only on the contraction γ. Also, the contraction obtained here is similar to the one constructed in [7]. Later we will argue that the conditions we used to construct γ are in fact equivalent to the data of a *nice* Morse function.

Example 5.4.5. Let us illustrate the above contraction with a simple example on rank one. Suppose (W, S) is the infinite dihedral group, so that the Coxeter complex (E, Σ) is the tessellation of $E = \mathbb{R}$ by length 1 intervals with integer endpoints and $W_0 = \{1, s\}$ with s the reflection around the origin. In this case, the vertices are $[n] \in \Sigma$ and the chambers are $[n, n+1] \in \Sigma$ with $n \in \mathbb{Z}$. For each $[n]$, we have that $|\operatorname{St}([n])| = [n-1, n+1]$. For any non-zero element in $\xi \in \mathbb{R}^+$, we have the Morse function $f(x) = \xi|x|$. It follows that, for $\sigma = [n]$, with $n > 0$, $y_\sigma = [n-1]$,

$$\gamma([n]) = \sum_{j=0}^{n-1} [j, j+1]$$

and $\operatorname{supp}(\gamma([n])) \subseteq [0, n]$. If $n < 0$ we have $\operatorname{supp}(\gamma([n])) \subseteq [n, 0]$.

5.4.2 The polysimplicial case

Given a simplicial complex $(X, \mathcal{P}(X))$ endowed with a Morse function φ satisfying properties (a), (b) and (c) as above, we saw in Theorem 5.4.3 that the augmented chain complex

$$\mathbb{Z} \leftarrow (C_*(X), \partial_*) \tag{5.12}$$

with the augmentation map $\epsilon[x] = [\emptyset]$ is acyclic, with contraction map γ as given in (5.11). We will denote such an augmented acyclic chain complex by $(C_*(X), \partial_*, \epsilon, \gamma)$. Now, suppose we are given a finite set of pairs $\{(X_{(i)}, \varphi_{(i)})\}$ each of which consists of a simplicial complex endowed with a Morse function satisfying the properties (a), (b) and (c) as above, so that we have a finite set of augmented acyclic chain complexes $(C_*(X_{(i)}), \partial_{*,(i)}, \epsilon_{(i)}, \gamma_{(i)})$. We wish to construct a contraction on the product using the data of each factor. From the classical Eilenberg-Zilber Theorem [27], we express the chain groups of the product in terms of the tensor product of each chain complex, and from Künneth's Formula [34, Theorem 3.15], the tensor product of these chain complexes is acyclic. But the whole point is that we wish to write down an explicit contraction with good bounds on the coefficients. This will be the case in each factor, if we assume conjecture 5.4.4 to be true.

Proposition 5.4.6. *If $(C_*(X), \partial_*, \epsilon, \gamma)$ and $(C_*(X'), \partial'_*, \epsilon', \gamma')$ are augmented acyclic chain complexes then we may construct a contraction Γ on*

$$\mathbb{Z} \xleftarrow{\varepsilon} (C_*(X) \otimes C_*(X'), \delta_*),$$

with $\varepsilon = \epsilon \otimes \epsilon'$, via

$$\Gamma(\sigma \otimes \tau') = \begin{cases} \gamma\sigma \otimes \tau', & \text{if } \dim\sigma = r > 0 \\ \gamma\sigma \otimes \tau' + \gamma[\emptyset] \otimes \gamma'\tau', & \text{if } \dim\sigma = 0. \end{cases}$$

Proof. Recall that $\delta(\sigma \otimes \tau') = \partial\sigma \otimes \tau' + (-1)^r \sigma \otimes \partial'\tau'$ and if $\dim\sigma = 0$, it holds that $\delta(\sigma \otimes \tau') = \sigma \otimes \partial'\tau'$. For $r = \dim\sigma > 1$, $\Gamma = \gamma \otimes id'$, so it is straightforward that $\delta\Gamma + \Gamma\delta = id$. If $r = 1$, on the one hand

$$\delta\Gamma(\sigma \otimes \tau') = \partial\gamma\sigma \otimes \tau' + \gamma\sigma \otimes \partial'\tau',$$

whereas on the other, denoting $\sigma = [\sigma_i, \sigma_f]$ (as $\dim \sigma = 1$)

$$\begin{aligned}\Gamma\delta(\sigma \otimes \tau') &= \Gamma(\partial\sigma \otimes \tau' - \sigma \otimes \partial'\tau') \\ &= \gamma\sigma_f \otimes \tau' + \gamma[\emptyset] \otimes \gamma'\tau' - \gamma\sigma_i \otimes \tau' - \gamma[\emptyset] \otimes \gamma'\tau' - \gamma\sigma \otimes \partial'\tau',\end{aligned}$$

so that $(\delta\Gamma + \Gamma\delta)(\sigma \otimes \tau') = (\partial\gamma + \gamma\partial)\sigma \otimes \tau' = \sigma \otimes \tau'$. Finally, if $r = 0$, then σ is a vertex and we compute

$$\Gamma\delta(\sigma \otimes \tau') = \gamma\sigma \otimes \partial'\tau' + \gamma[\emptyset] \otimes \gamma'\partial'\tau'$$

and

$$\begin{aligned}\delta\Gamma(\sigma \otimes \tau') &= \delta(\gamma\sigma \otimes \tau' + \gamma[\emptyset] \otimes \gamma'\tau') \\ &= \partial\gamma\sigma \otimes \tau' - \gamma\sigma \otimes \partial\tau' + \gamma[\emptyset] \otimes \partial'\gamma'\tau',\end{aligned}$$

from which, using that $\partial\gamma(\sigma) = \sigma - \gamma\epsilon\sigma = \sigma - \gamma[\emptyset]$ (as $\partial = \epsilon$ in $C_0(X)$), we get

$$\Gamma\delta + \delta\Gamma(\sigma \otimes \tau') = (\sigma - \gamma[\emptyset]) \otimes \tau' + \gamma[\emptyset] \otimes (\gamma'\partial' + \partial'\gamma')\tau' = \sigma \otimes \tau',$$

as required. □

If $n > 2$, define $\gamma(\sigma_{(1)} \otimes \cdots \otimes \sigma_{(n)})$ by

$$\gamma_{(1)}\sigma_{(1)} \otimes \cdots \otimes \sigma_{(n)}, \tag{5.13a}$$

if $\dim \sigma_{(1)} > 0$ and, for $1 \leq i \leq n$, by

$$\sum_{j=1}^{i+1} \gamma_{(1)}[\emptyset_{(1)}] \otimes \cdots \otimes \gamma_{(j-1)}[\emptyset_{(j-1)}] \otimes \gamma_{(j)}\sigma_{(j)} \otimes \sigma_{(j+1)} \otimes \cdots \otimes \sigma_{(n)}, \tag{5.13b}$$

if $\dim \sigma_{(1)} = \cdots = \dim \sigma_{(i)} = 0$ and $\dim \sigma_{(i+1)} > 0$. An inductive argument on the number of factors together with Proposition 5.4.6 gives us that the above γ defines a contraction on the augmented chain complex $(C_*(X_{(1)}) \otimes \cdots \otimes C_*(X_{(n)}), \delta_*, \varepsilon)$.

Remark. In what follows, if $X = X_{(1)} \times \cdots \times X_{(n)}$, we abuse the notation and write $\vec{\sigma} = \sigma_{(1)} \otimes \cdots \otimes \sigma_{(n)} \in C_r(X)$. This is so because we will regard $(C_*(X), \partial_*)$ equal to $(C_*(X_{(1)}) \otimes \cdots \otimes C_*(X_{(n)}), \delta_*)$.

5.4.3 Contraction and type-preserving automorphisms

Back to the case of a locally finite, discrete irreducible Euclidean building $(\mathscr{B}, \mathcal{A})$ over (W, S), where we have a simplicial complex $(\mathscr{B}, \mathcal{P}(\mathscr{B}))$, we will denote by $\mathrm{Aut}(\mathscr{B}, \mathcal{P}(\mathscr{B}))$ the group of simplicial automorphisms of $\mathscr{B}$ that preserves $\mathcal{A}$, that is, whenever A is an apartment, gA is as well. We say, moreover, that

$g \in \mathrm{Aut}(\mathscr{B}, \mathcal{P}(\mathscr{B}))$ is **type-preserving** if, when restricted to an apartment $A = \phi_A(E)$, there is an element $w \in W$ such that

$$g|_A = \phi_{gA} \circ w \circ \phi_A^{-1}. \tag{5.14}$$

Further, by the **convex hull** of a set of simplices of $\mathscr{B}$, we mean the intersection of all apartments containing them. Proposition 5.4.6 and the next corollary summarize all the properties we need from a contraction in order to revisit the results of [87] and [71]. We emphasize that Conjecture 5.4.4 are also essential to obtain the results of [71].

Lemma 5.4.7. *If $H \subseteq \mathrm{Aut}(\mathscr{B}, \mathcal{P}(\mathscr{B}))$ is a group of type-preserving automorphisms that fixes the special point p_0, then φ is H-invariant. In particular, if y_σ is the minimum of $\varphi|_{|\,\mathrm{St}(\sigma)|}$, then $hy_\sigma = y_{h\sigma}$, for all $h \in H$.*

Proof. Given $p \in \mathscr{B}$, pick an apartment $A \supseteq \{p_0, p\}$. The assumptions on H imply that, for any $h \in H$, there is a $w \in W_0$ such that $\phi_{hA}^{-1}(hp) = w\phi_A^{-1}(p)$, and thus $\varphi(hp) = f(w\phi_A^{-1}p) = \varphi(p)$. The in particular follows from H-invariance and the uniqueness of the minimum of φ on $|\,\mathrm{St}(\sigma)|$. □

Corollary 5.4.8. *Given $(\mathscr{B}, \mathcal{P}(\mathscr{B}))$ endowed with the Morse function as in (5.7), let $H \subseteq \mathrm{Aut}(\mathscr{B}, \mathcal{P}(\mathscr{B}))$ be a group of type-preserving automorphism that fixes the special point p_0. The Morse contraction γ constructed in (5.11) satisfies:*

(a) γ is H-equivariant,

(b) $\mathrm{supp}\,\gamma(\sigma) \subseteq \mathrm{hull}(\sigma \cup \{p_0\})$, for any $\sigma \in \mathcal{P}(\mathscr{B})$.

Proof. Given a simplex $\sigma \in \mathcal{P}(\mathscr{B})$, let $y_\sigma = \min \varphi|_{|\,\mathrm{St}(\sigma)|}$. Lemma 5.4.7 implies (a). As for item (b), we must show that $\mathrm{supp}\,\gamma(\sigma) \subseteq A$, for all $A \supseteq (\{p_0\} \cup \sigma)$. Fix such an A. If $p_0 \in \sigma$ (or $\gamma(\sigma) = 0$) then the support is empty and the assertion is trivially true. Suppose $\gamma(\sigma) \neq 0$ and assume $\mathrm{supp}\,\gamma(\tau) \subseteq A'$ for any $A' \supseteq (\{p_0\} \cup \tau)$, for all $\tau < \sigma$ in the Morse ordering. By construction (see the Lemma inside the proof of Theorem 5.3.16), the minimum $y_\sigma \in A$, so that $y_\sigma * \sigma \subseteq A$, and item (b) follows by induction, as $y_\sigma * \partial\sigma \subseteq A$. □

5.5 A comparison with the constructions of [7]

In this section, we compare the construction of the Morse function given in this chapter with the one given in [7]. If $(X, \mathcal{P}(X))$ is a simplicial complex endowed with a Morse function φ, given any $\sigma \in \mathcal{P}(X)$ one has the notion of **descending link**, denoted $\mathrm{Lk}_\downarrow(\sigma)$, as the subcomplex of $\mathrm{Lk}(\sigma)$ whose vertices $[x] \in \mathrm{Lk}_\downarrow(\sigma)$ satisfy $\varphi(x) < \varphi(y)$ for all vertex $[y] \in \mathrm{Lk}(\sigma)$. M. Bestvina and G. Savin define the notion of *nice* Morse function as follows. First, a Morse function φ on X is **reasonable** if either $X = \emptyset$ or

(i) φ attains a unique minimum on a vertex $y_0 \in X$,

(ii) for all vertices $x \neq y_0$ it holds $\mathrm{Lk}_\downarrow([x]; X) \neq \emptyset$.

Then, φ is **nice** if it is reasonable on X and the restriction of φ to $|\,\mathrm{Lk}_\downarrow(\sigma)|$ is reasonable. In Section 5.4.1 we wrote that the conditions (a), (b) and (c) were necessary for writing down a contraction map. But the fact that the minimum value of φ is 0 is not essential. The essential is:

(b') φ has a unique global minimum on X.

(c) for every $\sigma \in \mathcal{P}(X)$, there exists a unique global minimum of the restriction of φ to $|\,\mathrm{St}(\sigma)|$. Moreover, any local minimum of φ on $|\,\mathrm{St}(\sigma)|$ is the global minimum of φ on $|\,\mathrm{St}(\sigma)|$.

Proposition 5.5.1. *The notion of a nice Morse function is equivalent to the data of a Morse function satisfying (b') and (c) above.*

Proof. Let φ be a Morse function on $X \neq \emptyset$ satisfying conditions (b') and (c) as above. We show first that φ is reasonable. Indeed, (b') is equivalent to (i). Now let $x \neq y_0$, with y_0 the minimum granted from (b'). Then, the existence part of condition (c) implies $\mathrm{Lk}_\downarrow([x]) \neq \emptyset$, so φ is reasonable. Now let $\sigma \in \mathcal{P}(X)$. We must show φ is reasonable on $|\,\mathrm{Lk}_\downarrow(\sigma)|$. If $y_0 \in \sigma$, then $\mathrm{Lk}_\downarrow(\sigma) = \emptyset$ and we are done. If not, the unique minimum of φ on $|\,\mathrm{Lk}_\downarrow(\sigma)|$ is y_σ, granted from (c), so (i) is satisfied. As for (ii), if $[y] \in \mathrm{Lk}_\downarrow(\sigma)$ with $y \neq y_\sigma$, then, from (c), y is not a local minimum of φ on $|\,\mathrm{St}(\sigma)|$. This means there is a 1-simplex $[y, z] \in \mathrm{St}(\sigma)$ with $\varphi(z) < \varphi(y)$. Hence, $\mathrm{Lk}_\downarrow([y]) \neq \emptyset$, and thus φ is a nice Morse function.

For the converse, suppose φ is nice. We must prove (c). Given $\sigma \in \mathcal{P}(X)$ there exists a unique minimum y_σ for φ on $|\,\mathrm{St}(\sigma)|$, from (i). Now y can can only be a minimum of φ on $|\,\mathrm{St}(\sigma)|$ if $[y] \in \mathrm{Lk}_\downarrow(\sigma)$. But if $y \neq y_\sigma$, since $\mathrm{Lk}_\downarrow([y]) \neq \emptyset$, it follows that y is not a local minimum. $\square$

Thus the notion of a nice Morse function is a sufficient condition for writing down a contraction of the simplicial complex X which, in view of Conjecture 5.4.4, should have good bounds on its coefficients. The construction of a Morse function in the building used in [7] is different than the one used here. There, the function they used is given by the following principle: fix a chamber σ_0 in the building (assumed to be associated to an almost simple p-adic group G) and a generic point $x_0 \in \sigma_0$ close enough to a special vertex of σ_0. Then the function $\varphi(x) = d(x, x_0)$ (with $d(\cdot, \cdot)$ as in (5.6)) is a nice Morse function, which is invariant with respect to a stabilizer of the chamber σ_0 (an Iwahori subgroup of G). In our case, the function is invariant with respect to a stabilizer of a special point (a maximal compact of G).

5.6 Contraction of the Bruhat-Tits building

Let $\underline{G}$ be a connected, reductive group over a local, non-archimedean field F. Let $\mathscr{B} = \mathscr{B}(\underline{G}, F)$ be the Bruhat-Tits building associated to $(\underline{G}, F)$. In general, the group G does not acts by type-preserving automorphisms on $\mathscr{B}$. But there is always a subgroup $G^+ \subseteq G$ of finite-index which does acts in the required way (see the survey [85, Section 3.2.3]). To guarantee that the group G of rational points acts by type-preserving automorphisms, we add the assumption that the derived group is *simply connected* (see [99, 1.13]). In this case, we have an almost direct product $\underline{G} = (\underline{G}, \underline{G}) \cdot R(\underline{G})$, with $R(\underline{G})$ the connected component of the center and $(\underline{G}, \underline{G}) = \underline{G}_1 \times \cdots \times \underline{G}_n$ [78, Theorems 2.4 and 2.6], a direct product with all factors simply connected. Further, if $\underline{T}_c$ denotes the maximal F-split torus in the center, then one has a canonical decomposition $\mathscr{B}(\underline{G}, F) \cong \mathscr{B}((\underline{G}, \underline{G}), F) \times \mathscr{B}(\underline{T}_c, F)$ (cf. [99], p. 44) that splits further as

$$\mathscr{B}(\underline{G}, F) \cong \mathscr{B}(\underline{G}_{(1)}, F) \times \cdots \times \mathscr{B}(\underline{G}_{(n)}, F) \times \mathscr{B}(\underline{T}_c, F),$$

in which $\mathscr{B}(\underline{T}_c, F)$ is an affine space. By means of a *choice* of a basis of the lattice of F-rational cocharacters of $\underline{T}_c$, $X_*(\underline{T}_c)$, we identify $\mathscr{B}(\underline{T}_c, F) \cong \mathbb{R} \otimes X_*(\underline{T}_c)$, with the product of $m = \mathrm{rk}(\underline{T}_c)$ rank 1 Coxeter complexes as in Example 5.4.5 (that is, the 1-simplices are the intervals $[n, n+1]$). We end up with a polysimplicial complex $(\mathscr{B}, \mathcal{P}(\mathscr{B}))$ with

$$\mathscr{B} \cong \mathscr{B}_{(1)} \times \cdots \times \mathscr{B}_{(n)} \times \mathscr{B}_{(n+1)} \times \cdots \times \mathscr{B}_{(N)}, \tag{5.15}$$

with $N = n+m$. In this setting, we will say that a point $p_0 \in \mathscr{B}(\underline{G}, F)$ is **special** if the projection of p_0 onto $\mathscr{B}((\underline{G}, \underline{G}), F)$ is a special point as in the definition of the Morse function φ. After suitable choices of origins for the factors of $\mathscr{B}(\underline{T}_c, F)$ and special points of the factors of $\mathscr{B}((\underline{G}, \underline{G}), F)$, each $\mathscr{B}_{(i)}$ admits a Morse function $\varphi_{(i)}$ as in Section 5.3. Hence, we can construct a contraction γ of the augmented complex $\mathbb{Z} \leftarrow (C_*(\mathscr{B}), \partial_*)$ using the data of each factor.

Proposition 5.6.1. *Fix $p_0 \in \mathscr{B}$ a special point. Then, the contraction γ of the augmented complex $\mathbb{Z} \leftarrow (C_*(\mathscr{B}), \partial_*)$ defined by (5.13a) and (5.13b) satisfies:*

(a) γ is G_{p_0}-equivariant, where $G_{p_0} \subseteq G$ is the stabilizer of p_0,

(b) $\operatorname{supp} \gamma(\vec{\sigma}) \subseteq \operatorname{hull}(\vec{\sigma} \cup \{p_0\})$, for any $\vec{\sigma} \in \mathcal{P}(\mathscr{B})$,

Proof. If we ignore the center and assume $\underline{G}$ semisimple and simply connected, we have that $p_0 = [p_{0,(1)}] \otimes \cdots \otimes [p_{0,(n)}]$, each $p_{0,(i)} \in \mathscr{B}_{(i)}$ is special and G_i acts by type-preserving automorphisms. Also, $G = G_1 \times \cdots \times G_n$ acts componentwise in the tensor product. Hence, item (a) follows from Corollary 5.4.8(a) applied to equations (5.13a) and (5.13b).

For item (b), given $\vec{\sigma} = \sigma_{(1)} \otimes \cdots \otimes \sigma_{(n)}$ and an apartment $A \supseteq (\{p_0\} \cup \vec{\sigma})$, we have $A = A_{(1)} \times \cdots \times A_{(n)}$. By Corollary 5.4.8(b), $\operatorname{supp} \gamma_{(i)}(\sigma_{(i)}) \subseteq A_{(i)}$ for each factor, and as $\gamma_{(i)}[\emptyset_{(i)}] = [p_{0,(i)}] \in A_{(i)}$, it follows that $\operatorname{supp} \gamma(\vec{\sigma}) \subseteq A$.

Taking the center into account, for (a), the action of the center is by translations and it preserves apartments, so its intersection with G_{p_0} must act trivially, so the proof of (a) is similar. As for (b) following [71] (see the proof of their Theorem 2.2) we define $\operatorname{hull}(\vec{\sigma} \cup \{p_0\})$ to be the product of the hull of the projection of $\vec{\sigma} \cup \{p_0\}$ to $\mathscr{B}((\underline{G}, \underline{G}), F)$ and the smallest box $\prod_{i=1}^{m} [a_{(i)}, b_{(i)}]$, with one of $a_{(i)}$ or $b_{(i)}$ being 0, that contains the projection of $\vec{\sigma} \cup \{p_0\}$ onto $\mathscr{B}(\underline{T}_c, F)$. It follows that (b) also holds in this case (cf. Example 5.4.5). $\square$

Assuming Conjecture 5.4.4, then, in the proposition above, we also have:

(c) $\gamma(\vec{\sigma}) = \sum_{\vec{\tau}} n_{\vec{\tau}\vec{\sigma}} \vec{\tau}$ with $|n_{\vec{\tau}\vec{\sigma}}| \leq 1$ for all $\vec{\tau}, \vec{\sigma} \in \mathcal{P}(\mathscr{B})$.

Proof of (c). Equations (5.13a) and (5.13b) yield that $\gamma(\vec{\sigma})$ is a finite linear combination of elements of the type

$$\tau_{(1)} \otimes \cdots \otimes \sigma_{(n)}, \quad [p_{0,(1)}] \otimes \cdots \otimes [p_{0,(j-1)}] \otimes \tau_{(j)} \otimes \cdots \otimes \sigma_{(n)}. \tag{5.16}$$

If we assume that each $\gamma_{(j)}$ satisfy the property that $n_{\tau(j)\sigma(i)} \in \{-1, 0, 1\}$, for all j, the elements in (5.16) are all linearly independent, and thus the result. $\square$

5.7 A remark on minimal coset representatives

This section is completely independent from the rest of the chapter. During the early stages of the development of a systematic approach to construct Morse functions on an Euclidean building, the author was certain that such a Morse function should be explicitly related to a notion of *distance* on Coxeter complexes. This belief was partly motivated by the following passage of Milnor's beautiful book [64, §6, p. 32], when the problem of existence of good Morse functions on a manifold M embedded in $\mathbb{R}^n$ is addressed:

> " In fact, if for fixed $p \in \mathbb{R}^n$ define the function $L_p : M \to \mathbb{R}$ by $L_p(q) = \|p - q\|^2$. It will turn out that for almost all p, the function L_p has only non-degenerate critical points."

Moreover, the *combinatorial distance* was intended to be a key ingredient in the construction of Morse functions. The end result, described in Section 5.3, is in terms of inner products with respect to a fixed vector, a *height* function, so it turned out to be different than the initial guess. Nevertheless, it made possible the remark we wish to make in this section, in Proposition 5.7.2, below. As we remarked in the previous section, in [7] this approach suggested by the quote on Milnor's book was elegantly carried out by M. Bestvina and G. Savin.

Let (E, Σ) be the Coxeter complex associated to an affine Coxeter system (W, S), attached to the pair (Ψ_0, Π_0) consisting of a root system on E and a choice of simple roots, as described in Example 5.2.4 and let $C_\emptyset$ be the corresponding fundamental chamber. Given a vertex x of Σ, we can always write $x = vx_0$ with $x_0 \in \overline{C_\emptyset}$ and $v \in W$. The choice of v is unique up to right multiplication by an element of the stabilizer of x_0. It is well-known that there exists a maximal proper subgroup $J \subsetneq S$ such that the stabilizer is the parabolic subgroup W_J generated by the reflections in J. It follows that the chambers in $\mathrm{St}([x])$ are $\{w\overline{C_\emptyset} \mid w \in vW_J\}$, indexed by the coset vW_J.

The **combinatorial distance from $C_\emptyset$ to x**, denoted by $d(C_\emptyset, x)$, is defined to be the length of any minimal gallery connecting $C_\emptyset$ to x. Such a length is independent of the minimal gallery chosen, and, in fact, it equals the length of the minimal coset representative of vW_J. Minimal coset representatives were extensively studied in, for example [97]. Let w^x denote this minimal coset representative and let $\ell : W \to \mathbb{N}$ denote the length function on W. If we were to use the combinatorial distance to create a Morse function on $\Sigma(W, S)$, it would important to be able to compute it only using information about the neighborhood of x, that is, the $\mathrm{St}([x])$. The next proposition will give an answer to how to determine the length of a minimal coset representative of a vertex in the Coxeter complex. Although we believe such a result is not in the literature, it is too simple to be coined a theorem. But before that, we need a preliminary result. Let $\mathcal{S} \subseteq W$ be a subset of W, not necessarily a subgroup. Denote by $P_\mathcal{S}(t)$ the **Poincaré polynomial** of $\mathcal{S}$,

$$P_\mathcal{S}(t) = \sum_{w \in \mathcal{S}} t^{\ell(w)}. \tag{5.17}$$

The Poincaré polynomial carries some information about the set $\mathcal{S}$, for example:

Lemma 5.7.1. *Let $P_\mathcal{S}(t)$ and $P'_\mathcal{S}(t)$ be the the Poincaré polynomial of $\mathcal{S}$ and its derivative, respectively, with $|\mathcal{S}|$ finite. Then,*

- $|\mathcal{S}| = P_\mathcal{S}(1)$;
- $\sum_{w \in \mathcal{S}} \ell(w) = P'_\mathcal{S}(1)$. □

Proposition 5.7.2. *Let x be a vertex of $\Sigma(W, S)$, with $x = vx_0$ for $v \in W$ and a unique vertex x_0 of $C_\emptyset$. Let also w^x be the minimal coset representative of vW_J, where W_J is the stabilizer of x_0. Then,*

$$\ell(w^x) = \frac{P'_{vW_J}(1) - P'_{W_J}(1)}{P_{W_J}(1)}. \tag{5.18}$$

Proof. We can write $v = w^x v_J$, for a unique $v_J \in W_J$. In fact, any $u \in vW_J$ can be written as $u = w^x u_J$ for a unique $u_J \in W_J$ and, moreover, it holds that

$\ell(u) = \ell(w^x) + \ell(u_J)$ [42, Section 5.12]. Hence,

$$\begin{aligned} P'_{vW_J}(1) &= \sum_{w \in vW_J} \ell(w) \\ &= \sum_{w \in W_J} (\ell(w^x) + \ell(w)) \\ &= |W_J|\ell(w^x) + \sum_{w \in W_J} \ell(w). \end{aligned}$$

From Lemma 5.7.1, $|W_J| = P_{W_J}(1)$ and $\sum_{w \in W_J} \ell(w) = P'_{W_J}(1)$. Thus,

$$P'_{vW_J}(1) = P_{W_J}(1)\ell(w^x) + P'_{W_J}(1),$$

and thus the result. $\square$

Chapter 6

Some remarks on triple spaces

In this brief chapter, we propose to study the following: let $\underline{G}$ be a split, connected, reductive linear algebraic group over an arbitrary field F, $\underline{T}$ a maximal torus and $\underline{B} \subseteq \underline{G}$ a Borel containing $\underline{T}$. This data determines a root datum $\mathcal{R}(\underline{G}, \underline{B}, \underline{T})$ with root system Φ_0, set of simple roots Δ_0 and finite Weyl group W_0. Let also $\underline{P}_1, \underline{P}_2$ and $\underline{P}_3$ be three parabolic subgroups of $\underline{G}$ containing $\underline{B}$. Let us denote the groups of F-rational points without the underline. Recall that because the P_i's are standard parabolics of G there exist subsets $J_i \subseteq \Delta_0$ that generates subgroups W_i of W_0, for $i = 1, 2, 3$, in such a way that $P_i = BW_iB$ (note that $W_i = \{e\}$ if $P_i = B$). Recall also that the double coset spaces $P_i \backslash G/P_j$, for $i, j \in \{1, 2, 3\}$ can be represented by elements in $W_i \backslash W_0/W_j$ (see, for example, [19, Proposition 1.3.1]). Now, consider the triple space

$$\mathbb{T} := G\backslash(G/P_1 \times G/P_2 \times G/P_3),$$

where G acts diagonally. In the next proposition, we write a square brackets to indicate a representative in a quotient.

Proposition 6.0.1. *The set-theoretic map* $f\colon G\backslash(G/P_1 \times G/P_2 \times G/P_3) \to P_3\backslash G/P_1 \times P_3\backslash G/P_2$, *given by*

$$f([x_1, x_2, x_3]) = [x_3^{-1}x_1, x_3^{-1}x_2],$$

induces a surjection from $\mathbb{T} \to W_3\backslash W_0/W_1 \times W_3\backslash W_0/W_2$. *The inverse image of* $[v, w] \in W_3\backslash W_0/W_1 \times W_3\backslash W_0/W_2$ *is given by*

$$f^{-1}([v, w]) = \{[v, bw, 1]; b \in (P_3)_{v,w} := (P_3 \cap vP_1v^{-1})\backslash P_3/(P_3 \cap wP_2w^{-1})\}.$$

Corollary 6.0.2. *A complete set of representatives of the orbit space* $\mathbb{T} =$

$G\backslash(G/P_1 \times G/P_2 \times G/P_3)$ is given by

$$\bigcup_{(v,w)} \left(\bigcup_{b \in (P_3)_{v,w}} (v, bw, 1) \right),$$

where (v,w) runs through representatives of $W_3\backslash W_0/W_1 \times W_3\backslash W_0/W_2$.

Corollary 6.0.3. *Let $F_\bullet = \mathbb{F}_q((\varpi))$, $G_\bullet$, $I_\bullet$ and $W = X_*(\underline{T}) \rtimes W_0$ be as in the Introduction. A complete set of representatives of the orbit space $G_\bullet\backslash G_\bullet^3/I_\bullet^3$ is given by*

$$\bigcup_{[v,w]} \left(\bigcup_{b \in (P_3)_{v,w}} (v, bw, 1) \right),$$

where $(v,w) \in W_0 \times W_0$.

6.1 Proofs

The corollaries are immediate from the Proposition. As for the Proposition, first notice that f is well defined because the multiplication $x_3^{-1}x_i$ cancels the diagonal factor of G. Given v, w, choose lifts in G that represents them, and note that $f([v,w,1]) = [v,w]$. All is left is to determine the fiber. Given $[x_1, x_2, x_3] \in \mathbb{T}$, the condition that $f([x_1, x_2, x_3]) = [v,w]$ implies that $x_1 \in x_3P_3vP_1$ and $x_2 \in x_3P_3wP_2$. Hence, any element on the fiber of $[v,w]$ can be written as

$$[x_1, x_2, x_3] = [v, bw, 1],$$

for an element $b \in P_3$. This gives us that

$$f^{-1}([v,w]) = \bigcup_{b \in P_3} [v, bw, 1].$$

Now, given $b, b' \in P_3$, the condition $[v, bw, 1] = [v, b'w, 1]$ implies that there exists $g \in G$, $p_i \in P_i$ such that

$$(v, bw, 1) = (gvp_1, gb'wp_2, gp_3). \tag{6.1}$$

The third coordinate of equation (6.1) implies $g \in P_3$, whereas the first coordinate implies that $g \in vP_1v^{-1}$, hence $g \in P_3 \cap (vP_1v^{-1})$. Finally, the middle coordinate equation gives $wp_2w^{-1} \in P_3$ and $b = g\, b'\, (wp_2w^{-1})$, which means

$$b = b' \;\text{ on }\; (P_3 \cap vP_1v^{-1})\backslash P_3/(P_3 \cap wP_2w^{-1}).$$

6.2 Examples

Results towards a classification of triple flag varieties $\mathbb{T}$ with finite orbits (called of *finite type*) when $\underline{G} = GL_n$ were given in [60] and [61] in terms of quiver representations. Also, in [62], Matsuki describes some triple flag varieties of finite type for orthogonal groups. In the above mentioned papers the authors used the language of flags in a vector space to describe the orbits. Here, we wish to give a description of the orbits in terms of elements of the group.

Example 6.2.1. Let $\underline{G} = GL_2$ and let $\underline{B}$ be the set of upper triangular matrices, $\underline{T}$ the diagonal matrices and $F = \mathbb{C}$. Let us consider the triple space

$$\mathbb{T} := G\backslash(G/B \times G/B \times G/B).$$

Applying the theorem, we get that $\mathbb{T}$ is in bijection with the set

$$\bigcup_{v,w\in W_0} \{(v, bw, 1); b \in B_{v,w}\}.$$

The Weyl group is $W_0 = \{e, s\}$, with $s = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and we have to analyze the sets $B_{v,w}$. It is clear that if one of v or w is the identity, the double coset

$$B_{v,w} = (B \cap vBv^{-1}\backslash B/B \cap wBw^{-1})$$

will be be represented by the identity matrix, so the only relevant case is for $(v, w) = (s, s)$. In this case $B_{s,s}$ is equal to $T\backslash B/T$, which has as a set of representatives

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, u := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Indeed, observe that we have

$$B = \bigcup_{\xi\in F} \left(\begin{smallmatrix} 1 & \xi \\ 0 & 1 \end{smallmatrix}\right) T,$$

and hence, taking left cosets of T, yields to only consider $\xi = 0$ or $\xi \neq 0$. Explicitly, the 5 representatives of the orbit space are

$$\{(e, e, e), (e, s, e), (s, e, e), (s, s, e), (s, us, e)\}.$$

Example 6.2.2. Similarly to the example above, let us consider $\mathbb{T} := G\backslash(G/B\times G/B \times G/B)$ but for $G = SL_2(\mathbb{R})$ and B, T as before. Only the fiber over (s, s) is relevant ($s = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in this case), and we should understand $T\backslash B/T$. Just like in the last example, we have

$$B = \bigcup_{\xi\in F} \left(\begin{smallmatrix} 1 & \xi \\ 0 & 1 \end{smallmatrix}\right) T.$$

However in this time, since $T \cong \mathbb{R}^\times$ is disconnected the sign matters. Indeed, if $\xi > 0$, then

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} ab & ab^{-1}\xi \\ 0 & a^{-1}b^{-1} \end{pmatrix} \in B,$$

implies that all entries have the same sign, and hence the double coset $T\backslash G/T$ has three representatives

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, u^+ := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, u^- := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

and the 6 representatives of the orbit space in this case are

$$\{(e,e,e),(e,s,e),(s,e,e),(s,s,e),(s,u^+s,e),(s,u^-s,e)\}.$$

Things start to become complicated already when trying to compute the orbit spaces for GL_3, but it is still manageable "by hand". Note that from [60, Theorem 2.2], the triple space $\mathbb{T} = GL_n\backslash GL_n^3/B^3$ is *not* of finite type for $n > 3$.

Example 6.2.3. Now, let $G = GL_3$, $F = \mathbb{C}$, B the upper triangular matrices, T the diagonal maximal torus, $W_0 = \langle s_1, s_2\rangle \cong S_3$ with $s_1 = \left(\begin{smallmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{smallmatrix}\right)$ and $s_2 = \left(\begin{smallmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{smallmatrix}\right)$ and P the parabolic related with $W_2 = \langle s_2\rangle$. This parabolic has the shape $\left(\begin{smallmatrix} * & * & * \\ & * & * \\ & * & * \end{smallmatrix}\right)$. We are interested now in

$$\mathbb{T} = G\backslash(G/B \times G/B \times G/P).$$

From our result the orbit space is represented by

$$\bigcup_{v,w}\{(v,bw,1); b \in (P\cap vBv^{-1})\backslash P/(P\cap wBw^{-1})\},$$

with v, w running over $\{e, s_1, s_1s_2\}$, a set of representatives for $W_2\backslash W_0$. There are 9 cases to consider the fibers and the total number of orbits is 28. We give the answer, but we omit the details.

1. Case (e,e): 2 orbits $\{(e,e,e),(e,s_2,e)\}$.

2. Case (e,s_1): 2 orbits $\{(e,s_1,e),(e,s_2s_1,e)\}$.

3. Case (s_1,e): 2 orbits $\{(s_1,e,e),(e,s_2,e)\}$.

4. Case (e,s_1s_2): 2 orbits $\{(e,s_1s_2,e),(e,s_2s_1s_2,e)\}$.

5. Case (s_1s_2,e): 2 orbits $\{(s_1s_2,e,e),(s_2s_1,s_2,e)\}$.

To continue with the next cases, we will denote by $u_{ij}(\xi)$ the element that has 1 in the diagonal entries and $\xi \in F$ in the entry (i,j).

6. Case $((s_1,s_1)$: 4 orbits

$$\{(s_1,s_1,e),(s_1,u_{12}(1)s_1,e),(s_1,s_2s_1,e),(s_1,u_{12}(1)s_2s_1,e)\}.$$

7. Case (s_1,s_1s_2): 4 orbits

$$\{(s_1,s_1s_2,e),(s_1,u_{12}(1)s_1s_2,e),(s_1,s_2s_1s_2,e),(s_1,u_{12}(1)s_2s_1s_2,e)\}.$$

8. Case (s_1s_2, s_1): 4 orbits

$$\{(s_1s_2, s_1, e), (s_1s_2, u_{12}(1)s_1, e), (s_1s_2, s_2s_1, e), (s_1s_2, u_{12}(1)s_2s_1, e)\}.$$

9. Case (s_1s_2, s_1s_2): 6 orbits

$$\begin{aligned}\{&(s_1s_2, s_1s_2, e), (s_1s_2, u_{12}(1)s_1s_2, e), (s_1s_2, u_{13}(1)s_1s_2, e),\\ &\quad (s_1s_2, s_2s_1s_2, e), (s_1s_2, u_{12}(1)s_2s_1s_2, e), (s_1s_2, u_{13}(1)s_2s_1s_2, e)\}.\end{aligned}$$

Bibliography

[1] P. Abramenko and K. S. Brown. *Buildings. Theory and Applications*, volume 248 of *Graduate Texts in Mathematics*. Springer, New York, USA, 2008.

[2] J. Arthur. Linear Algebraic Groups. In Societé Mathématique de France, editor, *Orbites Unipotentes et Répresentations, II*, volume 171-172, pages 13–71. Astérisque, 1989.

[3] T. Banchoff. Critical points and curvature for embedded polyhedra. *J. Diff. Geom.*, 1(3-4):245–256, 1967.

[4] D. Barbasch and D. Ciubotaru. Spherical unitary principal series. *Pure Appl. Math. Q.*, 1(4):755–789, 2005. Special Issue: In memory of Armand Borel.

[5] M. Bestvina. PL Morse theory. *Math. Comm.*, 13(2):149–162, December 2008.

[6] M. Bestvina and G. Savin. Affine buildings for SL_n via PL Morse theory. Unpublished manuscript, 2012.

[7] M. Bestvina and G. Savin. Nice Morse functions for affine buildings. Unpublished manuscript, 2015.

[8] A. Borel. Some finiteness properties of adele groups over number fields. *Publ. Math. de l'IHES*, 16:5–30, 1963.

[9] A. Borel and H. Jacquet. Automorphic forms and automorphic representations. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 189–207, Corvalis, USA, 1979. Amer. Math. Soc., Providence, R.I.

[10] N. Bourbaki. *Groupes et algèbres de Lie, Chapitres 4, 5 et 6.* Hermann, Paris, France, 1968.

[11] G. E. Bredon. *Topology and Geometry*, volume 139 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, USA, 1991.

[12] K. S. Brown. *Buildings*. Springer, New York, USA, 1989.

[13] F. Bruhat and J. Tits. Groupes réductifs sur un corps local: I. donées radicielles valuées. *Publications Mathématiques de l'IHÉS*, 41:5–251, 1972.

[14] D. Bump. *Automorphic forms and representations*, volume 55 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, UK, 1997.

[15] D. Bump and Y. Choie. Schubert eisenstein series. *Amer. Jour. Math.*, 136(6):1581–1608, December 2014.

[16] R.W. Carter. *Finite Groups of Lie Type*. Wiley series in pure and applied mathematics. John Wiley and Sons, New York, USA, 1985.

[17] P. Cartier. Representatons of p-adic groups: A survey. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 111–155, Corvalis, USA, 1979. Amer. Math. Soc., Providence, R.I.

[18] W. Casselman. The unramified principal series of p-adic groups. I. The spherical function. *Compositio Math.*, 40(3):387–406, 1980.

[19] W. Casselman. Introduction to the theory of admissible representations of p-adic reductive groups. Draft: 1 May 1995, 1995.

[20] C. Chevalley. Sur certains groupes simples. *Tôhoku Math. J.*, 7(1-2):14–66, 1955.

[21] D. Ciubotaru, D. Kato, and S. Kato. On characters and formal degrees for classical affine Hecke algebras. *Invent. Math.*, 187(3):589–635, July 2012.

[22] L. Clozel. Spectral theory of automorphic forms. In P. Sarnak and F. Shahidi, editors, *Automorphic Forms and Applications*, pages 41–94. AMS/Institute for Advanced Study, 1997.

[23] J. Cogdell. Lectures on L-functions. converse theorems and functoriality of $GL(N)$. In *Lectures on Automorphic L-functions*, number 20 in Fields Institute Monographs, pages 5–97. AMS, Providence, USA, 2004.

[24] D. H. Colingwood and W. M. McGovern. *Nilpotent Orbits In Semisimple Lie Algebra*. Van Nostrand Reinhold, New York, USA, 1993.

[25] P. Delorme and E. Opdam. The Schwartz algebra of an affine Hecke algebra. *J. Reine Angew. Math.*, 625:59–114, 2008.

[26] E. Dynkin. Semisimple subalgebras of semisimple Lie algebras. *Amer. Math. Soc. Transl.*, 6:111–245, 1957.

[27] S. Eilenberg and J. A. Zilber. On products of complexes. *Amer. Jour. Math.*, 75(1):200–204, 1953.

[28] A. Borel et al. *Seminar on algebraic groups and related finite groups*. Lecture Notes in Mathematics. Springer-Verlag, Berlin-Heidelberg, Germany, 1970.

[29] D. Flath. Decomposition of representations into tensor products. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 179–183, Corvalis, USA, 1979. Amer. Math. Soc., Providence, R.I.

[30] P. Garrett. *Buildings and Classical Groups*. Chapman & Hall, London, UK, 1997.

[31] S. G. Gindikin and F. I. Karpelevič. Plancherel measure for symmetric Riemannian spaces of non-positive curvature. *Dokl. Akad. Nauk SSSR*, 145:252–255, 1962.

[32] E. Gutkin. Integrable systems with delta function potential. *Duke Math. J.*, 49:1–12, 1982.

[33] Harish-Chandra. *Automorphic forms on semi-simple Lie groups*. Number 62 in Lecture Notes in Mathematics. Springer, Heidelberg, Germany, 1968.

[34] A. Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, UK, 2002.

[35] G. J. Heckman and E. M. Opdam. Harmonic analysis for affine Hecke algebras. In R. Bott, A. Jaffe, D. Jerison, G. Lusztig, I. Singer, and S.-T. Yau, editors, *Current Developments in Mathematics*, pages 37–60. International Press, 1996.

[36] G. J. Heckman and E. M. Opdam. Yang's system of particles and Hecke algebras. *Annals of Mathematics*, 1(145):139–173, Jan 1997.

[37] V. Heiermann. Décomposition spectrale et représentations spéciales d'un groupe réductif p-adique. *J. Inst. Math. Jussieu*, 3(3):327–395, 2004.

[38] V. Heiermann. Orbites unipotentes et pôles d'ordre maximal de la fonction μ de Harish-Chandra. *Canad. J. Math.*, 58:1203–1228, 2006.

[39] S. Helgason. *Groups and Geomretric Analysos*. Pure and Aplied Mathematics. Academic Press, Orlando, USA, 1984.

[40] K. Hiraga, K. Ichino, and T. Ikeda. Formal degrees and adjoint gamma factors (and errata). *J. Amer. Math. Soc.*, 21(1):283–304, 2008.

[41] L. Hörmander. *The analysis of partial differential operators I*. Springer-Verlag, Heidelberg, Germany, 1990.

[42] J. E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge University Press, Cambridge, UK, 1990.

[43] J. E. Humphreys. *Linear Algebraic Groups*, volume 21 of *GTM*. Springer-Verlag, New York, USA, 1991.

[44] J. E. Humphreys. *Conjugacy Classes in Semisimple Algebraic Groups*, volume 43 of *Mathematical Surveys and Monographs*. AMS, Providence, USA, 1995.

[45] J.E. Humphreys. *Arithmetic Groups*, volume 189 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-Heidelberg, Germany, 1980.

[46] H. Jacquet. On the residual spectrum of $gl(n)$,. In R. Herb, S. Kudla, R. Lipsman, and J. Rosenberg, editors, *Lie groups and Representations II, Proceedings of the Special Year held at the University of Maryland, College Park 1982–1983*, volume 1041 of *Lecture Notes in Math*, pages 185–208. Springer-Verlag, 1984.

[47] J. Jorgenson and S. Lang. *The heat kernel and theta inversion on* $SL2(\mathbb{C})$. Springer Monographs in Mathematics. Springer-Verlag, New York, USA, 2008.

[48] D. Kazhdan and G. Lusztig. Proof of the Deligne-Langlands conjecture for Hecke algebras. *Invent. Math.*, 87:153–215, 1987.

[49] H. H. Kim. The residual spectrum of G_2. *Canad. J. Math.*, 48:1245–1272, 1996.

[50] H. H. Kim. Euler-Poincaré pairings and elliptic representations of Weyl groups and p-adic groups. *Compositio Math*, 129(2):149–182, 2001.

[51] H. H. Kim. Residual spectrum of odd orthogonal groups. *Int. Math. Res. Notices*, 17:873–906, 2001.

[52] L. Kramer. Metric properties of euclidean buildings. In *Global Differential Geometry*, number 17 in Proceedings in Mathematics, pages 147–159. Springer, Heidelberg, Germany, 2012.

[53] C. Kriloff and A. Ram. Representations of graded Hecke algebras. *Duke Math. Journal*, 6:31–69, May 2002.

[54] R. P. Langlands. The volume of the fundamental domain for some arithmetical subgroups of Chevalley groups. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, Proc. Sympos. Pure Math., IX, pages 143–148. Amer. Math. Soc., Providence, R.I., 1966.

[55] R. P. Langlands. *On the functional equations satisfied by Eisenstein series*, volume 544 of *Lecture Notes in Mathematics*. Springer-Verlag, New York, USA, 1976.

[56] J. M. Lee. *Introduction to topological manifolds*, volume 202 of *GTM*. Springer, New York, USA, 2nd edition, 2011.

[57] G. Lusztig. Affine Hecke algebras and their graded version. *J. Am. Math. Soc.*, 2(3):599–635, July 1989.

[58] G. Lusztig. Cells in affine Weyl groups IV. *J. Fac. Sci. Tokyo Un.*, 36:297–328, 1989.

[59] I. G. MacDonald. *Spherical functions on a group of p-adic type.* Publ. Ramanujan Institute, Madras, India, 1971.

[60] P. Magyar, J. Weyman, and A. Zelevinsky. Multiple flag varieties of finite type. *Adv. Math.*, 141(1):97–118, January 1999.

[61] P. Magyar, J. Weyman, and A. Zelevinsky. Symplectic multiple flag varieties of finite type. *J. Alg.*, 230(1):245–265, August 2000.

[62] T. Matsuki. An example of orthogonal triple flag variety of finite type. *J. Alg.*, 375(1):148–187, February 2013.

[63] S. D. Miller. Residual automorphic forms and spherical unitary representations of exceptional groups. *Annals of mathematics*, 177:1169–1179, 2013.

[64] J. Milnor. *Morse Theory.* Princeton University Press, Princeton, NJ, 1963.

[65] C. Moeglin. Orbites unipotentes et spectre discret non ramifie. Le cas des groupes classiques déployés. *Compositio Math*, 77:1–54, 1991.

[66] C. Moeglin and J.-L. Waldspurger. Le spectre résiduel de $gl(n)$. *Ann. Sci. École Norm. Sup.*, 22:605–674, 1989.

[67] C. Moeglin and J.-L. Waldspurger. *Spectral Decomposition and Eisenstein Series*, volume 113 of *Cambridge tracts in Mathematics*. Cambridge University Press, Cambridge, UK, 1995.

[68] J. R. Munkres. *Topology*. Prentice Hall, New Jersey, USA, 2nd edition, 2000.

[69] J. Neukirch. *Algebraic Number Theory*, volume 322 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin-Heidelberg, Germany, 1999.

[70] E. Opdam and M. Solleveld. Homological algebra for affine Hecke algebras. *Adv. in Math*, 220(5):1549–1601, March 2009.

[71] E. Opdam and M. Solleveld. Resolutions of tempered representations of reductive p-adic groups. *J Func. Anal.*, 265(1):108–134, July 2013.

[72] E. M. Opdam. A generating function for the trace of the Iwahori-Hecke algebra. *Prog. Math.*, 210:301–323, 2003.

[73] E. M. Opdam. On the spectral decomposition of affine Hecke algebras. *J. Inst. Math. Jussieu*, 3(4):531–648, 2004.

[74] E. M. Opdam. The central support of the Plancherel measure of an affine Heke algebra. *Moscow Mathematical Journal*, 7(4):723–741, October-December 2007.

[75] E. M. Opdam. Spectral correspondences for affine Hecke algebras. arXiv:1310.7193, 2013.

[76] E. M. Opdam. Spectral transfer morphisms for unipotent affine Hecke algebras. arXiv:1310.7790, 2013.

[77] A. Parreau. *Dégénérescences des sous-groupes discrets de groupes de Lie semisimples et actions de groupes sur des immeubles affines*. PhD thesis, Université Paris-Sud, Orsay, 2000.

[78] V. Platonov and A. Rapinchuk. *Algebraic Groups and Number Theory*, volume 139 of *Pure and applied mathematics*. Academic Press, San Diego, USA, 1994.

[79] A. Prasad. *The almost unramified discrete spectrum for split semisimple groups over a rational function field*. PhD thesis, University of Chicago, 2001.

[80] A. Prasad. Almost unramified discrete spectrum for split groups over $\mathbb{F}_q(t)$. *Duke Math. Journal*, 113(2):237–257, 2002.

[81] A. Prasad. Almost unramified automorphic representations for split groups over $\mathbf{F}_q(t)$. *J. Algebra*, 262(1):253–261, 2003.

[82] D. Ramakrishnan and R. J. Valenza. *Fourier Analysis on Number Fields*, volume 186 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, USA, 1999.

[83] M. Reeder. Whittaker functions, prehomogeneous vector spaces and standard representations of p-adic groups. *J. Reine Angew Math*, 450:83–121, 1994.

[84] M. Reeder. Formal degrees and l-packets of unipotent discrete series representations of exceptional p-adic groups, with an appendix by frank luebeck. *Crelle's Journal*, 520:37–93, 2000.

[85] B. Rémy, A. Thuillier, and A. Werner. Bruhat-Tits buildings and analytic geometry. In A. Ducros, C. Favre, and J. Nicaise, editors, *Proceedings of the Paris summer school Berkovich spaces*, number 2119 in Lecture Notes in Mathematics, pages 141–202. Springer, Cham, Switzerland, 2015.

[86] P. Schneider and U. Stuhler. Resolutions for smooth representations of the general linear group over a local field. *J. Reine Angew. Math.*, 436:19–32, 1993.

[87] P. Schneider and U. Stuhler. Representation theory and sheaves on the Bruhat-Tits building. *Inst. Hautes Études Sci. Publ. Math.*, 85:97–191, 1997.

[88] F. Shahidi. *Eisenstein series and automorphic L-functions*, volume 58 of *American Mathematical Society Colloquium Publications*. AMS, Providence, USA, 2010.

[89] C. L. Siegel. The volume of the fundamental domain for some infinite groups. *Trans. Amer. Math. Soc.*, 39:195–202, 1936.

[90] M. Solleveld. Parabolically induced representations of graded Hecke algebras. *Algebr. Represent. Theor.*, 15(2):233–271, April 2012.

[91] T. A. Springer. Some arithmetical results on semi-simple lie algebras. *Publ. Math. de l'IHES*, 30(1):115–141, December 1966.

[92] T. A. Springer. Reductive groups. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 3–27, Corvalis, USA, 1979.

[93] T. A. Springer. Linear Algebraic Groups. In A. N. Parshin and I. R. Shafarevich, editors, *Algebraic Geometry IV. Linear Algebraic Groups Invariant Theory*, volume 55 of *Encyclopaedia of Mathematical Sciences*, pages 1–122. Springer-Verlag, Berlin-Heidelberg, Germany, 1994.

[94] T. A. Springer. *Linear Algebraic Groups*, volume 9 of *Progress in Mathematics*. Birkhauser, Boston, USA, 2nd edition, 1998.

[95] W. Stas. On the order of the Dedekind zeta-function near the line $\sigma = 1$. *Acta Arith.*, 35:195–202, 1979.

[96] R. Steinberg. On a theorem of Pittie. *Topology*, 14:173–177, 1975.

[97] F. Stumbo. Minimal length coset representatives for quotients of parabolic subgroups in Coxeter groups. *Boll. Unione Mat. Ital.*, 3-B(3):699–715, 2000.

[98] J. Tate. Fourier analysis in number fields and Hecke's zeta-functions. In J. W. S. Cassels and A. Frolich, editors, *Algebraic Number Theory*, pages 305–347. Academic Press, London, UK, 1967.

[99] J. Tits. Reductive groups over local fields. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 29–69, Corvalis, USA, 1979.

[100] J. Tits. Immeubles de type affine. In Luigi A. Rosati, editor, *Buildings and the Geometry of Diagrams*, number 1181 in Lecture Notes in Mathematics, pages 159–190. Springer-Verlag, Berlin-Heidelberg, Germany, 1986.

[101] E. P. van den Ban and H. Schlichtkrull. A residue calculus for root systems. *Compos. Math.*, 123:27–82, 2000.

[102] N. R. Wallach. *Real reductive groups I*, volume 132 of *Pure and Applied Mathematics*. Academic Press, San Diego, USA, 1988.

[103] A. Weil. *Basic number theory*. Springer-Verlag, New York-Berlin, 3rd edition, 1974. Die Grundlehren der Mathematischen Wissenschaften, Band 144.

[104] J. H. C. Whitehead. Combinatorial homotopy. i. *Bull. Amer. Math. Soc.*, 55:213–245, 1949.

[105] C. N. Yang. Some exact results for the many-body problem in one dimension with repulsive delta function interaction. *Phys. Rev. Letters*, 19:1312–1314, 1967.

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Samenvatting

Over het onvertakte sferische automorfe spectrum

De afgelopen jaren waren we geïnteresseerd in de volgende problemen: zij $\underline{G}$ een samenhangende reductieve lineaire algebraïsche groep, gedefiniëerd over een lichaam F, waarvan we voor het gemak aannemen dat het semisimpel is. Veronderstel eerst dat $\underline{G}$ een chevalleygroep is en F een globaal lichaam. Zoals gebuikelijk laten we, voor een plaats v van F, F_v de lokale completering zijn van F bij v. Voor elke niet-archimedische lokale plaats v krijgen we een maximale compacte (open) ondergroep $K_v := \underline{G}(\mathfrak{o}_v)$ van $G_v := \underline{G}(F_v)$, waar $\mathfrak{o}_v$ de de ring van gehelen van F_v is. Als v een archimedische plaats is, kiezen we een maximale compacte (gesloten) ondergroep K_v van G_v. Zij $\underline{G}(\mathbb{A})$ de groep van adelische punten van $\underline{G}$. Dit is aan lokaal compacte groep en de homogene ruimte $\underline{G}(F)\backslash\underline{G}(\mathbb{A})$ heeft eindig volume. Het eerste deel van de scriptie is gewijd aan het volgende klassieke thema in automorferepresentatietheorie:

(a) voor F een globaal lichaam, zij S de eindige verzameling van plaatsen die *niet* archimedisch zijn. Als $v \notin S$, zij $K'_v = K_v$ zoals hierboven. Als $v \in S$, zij $K'_v \subseteq K_v$ een compact-open ondergroep. We kunnen bijvoorbeeld $K'_v = I_v$ nemen, de iwahoriondergoep van G_v. Bekijk de compacte ondergroep $\mathbf{K}' = \prod_v K'_v$ van $\underline{G}(\mathbb{A})$. Wou zouden graag de ruimte $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}'}$ van $\mathbf{K}'$-invariante kwadratisch integreerbare functies op $\underline{G}(F)\backslash\underline{G}(\mathbb{A})$ willen bestuderen als moduul voor een globale heckealgebra $\mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K}')$, en ook de $\underline{G}(\mathbb{A})$-ruimte gegenereerd door deze $\mathbf{K}'$-invarianten. In het bijzonder willen we de representaties bepalen die disceet voorkomen in deze ruimtes.

Voor ons tweede probleem laten we de aanname dat $\underline{G}$ een chevalleygroep is vallen, maar we eisen nog steeds dat $\underline{G}$ samenhangend is. We bespreken:

(b) voor F een lokaal niet-archimedisch lichaam willen een een expliciete samentrekking van zijn bruhat-titsgebouw $\mathscr{B}(\underline{G}, F)$ construeren door middel van stuksgewijs lineaire morsetheorie.

Het eerste probleem is ingebed in een algemener raamwerk waar $\underline{G}$ mogelijk niet splijt ver het globale lichaam F. De ruimte $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))$ van kwadratisch integreerbare automorfe vormen is een centraal object in de theorie van automorfe vormen en zijn relatie met representatietheorie en getaltheorie via het langlandsprogramma. Langlands gaf een abstracte decompositie van deze ruimte in termen van klassen van cuspidal data $\mathfrak{X}$, waarin elke $\mathfrak{X} := [\underline{M}, \mathfrak{P}]$ een equivalentieklasse is van paren $(\underline{M}, \mathfrak{P})$, een parabolisch datum genaamd, met $\underline{M}$ een F-leviondergroep van $\underline{G}$ en $\mathfrak{P}$ een baan van zekere karakterdraaiïngen van een cuspidal automorfe representatie van $\underline{M}$:

$$L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A})) = \hat{\oplus}_{\mathfrak{X}}\, L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}.$$

Deze ruimes hebben een actie van de groep $\underline{G}(\mathbb{A})$ door middel van de rechter reguliere representatie. Een fundamenteel probleem in de theorie van automorfe vormen is het verder ontbinden van elke sommand $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}$ als een hilbertintegraal van irreducibele, unitaire $\underline{G}(\mathbb{A})$-representaties. Als we een compacte ondergroep $\mathbf{K}' \subseteq \underline{G}(\mathbb{A})$ kiezen als hierboven in de decompositie van $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}$, kunnen we $\mathbf{K}'$-bolvormige representaties bestuderen die voorkomen in de drager van de spectrale maat voor de decompositie van deze $\underline{G}(\mathbb{A})$-ruimte. Dit kan gedaan worden door de ruimte $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}^{\mathbf{K}'}$ te bekijken, welke een moduul is voor een globale heckealgebra die gerelateerd is aan het paar $(\underline{G}(\mathbb{A}), \mathbf{K}')$.

De irreducibele gesloten deelruimtes van $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))$ heten discretereeksrepresentaties. Onder de discretereeksrepresentaties bevinden zich de cuspidal automorfe representaties (die waarvan de ruimte een cuspidal automorfe vorm van $\underline{G}(\mathbb{A})$ bevat, oftewel ze komen voor in een $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}$ met $\mathfrak{X} = [\underline{G}, \mathfrak{P}]$ en $\mathfrak{P}$ een éénpuntsverzameling) en het residuele spectrum (die discretereeksrepresentaties die voorkomen in een $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{[\underline{M},\mathfrak{P}]}$ met $\underline{M} \neq \underline{G}$). Eén van de belangrijkste successen van Langlanss was het bewijs dat de representaties in het residuele spectrum overeenkomen met bepaalde residuen van eisensteinreeksen via een ingewikkelde contourverschuiving.

Laat ons nu de resultaten beschrijven die we hebben gevonden. Beschouw bij probleem (a) een getallenlichaam F, een chevalleygroep $\underline{G}$ en zij $\mathbf{K}' = \mathbf{K} = \prod_v K_v$, met $K_v = \underline{G}(\mathfrak{o}_v)$ voor alle niet-archimedische plaatsen v. Beschouw in deze situatie de klasse van cuspidal data $\mathfrak{X} = [\underline{T}, \imath]$, waar $\underline{T}$ een vaste maximale torus is die splijt over F en $\imath$ een baan van het triviale karakter van $\underline{T}(F)\backslash\underline{T}(\mathbb{A})$ is. We zijn geïnteresseerd in het residuele spectrum gerelateerd aan $[\underline{T}, \imath]$. Zij $X^*(\underline{T})$ het karakterrooster van $\underline{T}$ en zij a_T^* de ruimte $\mathbb{R} \otimes X^*(\underline{T})$, met wortelsysteem $\Phi_0 = \Phi(\underline{G}, \underline{T}) \subseteq a_T^*$. Zij ook $a_{T,\mathbb{C}}^*$ zijn complexificatie en W_0 de weylgroep van Φ_0. De belangrijkste resultaten over probleem (a) gaan over de ontbinding van de automorfe ruimte $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{[\underline{T},\imath]}^{\mathbf{K}}$, in termen van de ontbinding van een lokaal moduul. In meer detail, zij ${}^L G$ de complexe (samenhangende) duale groep van $\underline{G}$ en zij ${}^L\mathfrak{g}$ zijn liealgebra. Kies een vaste cartaninvolutie θ en een corresponderend maximale compacte ${}^L K$ van ${}^L G$ zo dat ia_T^* de liealgebra is van een maximale torus van een compacte vorm van ${}^L\mathfrak{g}$. Dan hebben we dat θ als de identiteit werkt op ia_T^* en als min de identiteit op a_T^*, waar $a_{T,\mathbb{C}}^* = a_T^* + ia_T^*$. Bekijk nu de ruimte P^{su}, die we met reden de ruimte van "bolvormige niet-geramificeerde parameters" noemen, die bestaat uit ${}^L G$-conjugatieklassen van paren $\{(\phi, \lambda)\}$ die de volgende eigenschappen hebben: (1) $\phi : \mathfrak{sl}_2 \to {}^L\mathfrak{g}$ is een liealgebrahomomorfisme, (2) $\lambda \in {}^L\mathfrak{g}$ is een semisimpel element, (3) $[\lambda, \phi(x)] = 0$ voor alle $x \in \mathfrak{sl}_2$, (4) $\theta(\lambda) = \lambda$ en (5) $\theta \circ \phi \circ \theta_{\mathfrak{sl}_2} = \phi$, waar $\theta_{\mathfrak{sl}_2}$ de cartaninvolutie $x \mapsto -x^T$ is van $\mathfrak{sl}_2$. We beargumenteren dat deze ruimte in bijectie is met een vereniging van affiene deelruimtes van $a_{T,\mathbb{C}}^*$ modulo de werking van W_0 en we definiëren op de ruimte P^{su} een expliciete maat μ_{aut}. Dan hebben we:

Stelling A. *We hebben een isometrisch isomorfisme*

$$L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}}_{[\underline{T},\mathbf{1}]} \cong L^2(P^{\mathrm{su}}, \mu_{\mathrm{aut}}).$$

Bovendien is dit isomorfisme expliciet gegeven in termen van een fouriertransformatie die de natuurlijke werking van de globale bolvormige heckealgebra $\mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K})$, het tensorproduct van de lokale bolvormige heckealgebra's $\mathcal{H}(G_v, K_v)$ in elke plaats v, diagonaliseert.

Om het bovenstaande isomorfisme te verkrijgen, moeten we een contourverplaatsing zoals die van Langlands gebruiken. We worden gedwongen rekening te houden met polen die geïnduceerd worden door de nulpunten van de dedekindzetafunctie, een globaal ingrediënt. Desalniettemin kunnen we, in het huidige niet-geramificeerde en **K**-bolvormige geval, het probleem reduceren tot een residuberekening in een lokale context, namelijk die van de spectrale decompositie van het yangsysteem, een moduul over de gegradeerde affiene heckealgebra. Dit is bestudeerd door G. Heckman en E. Opdam.

De ruimte $P^{\mathrm{su}}_{\mathrm{dist}}$ van geïsoleerde punten op P^{su} bestaat uit paren $(\phi, 0)$, waar ϕ het jacobson-morozov-$\mathfrak{sl}_2$-homomorfisme is van een achtenswaardige baan $\mathbf{o}$ van ${}^L G$. Hieruit volgt dat het discrete deel van $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}}_{[\underline{T},\mathbf{1}]}$ geparametriseerd wordt door die achtenswaardige unipotente banen van de duale groep. Voor zo een gegeven baan, zij $2\lambda(\mathbf{o}) \in a_T^*$ het bijbehorende gewogen dynkindiagram. Laat, voor elke lokale plaats v, $\pi_{v,\lambda(\mathbf{o})}$ de unieke irreducibele bolvormige onderquotient van de niet-geramificeerde minimale hoofdreeks van G_v bij de parameter $\lambda(\mathbf{o})$ zijn en zet $\pi_{\lambda(\mathbf{o})} = \otimes_v \pi_{v,\lambda(\mathbf{o})}$. Uit stelling A kunnen we concluderen dat:

Stelling B. *Elke $\underline{G}(\mathbb{A})$-representatie $\pi_{\lambda(\mathbf{o})}$ komt voor in het discrete spectrum van $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{[\underline{T},\mathbf{1}]}$ met multipliciteit één.*

Stelling C. *De bolvormige irreducibele representaties $\pi_{v,\lambda(\mathbf{o})}$ zijn unitariseerbaar voor elke lokale plaats v.*

Ook kunnen we nu met behulp van de isometry van Stelling A) normen van vectoren aan beide kanten vergelijken om een aantal niet-triviale identiteiten de verkrijgen. Eén hiervan reproduceert een resultaat van Langlands over het volume van $\underline{G}(F)\backslash\underline{G}(\mathbb{A})$ en een ander de kwadratische norm van een zekere $\mathbf{K}$-bolvormige vectoren gegeven als de kwadratisch-integreerbare residuen van eisensteinreeksen op de punten $\lambda(\mathbf{o})$ die hierboven genoemd zijn. We merken op dat in veel speicale klassen van reductieve groepen vergelijkbare resultaten on het $\mathbf{K}$-bolvormige automorfe spectrum zijn behaald door H. Jacquet, H. Kim, R. Langlands, S. Miller, C. Moeglin en J.-L. Waldspurger.

Om uit het formule van Stelling A betekenisvolle identiteiten te halen bij het vergelijken van normen aan beide kanten, moeten alle constanten in de spectrale maten expliciet gemaakt worden. In het bijzonder moeten we de exacte constanten van ν_P, de plancherelmaat van het yangsysteem berekenen, die nog niet beschikbaar waren toen G. Heckman en E. Opdam hun artikel schreven, maar die met een limietprocedure kunnen worden afgeleid uit de expliciete formele graadformules voor discretereekskarakter van iwohori-heckealgebra's. Deze limietprocedure wordt hier gedaan. Deze berekening is cruciaal voor de uitdrukking van de maat μ_{aut} van stelling A.

Stelling D. *Voor elke achtenswaardige baan* $\mathbf{o}$ *voldoet de plancherelmaat van het yangsysteem aan*

$$\nu_P(\{\lambda(\mathbf{o})\}) = \left| \frac{1}{\operatorname{vol}(X_*(\underline{T}))} \frac{|W_{\lambda(\mathbf{o})}|}{|A_{\mathbf{o}}|} \frac{\prod'_{\alpha\in\Phi_0}\langle\alpha^\vee,\lambda(\mathbf{o})\rangle}{\prod'_{\alpha\in\Phi_0}\langle\alpha^\vee,\lambda(\mathbf{o})\rangle+1} \right|.$$

Hier is $A_{\mathbf{o}}$ *de componentgroep van de centralisator van een element van* $\mathbf{o}$ *en* $W_{\lambda(\mathbf{o})} \subseteq W_0$ *is de stabilisator van* $\lambda(\mathbf{o})$.

Over probleem (b): het gebouw van een reductieve group $\underline{G}$ over een lokaal niet-archimedisch lichaam F is een belangrijk object dat gebruikt wordt om de repretentatietheorie van $G := \underline{G}(F)$ te begrijpen. Het is ook een studieobject an sich vanwege zijn vele symmetrieën en meetkundige eigenschappen. Zo is het een celcomplex (met als cellen polysimplices: eindige producten van simplices) met daarop een natuurlijke actie van G en is het een samentrekbare, complete metrische ruimte.

Door deze eigenschappen te verkennen, bestudeerden P. Schneider en U. Stuhler homologische resoluties voor gladde representaties van G. Bovendien bestudeerden E. Opdam en M. Solleveld resolutiens voor getemperde representaties met behulp van een expliciete samentrekking van het coxetercomplex.

In navolging van een idee van G. Savin en M. Bestvina voor het gebouw van SL_n, stellen we voor euclidische gebouwen te bestuderen, en in het bijzonder de complexen van hierboven, door middel van stuksgewijs lineaire morsetheorie. Zij $\mathscr{B} := \mathscr{B}(\underline{G}, F)$ het bruhat-titsgebouw van een samenhangende reductieve algebraische group gedefinieerd over een lokaal niet-archimedisch lichaam F. Gegeven een (poly-)simplex σ van $\mathscr{B}$, de ster van σ, $\operatorname{St}(\sigma)$, is de vereniging van ale (poly)simplices van $\mathscr{B}$ die σ bevatten, samen met hun zijvlakken. We laten zien dat:

Stelling E. *Op $\mathscr{B}$ bestaat een morsefunctie $\varphi : \mathscr{B} \to \mathbb{R}$ die een uniek globaal minimum heeft en en zodat voor elk (poly)simplex σ van $\mathscr{B}$, elk locaal minimum van φ op $\mathrm{St}(\sigma)$ is het globale minimum van φ op $\mathrm{St}(\sigma)$.*

De eigenschap in stelling E laat ons op een unieke wijze de richting van steilste afdaling te definiëren rond elke cel van $\mathscr{B}$ richting een gegeven speciaal punt in het gebouw. We zouden deze morsetheorieconstructie later willen toepassen om enige resultaten van P. Schneider en U. Stuhler, en E. Opdam en M. Solleveld, over Ext-groepen van getemperde representaties opnieuw te bekijken. Maar om dit te doen moet de samentrekking die we hier construeren goede grenzen in zijn coëfficienten hebben. Wij spreken het vermoeden uit dat dit inderdaad het geval is met de gemaakte constructies.

Trefwoorden: automorfe vormen, spectrale decompositie, residuberekening, harmonische analyse, affiene heckealgebra.

Résumé

Sur le spectre automorphe sphérique non-ramifié

Nous nous sommes intéressés au cours des dernières années aux problèmes suivants: soit $\underline{G}$ un groupe linéaire algébrique réductif et connexe défini sur un corps F, que nous supposons semi-simple, pour des raisons de simplicité. Supposons d'abord que $\underline{G}$ est un groupe de Chevalley et F est un corps global. Pour chaque place v de F, soit F_v le complété de F en v. Pour chaque place locale non archimédienne v, on obtient un sous-groupe compact maximal (et ouvert) $K_v := \underline{G}(\mathfrak{o}_v)$ de $G_v := \underline{G}(F_v)$, où $\mathfrak{o}_v$ désigne l'anneau des entiers de F_v. Si v est une place archimédienne, on fixe un sous-groupe compact maximal (et fermé) K_v de G_v. Soit $\underline{G}(\mathbb{A})$ le groupe des points adéliques de $\underline{G}$. C'est un groupe localement compact et l'espace homogène $\underline{G}(F)\backslash\underline{G}(\mathbb{A})$ a un volume fini. La première partie de la thèse est consacrée au thème classique suivant dans la théorie des représentations automorphes:

(a) pour F un corps global, soit S un ensemble fini de places non archimédiennes. Si $v \notin S$ on pose $K'_v = K_v$ comme ci-dessus. Si $v \in S$ on a $K'_v \subseteq K_v$ un sous-groupe compact-ouvert. Par exemple, nous pourrions prendre $K'_v = I_v$, le sous-groupe d'Iwahori de G_v. Considérons le sous-groupe compact $\mathbf{K}' = \prod_v K'_v$ de $\underline{G}(\mathbb{A})$. Nous voulons étudier l'espace $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}'}$ des fonctions $\mathbf{K}'$-invariantes et de carré intégrable sur $\underline{G}(F)\backslash\underline{G}(\mathbb{A})$ comme un module pour une algèbre de Hecke globale $\mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K}')$, ainsi que le $\underline{G}(\mathbb{A})$-espace engendré par ces fonctions $\mathbf{K}'$-invariants. En particulier, nous souhaitons déterminer les représentations intervenant discrètement dans ces espaces.

Quant à notre deuxième problème, soit $\underline{G}$ un groupe réductif et connexe. Nous discutons:

(b) pour F un corps local non archimédien, nous voulons construire une contraction explicite de l'immeuble de Bruhat-Tits $\mathscr{B}(\underline{G}, F)$ au moyen de la théorie de Morse linéaire par morceaux.

Le première problème s'insère dans un cadre plus général où $\underline{G}$ est non nécéssairement déployé sur le corps global F. L'espace $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))$ des formes automorphes de carré intégrable est un objet centrale dans la théorie des formes automorphes et en relation avec la théorie des représentations et la théorie des nombres via le programme de Langlands. Langlands a décrit une décomposition abstraite de cet espace en termes de classes de données cuspidales $\mathfrak{X}$, dans laquelle chaque $\mathfrak{X} := [\underline{M}, \mathfrak{P}]$ est une classe d'équivalence de paires $(\underline{M}, \mathfrak{P})$, que l'on appelle donnée cuspidale, avec $\underline{M}$, un sous groupe de Levi sur F de $\underline{G}$ et $\mathfrak{P}$ l'orbite de certaines représentations cuspidales tordues par des caractères de $\underline{M}$:

$$L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A})) = \hat{\oplus}_{\mathfrak{X}}\, L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}.$$

Le groupe $\underline{G}(\mathbb{A})$ agit par action régulière àï£¡ droite sur ces espaces. Un problème fondamental dans la théorie des formes automorphes consiste en la décomposition de chaque composante $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}$ sous la forme d'une intégrale de Hilbert de représentations irréductibles, unitaires de $\underline{G}(\mathbb{A})$. Si l'on choisit un sous-groupe compact $\mathbf{K}' \subseteq \underline{G}(\mathbb{A})$ comme ci-dessus; dans la décomposition de $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}$, on peut étudier le sous-espace des représentations $\mathbf{K}'$-invariantes qui apparaissent dans le support de la mesure spectrale pour la décomposition de ce $\underline{G}(\mathbb{A})$-espace. Pour le faire l'on considère l'espace $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}^{\mathbf{K}'}$ qui est un module pour l'algèbre de Hecke associée àï£¡ la paire $(\underline{G}(\mathbb{A}), \mathbf{K}')$.

Les sous-espaces irréductibles fermés de $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))$ sont les séries discrètes. Parmi les séries discrètes l'on trouve les représentations cuspidales automorphes (celles dont l'espace de représentation contient une forme cuspidale automorphe de $\underline{G}(\mathbb{A})$; c'est-àï£¡-dire celles qui apparaissent dans $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}$ avec $\mathfrak{X} = [\underline{G}, \mathfrak{P}]$ et $\mathfrak{P}$ un singleton) et le spectre résiduel (les séries discrètes qui apparaissent dans $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{[\underline{M},\mathfrak{P}]}$ avec $\underline{M} \neq \underline{G}$). L'un des principaux résultats de Langlands est que les représentations apparaissant dans le spectre résiduel correspondent àï£¡ certains résidus de séries d'Eisenstein via un changement de contour intégral compliqué.

Décrivons maintenant nos résultats. Concernant le problème (a), considérons un corps de nombres F, un groupe de Chevalley $\underline{G}$ est posons $\mathbf{K}' = \mathbf{K} = \prod_v K_v$, avec $K_v = \underline{G}(\mathfrak{o}_v)$ pour toutes les places non-archimédiennes v. Considérons alors la classe de la donnée cuspidale $\mathfrak{X} = [\underline{T}, \imath]$ où $\underline{T}$ est un F-tore déployé maximal et $\imath$ l'orbite du caractère trivial de $\underline{T}(F)\backslash\underline{T}(\mathbb{A})$. On s'intérèsse au spectre résiduel associé àï£¡ $[\underline{T}, \imath]$. On note $X^*(\underline{T})$ le réseau de caractères de $\underline{T}$ et a_T^* l'espace $\mathbb{R} \otimes X^*(\underline{T})$, ainsi que Φ_0 le système de racines tel que $\Phi_0 = \Phi(\underline{G}, \underline{T}) \subseteq a_T^*$. On note $a_{T,\mathbb{C}}^*$ sa complexification et W_0 le groupe de Weyl de Φ_0. Les résultats principaux relatifs au problème (a) concernent la décomposition de l'espace automorphe $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{[\underline{T},\imath]}^{\mathbf{K}}$ en termes de décomposition de modules locaux.

Plus exactement, posons ${}^LG$ le groupe dual complexe (connexe) de $\underline{G}$ et ${}^L\mathfrak{g}$ son algèbre de Lie. On se fixe un involution θ et un compact maximal correspondant ${}^LK$ de ${}^LG$, de telle façon que ia_T^* est l'algèbre de Lie d'un tore maximal de la forme compacte de ${}^L\mathfrak{g}$. On remarque alors que θ agit par l'identité sur ia_T^* et par (-1) sur a_T^*, où $a_{T,\mathbb{C}}^* = a_T^* + ia_T^*$. Maintenant, considérons l'espace P^{su}, que l'on appelle l'espace des paramètres sphériques non ramifiés, consistant en les ${}^LG$-classes de conjugaison de paires $\{(\phi, \lambda)\}$ satisfaisant les propriétés (1) $\phi : \mathfrak{sl}_2 \to {}^L\mathfrak{g}$ est un homorphisme d'algèbres de Lie, (2) $\lambda \in {}^L\mathfrak{g}$ est un élément semi-simple, (3) $[\lambda, \phi(x)] = 0$ pour tous les $x \in \mathfrak{sl}_2$, (4) $\theta(\lambda) = \lambda$ et (5) $\theta \circ \phi \circ \theta_{\mathfrak{sl}_2} = \phi$, où $\theta_{\mathfrak{sl}_2}$ est l'involution de Cartan $x \mapsto -x^T$ de $\mathfrak{sl}_2$. Nous montrons que cet espace est en bijection avec l'union des sous-espaces affines de $a_{T,\mathbb{C}}^*$ modulo l'action de W_0 et nous munissons cet espace P^{su} d'une mesure μ_{aut}. On a alors:

Théorème A. *On a un isomorphisme isométrique:*

$$L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}}_{[\underline{T},\mathbf{1}]} \cong L^2(P^{\mathrm{su}}, \mu_{\mathrm{aut}}).$$

De plus, cet isomorphisme est explicitement donné en termes d'une transformée de Fourier qui diagonalise l'action naturelle de l'algèbre de Hecke sphérique globale $\mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K})$ - produit tensoriel des algèbres de Hecke sphériques locales $\mathcal{H}(G_v, K_v)$ pour chaque place v.

Pour obtenir l'isomorphisme ci-dessus, l'on procède àï£¡ un changement de contour similaire àï£¡ celui proposé par Langlands. L'on doit prendre en compte les pôles induits par les zéros de la fonction zeta de Dedekind, qui est un ingrédient global. Cependant, dans le cas non-ramifié $\mathbf{K}$-sphérique considéré ici, on peut réduire le problème àï£¡ un calcul de résidus dans un contexte local, celui de la décomposition spectrale du système de Yang, ce dernier étant un module sur l'algèbre de Hecke affine. Celle-ci a été étudiée et décrite par G. Heckman et E. Opdam.

L'espace $P^{\mathrm{su}}_{\mathrm{dist}}$, des points isolés de P^{su}, consiste en des paires $(\phi, 0)$, pour lesquelles ϕ est le $\mathfrak{sl}_2$-homomorphisme de Jacobson-Morozov d'une orbite distinguée $\mathbf{o}$ de ${}^LG$. Il s'ensuit que la partie discrète de $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}}_{[\underline{T},\mathbf{1}]}$ est paramétrisée par les orbites unipotentes distinguées du groupe dual. Etant donnée une telle orbite, l'on pose $2\lambda(\mathbf{o}) \in a_T^*$ le weighted Dynkin diagram lui correspondant. Pour chaque place locale v, $\pi_{v,\lambda(\mathbf{o})}$ est l'unique sous-quotient irréductible sphérique de la série principale non-ramifiée minimale de G_v au paramètre $\lambda(\mathbf{o})$ et l'on pose $\pi_{\lambda(\mathbf{o})} = \otimes_v \pi_{v,\lambda(\mathbf{o})}$. A partir du Théorème A on peut conclure:

Théorème B. *Chaque $\underline{G}(\mathbb{A})$-représentation $\pi_{\lambda(\mathbf{o})}$ apparaît dans le spectre discret de $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{[\underline{T},\mathbf{1}]}$ sans multiplicité.*

Théorème C. *Les représentations irréductibles sphériques $\pi_{v,\lambda(\mathbf{o})}$ sont unitarisables, pour chaque place locale v.*

Par ailleurs, en utilisant l'isométrie du Théorème A, on peut comparer les normes des deux côtés pour obtenir des identités non triviales. Parmi celles-ci, l'on retrouve un résultat de Langlands sur le volume de $\underline{G}(F)\backslash\underline{G}(\mathbb{A})$ ou encore l'identité donnant la norme de certains vecteurs $\mathbf{K}$-sphériques au carré comme résidus de carré intégrables de séries d'Eisenstein au points $\lambda(\mathbf{o})$ mentionnés cidessus. On remarque que dans de nombreux cas particuliers de groupes réductifs des résultats similaires concernant le spectre automorphe $\mathbf{K}$-sphérique ont été obtenus par H. Jacquet, H. Kim, R. Langlands, S. Miller, C. Moeglin and J.-L. Waldspurger.

Pour que la formule du Théorème A produisent de telles identités lorsque l'on compare les normes de deux côtés, l'on doit expliciter toutes les constantes qui apparaissent dans les expressions de mesures spectrales. En particulier, l'on doit calculer les constantes dans l'expression de ν_P, la mesure de Plancherel du système de Yang, qui n'étaient pas disponibles lorsque G. Heckman et E. Opdam ont écrit leur article mais qui peuvent être calculées par un calcul de limite àï£¡ partir des expressions explicites des degrés formels pour les caractères des séries discrètes des algèbres de Hecke-Iwahori. L'on réalise cette procédure limite ici. Ce calcul est crucial pour dériver l'expression de la mesure μ_{aut} du Théorème A.

Théorème D. *Pour chaque orbite distinguée $\mathbf{o}$, la mesure de Plancherel du système de Yang satisfait:*

$$\nu_P(\{\lambda(\mathbf{o})\}) = \left| \frac{1}{\operatorname{vol}(X_*(\underline{T}))} \frac{|W_{\lambda(\mathbf{o})}|}{|A_{\mathbf{o}}|} \frac{\prod'_{\alpha\in\Phi_0}\langle\alpha^\vee,\lambda(\mathbf{o})\rangle}{\prod'_{\alpha\in\Phi_0}\langle\alpha^\vee,\lambda(\mathbf{o})\rangle+1} \right|.$$

Ici, $A_{\mathbf{o}}$ est le groupe des composantes connexes du centralisateur d'un élément de $\mathbf{o}$ et $W_{\lambda(\mathbf{o})} \subseteq W_0$ est le stabilisateur de $\lambda(\mathbf{o})$.

Quant au problème (b), l'immeuble d'un groupe réductif $\underline{G}$ sur un corps local non-archimédien F est un objet important utilisé pour comprendre la théorie des représentations de $G := \underline{G}(F)$. C'est également un objet d'étude en soi de part ses multiples propriétés géométriques et symétries. Parmi ces propriétés, celle d'être un complexe cellulaire (dont les cellules sont des polysimplexes, c'est-à-dire un produit fini de simplexes), muni d'une action naturelle de G et d'un espace métrique complet contractible.

En étudiant ces propriétés, P. Schneider et U. Stuhler ont étudié les résolutions homologiques des représentations lisses de G. En suivant, E. Opdam et M. Solleveld ont étudié ces résolutions pour des représentations tempérées en utilisant une contraction explicite du complexe de Coxeter.

En s'inspirant d'une idée de G. Savin et M. Bestvina pour les immeubles de SL_n, l'on se propose d'étudier les immeubles euclidiens, et en particulier les complexes ci-dessus, en utilisant la théorie de Morse linéaire par morceaux. Soit $\mathscr{B} := \mathscr{B}(\underline{G}, F)$ l'immeuble de Bruhat-Tits d'un groupe algébrique connexe défini sur un corps non-archimédien F. Etant donné un (poly)simplexe σ de $\mathscr{B}$, l'étoile de σ, notée $\mathrm{St}(\sigma)$, est l'union de tous les poly-simplexes de $\mathscr{B}$ qui contiennent σ, avec leurs facettes. On montre que:

Théorème E. *On peut munir $\mathscr{B}$ d'une fonction de Morse $\varphi : \mathscr{B} \to \mathbb{R}$ qui a un unique minimum global et satisfait à propriété suivante:*
Pour chaque (poly)simplexe σ de $\mathscr{B}$, chaque minimum local de φ sur $\mathrm{St}(\sigma)$ est l'unique minimum global de φ sur $\mathrm{St}(\sigma)$.

La propriété mentionnée dans le Théorème E nous permet de définir de façon unique la direction de descente la plus rapide autour d'une cellule quelconque de $\mathscr{B}$ vers un point fixe spécial dans l'immeuble. Nous souhaiterions prochainement appliquer cette construction donnée par la théorie de Morse afin de revisiter certains résultats de P. Schneider et U. Stuhler, ainsi que de E. Opdam et M. Solleveld, concernant les Ext-groupes de représentations tempérées. Cependant, pour réaliser cette application il nous faudra montrer que la contraction que nous avons construite dans notre travail ait certaines bonnes bornes pour ces coefficients. On conjecture que c'est en effet le cas pour les constructions que nous avons réalisées.

Mots clés: formes automorphes, décomposition spectrale, calcul résiduel, analyse harmonique, algèbre de Hecke affine.

Summary

On the unramified spherical automorphic spectrum

In the past years, we were interested in the following problems: let $\underline{G}$ be a connected reductive linear algebraic group defined over a field F, which we will assume to be semisimple for simplicity. Suppose first that $\underline{G}$ is a Chevalley group and F is a global field. As usual, for a place v of F we let F_v be the local completion of F at v. For each non-Archimedean local place v, we obtain a maximal compact (and open) subgroup $K_v := \underline{G}(\mathfrak{o}_v)$ of $G_v := \underline{G}(F_v)$, where $\mathfrak{o}_v$ denotes the ring of integers of F_v. If v is an Archimedean place, we fix a maximal compact (and closed) subgroup K_v of G_v. Let $\underline{G}(\mathbb{A})$ be the group of adelic points of $\underline{G}$. This is a locally compact group and the homogeneous space $\underline{G}(F)\backslash\underline{G}(\mathbb{A})$ has finite volume. The first part of the thesis is devoted to the following classical theme in automorphic representation theory:

(a) for F a global field, let S be a finite set of places *not* including the Archimedean ones. If $v \notin S$ let $K'_v = K_v$ as above. If $v \in S$ let $K'_v \subseteq K_v$ be a compact-open subgroup. For example, we could take $K'_v = I_v$, the Iwahori subgroup of G_v. Consider the compact subgroup $\mathbf{K}' = \prod_v K'_v$ of $\underline{G}(\mathbb{A})$. We wish to study the space $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}'}$ of $\mathbf{K}'$-invariant square integrable functions on $\underline{G}(F)\backslash\underline{G}(\mathbb{A})$ as a module for a global Hecke algebra $\mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K}')$, as well as the $\underline{G}(\mathbb{A})$-space generated by these $\mathbf{K}'$-invariants. In special, we wish to determine the representations occurring discretely in these spaces.

As for our second problem, we drop the assumption that $\underline{G}$ is a Chevalley group but we still ask for $\underline{G}$ to be connected. We discuss:

(b) for F a local non-Archimedean field, we wish to construct an explicit contraction of its Bruhat-Tits building $\mathscr{B}(\underline{G}, F)$ by means of piecewise linear Morse theory.

The first problem is inserted in a more general framework where $\underline{G}$ is not necessarily split over the global field F. The space $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))$ of square-integrable automorphic forms is a central object in the theory of automorphic forms and its relation to representation theory and number theory via the Langlands program. Langlands gave an abstract decomposition of this space in terms of classes of cuspidal data $\mathfrak{X}$, in which each $\mathfrak{X} := [\underline{M}, \mathfrak{P}]$ is an equivalence class of pairs $(\underline{M}, \mathfrak{P})$, called cuspidal datum, with $\underline{M}$ an F-Levi subgroup of $\underline{G}$ and $\mathfrak{P}$ an orbit of certain character twists of a cuspidal automorphic representation of $\underline{M}$:

$$L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A})) = \hat{\oplus}_{\mathfrak{X}}\, L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}.$$

These spaces are acted upon by the group $\underline{G}(\mathbb{A})$ by means of the right regular representation. A fundamental problem in the theory of automorphic forms is to decompose further each summand $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}$ as a Hilbert integral of irreducible, unitary $\underline{G}(\mathbb{A})$-representations. If we fix a compact subgroup $\mathbf{K}' \subseteq \underline{G}(\mathbb{A})$ as above, in the decomposition of $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}$, one can study the subset of $\mathbf{K}'$-spherical representations occurring in the support of the spectral measure for the decomposition of this $\underline{G}(\mathbb{A})$-space. This is done by considering the space $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}^{\mathbf{K}'}$, which is a module for a global Hecke algebra associated to the pair $(\underline{G}(\mathbb{A}), \mathbf{K}')$.

The irreducible closed subspaces of $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))$ are called discrete series representations. Among the discrete series representations are the cuspidal automorphic representations (those whose space contains a cuspidal automorphic form of $\underline{G}(\mathbb{A})$, that is, they appear in an $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{\mathfrak{X}}$ with $\mathfrak{X} = [\underline{G}, \mathfrak{P}]$ and $\mathfrak{P}$ a singleton) and the residual spectrum (that is, those discrete series representations which appear in an $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{[\underline{M},\mathfrak{P}]}$ with $\underline{M} \neq \underline{G}$). One of the main achievements of Langlands was to show that the representations in the residual spectrum correspond to certain residues of Eisenstein series via a complicated contour shift.

Let us now describe some results that we found. About problem (a), consider F a number field, $\underline{G}$ a Chevalley group and let $\mathbf{K}' = \mathbf{K} = \prod_v K_v$, with $K_v = \underline{G}(\mathfrak{o}_v)$ for all non-Archimedean places v. In this situation, consider the class of cuspidal data $\mathfrak{X} = [\underline{T}, \mathfrak{1}]$ where $\underline{T}$ is a fixed F-split maximal torus and $\mathfrak{1}$ denotes the orbit of the trivial character of $\underline{T}(F)\backslash\underline{T}(\mathbb{A})$. We are interested in the residual spectrum associated with $[\underline{T}, \mathfrak{1}]$. Let $X^*(\underline{T})$ denote the character lattice of $\underline{T}$ and let a_T^* denote the space $\mathbb{R} \otimes X^*(\underline{T})$, with the root system $\Phi_0 = \Phi(\underline{G}, \underline{T}) \subseteq a_T^*$. Let also $a_{T,\mathbb{C}}^*$ be its complexification and W_0 be the Weyl group of Φ_0. The main results concerning problem (a) are about the decomposition of the automorphic space $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{[\underline{T},\mathfrak{1}]}^{\mathbf{K}}$, in terms of the decomposition of a local module. In more of details, let ${}^L G$ be the complex (connected) dual group of $\underline{G}$ and let ${}^L\mathfrak{g}$ be its Lie algebra. Fix a Cartan involution θ and a corresponding maximal compact ${}^L K$ of ${}^L G$ in such a way that ia_T^* is the Lie algebra of a maximal torus of a compact form of ${}^L\mathfrak{g}$. We then have that θ acts as the identity on ia_T^* and as minus the identity on a_T^*, where $a_{T,\mathbb{C}}^* = a_T^* + ia_T^*$. Now consider the space P^{su}, which we have reasons to call the space of "spherical unramified parameters", that consists of ${}^LG$-conjugacy classes of pairs $\{(\phi, \lambda)\}$ satisfying the properties (1) $\phi : \mathfrak{sl}_2 \to {}^L\mathfrak{g}$ is a Lie algebra homomorphism, (2) $\lambda \in {}^L\mathfrak{g}$ is a semisimple element, (3) $[\lambda, \phi(x)] = 0$ for all $x \in \mathfrak{sl}_2$, (4) $\theta(\lambda) = \lambda$ and (5) $\theta \circ \phi \circ \theta_{\mathfrak{sl}_2} = \phi$, where $\theta_{\mathfrak{sl}_2}$ is the Cartan involution $x \mapsto -x^T$ of $\mathfrak{sl}_2$. We argue that this space is in bijection with a union of affine subspaces of $a^*_{T,\mathbb{C}}$ modulo the action of W_0 and we endow the space P^{su} with an explicit measure μ_{aut}. We then have:

Theorem A. *We have an isometric isomorphism*

$$L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}}_{[\underline{T},\mathbf{1}]} \cong L^2(P^{\mathrm{su}}, \mu_{\mathrm{aut}}).$$

Moreover, this isomorphism is explicitly given in terms of a Fourier transform which diagonalizes the natural action of the global spherical Hecke algebra $\mathcal{H}(\underline{G}(\mathbb{A}), \mathbf{K})$, the tensor product of the local spherical Hecke algebras $\mathcal{H}(G_v, K_v)$ in each place v.

To obtain the above isomorphism, we must deal with a contour shift as the one considered by Langlands. We are forced to take into consideration poles induced by zeroes of the Dedekind zeta function, which is a global ingredient. However, in the present unramified and **K**-spherical case, we can reduce the problem to a residue calculus in a local context, namely that of the spectral decomposition of the Yang system, a module over the graded affine Hecke algebra. This was studied by G. Heckman and E. Opdam.

The space $P^{\mathrm{su}}_{\mathrm{dist}}$, of isolated points of P^{su}, consists of the pairs $(\phi, 0)$ in which ϕ is the Jacobson-Morozov $\mathfrak{sl}_2$-homomorphism of a distinguished orbit $\mathbf{o}$ of ${}^LG$. It follows that the discrete part of $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))^{\mathbf{K}}_{[\underline{T},\mathbf{1}]}$ is parametrized by those distinguished unipotent orbits of the dual group. Given such an orbit, let $2\lambda(\mathbf{o}) \in a^*_T$ be the corresponding weighted Dynkin diagram. To each local place v, let $\pi_{v,\lambda(\mathbf{o})}$ be the unique irreducible spherical subquotient of the unramified minimal principal series of G_v at the parameter $\lambda(\mathbf{o})$ and put $\pi_{\lambda(\mathbf{o})} = \otimes_v \pi_{v,\lambda(\mathbf{o})}$. We are able to conclude from Theorem A that:

Theorem B. *Each $\underline{G}(\mathbb{A})$-representation $\pi_{\lambda(\mathbf{o})}$ occurs in the discrete spectrum of $L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))_{[\underline{T},\mathbf{1}]}$ with multiplicity one.*

Theorem C. *The spherical irreducible representations $\pi_{v,\lambda(\mathbf{o})}$ are unitarizable, for each local place v.*

Also, now using the isometry of Theorem A we can compare norms of vectors on both sides to obtain some nontrivial identities. Among them, we recover a result of Langlands on the volume of $\underline{G}(F)\backslash\underline{G}(\mathbb{A})$ as well as the square norm of some **K**-spherical vectors given as the square integrable residues of Eisenstein series at the points $\lambda(\mathbf{o})$ mentioned above. We remark that in many special classes of reductive groups similar results on the **K**-spherical automorphic spectrum have been obtained by H. Jacquet, H. Kim, R. Langlands, S. Miller, C. Moeglin and J.-L. Waldspurger.

In order for the equation of Theorem A to yield meaningful identities when comparing norms on both sides, one needs to make explicit all the constants involved in the spectral measures. In particular, we need to compute the precise constants of ν_P, the Plancherel measure of the Yang system, which were not yet available at the time that G. Heckman and E. Opdam wrote their article but which can be derived by a limit procedure from the explicit formal degree formulas for discrete series characters of Iwahori-Hecke algebras. This limit procedure is done here. This computation is crucial for the expression of the measure μ_{aut} of Theorem A.

Theorem D. *For each distinguished orbit* $\mathbf{o}$*, the Plancherel measure of the Yang system satisfies*

$$\nu_P(\{\lambda(\mathbf{o})\}) = \left| \frac{1}{\mathrm{vol}(X_*(\underline{T}))} \frac{|W_{\lambda(\mathbf{o})}|}{|A_{\mathbf{o}}|} \frac{\prod'_{\alpha\in\Phi_0} \langle \alpha^\vee, \lambda(\mathbf{o})\rangle}{\prod'_{\alpha\in\Phi_0} \langle \alpha^\vee, \lambda(\mathbf{o})\rangle + 1} \right|.$$

Here, $A_{\mathbf{o}}$ *is the component group of the centralizer of an element of* $\mathbf{o}$ *and* $W_{\lambda(\mathbf{o})} \subseteq W_0$ *is the stabilizer of* $\lambda(\mathbf{o})$*.*

As for the problem (b), the building of a reductive group $\underline{G}$ over a local non-Archimedean field F is an important object used to understand the representation theory of $G := \underline{G}(F)$. It is also an object of study per se due to its many symmetries and geometrical properties. Amongst them, it is a cell complex (whose cells are polysimplices, that is, a finite product of simplices) endowed with a natural action of G and a contractible, complete metric space.

By exploring these properties, P. Schneider and U. Stuhler studied homological resolutions for smooth representations of G. Further, E. Opdam and M. Solleveld studied resolutions for tempered representations using a explicit contraction of the Coxeter complex.

Following an idea of G. Savin and M. Bestvina for the building of SL_n, we propose to study Euclidean buildings, and in particular the complexes above, by means of piecewise linear Morse theory. Let $\mathscr{B} := \mathscr{B}(\underline{G}, F)$ be the Bruhat-Tits building of a connected reductive algebraic group defined over a local non-Archimedean field F. Given a (poly)simplex σ of $\mathscr{B}$, the star of σ, denoted $\mathrm{St}(\sigma)$, is the union of all (poly)simplices of $\mathscr{B}$ that contain σ, together with their faces. We show that:

Theorem E. *One can endow* $\mathscr{B}$ *with a Morse function* $\varphi : \mathscr{B} \to \mathbb{R}$ *that has a unique global minimum and satisfies the following property: for each (poly)simplex* σ *of* $\mathscr{B}$*, any local minimum of* φ *on* $\mathrm{St}(\sigma)$ *is the unique global minimum of* φ *on* $\mathrm{St}(\sigma)$*.*

The property mentioned in the statement of Theorem E allow us to uniquely define the direction of steepest descent around any cell of $\mathscr{B}$ towards a fixed special point in the building. We would later like to apply this construction given by Morse theory to revisit some results of P. Schneider and U. Stuhler, and E. Opdam and M. Solleveld, about Ext-groups of tempered representations. However, in order to do so, we must have that the contraction we construct here have good bounds in its coefficients. We conjecture that this is indeed the case with constructions made.

Keywords: automorphic forms, spectral decomposition, residue calcules, harmonic analysis, affine Hecke algebra.