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Berezin–Toeplitz quantization and noncommutative geometry

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“ *The reason I progress too slowly in knowledge is perhaps that I hate too little my own ignorance* ”

Kurt Gödel

“ *De deux choses lune*
l’autre c’est le soleil ”

Jacques Prévert

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Abstract/Résumé

Abstract The results of this thesis show links between the Berezin–Toeplitz quantization and noncommutative geometry.

We first give an overview of the three different domains we handle: the theory of Toeplitz operators (classical and generalized), the geometric and deformation quantizations and the principal tools we use in noncommutative geometry.

The first step of the study consists in giving examples of spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ involving algebras of Toeplitz operators acting on the Hardy and weighted Bergman spaces over a smoothly bounded strictly pseudoconvex domain Ω of $\mathbb{C}^n$, and also on the Fock space over $\mathbb{C}^n$. It is shown that resulting noncommutative spaces are regular and of the same dimension as the complex domain. We also give and compare different classes of operator $\mathcal{D}$, first by transporting the usual Dirac operator on $L^2(\mathbb{R}^n)$ via unitaries, and then by considering the Poisson extension operator or the complex normal derivative on the boundary.

Secondly, we show how the Berezin–Toeplitz star product over Ω naturally induces a spectral triple of dimension $n+1$ whose construction involves sequences of Toeplitz operators over weighted Bergman spaces. This result led us to study more generally to what extent a family of spectral triples can be integrated to form another spectral triple. We also provide an example of such a triple.

Résumé Cette thèse montre en quoi la quantification de Berezin–Toeplitz peut être incorporée dans le cadre de la géométrie non commutative.

Tout d'abord, nous présentons les principales notions abordées : les opérateurs de Toeplitz (classiques et généralisés), les quantifications géométrique et par déformation, ainsi que quelques outils de la géométrie non commutative.

La première étape de ces travaux a été de construire des triplets spectraux $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ utilisant des algèbres d'opérateurs de Toeplitz sur les espaces de Hardy et Bergman pondérés relatifs à des ouverts Ω de $\mathbb{C}^n$ à bord régulier et strictement pseudoconvexes, ainsi que sur l'espace de Fock sur $\mathbb{C}^n$. Nous montrons que les espaces non commutatifs induits sont réguliers et possèdent la même dimension que le domaine complexe sous-jacent. Différents opérateurs $\mathcal{D}$ sont aussi présentés. Le premier est l'opérateur de Dirac usuel sur $L^2(\mathbb{R}^n)$ ramené sur le domaine par transport unitaire, d'autres sont formés à partir de l'opérateur d'extension harmonique de Poisson ou de la dérivée normale complexe sur $\partial\Omega$.

Dans un deuxième temps, nous présentons un triplet spectral naturel de dimension $n+1$ construit à partir du produit star de la quantification de Berezin–Toeplitz. Les éléments de l'algèbre correspondent à des suites d'opérateurs de Toeplitz dont chacun des termes agit sur un espace de Bergman pondéré. Plus généralement, nous posons des conditions pour lesquelles une somme infinie de triplets spectraux forme de nouveau un triplet spectral, et nous en donnons un exemple.

Résumé substantiel

En physique mathématique, la *quantification par déformation* est une catégorie de processus par lesquels les observables f d'un système dynamique classique sont transformées en opérateurs autoadjoints Q_f agissant sur des espaces de Hilbert, c'est à dire les observables quantiques. Son principe est de rendre l'algèbre des observables classiques non commutatives grâce à un produit star $\star$ issu d'une relation de type $Q_f Q_g = Q_{f \star g}$.

On s'intéresse ici à la *quantification de Berezin–Toeplitz* qui met en jeu des *opérateurs de Toeplitz* agissant sur des espaces de Hilbert tels que l'espace de Hardy et Bergman pondéré d'une part, qui sont liés à des domaines ouverts Ω de $\mathbb{C}^n$ *strictement pseudoconvexes* à bords lisses, et d'autre part l'espace de Fock relatif à $\mathbb{C}^n$. La théorie des opérateurs de Toeplitz généralisés offre un cadre plus commode pour manipuler les objets en présentant des propriétés très similaires (mais néanmoins différentes) de celles des opérateurs pseudodifférentiels classiques sur une variété compacte: calcul symbolique, relation d'ordre, formule de Weyl, etc.

De son côté, la *géométrie non commutative* offre un cadre mathématique permettant the décrire une géométrie de manière totalement spectrale. L'idée est de considérer que si un espace topologique peut être entièrement caractérisé par l'algèbre commutative des fonctions qui sont définies sur cet espace, une algèbre qui n'est plus commutative décrirait par analogie un espace d'une autre nature : un *espace non commutatif*. On remarque alors que les outils mathématiques utilisés (C^*-algèbres, espaces de Hilbert, opérateurs, etc.) sont en lien étroit avec la mécanique quantique où la noncommutativité est omniprésente, et a fortiori, le processus de quantification.

Le travail de la thèse consiste à étudier dans quelles mesures la quantification de Berezin-Toeplitz peut être abordée avec le formalisme de la géométrie non commutative.

En premier lieu, nous exposons en détails les propriétés des opérateurs de Toeplitz classiques et leurs liens avec les opérateurs de Toeplitz généralisés agissant sur l'espace de Hardy relatif à $\partial\Omega$. Dans le cas de la boule unité, nous observons que les opérateurs de Toeplitz admettent des propriétés intéressantes et qu'ils sont étroitement reliés à l'algèbre de Lie du groupe de Heisenberg de dimension $2n+1$. Nous présentons aussi quelques exemples d'opérateurs dont les propriétés serviront plus loin dans la construction de triplets spectraux.

Dans un second temps, nous exposons les principes de deux méthodes de quantifications: la quantification géométrique sur des espaces de type Kähler, ainsi que la quantification par déformation sur des domaines strictement pseudoconvexes. On observe que la théorie des opérateurs de Toeplitz généralisés est impliquée dans la plupart des constructions considérées.

Puis, nous présentons les principaux outils de la géométrie non commutative que nous utilisons, à savoir la notion de triplet spectral formé d'une algèbre $\mathcal{A}$ qui se représente fidèlement sur un espace de Hilbert

$\mathcal{H}$, ainsi que d'un opérateur $\mathcal{D}$ autoadjoint à résolvante compacte et tel que son commutateur avec un élément de l'algèbre soit borné sur $\mathcal{H}$. Les notions de régularité, de dimension spectrale ainsi que d'action spectrale sont aussi abordées.

Enfin, nous présentons les résultats sur les triplets spectraux obtenus à l'aide des diverses algèbres d'opérateurs de Toeplitz. Le résultat principal concerne l'espace de Hardy et utilise la théorie des opérateurs de Toeplitz généralisés. Des triplets sur les espaces de Bergman avec poids ainsi que sur Fock ont aussi été obtenus. Une astuce permettant de contourner la trivialité induite par la positivité de certains opérateurs $\mathcal{D}$ est aussi donnée. Nous étudions aussi le cas de la boule unité où la géométrie induit de grandes simplifications qui nous permettent de calculer explicitement l'action spectrale ainsi qu'exhiber un exemple de structure de réalité dans le cas d'une algèbre commutative.

Le triplet spectral obtenu à partir de la quantification de Berezin–Toeplitz possède une dimension surnuméraire qui peut être expliquée par le fait que les éléments de l'algèbre correspondent à des sommes directes d'opérateurs: c'est cette sommation qui apporte ce degré de liberté supplémentaire. La construction de ce triplet spectral nous a enfin amené à étudier les conditions pour lesquelles une somme directe de triplets spectraux formait elle-même un triplet spectral. Un exemple utilisant des opérateurs de Toeplitz à symboles polynomiaux illustre cette motivation.

Plusieurs perspectives peuvent être considérées. Tout d'abord, il serait intéressant d'obtenir un résultat plus général pour établir un triplet spectral à l'aide d'opérateurs de Toeplitz généralisés en utilisant leur définition abstraite valide sur toute variété compacte admettant un cône symplectique (le cas du bord d'une variété strictement pseudoconvexe en est un exemple). Le résultat sur le triplet spectral utilisant la quantification de Berezin–Toeplitz est un exemple supplémentaire pour mieux comprendre les liens existants entre le processus de quantification et la géométrie non commutative. De manière analogue, la quantification de Weyl donne lieu naturellement à un triplet spectral [68]. Il serait donc intéressant d'étudier dans quelles mesures un triplet spectral peut-être canoniquement associé à un processus de quantification. On peut aussi penser à étendre le dernier résultat de cette thèse et établir les conditions pour qu'un triplet spectral puisse être désintégrable en une somme infinie de triplets spectraux. Enfin, dans la dynamique du programme de Fefferman [65], un autre axe de recherche consisterait à obtenir des invariants locaux supplémentaires sur des variétés strictement pseudoconvexes à partir de triplets spectraux impliquant les opérateurs de Toeplitz généralisés.

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Introduction

Tea and Toeplitz operators

Unexpectedly, Toeplitz operations can be experienced in the everyday life, as the following scene shows. Suppose you have a cup full of hot water. You can either drink it, assuming the temperature is acceptable, or prepare a tastier beverage by adding some tea leaves and, after the infusion, filtering water in order to obtain a different drink. Schematically, you transform the water and filter the resulting liquid to get something drinkable.

In mathematical terms, the water is replaced by an element ϕ in some Hilbert space $\mathcal{H} \subset L^2(X)$ of functions over a domain X, adding the tea consists in multiplying an element by some other general (and sufficiently nice) function u on X, and the filter is represented by a projector $\Pi : L^2(X) \to \mathcal{H}$. The whole process can be concisely written as $\Pi(u\,\phi)$, which is by definition the action of the Toeplitz operator T_u on ϕ. Note that the presence of the projector is crucial to stay in the right Hilbert space after the multiplication by u (and to avoid drinking the leaves).

The main reasons for which a great interest is taken in the study of Toeplitz operators are that they intertwine various domains of mathematics such as operator theory, Banach algebra or analysis, but also offer a rich framework in the study of some linear systems appearing in various problems like stochastic processes, numerical analysis or least square approximation [9, 30, 78] and in theoretical physics.

Brown and Halmos gathered in [32] algebraic properties about Toeplitz operators over the unit circle and showed that the necessary and sufficient condition in order to obtain the identity $T_u T_v = T_{uv}$, for any two symbols $u, v \in L^\infty(\mathbb{S}^1, d\mu)$, is that either $\bar{u}$ or v lies in $H^2(\mathbb{S}^1)$. This result gives an analytic characterization of the obstruction for the map $u \mapsto T_u$ to be multiplicative, and we can see that this condition is very restrictive. A generalization of this result establishes a necessary [5] and sufficient [148] condition on the symbols for the difference $T_u T_v - T_{uv}$ to be compact. Conditions of normality, inversibility and also connectedness of the spectrum of Toeplitz operators have also been studied in this paper. For a good overview of Toeplitz operators over the unit circle, the reader can refer to [29], while the spectral and algebraic properties can be found in [86, Chapter 25] and also [88, 50, 46, 138, 74, 7].

The study of Toeplitz operators has then been extended to Hardy spaces over the boundary of more general domain (unit sphere $\mathbb{S}^n$, unit torus $\mathbb{S}^1 \times \mathbb{S}^1$, the smooth boundary of strictly pseudoconvex domains, etc.), to Bergman spaces (over the unit disk $\mathbb{D}$, bounded symmetric domains of $\mathbb{C}^n$, Kähler manifolds, etc.) and their weighted versions, and also to Fock space over $\mathbb{C}^n$ with gaussian weights [16, 55, 37]. The case of the unit ball of $\mathbb{C}^n$ has been largely investigated and the symmetries allow various interesting properties such as the representation of generators of the Heisenberg Lie algebra as Toeplitz operators [95] or formulae for Dixmier traces of products and commutators of Toeplitz (and Hankel) operators [58].

The so-called Generalized Toeplitz Operators (GTOs), introduced by Boutet de Monvel and Guillemin [20, 21, 22], have been set up as an extension of the classical Toeplitz operators on Hardy spaces. The theory is based on two principal objects: a symplectic closed conic subset Σ of the cotangent space of some smooth compact manifold Ω, and a related endomorphism Π_Σ on $L^2(\Omega)$ which admits microlocal structure very similar to the one of the corresponding Szegö projector. Then, operators of the form $\Pi_\Sigma\, P\, \Pi_\Sigma$, where P is a pseudodifferential operator on $L^2(\Omega)$, are called GTOs. Generalizing, in a sense, the theory of pseudodifferential operators, they enjoy similar properties which will be of great interest for us (symbolic calculus, a Weyl law, etc.).

Classical and generalized Toeplitz operators also play an important role in quantum mechanics, and particularly in the process of *quantization*. But first, let us recall some facts about the formalism of quantum mechanics to understand this concept.

Zooming into matter

The foundations of the principles of quantum physics as we know it today took place in the first quarter of the past century. The studies of the black-body radiation phenomenon [116, 117], the photoelectric effect [53], Young's [31] and Aspect's [3] experiments, to mention only these, consisted in the first step in understanding the world of quantum mechanics. This new physics implied deep philosophical questionings about the understanding of the world: discrete levels of energy, intrinsic indeterminism, non-locality, non-separability and noncommutativity of the measures were new concepts, incompatible with the current theories at this time. Also, the lack of analogous phenomena in the macroscopic world was (and stays) an additional difficulty for the human mind to get a good intuition of what happens at the quantum level. Nonetheless, a mathematical formalism of quantum mechanics has been set up [111], predicting with good precision the results of the physical experiments.

One of the main obstructions to getting a universal description of classical and quantum physics lies in the fundamental difference between the nature of the corresponding observables. A macroscopic system, subject to the laws of classical mechanics, can be reasonably considered as totally insensible to any (passive) measures on it: looking at a moving car to evaluate its position or speed will not alter its trajectory. Moreover, measuring its position before its speed is the same as doing it the other way around. It means that the observables, the quantities which can be measured, are *commutative* (1). At the quantum level however, things are drastically different. A measurement on a quantum system induces perturbations to the system itself which can definitely not be neglected (2). The system is in a superposition of many different states until it interferes with its environment, or is subject to a measurement, which causes the reduction of its states to a unique one: this phenomena is called the *decoherence*. So a subsequent measurement will be performed on this precise state only. As a consequence, the observations on a quantum system depend directly on the order of measurements: this is the *noncommutativity* of quantum observables (3).

So we have on one side the classical level, with commuting observables, on the other side the quantum world, with noncommutative observables, and each of them possesses their own laws. What we are

(1) Commutativity of numbers became a triviality as soon as we learnt that $2 \times 3 = 3 \times 2$, a fact which could seem somewhat surprising with a little hindsight.

(2) Actually, some observables of particular quantum systems can be measured without altering them. This is part of the field of Quantum Non Destructive measurements which has been investigated since the 70's [142]

(3) Quantum observables are more precisely *not necessarily* noncommutative, which makes noncommutativity a generalization of commutativity, not its opposite. Note also that noncommutative phenomena are also frequent in the everyday life: using the previous example, putting tea leaves in some hot water before drinking it is not the same as the other way around.

looking for is a unique mathematical formalism which describes both of them. The main approach which has been considered since the establishment of quantum mechanics is the process of *quantization*, in which a quantum system is assigned to a classical one. In this way, the quantum theory is obtained by translating the mathematical formulation of the original classical system, which is thought as its limit, in a certain sense. The opposite procedure, which starts from the quantum theory to build a classical one is called *dequantization*. To understand quantization, let us now give more details on how observables (classical and quantum) are mathematically formalized.

The Hamiltonian formalism of the dynamics of a classical system with n variables is defined over a phase space, which is the cotangent bundle of the configuration space. It inherits naturally the structure of a symplectic manifold (Ω, ω), and a point $(q,p) \in \Omega$ represents a state of the system, given by its position p and its momentum p. The set of classical observables, wich are real valued smooth functions on Ω, forms a commutative algebra, with usual addition scalar multiplication and pointwise multiplication. The properties of the bracket induced by ω (see (A.2), (A.3)) makes $(C^\infty(\Omega), \{\,.\,,\,.\,\})$ a Poisson algebra.
The dynamics of the system are encoded in a particular function $H = H(q,p,t)$ called the Hamiltonian (in general the total energy of the system). The equations of motion of the system are derived from H, using the Poisson bracket:

$$\partial_t q_j = \{\, q_j \,,\, H \,\}\,, \qquad \text{and} \qquad \partial_t p_j = \{\, p_j \,,\, H \,\}\,, \qquad \text{for any } j = 1, \ldots, n.$$

Now let us recall that in quantum mechanics, the state of a system is not a point of the phase space, but is represented as an element of some Hilbert space $\mathcal{H}$, endowed with an hermitian inner product $\langle\,.\,,\,.\,\rangle$. This vector encodes the probability for the system to be in a certain state, which is the only information we can get from a quantum system: this is the *quantum indeterminacy*. More precisely, the probability for a state ψ_1 to be in another state ψ_2 is given by the Born rule

$$\mathrm{Prob}(\psi_1, \psi_2) := \tfrac{|\langle\, \psi_1 \,,\, \psi_2 \,\rangle|^2}{\langle\, \psi_1 \,,\, \psi_1 \,\rangle \langle\, \psi_2 \,,\, \psi_2 \,\rangle}\,.$$

As a consequence, for any $\lambda \in \mathbb{C}\backslash\{0\}$, ψ and $\lambda\psi$ represent the same entity: the physical quantum state of a system is an element of the projective Hilbert space $P\mathcal{H}$, rather than $\mathcal{H}$ itself.
The time evolution of any non-relativistic quantum system with state $\psi = \psi(x,t)$ verifies the Schrödinger equation

$$H\psi = i\hbar\, \partial_t \psi\,,$$

where H is a selfadjoint operator on H, called the Hamiltonian of the system, and $\hbar := \frac{h}{2\pi}$ is the reduced Planck constant [(4)]. Analogously as the classical system, H encodes the total energy of the system. Equivalently, we can say that the evolution is described by a one parameter group of unitary operators $U(t)$ subject to the relation $HU(t) = i\hbar\, \partial_t U(t)$. Here, the unitarity of $U(t)$ comes from the theoretic determinism of time evolution of a quantum state. Indeed, we must underline the fact that if the measurements on a quantum system is of statistic nature, the quantum state itself is well defined, as a fixed element of the previous Hilbert space. Its evolution, independently of any measurement, is the result of the action of a linear operator on its state, which preserves its normalization. If $\psi(t_1)$ and $\psi(t_2)$ are the states at times t_1 and t_2, we want $|\langle\, \psi(t_1) \,,\, \psi(t_1) \,\rangle|^2 = |\langle\, \psi(t_2) \,,\, \psi(t_2) \,\rangle|^2$, so from the Wigner's theorem, the linear operator $U_{t_2 t_1}$ which sends $\psi(t_1)$ to $\psi(t_2)$ is necessarily unitary (antiunitary is rejected from the fact that for any $t \in \mathbb{R}$, $U_t = U_{t/2} U_{t/2}$).

(4)The Planck constant h, whose symbol comes from the German word “ Hilfsgröße ” (auxiliary variable), was introduced in [117] in order to solve the problem of the radiation of a black body. The fact that “ Hilfe ” stands for “ help ” in German also reflects the level of desperation this problem caused among the scientific community at that time.

The difficulty of quantization is to find a “ good ” recipe to assign a quantum system to a classical one. More precisely, we are looking for an application Q, called a *quantization map*, which sends a classical observable $f \in (C^\infty(\Omega), \{\,.\,,\,.\,\})$ [(5)] to some selfadjoint [(6)] operator $Q_f \in \mathrm{End}(\mathcal{H})$, with respect to reasonable constraints on it. We can consider a priori many recipes to quantize a system and there is apparently no reason to prefer a particular quantization map than another one. Physical considerations lead to the following reasonable conditions in the case of the flat phase space $\Omega = \mathbb{R}^n \times \mathbb{R}^n \ni (q,p)$ [132]:

Conditions 0:

i) $f \mapsto Q_f$ is linear,

ii) for any polynomial ϕ, $Q_{\phi(f)} = \phi(Q_f)$ (von Neumann rule),

iii) $[\,Q_f,\, Q_g\,] = -i\hbar\, Q_{\{\,f\,,g\,\}}$ (canonical commutation relation),

iv) $Q_{q_j} = q_j$ and $Q_{p_j} = -i\hbar\,\partial_{q_j}$ (canonical quantization).

Unfortunately, there is no quantization map verifying the four conditions simultaneously [56, 1] and some axioms must be abandoned or at least relaxed. For instance, one can restrict ii) by considering a polynomial ϕ of order at most 1, which allows to quantize a space of classical observables.

The *geometric quantization* program was introduced in the 70’s by Kirillov [97], Kostant [101] and Souriau [136] to solve this problem (among others) and became a branch of mathematics of its own. It provides a quantization map which fulfils i), iii) and iv) of Conditions 0 (the von Neumann rule is actually very restrictive and must be forgotten in this context). The idea is to consider additional geometric structures over the phase space of the system and propose a natural quantization map. The three steps are the *prequantization* (endow the phase space Ω with a particular complex line bundle), *polarization* (reduce the number of variables by choosing a distribution in the tangent space of the domain), and *metaplectic correction* (needed to define a correct space of quantum states).

The other approach we will focus on is called *deformation quantization*, which has been thought to solve some drawbacks of the previous program (e.g. lack of physical signification of the geometric structures, too few quantizable observables [1, Section 3.7]). First, consider a body moving at high speed v, and whose dynamics is described by Einstein’s theory. When the ratio v/c, where c denotes the speed of light in vacuum, becomes negligible, the laws of general relativity reduce to the ones of Newton’s classical mechanics. In these terms, the theory takes into account a continuous parameter which determines whether a system can be characterized as classical or relativistic. Now At the atomic level, the ubiquitous constant is the (reduced) Planck constant $\hbar$. On one side, we have classical mechanics and on the other side the quantum theory, which depends on $\hbar$: one can see the latter as a continuous scale cursor which indicates the “ degree of quantumness ” of the system. We are thus looking for a mathematical description of the system involving $\hbar$ as a real parameter, so that when $\hbar \neq 0$ is significant, the observables do not commute, whereas as $\hbar$ tends to vanish, we recover the classical properties of the system. This procedure is called the *semi-classical limit* and can be interpreted as a mathematical formulation of the *correspondence principle*. A solution consists in deforming the pointwise product on the set of classical variables into a so-called *star product*. The idea is to consider a quantization $f \mapsto Q_f$, and construct a product $\star$

(5) We will see that the existing quantizations do not concern all smooth observables, and some constraints must be considered on them.

(6) Or equivalently antiselfadjoint.

subject to a relation of the form $Q_f Q_g = Q_{f\star g}$ [(7)], where $f \star g = f\,g + O(\hbar)$ can be expressed as a power series in $\hbar$ with smooth coefficients depending on the two functions f and g. This is why we speak of *deformation* quantization: the product $\star$ is the usual product together with additional terms which breaks the commutativity and also tend to vanish when $\hbar$ goes to 0. In this way, the initial commutative algebra of classical observables endowed with the product $\star$, becomes the noncommutative algebra of quantum observables.

At this point, it becomes intuitively clear that Toeplitz operators are suitable in the context of quantization. Indeed, the set of square integrable functions over the phase space (or sections on some bundle over it) is in general too large to define reasonable quantum states and we consider a smaller Hilbert space: we consider the orthogonal projector from the first space to this Hilbert space. Secondly, given a function (classical observable), a natural idea to define an operator (a quantum observable) acting on the quantum states is to consider simple pointwise multiplication by the function. Then, to ensure that the result states in the correct Hilbert space, one can roughly apply the previous projector on it, and the resulting operator is a Toeplitz. In the context of geometric quantization over Kähler manifolds, the particular direction (polarization) that must be chosen in the tangent space of the manifold in order to define the right Hilbert space of quantum states, is canonically given by the set of holomorphic sections of a line bundle (Kähler polarization). The quantization map proposed by the initial program, involving a connection on the bundle, happens to be a Toeplitz operator, thanks to the Tuynman's relation [143]. But there is more: any bounded linear operator acting on the space of holomorphic sections over tensor powers of the line bundle is actually Toeplitz [28]. In the framework of deformation quantization, results show that the Toeplitz quantizations $u \mapsto T_u$ over nice domains also induce a star product.

A new geometry

The success of quantum mechanics and of general relativity both revolutionized the field of theoretical physics and laid the foundations of the mathematical formalisms we still use today. They brought a new perspective to our understanding of the structure of matter on one side, and the one of the universe at cosmological scale on the other side. We have seen that quantization aims to merge classical and quantum mechanics, while general relativity unifies the classical Newton's law together with Maxwell's equations and also gravitation. These theories concern very different fields of physics, but they can be intersected in several ways. First, the relativity seems to predict that the universe has an origin, the Big Bang, and near this singularity, during the so-called Planck epoch, the quantum phenomena cannot be neglected and must be taken into account to describe the evolution of space-time. Secondly, the very nature of space-time would be quantum at small scales, just like the matter is.

During the second half of the twentieth century, different approaches were investigated in order to get a global theory which would encompass quantum mechanics and relativity: the so-called *quantum gravity*. This problem gave rise to a multitude of branches in theoretical physics: string theory, loop quantum gravity, supersymmetry, applications of noncommutative geometry, to name only few.

The one on which we are focusing here is of course noncommutative geometry. To understand the idea, let us make briefly comment on how a space and the functions defined on it are related. First, the set of functions that can be defined over a space X depends on the nature of the space. For instance, topological spaces are related to the notions of neighborhood or compactness, and the related functions are the continuous ones. For Riemannian manifolds, which allows to define the notions of differentiation and

[(7)] The symbol " $=$ " actually refers to an asymptotical relation.

smoothness, the natural class of functions we can consider is the space of smooth functions. In both cases, the set of functions, $C(X)$ or $C^\infty(X)$, naturally forms a commutative algebra. Now, given a fixed point x of the space, on can look at the set of all values $f(x)$ when f lies in the corresponding algebra: the role of the variable is then played by the function f. In this way, the initial space of points is exchanged for the space of functions. A fundamental result of Gelfand and Naimark [70], recalled further, ensures that this matching keeps all the information about the space, and reciprocally, to any commutative C^*-algebra is associated some space. In other words, talking about the points of a space or the functions defined on it is the same. As a consequence, the topology or geometry of the space can be entirely characterized by the corresponding set of functions: the relations between the points are of the same nature than those between elements of a commutative algebra. The advantage of this viewpoint is twofold: first, an algebra carries a richer structure than a simple set, and secondly, this allows to use all the machinery of analysis.

This is where the noncommutativity, omnipresent in quantum mechanics, comes into the picture. By analogy with the previous correspondence, if the commutative algebra of classical observables describes the domain they are defined on, the noncommutative algebra of quantum observables, modelled by operators acting on Hilbert spaces, should correspond to a space of a totally different nature: a so-called *noncommutative space*. The well-known GNS construction ensures that we can replace this space of bounded operators by a general C^*-algebra. The noncommutative geometry, introduced by Connes [45, 41, 43, 35], aims to describe these spaces by manipulating a noncommutative involutive algebra $\mathcal{A}$ represented faithfully by bounded operators acting on a Hilbert spaces $\mathcal{H}$. To obtain a bit more information on the noncommutative space, we also consider an operator $\mathcal{D}$ acting on $\mathcal{H}$, describing spectrally the underlying metric and smoothness. With some compatibility conditions, the triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ constitutes one of the principal objects in the theory: a so-called *spectral triple*. Other geometric quantities are then translated in terms of algebraic structures, as the following non-exhaustive list shows:

Topology/Geometry	**Algebra**
(Compact) Topological space	(Unital) C^*–algebra
Topology	Set of ideals
Compactification	Unitization
Symmetries	Automorphisms
Infinitesimal	Compact operator
Metric	“ Dirac like ” operator $\mathcal{D}$
Integral	Trace
Derivation	Commutator with $\mathcal{D}$
Vector bundle	Finitely generated projective module

The structure of the thesis

Let us now relate these three different aspects of mathematics and theoretical physics we want to connect: Toeplitz operators, the quantization procedure and the noncommutative geometry. We briefly mentioned above to what extent Toeplitz operators and geometric and deformation quantizations are related. This point will be presented under different aspects and the main results recalled. We propose in this thesis to strengthen the link between Toeplitz operators and noncommutative geometry on the one hand, and, on the other hand, between deformation quantization and noncommutative geometry. This leads to a natural organisation of the work into two different parts:

- we first build several classes of spectral triples based on algebras of Toeplitz operators and study some aspects of the underlying geometry,
- then, we show that a spectral triple can be associated in the context of the Berezin–Toeplitz quantization on strictly pseudoconvex domains of $\mathbb{C}^n$.

This thesis gathers the results obtained during a long collaboration with my supervisor Bruno Iochum and Miroslav Engliš [61] together with additional subsequent work. For consistency reasons, we chose to respect previous order to expose the different notions.

Chapter 1 is concerned with the theory of Toeplitz operators and contains several results of [61]. We recall first in Section 1.1 the definitions of the different Hilbert spaces we use (Hardy, Bergman and Fock), the related operators and their links with the Heisenberg Lie algebra. Section 1.2 fixes some definitions and notations about classical Toeplitz operators and also describes the class of GTOs when the corresponding domain is a smoothly bounded strictly pseudoconvex domain of $\mathbb{C}^n$. Particular attention is paid to the definitions of order and symbol of a GTO, and also their relations with pseudodifferential operators. Section 1.3 is devoted to describing the interactions between Toeplitz operators of different kinds. First, classical Toeplitz operators are closely related to GTOs through explicit unitaries which involve the Poisson extension operator from the boundary to the interior of the domain. Secondly, in the case of the unit ball of $\mathbb{C}^n$, we will see that they can be seen as the representation of elements of the $2n+1$ dimensional Heisenberg Lie algebra. Finally, Section 1.4 presents different classes of Toeplitz operators which will be used in Chapter 4.

Chapter 2 gives a quick overview of two principal quantization procedures. We present in Section 2.1 the main concepts of geometric quantization. After a quick presentation of its construction, we recall how Toeplitz operators appears in this context and show as an example how to recover the Fock space on $\mathbb{C}^n$. We also present a result on Kähler manifolds which is related to the following chapter. Section 2.2 presents the procedure of deformation quantization and how to construct a star product in order to get a phase space formulation of quantum mechanics. Two methods are introduced: the Berezin and the Berezin–Toeplitz quantization. The latter, which uses Toeplitz operators, will be of particular interest for us.

In Chapter 3, we present the principal tools of noncommutative geometry we consider. We underline the link between topology and algebra in Section 3.1 in the light of two classical results in C^*-algebra theory: the celebrated Gelfand–Naimark theorems. Section 3.2 is devoted to present the origins, the generalizations and the main properties of the Dirac operator. The notions of spectral triple, spectral dimension, regularity, real structure and spectral action are defined in Section 3.3.

Chapter 4 presents the spectral triples obtained from the different algebras of Toeplitz operators. The general result on the Hardy space is shown in Section 4.1 using the powerful machinery of GTOs. In Section 4.2 we present spectral triples over weighted Bergman spaces on smoothly bounded strictly pseudoconvex domains. A trick to get around a triviality induced by the negativity of the operator $\mathcal{D}$ is proposed. We also investigate the case of the unit ball whose underlying geometry allows for simplifications. An example of commutative spectral triple together with a real structure and a computation of a spectral action are presented. We make some remarks in Section 4.3 about the use of the Dixmier trace in the context of spectral triples built on algebras related to Toeplitz operators. For completion, Section 4.4 presents spectral triples using Toeplitz operators on the Fock space. Finally, we investigate in Section 4.5 how the deformation quantization could be incorporated in the context of noncommutative geometry. First, we show in Section 4.5.1 that a natural spectral triple is associated to this quantization. Some remarks are given about its dimension. This result, together with the proof of the Berezin–Toeplitz quantization over pseudoconvex domains led to study to what extent a direct sum of spectral triples is a spectral triple again. The conditions are presented in Section 4.5.2, and an example is also given.

The section Notations and symbols at the end may be also helpful for the reader.

Chapter 1

Toeplitz operators

To understand the idea behind the definition of Toeplitz operators, let us describe the class of matrices from which they emerged.

Consider functions of the form $u : z \in \mathbb{S}^1 \mapsto \sum_{j\in\mathbb{Z}} u_j z^j \in \mathbb{C}$, with absolutely convergent Fourier series: $\| u \|_1 := \sum_{j\in\mathbb{Z}} |u_j| < \infty$. The corresponding *Toeplitz matrix* $T(u)$, named after the German mathematician O. Toeplitz, is constructed such that for any $j \in \mathbb{Z}$, its j^{th} parallel to the diagonal contains only the coefficient u_j:

$$T(u) := \begin{bmatrix} u_0 & u_{-1} & u_{-2} & \cdots \\ u_1 & u_0 & u_{-1} & \ddots \\ u_2 & u_1 & u_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix} .$$

As an operator on $\ell^2(\mathbb{S}^1)$, one can check that $T(u)$ is bounded with $\| T(u) \| \le \| u \|_1$ [(1)] and also that $T(u)^* = T(\bar{u})$. A quick calculation also shows that in general, the product of two Toeplitz matrices is not Toeplitz.

The space $H^2(\mathbb{S}^1) := \{\phi = \sum_{j=0}^{\infty} \phi_j z^j \in L^2(\mathbb{S}^1, d\mu)\}$, where $d\mu$ is the usual Lebesgue measure on $\mathbb{S}^1$, is a closed subspace of $L^2(\mathbb{S}^1, d\mu)$ and is called the *Hardy space* over the unit circle of $\mathbb{C}$. Denote Π the orthogonal projection from $L^2(\mathbb{S}^1, d\mu)$ to $H^2(\mathbb{S}^1)$, called the *Szegö* projector, and define the map $\Upsilon : \phi = \sum_{j=0}^{\infty} \phi_j z^j \in H^2(\mathbb{S}^1) \mapsto \Upsilon(\phi) := (\phi_j)_{j\in\mathbb{N}} \in \ell^2(\mathbb{S}^1)$. Then we get the identity

$$\Upsilon^{-1} T(u) \Upsilon : \phi \in H^2(\mathbb{S}^1) \mapsto \Upsilon^{-1} T(u) \Upsilon(\phi) = \Pi(u\,\phi) =: T_u(\phi) \in H^2(\mathbb{S}^1) .$$

In other words, the matrix $T(u)$ corresponds to an operator T_u acting on $H^2(\mathbb{S}^1)$, which multiplies a vector by u and keeps only the Fourier coefficients of the result: such operators T_u are called *Toeplitz operators*. The operator $H_u : \phi \in H^2(\mathbb{S}^1) \mapsto (I - \Pi)(u\,\phi) \in L^2(\mathbb{S}^1, d\mu) \ominus H^2(\mathbb{S}^1)$, seen as the complementary of T_u, is called a *Hankel operator*. In the rest of the thesis, we will not make use of Hankel operators and we refer to [6, 115] for the relations between Toeplitz and Hankel operators.

As mentioned in the Introduction, the definition of Toeplitz operators can be extended to more general domains, and we focus here on a class of manifolds which possess interesting properties: the case of open bounded *pseudoconvex manifolds* (see Appendix A.5). We will also consider Toeplitz operators all over $\mathbb{C}^n$.

[(1)] It is actually the case for $\ell^p(\mathbb{S}^1)$, with $1 \le p \le \infty$.

1.1 Bergman, Hardy and Fock spaces, Heisenberg algebra

For the rest of this section, Ω is a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with defining function r and $d\mu$ denotes the usual Lebesgue measure. Recall that, given a weight on some domain Ω (a non-negative measurable function $w : \overline{\Omega} \to \mathbb{R}^+$), the space $L^2(\Omega, w)$ is the set of square integrable functions f on Ω with respect to w:

$$\| f \|_w := \Big(\int_\Omega |f(z)|^2 \, w(z) \, d\mu(z) \Big)^{1/2} < +\infty \,. \tag{1.1}$$

The corresponding weighted inner product is defined as $\langle \, f \, , \, g \, \rangle_w := \int_\Omega f \, \bar{g} \, w \, d\mu$.

From (A.8), the 1-form

$$\eta := \tfrac{1}{2i}(\partial r - \bar{\partial} r)|_{\partial\Omega} \,, \tag{1.2}$$

is a contact form and

$$\nu := \eta \wedge (d\eta)^{n-1} \tag{1.3}$$

is a volume form on $\partial\Omega$.

1.1.1 Bergman and Hardy spaces

Hilbert spaces of Bergman and Hardy type are defined respectively over the strictly pseudoconvex domain Ω and its boundary $\partial\Omega$. Depending on the cases, we can choose on $\partial\Omega$ either the Lebesgue measure $d\mu$ or the measure induced by ν in (1.3). On Ω, we make use of weighted measures. To control the behaviour of the weight at the boundary of the domain Ω, we consider from now on weights of the form

$$w_m = (-r)^m \, \chi \,, \tag{1.4}$$

where $m > -1$ is a real number and $\chi \in C^\infty(\overline{\Omega})$ is such that $\chi|_{\overline{\Omega}} > 0$. Indeed, from Remark A.3.5, the dependence of w_m on r and χ is weak, whereas the vanishing order of w_m at the boundary, which is exactly m here, will play a crucial role later on. Note also that the next definitions remain valid for general weights.

The notation is as follows: we choose to underline the dependence on m of the related operators and spaces, and in general bold letters refer to operators acting on spaces defined on Ω whereas the regular roman ones concern those over $\partial\Omega$.

Definition 1.1.1. *The weighted Bergman space is*

$$A^2_m(\Omega) := A^2_m := \{ f \in L^2(\Omega, w_m), \ f \text{ is holomorphic on } \Omega \} \,,$$

endowed with the norm (1.1).
When $w_m = 1$, the space $A^2(\Omega)$ is called the unweighted Bergman space.
Denote $\mathbf{\Pi}_m$ the orthogonal projection from $L^2(\Omega, w_m)$ to $A^2_m(\Omega)$,

Let $(v_{m,\alpha})_{\alpha \in \mathbb{N}^n}$ be an orthonormal basis of $A^2_m(\Omega)$. When $\Omega = \mathbb{B}^n$ is the unit ball of $\mathbb{C}^n$, if r is radial (i.e. $r(z) = r(|z|)$), and with the weight $w_m = (-r)^m$, $m \in \mathbb{N}$, we have the following orthonormal basis for $A^2_m(\mathbb{B}^n)$ [79, Corollary 2.5]:

$$v_{m,\alpha}(z) := b_\alpha \, z^\alpha := \Big(\int_{\mathbb{B}^n} z^\alpha \bar{z}^\alpha \, w_m(|z|) \, d\mu(z) \Big)^{-1/2} z^\alpha \,, \quad \alpha \in \mathbb{N}^n. \tag{1.5}$$

In particular, an orthonormal basis of $A^2(\mathbb{B}^n)$ is given by the family [157, Lemma 1.11]

$$v_\alpha(z) = b_\alpha\, z^\alpha := \left(\tfrac{(|\alpha|+n)!}{n!\,\alpha!\,\mu(\mathbb{B}^n)}\right)^{1/2} z^\alpha\,. \tag{1.6}$$

Clearly, $(v_\alpha|_{\mathbb{S}^{2n-1}})_{\alpha\in\mathbb{N}^n}$ is an orthogonal basis of $H^2(\mathbb{S}^{2n-1})$. We denote $\mathbf{\Pi}$ the orthogonal projection from $L^2(\mathbb{B}^n)$ to $A^2(\mathbb{B}^n)$.

For $s\in\mathbb{R}$, we denote $W^s(\Omega)$ and $W^s(\partial\Omega)$ the usual Sobolev spaces on Ω and $\partial\Omega$ respectively (see [109, Section 1] or [80, Appendix]).

Definition 1.1.2. *The holomorphic (resp. harmonic) Sobolev space on Ω of order $s\in\mathbb{R}$ is defined by*

$$W^s_{hol}(\Omega)\ \big(resp.\, W^s_{harm}(\Omega)\big) := \{f\in W^s(\Omega),\ f \text{ is holomorphic (resp. harmonic) on } \Omega\}.$$

Thus $W^0_{hol}(\Omega) = A^2(\Omega)$.
The set of harmonic functions in $L^2(\Omega, w_m)$ is denoted $L^2_{harm}(\Omega, w_m)$.

Remark 1.1.3. *In our case, Ω has a smooth boundary and we have the following characterizations for holomorphic Sobolev spaces: if $s<\frac{1}{2}$, $W^s_{hol}(\Omega)$ is exactly $L^2_{hol}(\Omega, w_{-2s})$ with equivalent norms [108, Section 4, Remark 1], and for any $s\in\mathbb{R}$, $f\in W^s_{hol}(\Omega)$ if and only if $\partial^\alpha f\in W^{s-m}_{hol}(\Omega)$ for any $m>s-1/2$ and $\alpha\in\mathbb{N}^n$ such that $|\alpha|\le m$ [13]. See also [60, Section 1] for more details.*

The link between the boundary $\partial\Omega$ and the interior Ω is given by the following extension operator:

Definition 1.1.4. *The Poisson operator K is the harmonic extension operator which solves the Dirichlet problem:*

$$\Delta\, Ku = 0 \ \text{ on } \Omega\,, \qquad \text{and} \qquad Ku = u\,, \text{ on } \partial\Omega\,.$$

Here, $\Delta = \partial\bar\partial$ denotes the complex Laplacian.

In a small neighborhood of a point $z\in\partial\Omega$, we work with the coordinates $(x,t)\in\mathbb{R}^{2n-1}\times\mathbb{R}^+$. One can check that in the case of the half-plane, the action of K on a function $u\in C^\infty(\mathbb{R}^{2n-1})$ is given in this coordinate system by

$$Ku\,(x,t) = (2\pi)^{-(2n-1)/2}\int_{\mathbb{R}^{2n-1}} e^{ix\xi}\, e^{-t|\xi|}\, \hat u(\xi)\, d\xi\,. \tag{1.7}$$

By elliptic regularity theory [109], the operator K extends to a continuous map from $W^s(\partial\Omega)$ onto $W^{s+1/2}_{harm}(\Omega) = W^{s+(m+1)/2}_{harm}(\Omega, w_m)$, for all $s\in\mathbb{R}$. In particular $K : C^\infty(\partial\Omega)\to C^\infty_{harm}(\overline{\Omega})$.
We denote by K_m the operator K considered as acting from $L^2(\partial\Omega)$ into $L^2(\Omega, w_m)$, and let K_m^* be its Hilbert space adjoint. A simple computation shows K_m^* is related to K^* $(= K^*_{r0})$ through the identity $K^*_m u = K^*(w_m u)$. Note also that K_m is injective since $0 = K_m u \Rightarrow K_m u|_{\partial\Omega} = 0 \Leftrightarrow u = 0$.
Now consider the operator $\Lambda_m : L^2(\partial\Omega)\to L^2(\partial\Omega)$, also denoted Λ_{w_m} and simply Λ in the unweighted case, defined as

$$\Lambda_m := K^*_m K_m = K^*\, w_m\, K\,. \tag{1.8}$$

Lemma 1.1.5. *With a weight of the form (1.4), the operator Λ_w is an elliptic selfadjoint pseudodifferential operator of order $-(m+1)$ on $\partial\Omega$ with principal symbol locally given by*

$$\sigma(\Lambda_m)(x,\xi) = \tfrac{1}{2}\Gamma(m+1)\,\chi(x,0)|\eta_x|^m\,|\xi|^{-(m+1)}\,, \quad (x,\xi)\in\mathbb{R}^{2n-1}\times\mathbb{R}^{2n-1}\,, \tag{1.9}$$

so, when $m\in\mathbb{N}$,

$$\sigma(\Lambda_m)(x,\xi) = 2^{-(m+1)}\,(\partial_t^m w_m)(x,0)\,|\xi|^{-(m+1)}\,, \quad (x,\xi)\in\mathbb{R}^{2n-1}\times\mathbb{R}^{2n-1}\,. \tag{1.10}$$

This is actually a subject of the extensive theory of calculus of boundary pseudodifferential operators due to Boutet de Monvel [19]. We chose here to give a more detailed proof as it can be found in the literature. The following remains valid for any real number $m > -1$.

Proof. For any $u \in L^2(\partial\Omega)$, $z \in \mathbb{R}^{2n-1}$, we have from (1.7)

$$\begin{aligned}(\Lambda_m u)(z) =& (2\pi)^{-(2n-1)} \int dx dt d\xi \, e^{i\xi(z-x)} e^{-t(|\xi|)} \, \chi(x,t) \, (-r)^m(x,t) \, (Ku)(x,t) \\ =& (2\pi)^{-2(2n-1)} \int dx dt d\xi dy d\zeta \, e^{i\xi(z-x)} e^{-t|\xi|} e^{-t|\zeta|} e^{i\zeta(x-y)} \, \chi(x,t) \, (-r)^m(x,t) \, u(y) \\ =& (2\pi)^{-2(2n-1)} \int dx dy d\xi d\zeta dt \, e^{i\xi(z-x)+i\zeta(x-y)} e^{-t(|\xi|+|\zeta|)} \, \chi(x,t) \, (-r)^m(x,t) \, u(y) \,,\end{aligned}$$

with $(x,y,\xi,\zeta,t) \in \mathbb{R}^{2n-1} \times \mathbb{R}^{2n-1} \times \mathbb{R}^{2n-1} \times \mathbb{R}^{2n-1} \times \mathbb{R}_{>0}$. This can be written as

$$(\Lambda_m u)(z) = \Big(\int_{\mathbb{R}^+} dt \, A_t \, B_t \, f \Big)(z) \,,$$

where A_t and B_t are two pseudodifferential operators depending on the parameter $t \in \mathbb{R}_{>0}$ with respective total symbol $\sigma_{tot}(A_t)(z,\xi) = e^{-t|\xi|}$ and $\sigma_{tot}(B_t)(x,\zeta) = e^{-t|\zeta|} \, \chi(x,t) \, (-r)^m(x,t)$. From the relation $\sigma(A_t B_t)(x,\xi) = \sigma(A_t)(x,\xi) \, \sigma(B_t)(x,\xi) = e^{-2t|\xi|} \, \chi(x,t) \, (-r)^m(x,t)$, we get for the principal symbol of Λ_m

$$\sigma(\Lambda_m)(x,\xi) = \int_{\mathbb{R}^+} dt \, e^{-2t|\xi|} \, \chi(x,t) \, (-r)^m(x,t) = \int_{\mathbb{R}^+} dt \, \tfrac{1}{2|\xi|} e^{-t} \, \chi(x, \tfrac{t}{2|\xi|}) \, (-r)^m(x, \tfrac{t}{2|\xi|}) \,.$$

The Taylor series of the term $\chi \, (-r)^m$ in the variable t near 0 gives

$$\begin{aligned}\chi(x, \tfrac{t}{2|\xi|}) \, (-r)^m(x, \tfrac{t}{2|\xi|}) &= \Big(- \sum_{j=1}^{\infty} \big(\tfrac{t}{2|\xi|}\big)^j \tfrac{1}{j!} \partial_t^j r(x,0) \Big)^m \Big(\sum_{j=0}^{\infty} \big(\tfrac{t}{2|\xi|}\big)^j \tfrac{1}{j!} \partial_t^j \chi(x,0) \Big) \\ &\underset{|\xi|\to\infty}{\sim} \chi(x,0) \, (-\partial_t r)^m(x,0) (2|\xi|)^{-m} \, t^m = \chi(x,0) \, |\eta_x|^m |\xi|^{-m} \, t^m \,,\end{aligned}$$

which leads to

$$\sigma(\Lambda_\omega)(x,\xi) = \tfrac{1}{2} \, \chi(x,0) \, |\eta_x|^m |\xi|^{-(m+1)} \int_{\mathbb{R}^+} dt \, e^{-t} t^m = \tfrac{1}{2} \Gamma(m+1) \, \chi(x,0) \, |\eta_x|^m |\xi|^{-(m+1)} \,.$$

The relation (1.10) is direct. □

The inverse operator Λ_m^{-1} is well defined on $\mathrm{Ran}(K_m^*)$, thus we have

$$\Lambda_m^{-1} \, K_m^* K_m = I_{L^2(\partial\Omega)} \,, \qquad \text{and} \qquad K_m \, \Lambda_m^{-1} \, K_m^* = \mathbf{\Pi}_{m,harm} \,, \tag{1.11}$$

where $\mathbf{\Pi}_{m,harm}$ is the orthogonal projection from $L^2(\Omega, w_m)$ onto $L^2_{harm}(\Omega, w_m)$. To get the second equality, apply K_m on both sides of the first one, and deduce that $K_m \Lambda_m^{-1} K_m^*$ is the identity on $\overline{\mathrm{Ran}(K_m)}$ which is the closure of $W^{1/2}_{harm}(\Omega)$ in $L^2(\Omega, w_m)$, i.e. $L^2_{harm}(\Omega, w_m)$. Then conclude by observing that $K_m \Lambda_m^{-1} K_m^*$ vanishes on $\overline{\mathrm{Ran}(K_m)}^{\perp} = \mathrm{Ker}(K_m^*)$.

We can now introduce the left inverse of K_m, which takes the boundary value of any function f in $L^2_{harm}(\Omega, w_m)$.

Definition 1.1.6. *The trace operator $\gamma_m : L^2(\Omega, w_m) \to L^2(\partial\Omega)$ is defined by*

$$\gamma_m := \Lambda_m^{-1} K_m^* \,.$$

In particular, (1.11) gives

$$K_m\,\gamma_m|_{L^2_{harm}(\Omega,w_m)} = I_{L^2_{harm}(\Omega,w_m)}\,, \quad \text{and} \quad \gamma_m\,K_m = I_{L^2(\partial\Omega)}\,. \tag{1.12}$$

The operator γ_m extends continuously to $\gamma_m : W^s_{harm}(\Omega) \to W^{s-1/2}(\partial\Omega)$ for any $s \in \mathbb{R}$.

Definition 1.1.7. *The Hardy space is*

$$H^2(\partial\Omega) := H^2 := W^0_{hol}(\partial\Omega)\,,$$

where, for any $s \in \mathbb{R}$,

$$W^s_{hol}(\partial\Omega) := \{u \in W^s(\partial\Omega),\ Ku \text{ is holomorphic on } \Omega\}\,.$$

Denote Π the orthogonal projection from $L^2(\partial\Omega)$ into $H^2(\partial\Omega)$, also called the Szegö projection.

Note that the Hardy space can be equivalently defined as the closure of $C^\infty_{hol}(\partial\Omega)$ in $L^2(\partial\Omega)$, or as the boundary values of holomorphic functions that are square integrable up to the boundary.

1.1.2 Fock space

We use a formal script font for operators related to the Fock space.

For $\mathfrak{m} \in \mathbb{R}$, the spaces $\mathrm{GLS}^{\mathfrak{m}}(\mathbb{C}^n)$ and $\mathscr{S}^{\mathfrak{m}}(\mathbb{C}^n)$ correspond respectively to the spaces $\mathrm{GLS}^{\mathfrak{m}}(\mathbb{R}^n \times \mathbb{R}^n)$ (Definition C.1.11) and $\mathscr{S}^{\mathfrak{m}}(\mathbb{R}^n \times \mathbb{R}^n)$ (Definition C.1.12) by replacing in the definitions x and ξ with z and $\bar{z}$. These complex versions has been considered in [24, Section 3].

Definition 1.1.8. *Let $\mathfrak{m} \in \mathbb{R}$ and $\rho_{\mathfrak{m}}$ be a strictly positive function in $\mathscr{S}^{\mathfrak{m}}(\mathbb{C}^n)$. The Fock space is*

$$\mathscr{F}_{\mathfrak{m}}(\mathbb{C}^n) := \mathscr{F}_{\mathfrak{m}} := \{\varphi \in L^2(\mathbb{C}^n, \rho_{\mathfrak{m}}(z)e^{-|z|^2}\,d\mu(z))\,, \varphi \text{ holomorphic on } \mathbb{C}^n\}\,.$$

Denote $\mathscr{F}(\mathbb{C}^n)$ or $\mathscr{F}$ when $\rho_{\mathfrak{m}} = 1$.
Let also $\mathscr{P}_{\mathfrak{m}}$ and $\mathscr{P}$ be the orthogonal projections from $L^2(\mathbb{C}^n)$ to $\mathscr{F}_{\mathfrak{m}}$ and $\mathscr{F}$ respectively.

Remark 1.1.9. *The Fock space $\mathscr{F}$, introduced in [11, 130], is sometimes called the Bargmann–Segal space. It is isomorphic [11] to the* bosonic Fock space*, defined in the context of quantum mechanics as the Hilbert space completion of*

$$\bigoplus_{j\in\mathbb{N}} \mathrm{Sym}(\mathcal{H}^{\otimes j})\,,$$

where Sym *is the operator that symmetrizes a tensor product, and $\mathcal{H}$ is an Hilbert space representing all states of a single particle.*

The family of functions $(u_\alpha)_{\alpha\in\mathbb{N}^n}$, where

$$u_\alpha(z) := a_\alpha\, z^\alpha := (\pi^n \alpha!)^{-1/2}\, z^\alpha\,, \tag{1.13}$$

forms an orthonormal basis of $\mathscr{F}$. Recall that the Bargmann transform $\mathscr{B} : L^2(\mathbb{R}^n) \to \mathscr{F}$, defined as

$$(\mathscr{B}f)(z) := \pi^{-3n/4} \int_{\mathbb{R}^n} e^{-\frac{1}{2}(z^2 - 2\sqrt{2}\,xz + x^2)}\, f(x)\, d\mu(x)\,, \tag{1.14}$$

is a unitary, with inverse

$$(\mathscr{B}^{-1}\varphi)(z) := \pi^{-3n/4} \int_{\mathbb{C}^n} e^{-\frac{1}{2}(\bar{z}^2 - 2\sqrt{2}\,x\bar{z} + x^2)}\, e^{-|z|^2} \varphi(z)\, d\mu(z)\,.$$

We have the relations [24, (14)]

$$\mathscr{B}^{-1}\, z_j\, \mathscr{B} = W_{x_j + i\xi_j}\,, \quad \text{and} \quad \mathscr{B}^{-1}\, \partial_{z_j}\, \mathscr{B} = W_{x_j - i\xi_j}\,,$$

where W denotes the Weyl operator on $L^2(\mathbb{R}^n \times \mathbb{R}^n)$, see Definition (C.4). As a consequence, the space $\mathscr{S}^s(\mathbb{R}^n \times \mathbb{R}^n)$ is identified with $\mathscr{S}^s(\mathbb{C}^n)$ for any $s \in \mathbb{R}$. Weyl operators can thus be defined naturally on the Fock space $\mathscr{F}$:

Definition 1.1.10. *Let $s \in \mathbb{R}$. The complex Weyl operator $\mathscr{W}_\sigma : \mathscr{F} \to \mathscr{F}$, where $\sigma \in \mathscr{S}^s(\mathbb{C}^n)$ is defined by the relation*

$$\mathscr{B}^{-1}\, \mathscr{W}_\sigma \mathscr{B} = W_{\mathscr{B}^{-1}\sigma\mathscr{B}}\,.$$

1.1.3 The Heisenberg Lie algebra and its representations

We will see in Section 1.3 that Toeplitz operators on $A^2(\mathbb{B}^n)$, where $\mathbb{B}^n$ denotes again the unit ball of $\mathbb{C}^n$, can be seen as the realization of elements of the enveloping algebra of the Heisenberg Lie algebra. This section also sets up some useful unitaries between the previous Hilbert spaces. In the following notations, the indices A, F, H and L refer respectively to the spaces $A^2_m(\Omega)$, $\mathscr{F}(\mathbb{C}^n)$, $H^2(\partial\Omega)$ and $L^2(\mathbb{R}^n)$.
Recall that the Heisenberg group $\mathbb{H}^n$ is the set $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ endowed with the product:

$$(q,p,t)\,(q',p',t') = \big(q+q', p+p', t+t' + \tfrac{1}{2}(qp' - pq')\big)\,.$$

The unit element is $(0,0,0)$ and $(q,p,t)^{-1} = (-q,-p,-t)$.
For $j \in \{1,\cdots,n\}$, denote $\mathbb{Q}_j$, $\mathbb{P}_j$ and $\mathbb{T}$ the generators of $\mathbb{H}^n$:

$$\begin{aligned} \exp(\mathbb{Q}_j) &:= \ (1_j, (0,\ldots,0), 0)\,, \\ \exp(\mathbb{P}_j) &:= \ \big((0,\ldots,0), 1_j, 0\big)\,, \\ \exp(\mathbb{T}) &:= \ \big((0,\ldots,0), (0,\ldots,0), 1\big)\,, \end{aligned} \tag{1.15}$$

where 1_k denotes the multiindex of $\mathbb{N}^n$ being zero everywhere and 1 at the k^{th} position.
These generators form a basis of the Lie algebra $\mathfrak{h}^n$ of $\mathbb{H}^n$. The only non null commutation relations are $[\,\mathbb{Q}_j,\, \mathbb{P}_k\,] = \delta_{j,k}\,\mathbb{T}$. Let

$$a_j^+ := \tfrac{1}{\sqrt{2}}(\mathbb{Q}_j - i\,\mathbb{P}_j) \quad \text{and} \quad a_j^- := \tfrac{1}{\sqrt{2}}(\mathbb{Q}_j + i\,\mathbb{P}_j)\,, \tag{1.16}$$

be the j^{th} creation and annihilation elements of $\mathfrak{h}^n$, which verify

$$[\,a_j^-,\, a_k^+\,] = -\tfrac{i}{2}(\,[\,\mathbb{Q}_j,\, \mathbb{P}_k\,] + [\,\mathbb{Q}_k,\, \mathbb{P}_j\,]\,) = -i\delta_{j,k}\,\mathbb{T}\,.$$

We will also use the following element $\mathfrak{N}$ of the universal enveloping algebra $\mathrm{Env}(\mathfrak{h}^n)$ of $\mathfrak{h}^n$:

$$\mathfrak{N} := \tfrac{1}{2}\sum_{j=1}^n a_j^+ a_j + a_j a_j^+\,.$$

Definition 1.1.11. *The representation π_L of $\mathfrak{h}^n$ on $L^2(\mathbb{R}^n)$ (also called the Schrödinger representation) is defined by*

$$\pi_L(\mathbb{Q}_j)f(x) := x_j\, f(x)\,, \quad \pi_L(\mathbb{P}_j)f(x) := -i\,\partial_{x_j} f(x)\,, \quad \pi_L(\mathbb{T})f(x) := i\, f(x)\,.$$

We have also

$$\pi_L(a_j^+) = \tfrac{1}{\sqrt{2}}(x_j - \partial_{x_j})\,, \quad \pi_L(a_j^-) = \tfrac{1}{\sqrt{2}}(x_j + \partial_{x_j})\,, \quad \pi_L(\mathfrak{N}) = \tfrac{1}{4}\sum_{j=1}^{n}(x_j^2 - \partial_{x_j}^2)\,.$$

Remark 1.1.12. *Since the role of $\hbar$ is not relevant in this section, we chose to set $\hbar = 1$ in the previous representation.*

We use the Bargmann transform $\mathscr{B}$ in order to get a unitary representation of $\mathfrak{h}^n$ on the Fock space $\mathscr{F}$:

Definition 1.1.13. *The Fock representation π_F of an element $h \in \mathfrak{h}^n$ on $\mathscr{F}$ is*

$$\pi_F(h) := \mathscr{B}\,\pi_L(h)\,\mathscr{B}^{-1}.$$

Proposition 1.1.14 ([61, Proposition 3.7])**.** *The explicit actions on the basis (1.13) of $\mathscr{F}$ are given by*

$$\begin{aligned}
&\pi_F(\mathbb{Q}_j)\,u_\alpha = \tfrac{1}{\sqrt{2}}\left(\alpha_j^{1/2}\,u_{\alpha-1_j} + (\alpha_j+1)^{1/2}\,u_{\alpha+1_j}\right),\\
&\pi_F(\mathbb{P}_j)\,u_\alpha = -\tfrac{i}{\sqrt{2}}\left(\alpha_j^{1/2}\,u_{\alpha-1_j} - (\alpha_j+1)^{1/2}\,u_{\alpha+1_j}\right),\\
&\pi_F(\mathbb{T})\,u_\alpha = i\,u_\alpha\,,\\
&\pi_F(a_j^+)\,u_\alpha = (\alpha_j+1)^{1/2}\,u_{\alpha+1_j}\,, \quad \pi_F(a_j^-)\,u_\alpha = \alpha_j^{1/2}\,u_{\alpha-1_j}\,, \quad \pi_F(\mathfrak{N})\,u_\alpha = (|\alpha| + \tfrac{n}{2})\,u_\alpha\,.
\end{aligned}$$

Proof. Differentiating (1.14), we get $\partial_{z_j}(\mathscr{B}f) = \mathscr{B}\left((-z_j + \sqrt{2}\,x_j)f\right)$, so $(z_j + \partial_{z_j})\,\mathscr{B} = \mathscr{B}\,(\sqrt{2}\,x_j)$, and

$$\mathscr{B}\,x_j\,\mathscr{B}^{-1} = \tfrac{1}{\sqrt{2}}\,(z_j + \partial_{z_j})\,,$$

while integration by parts in (1.14) gives $\mathscr{B}\,(\partial_{x_j} f) = \mathscr{B}\left((x_j - \sqrt{2}\,z_j)f\right)$, so we obtain $\mathscr{B}\,(x_j - \partial_{x_j}) = (\sqrt{2}\,z_j)\,\mathscr{B}$, or

$$\mathscr{B}\,\partial_{x_j}\,\mathscr{B}^{-1} = \tfrac{1}{\sqrt{2}}(-z_j + \partial_{z_j})\,.$$

We get the result from (1.13). □

For the peculiar case $\Omega = \mathbb{B}^n$, define the unitaries $U_{A_mF} : \mathscr{F} \to A^2_m(\mathbb{B}^n)$ and $U_{A_mL} : L^2(\mathbb{R}^n)$ to $A^2_m(\mathbb{B}^n)$ as

$$U_{A_mF}\,(u_\alpha) := v_{m,\alpha}\,, \quad U_{A_mL} := U_{A_mF}\,\mathscr{B}\,,$$

where $v_{m,\alpha}$ are defined in (1.6).

Definition 1.1.15. *The Bergman representation π_{A_m} of an element $h \in \mathfrak{h}^n$ on $A^2_m(\mathbb{B}^n)$ is given by*

$$\pi_{A_m}(h) := U_{A_mL}\,\pi_L(h)\,U_{A_mL}^{-1}\,.$$

We denote by π_A this representation in the case $w = r^0 = 1$.

The following results are straightforward, since the Bergman representation differs from the Fock representation by a simple change of basis:

Proposition 1.1.16 ([61, Proposition 3.9])**.** *The representation π_{A_m} has the following properties:*

$$\begin{aligned}
&\pi_{A_m}(\mathbb{Q}_j)\, v_{m,\alpha} = \tfrac{1}{\sqrt{2}}\left(\alpha_j^{1/2}\, v_{m,\alpha-1_j} + (\alpha_j+1)^{1/2}\, v_{m,\alpha+1_j}\right),\\
&\pi_{A_m}(\mathbb{P}_j)\, v_{m,\alpha} = -\tfrac{i}{\sqrt{2}}\left(\alpha_j^{1/2}\, v_{m,\alpha-1_j} - (\alpha_j+1)^{1/2}\, v_{m,\alpha+1_j}\right),\\
&\pi_{A_m}(\mathbb{T})\, v_{m,\alpha} = i\, v_{m,\alpha}\,,\\
&\pi_{A_m}(a_j^+)\, v_{m,\alpha} = (\alpha_j+1)^{1/2}\, v_{m,\alpha+1_j}\,, \quad \pi_{A_m}(a_j^-)\, v_{m,\alpha} = \alpha_j^{1/2}\, v_{m,\alpha-1_j}\,,\\
&\pi_{A_m}(\mathfrak{N})\, v_{m,\alpha} = (|\alpha| + \tfrac{n}{2})\, v_{m,\alpha}\,.
\end{aligned}$$

1.2 Toeplitz operators

1.2.1 Classical Toeplitz operators

Definition 1.2.1. *Let $\mathfrak{m} \in \mathbb{R}$, another $m > -1$, $\psi \in L^\infty(\mathbb{C}^n)$, $f \in L^\infty(\Omega)$ and $u \in L^\infty(\partial\Omega)$. The classical Toeplitz operators on $\mathscr{F}_\mathfrak{m}$, A^2_m and H^2 are defined respectively as*

$$\begin{array}{rcccl}
\mathscr{T}_g & : & \mathscr{F}_\mathfrak{m} & \to & \mathscr{F}_\mathfrak{m}\\
 & & \phi & \mapsto & \mathscr{T}_g(\phi) := \mathscr{P}_\mathfrak{m}(g\,\phi)\\[1em]
\mathbf{T}_f & : & A^2_m & \to & A^2_m\\
 & & \phi & \mapsto & \mathbf{T}_f(\phi) := \mathbf{\Pi}_m(f\,\phi)\\[1em]
T_u & : & H^2 & \to & H^2\\
 & & \phi & \mapsto & T_u(\phi) := \Pi(u\,\phi)\,,
\end{array}$$

where $\mathscr{P}_\mathfrak{m}$, $\mathbf{\Pi}_m$ and Π are the orthogonal projectors (see respectively Definition 1.1.1, Definition 1.1.7 and Definition 1.1.8). The functions g, f and u are called the symbols of the corresponding Toeplitz operators.

We will also consider smooth functions as symbols instead of just bounded ones. The notations $\mathscr{T}_\psi^{(\mathfrak{m})}$ and $\mathbf{T}_f^{(m)}$ are also used in order to specify that the corresponding Hilbert space depends on the weight $\rho_\mathfrak{m}$ and w_m respectively. The following properties for the Hardy case

$$u \mapsto T_u \text{ is linear,} \quad T_u^* = T_{\bar{u}}\,, \quad T_1 = I\,, \quad \|\, T_u \,\| \le \|\, u \,\|_\infty\,.$$

remain valid for the Fock and Bergman spaces. Remark that for two functions ϕ, ψ in the Hardy space, we have

$$\langle\, T_u\phi\,,\, \psi\,\rangle_{H^2} = \langle\, \Pi u\,\phi\,,\, \psi\,\rangle_{H^2} = \langle\, u\,\phi\,,\, \psi\,\rangle_{H^2} \ne \langle\, (\Pi u)\phi\,,\, \psi\,\rangle_{H^2} = \langle\, M_{\Pi u}\phi\,,\, \psi\,\rangle_{H^2}\,,$$

unless of course if $u \in H^2$ (again, the same holds for the other spaces). For any strictly positive function u in $L^\infty(\partial\Omega)$, T_u is a selfadjoint and positive definite operator on $H^2(\partial\Omega)$ since

$$\langle\, T_u\phi\,,\, \phi\,\rangle = \int_{\partial\Omega} u(z)|\phi(z)|^2\, d\mu(z) > 0\,, \qquad \text{for any } \phi \ne 0\,.$$

In particular, it is an injection, so there exists an inverse T_u^{-1}, which is densely defined on $H^2(\partial\Omega)$. The same is true in the Bergman case for $\mathbf{T}_f$, $f \in L^\infty(\Omega)$.
We will see in Section 2.1 that in the context of geometric quantization, Toeplitz operators can be defined as a sequence of operators acting on holomorphic sections of tensor powers of a line bundle over a Kähler manifold.

1.2.2 Generalized Toeplitz operators

We refer to Appendix C for generalities about pseudodifferential operators and Remark C.1.7 for the definition OPS, when S is a space of symbols.

1.2.2.1 Definition

The theory of GTOs, developed in [22] by Boutet de Monvel and Guillemin has been established in a very general framework and concerns smooth compact manifolds. The two principal objects are a symplectic closed subcone Σ of the cotangent space and a so-called *Toeplitz structure* Π_Σ, which acts on the Hilbert space of square summable half-densities on the manifold and has the same microlocal features as the previous Szegö projector Π (see [22, Definition 2.10] for the complete definition).

Actually, on a compact manifold M, " *pseudodifferential operators are Toeplitz operators in disguise* ", as Guillemin writes [83, Section 5]. Indeed, the corresponding symplectic cone Σ is just the cotangent bundle $T^*M\backslash\{0\}$, while the Toeplitz structure is simply the identity operator on $L^2(M)$.

For our purposes, we restrict to the case when Ω is a smoothly bounded strictly pseudoconvex domain of $\mathbb{C}^n$, as in Section 1.1 (see Appendix A.5 for details). Here, the compact manifold is the boundary $\partial\Omega$, which is also a contact manifold, hence carries a contact form η given by (1.2). The natural symplectic cone Σ we consider consists is all positive multiples of the contact form η (A.10), while the Toeplitz structure is just the Szegö projector Π (Definition 1.1.7).

Definition 1.2.2. *For a pseudodifferential operator P on $L^2(\partial\Omega)$ of order $s \in \mathbb{R}$, the generalized Toeplitz operator (GTO) $T_P : W^s_{hol}(\partial\Omega) \to H^2(\partial\Omega)$ is defined by*

$$T_P := \Pi\, P|_{W^s_{hol}(\partial\Omega)}\,.$$

One can alternatively extend the definition of $T_P : W^s(\partial\Omega) \to H^2(\partial\Omega)$ by taking $T_P = \Pi\, P\, \Pi$.

We see that the structure of GTOs is the one of classical Toeplitz operators on the Hardy space, but after replacing the multiplication by the function u with an operator $P \in \Psi\mathrm{DO}(\partial\Omega)$. Despite their apparent complex definition, it turns out that GTOs are far more convenient to work with than classical ones. First of all, the product of two GTOs is again a GTO, which was not the case for classical Toeplitz operators. Moreover, they enjoy very similar properties to the usual pseudodifferential operators, allowing to define the notions of order and principal symbol. However, the microlocal structure of GTOs generates some subtleties we must take care of. The two following sections are devoted to describe the relation between GTOs and pseudodifferential operators and also the definition of their order and symbol.

1.2.2.2 Microlocal structure

Techniques of *microlocal analysis* [134, 52] have been developed to characterize operators appearing in partial differential equations. Pseudodifferential operators are described by their symbol and act on functions by switching from local variables to dual ones via the Fourier transform. The smoothness of a function, with variables $x \in \mathbb{R}^n$, is characterized by the asymptotical behaviour of its Fourier transform $|\hat{f}(\xi)|$ as $|\xi| \to +\infty$. The idea of microlocal analysis is to work in a conic neighborhood of the cotangent bundle at some point x_0, instead of only looking at a small neighborhood of the point x_0.

Following [20, 23], we now present the microlocal structure of GTOs. We need to introduce operators of Hermite type to understand the relation between GTOs on $\partial\Omega$ and pseudodifferential operators on $\mathbb{R}^n$.

Definition 1.2.3. *A cone is a smooth principal bundle under the action of* $\mathbb{R}^+$ *over a smooth manifold.*

For instance, if X is an open set of $\mathbb{R}^n$, $T^*X\backslash\{0\}$ is a cone, whose elements are subject to the action $(x, \lambda\xi) = \lambda(x, \xi)$, $\lambda \in \mathbb{R}_{>0}$.

For the rest of this section, V denotes a conic neighborhood of some fixed point $(z, \eta_z) \in \Sigma$. Consider the following open cones (see [20, Section 1]):

- $U_0 := \{(x, y, \xi) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \times (\mathbb{R}^n\backslash\{0\})\}$,
- $U_1 =: \{(x, y, y', t) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}_{>0}\}$,
- $V_0 := \{(x, y, \xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \times (\mathbb{R}^n\backslash\{0\}) \times (\mathbb{R}^{n-1}\backslash\{0\})\}$,
- $V_1 := \{(x, y, y', y'', t) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}_{>0}\}$,
- $\Sigma_0 := \{(x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})\}$,
- $\Sigma_1 := \{(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}_{>0}\}$.

Definition 1.2.4 ([20, (1.3)])**.** *The space* $S^{m,k}(U_1, \Sigma_1)$ *is the set of all smooth functions on* U_1 *such that for any* $p \in \mathbb{N}$ *and multiindices* α, β_1, β_2*:*

$$|\partial_{y'}^{\alpha} \partial_x^{\beta_1} \partial_y^{\beta_2} \partial_t^p a|(x, y, y', t) \lesssim t^{m-p} \left(|y'|^2 + \tfrac{1}{t}\right)^{(k-|\alpha|)/2}.$$

Define also $\mathscr{H}^m(U_1, \Sigma_1) := \cap_{j\in\mathbb{N}} S^{m-j,-2j}(U_1, \Sigma_1)$*. According to [20, (5.2)],* $a \in \mathscr{H}^m(U_1, \Sigma_1)$ *if and only if*

$$|y'^{\alpha} \partial_{y'}^{\beta} \partial_x^{\gamma_1} \partial_y^{\gamma_2} \partial_t^p a|(x, y, y', t) \lesssim t^{m-p-|\alpha|/2+|\beta|/2}.$$

Note that this definition also makes sense for the couples (V_1, Σ_1) and (U_0, Σ_0).

Definition 1.2.5. *[20, Definition 5.6] A Hermite operator* H *on* U_0 *of order* $m \in \mathbb{R}$ *is an element of* $OP\mathscr{H}^{m+(n-1)/4}(U_0, \Sigma_0)$*, i.e. is of the form*

$$H = H' + R : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{2n-1}),$$

where R *is a smoothing operator and* H' *has the integral representation*

$$(H'f)(x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi}\, h(x, y, \xi)\, \hat{f}(\xi)\, d\xi,$$

where $h \in \mathscr{H}^{m+(n-1)/4}(U_0, \Sigma_0)$*.*
The set of such operators is denoted $\mathrm{Herm}^m(U_0, \Sigma_0)$*.*
The function h *is called the symbol of* H *which we denote by* $\sigma(H)$*.*

Finally, the space of Fourier Integral Operator of order m is denoted FIO^m, see [92] for a complete description of Fourier Integral Operators.
We recall the relations between the different classes of operators (see [20, (5.9)–(5.12) and p. 613] for the details):

$$\begin{aligned}
\mathrm{Herm}^m \circ \Psi\mathrm{DO}^{m'} &\in \mathrm{Herm}^{m+m'}\,, \\
(\mathrm{Herm}^m)^* \circ \mathrm{Herm}^{m'} &\in \Psi\mathrm{DO}^{m+m'}\,, \\
\mathrm{Herm}^m \circ (\mathrm{Herm}^{m'})^* &\in OP\mathscr{H}^{m+m'}\,, \\
OPS^{m,k} \circ \mathrm{Herm}^{m'} &\in \mathrm{Herm}^{m+m'-k/2}\,, \\
FIO^m \circ \mathrm{Herm}^{m'} &\in \mathrm{Herm}^{m+m'}\,.
\end{aligned} \tag{1.17}$$

The following result establishes the relation between $\Psi\mathrm{DO}^m(\mathbb{R}^{2n-1})$ and $OPS^{m,k}(V_0, \Sigma_0)$:

Proposition 1.2.6 ([20, Example 1.4]). *Let $P \in \Psi\mathrm{DO}^m(\mathbb{R}^{2n-1})$ whose total symbol p has in V_0 the asymptotic expansion*

$$p \sim \sum_{j\in\mathbb{N}} p_{m-j/2}\,, \tag{1.18}$$

with $p_{m-j/2}$ homogeneous smooth functions of degree $m - j/2$. Then $p \in S^{m,k}(V_0, \Sigma_0)$, $k \in \mathbb{N}$, if and only if for any $j \in \mathbb{N}$, $p_{m-j/2}$ vanishes to order (at least) $k - j$ on Σ_0 (there is no condition if $j \geq k$).

Definition 1.2.7. *Let X be a compact space and $\mathcal{C}$ be a cone in T^*X.
For a function $f \in C^\infty(T^*X)$, denote $v_{\mathcal{C}}(f)$ the vanishing order of f on $\mathcal{C}$. In local coordinates,*

$$v_{\mathcal{C}}(f) = \min\{k \in \mathbb{N}\ , \partial_x^\alpha f(x,\xi) \neq 0\ , |\alpha| = k\ , (x,\xi) \in \mathcal{C}\}\,.$$

Let P be a pseudodifferential operator on X, whose total symbol p given in some coordinate system, verifies (1.18). Define the quantity

$$k_P := \min_{j\in\mathbb{N}}\{v_{\mathcal{C}}(p_{m-j/2}) + j\}\,.$$

Remark 1.2.8. *Using the same expansion on V_0 for the total symbol of P, we get equivalently*

$$P \in \Psi\mathrm{DO}^m(\mathbb{R}^{2n-1}) \Leftrightarrow P \in OPS^{m,k_P}(V_0, \Sigma_0)\,,$$

In particular, if the total symbol of P vanishes to order $k \in \mathbb{N}$ on Σ_0, then $P \in OPS^{m,k}(V_0, \Sigma_0)$, and if the principal symbol of P does not vanish identically on Σ_0, then $P \in OPS^{m,0}(V_0, \Sigma_0)$.

Remark 1.2.9. *As in [20], we suppose that total symbols of pseudodifferential operators verify (1.18) instead of (C.1), i.e., they admit half integers homogeneous components. Thus for the classical pseudo-differential operators presented in Appendix C, the terms $p_{m-j/2}$ vanish for any $j \in 2\mathbb{N} + 1$.*

The model is given by the operators $D_j : \mathcal{S}(\mathbb{R}^{2n-1}) \to \mathcal{S}(\mathbb{R}^{2n-1})$, $j = 1, \ldots, n-1$, defined as

$$D_j := \partial_{y_j} + y_j|D_x|\,,$$

where $|D_x|$ is such that $\widehat{|D_x|f}(\xi,\eta) = |\xi|\hat{f}(\xi,\eta)$, $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^{n-1}$ being the dual variables of $(x,y) \in \mathbb{R}^n \times \mathbb{R}^{n-1}$. Let $\mathcal{H}_0$ be the L^2-closure of the space of functions $f \in \mathcal{S}(\mathbb{R}^{2n-1})$ solutions to $D_j f = 0$, for any $j = 1, \ldots, n-1$, and Π_0 the orthogonal projector from $L^2(\mathbb{R}^{2n-1})$ to $\mathcal{H}_0$. From

[20], there is a canonical transformation $\phi : V_0 \to V$ which defines a symplectic isomorphism between $V_0 \cap \Sigma_0$ and $V \cap \Sigma$. Modulo smoothing operators, there is an elliptic positive FIO F on V_0 of order 0, associated to ϕ, which transforms the left ideal of pseudodifferential operators generated by the D_j into the left ideal generated by the components of the boundary Cauchy–Riemann operator $\bar{\partial}_b$ (see Definition A.3.4). Moreover, it verifies

$$FF^* \sim I \text{ on } V \quad \text{ and } \quad F^*\Pi F \sim \Pi_0 \text{ on } V_0 \,,$$

(recall from Appendix C that $A \sim B$ if and only if they differ by a smoothing operator; this must not be confused with $A = B$, which is the equality between operators acting on some Hilbert space). According to [20, p. 603], the spaces $OPS^{m,k}(U_1, \Sigma_1)$ from Definition 1.2.4 are invariant under coordinate changes, which allows to define the classes $OPS^{m,k}(V, \Sigma)$. In other words, in $V \cap \partial\Omega$, the total symbol of $P \in \Psi\mathrm{DO}(\partial\Omega)$ depends on the choice of coordinate patch from $V \cap \partial\Omega$ to $V_0 \cap \mathbb{R}^{2n-1}$, but the classes $OPS^{m,k}(V, \Sigma)$ do not. Thus, in $V \cap \partial\Omega$, the vanishing order of the terms of the total symbol of P in the expansion (1.18) is independent of the choice of coordinate patch.
From [20, Proposition 3.12], $P \in OPS^{m,k}(V, \Sigma)$ if and only if $F^*PF \in OPS^{m,k}(V_0, \Sigma_0)$ (the original result is just an implication, but the reciprocal is direct by considering the FIO F' on V associated to ϕ^{-1}). Remark 1.2.8 can thus be shifted to the level of $\partial\Omega$:

Remark 1.2.10. *We have*

$$P \in \Psi\mathrm{DO}^m(\partial\Omega) \Leftrightarrow P \in OPS^{m,k_P}(V, \Sigma) \,,$$

with k_P as in Definition 1.2.7. In particular, if the total symbol of P vanishes to order $k \in \mathbb{N}$ on Σ, then $P \in OPS^{m,k}(V, \Sigma)$, and if the principal symbol of P does not vanish identically on Σ, then P belongs to $OPS^{m,0}(V, \Sigma)$.

Let $H_0 : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{2n-1})$ be the operator

$$(H_0 f)(x, y) = (2\pi)^{-n} \int_{\xi \in \mathbb{R}^n} e^{ix\xi - \frac{1}{2}y^2|\xi|} \, (\tfrac{|\xi|}{2\pi})^{(n-1)/4} \, \hat{f}(\xi) \, d\xi, \qquad (x, y) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \,.$$

It is an Hermite operator of order 0 which verifies $D_j H_0 = 0$ for any $j = 1, \ldots, n-1$.

Remark 1.2.11. *It corresponds to H_α in [20, Example 5.7] when $\alpha = 0$, to H_0 in [21, p. 253], or R in [22, (2.7)].*

It defines an isomorphism from $L^2(\mathbb{R}^n)$ to $\mathcal{H}_0$ [21, Proposition 2.2], and we have the relations [21, (2.8)]

$$H_0^* H_0 = I_{C^\infty(\mathbb{R}^n)} \,, \quad H_0 H_0^* = \Pi_0 \,.$$

Let $P \in \Psi\mathrm{DO}^m(\partial\Omega)$ with total symbol p. Denoting $k := v_\Sigma(p)$, $F^*PF \in OPS^{m,k}(V_0, \Sigma_0)$. From (1.17), F^*PFH_0 is an Hermite operator on $\mathbb{R}^n$ of order $m - k/2$ and $H_0^* F^* P F H_0$ is in $\Psi\mathrm{DO}^{m-k/2}(\mathbb{R}^n)$. Then, from (1.17) again, any operator A of the form

$$A \sim H_0^* F^* F H_0 : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

is a classical pseudodifferential operator of order 0 on $\mathbb{R}^n$ and moreover the following operator associated to A

$$H \sim F H_0 A^{-1/2} : C^\infty(\mathbb{R}^n) \to C^\infty(\partial\Omega) \tag{1.19}$$

is an Hermite operator of order 0 which verifies

$$H^* H \sim I \quad \text{and} \quad HH^* \sim \Pi \,. \tag{1.20}$$

Combining this to Remark 1.2.10, we obtain:

Proposition 1.2.12.

$$P \in \Psi\mathrm{DO}^m(\partial\Omega) \Leftrightarrow H^* P H \in \Psi\mathrm{DO}^{m-k_P/2}(\mathbb{R}^n)\,,$$

with k_P as in Definition 1.2.7.

Locally, the operator H (1.19) brings back any GTO T_P to a pseudodifferential operator Q on $\mathbb{R}^n$:

$$T_P = \Pi P \Pi \sim HH^*\,P\,HH^* \sim HQH^*, \quad Q \sim H^* P H \sim H^* T_P H\,.$$

According to [21, p. 253], the map $T_P \mapsto Q \sim H^* P H$ is an isomorphism [(2)], for we have the identity $\sigma_{tot}(Q) = \sigma_{tot}(P) \circ \phi$.

1.2.2.3 Order and symbol of a GTO

In [21, 22], the order of a GTO T_P is defined as the order of P (in the sense of pseudodifferential operators) and its symbol as the one of P evaluated on the symplectic cone Σ. However, this leads to an ambiguity since we can always construct another pseudodifferential operator Q on $\partial\Omega$ such that $\mathrm{ord}(P) \neq \mathrm{ord}(Q)$, $\sigma(P) \neq \sigma(Q)$ and $T_P \sim T_Q$. The construction is recursive: we choose some Q_0 such that $q_0 := \mathrm{ord}(Q_0) - \frac{1}{2}k_{Q_0} = \mathrm{ord}(P) - \frac{1}{2}k_P$ and the leading term in (1.18) of $\partial^\alpha \sigma_{tot}(Q_0)|_\Sigma$, $|\alpha| = k_{Q_0}$, equals to $\sigma(P)$ (there are plenty of such operators Q_0). Then $H^*(P - Q_0)H \in \Psi\mathrm{DO}^{q_0-1}(\mathbb{R}^n)$. Similarly, construct a family of operators $(Q_j)_{j\in\mathbb{N}}$, verifying $H^*(P - \sum_{j=0}^{N-1} Q_j)H \in \Psi\mathrm{DO}^{q_0-N}(\mathbb{R}^n)$ for any $N \in \mathbb{N}$, and take finally $Q = \sum_{j\in\mathbb{N}} Q_j$.
Thus we get $H^*(P - Q)H \in \Psi\mathrm{DO}^{-\infty}(\partial\Omega)$, hence $T_P \sim T_Q$. Moreover,

$$\mathrm{ord}(Q) = \mathrm{ord}(Q_0) = \mathrm{ord}(P) - \tfrac{1}{2}(k_P - k_{Q_0}) \quad \text{and} \quad \sigma(Q) = \sigma(Q_0)$$

can be chosen almost arbitrarily. The only two constraints on Q are $\mathrm{ord}(Q) \geq \mathrm{ord}(P) - \frac{1}{2}k_P$ and $\sigma(Q)|_\Sigma = \sigma(P)|_\Sigma$. This induces naturally the following definitions (already considered in [60])

Definition 1.2.13. *The order and symbol of a GTO T_P are defined as*

$$\widetilde{\mathrm{ord}}(T_P) := \min\{\mathrm{ord}(Q)\,, T_Q \sim T_P\}\,,$$
$$\widetilde{\sigma}(T_P) := \sigma(Q)|_\Sigma\,, \quad T_Q \sim T_P \ \textit{and}\ \mathrm{ord}(Q) = \widetilde{\mathrm{ord}}(T_P)\,.$$

The set of GTOs of order $m \in \mathbb{R}$ (resp. less or equal to m) is denoted GTO^m (resp. $\mathrm{GTO}^{\leq m}$).
A GTO T_P such that $\widetilde{\sigma}(T_P)$ vanishes nowhere is called elliptic.

Remarks 1.2.14.

i) The symbol of a GTO is the counterpart of the principal symbol of pseudodifferential operators. But the total symbol of a GTO is not well defined. Even the definition of the subprincipal symbol needs additional assumptions on Σ [22, (*) Section 11].

ii) Note also that with this definition, we cannot have $\widetilde{\sigma}(T_P) = 0$, except when $\widetilde{\mathrm{ord}}(T_P) = -\infty$,

iii) Symbols $\widetilde{\sigma}(T_P)$ can be equivalently seen as functions with variables $(z, t\,\eta_z) \in \Sigma \subset T^*\partial\Omega$, $t \in \mathbb{R}_{>0}$ or $(z,t) \in \partial\Omega \times \mathbb{R}_{>0}$.

[(2)] The isomorphism is local since the existence of the FIO F and the canonical relation ϕ is local.

The following proposition justifies the existence of the operator Q in Definition 1.2.13.

Proposition 1.2.15. *For any $T_P \in \mathrm{GTO}^m$, there exists $Q \in \Psi\mathrm{DO}^m(\partial\Omega)$ such that $T_Q \sim T_P$ and $\widetilde{\sigma}(T_P) = \sigma(Q)|_\Sigma$.*

Proof. First, we prove the existence of such Q and then show that its order is minimal (hence defines the order of T_P). The result is microlocal so we work again in the conic neighborhood V of some fixed point $(z, \eta_z) \in \Sigma$, and identify the neighborhood $V \cap \partial\Omega$ of the point $z \in \partial\Omega$ with an open set of $\mathbb{R}^{2n-1}$. We denote m' and k the integers verifying $P \in OPS^{m',k}(V_0, \Sigma_0)$ and, from Proposition 1.2.12, $m = m' - k/2$. From [20, (5.13)], there is a unique differential operator (acting on the spaces of classes of symbols)

$$s_\Sigma(P)(x, y, \xi) := \sum_{|\alpha|+|\beta|\leq k} a_{\alpha\beta}(x, \xi)\, y^\alpha\, (-i\partial_y)^\beta\,,$$

where $a_{\alpha\beta}$ are smooth homogeneous functions of degree $m' - k/2 + |\alpha|/2 - |\beta|/2$, such that the corresponding pseudodifferential operator $S(P)$, defined as

$$S(P)f(x, y) := \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{ix.\xi+iy.\eta} \sum_{|\alpha|+|\beta|\leq k} a_{\alpha\beta}(x, \xi)\, y^\alpha\, \eta^\beta\, \hat{f}(\xi, \eta)\, d\xi d\eta\,, \quad f \in C_0^\infty(\partial\Omega),$$

verifies $P - S(P) \in OPS^{m',k-1}(V_0, \Sigma_0)$. The terms $a_{\alpha\beta}$ are obtained by writing the Taylor expansion of the total symbol of P near Σ_0 (defined as the subset of V_0 with $y = \eta = 0$). According to [20, (5.10), (5.13), (5.14)], for any $(x, \xi) \in T^*\mathbb{R}^n$, the symbol of H^*PH is given by

$$\begin{aligned}\sigma(H^*PH)(x, \xi) &= \int_{\mathbb{R}^{n-1}} dy\, \overline{\sigma(H)}(x, y, \xi)\, \Big(\sum_{|\alpha|+|\beta|\leq k} a_{\alpha\beta}(x, \xi)\, y^\alpha(-i\partial_y)^\beta\Big) \circ \sigma(H)(x, y, \xi)\,,\\ &=: \sum_{|\alpha|+|\beta|\leq k} a_{\alpha\beta}(x, \xi)\, b_{\alpha\beta}(x, \xi)\,,\end{aligned}$$

where $b_{\alpha\beta}(x, \xi)$ contains all the terms after integration over the y variable. From Definition 1.2.4 and Proposition 1.2.12, $\sigma(H^*PH)$ is the principal symbol of a pseudodifferential operator of order $m = m' - k/2$. Now, consider a pseudodifferential operator Q_0 such that its principal symbol evaluated on Σ_0 is

$$\sigma(Q_0)(x, 0, \xi, 0) := \|\, \sigma(H)(x, ., \xi)\, \|_{L^2}^{-2} \sum_{|\alpha|+|\beta|\leq k} a_{\alpha\beta}(x, \xi)\, b_{\alpha\beta}(x, \xi)$$

(from (1.20), $\|\, \sigma(H)(x, ., \xi)\, \|_{L^2} \neq 0$). Then Q_0 belongs to $OPS^{m,0}(V_0, \Sigma_0)$ and the corresponding differential operator $s_\Sigma(Q_0)(x, y, \xi)$ is just the multiplication by $\sigma(Q_0)|_{\Sigma_0}$, which leads to

$$\sigma(H^*Q_0H)(x, \xi) = \sigma(Q_0)(x, 0, \xi, 0)\, \|\, \sigma(H)(x, ., \xi)\, \|_{L^2}^2 = \sigma(H^*PH)(x, \xi)\,.$$

So, $Q_0 \in \Psi\mathrm{DO}^m(\partial\Omega)$ is such that $H^*(P - Q_0)H \in \Psi\mathrm{DO}^{m-1}(\mathbb{R}^n)$. With the same reasoning, we construct a sequence $Q_j \in \Psi\mathrm{DO}^{m-j}(\partial\Omega)$, $j \in \mathbb{N}$, such that for any $N \in \mathbb{N}$, $H^*(P - \sum_{j=0}^{N-1} Q_j)H$ belongs to $\Psi\mathrm{DO}^{m-N}(\mathbb{R}^n)$. Then $Q := \sum_{j=0}^\infty Q_j \in \Psi\mathrm{DO}^m(\partial\Omega)$ is such that $H^*(P - Q)H$ is smoothing and $T_P = T_Q$, with $\widetilde{\sigma}(T_P) = \sigma(Q)|_{\Sigma_0}$.
Now we check that $\mathrm{ord}(Q)$ minimizes $\{\mathrm{ord}(R)\,, R \in \Psi\mathrm{DO}(\partial\Omega)\,, T_R = T_P\}$. By definition,

$$\begin{aligned}\widetilde{\mathrm{ord}}(T_P) &= \inf\{\mathrm{ord}(R)\,, R \in \Psi\mathrm{DO}(\partial\Omega)\,, T_R = T_P\}\\ &= \inf\{\mathrm{ord}(R)\,, R \in \Psi\mathrm{DO}(\partial\Omega)\,, H^*RH \sim H^*PH\}\,.\end{aligned}$$

Let m'' and k'' the integers such that $R \in OPS^{m'',k''}(U_0, \Sigma_0)$, that is $\mathrm{ord}(R) = m''$, then evaluating the order of both sides of the previous relation $H^*RH \sim H^*PH$, we have

$$m'' - \frac{k''}{2} = m' - \tfrac{1}{2}k\,, \quad \text{i.e.} \quad m'' = m' - \tfrac{1}{2}k + \tfrac{1}{2}k''\,,$$

which is minimized when $k'' = 0$. Thus, $\widetilde{\mathrm{ord}}(T_P) = m' - \frac{1}{2}k = \mathrm{ord}(Q)$. □

In many cases, the only information we get on $P \in \Psi\mathrm{DO}(\partial\Omega)$ is its principal symbol and it is a priori not possible to get the order of the corresponding GTO T_P. The following corollary is sometimes more useful:

Corollary 1.2.16. *Let P be a pseudodifferential operator on $\partial\Omega$ whose principal symbol does not vanish identically on Σ. Then $\widetilde{\mathrm{ord}}(T_P) = \mathrm{ord}(P)$ and $\widetilde{\sigma}(T_P) = \sigma(P)|_\Sigma$.*

Example 1.2.17. *The operator $\Lambda_m := K_m^* K_m$ of (1.8) belongs to $\Psi\mathrm{DO}^{-(m+1)}(\partial\Omega)$ with principal symbol given by (1.10), which does not vanish identically on Σ since, in the parametrization (A.11), $\sigma(\Lambda_m)|_\Sigma(x,t) = 2^{-(m+1)}(\partial_t^m w_m)(w,0)|t\,\eta_x|^{-(m+1)}$, for $(x,t) \in \partial\Omega \times \mathbb{R}_{>0}$. As a consequence, T_{Λ_m} lies in $\mathrm{GTO}^{-(m+1)}$.*

1.2.2.4 Main properties

The main results on GTOs are gathered here, coming from previous remarks and the original works [20, 21, 22]. They can be proved using the previous microlocal description.

Proposition 1.2.18.

(P1) [22, Proposition 2.13] For any T_P of order m, there exists $Q \in \Psi\mathrm{DO}^m(\partial\Omega)$ such that $T_Q = T_P$ and $[\,Q, \Pi\,] = 0$, i.e. $T_P = Q|_{H^2}$.

(P2) The set of GTOs forms an algebra which is, modulo smoothing operators, locally isomorphic to the algebra $\Psi\mathrm{DO}(\mathbb{R}^n)$.

(P3) We have:

$$\text{i) } \widetilde{\mathrm{ord}}(T_P T_Q) = \widetilde{\mathrm{ord}}(T_P) + \widetilde{\mathrm{ord}}(T_Q)$$

$$\text{ii) } \widetilde{\sigma}(T_P T_Q) = \widetilde{\sigma}(T_P)\,\widetilde{\sigma}(T_Q)$$

$$\text{iii) } \widetilde{\sigma}([\,T_P,\,T_Q\,]) = -i\{\,\widetilde{\sigma}(T_P)\,,\,\widetilde{\sigma}(T_Q)\,\}_\Sigma$$

(P4) If $P \in \Psi\mathrm{DO}^m$ and $\sigma(P)|_\Sigma = 0$, then there is $Q \in \Psi\mathrm{DO}^{\leq m-1}(\partial\Omega)$ such that $T_Q = T_P$.

(P5) Let $T_P \in \mathrm{GTO}^m$ and $Q \in \Psi\mathrm{DO}^m(\partial\Omega)$ such that $T_Q = T_P$. Then $\sigma(Q)|_\Sigma \neq 0$.

(P6) [20, Proposition 2.11] If $T_Q = T_P$, with $\mathrm{ord}(Q) = \mathrm{ord}(P)$, then $\sigma(Q)|_\Sigma = \sigma(P)|_\Sigma$.

(P7) If $T_P \in \mathrm{GTO}^m$, then $\widetilde{\sigma}(T_P)$ is a smooth function homogeneous of degree m in the dual variable. Using the parametrization (A.11) of Σ, then there is $u_P \in C^\infty(\partial\Omega)$ such that

$$\widetilde{\sigma}(T_P)(z,t) = t^m\,u_P(z)\,, \quad \textit{for any } z \in \partial\Omega, \textit{ and } t \in \mathbb{R}_{>0}\,.$$

(P8) Let $T_P, T_Q \in \mathrm{GTO}^m$. If $\widetilde{\sigma}(T_P) = \widetilde{\sigma}(T_Q)$, then $T_P - T_Q \in \mathrm{GTO}^{\leq m-1}$.

(P9) We get $\widetilde{\mathrm{ord}}(T_P) \leq 0 \Leftrightarrow T_P \in \mathcal{B}(L^2(\partial\Omega))$ *and* $\widetilde{\mathrm{ord}}(T_P) < 0 \Leftrightarrow T_P \in \mathcal{K}(L^2(\partial\Omega))$.

(P10) Any elliptic $T_P \in \mathrm{GTO}^m$ *admits a parametrix, i.e. there is* $T_Q \in \mathrm{GTO}^{-m}$ *such that* $T_P T_Q \sim T_Q T_P \sim I$*, with* $\widetilde{\sigma}(T_Q) = \widetilde{\sigma}(T_P)^{-1}$.

(P11) [60, Proposition 16] Let T *be an elliptic positive selfadjoint on* $H^2(\partial\Omega)$ *such that* $T \sim T_P$*, where* $T_P \in \mathrm{GTO}^m$*, with* $m \neq 0$ *and* $\widetilde{\sigma}(T_P) > 0$*. Then for any* $s \in \mathbb{C}$*, the power* T^s*, in the sense of the spectral theorem, is a GTO of order* ms *modulo smoothing operators. In particular, for* $s = -1$*, the inverse of* T_P *is a GTO of order* $-m$.

(P12) [22, Theorem 13.1] Let $T_P \in \mathrm{GTO}^1$ *be selfadjoint and elliptic with ordered eigenvalues* $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ *counting multiplicities. Then the counting function* N_{T_P} *has the following asymptotic behaviour*

$$N_{T_P}(\lambda) \underset{\lambda\to\infty}{\sim} \frac{\mathrm{vol}(\Sigma_{T_P})}{(2\pi)^n}\,\lambda^n\,, \tag{1.21}$$

where Σ_{T_P} *is the subset of* Σ *where* $\widetilde{\sigma}(T_P) \leq 1$ *and* $\mathrm{vol}(\Sigma_{T_P})$ *its symplectic volume.*

(P13) [62, Theorem 3] If $T_P \in \mathrm{GTO}^{-n}$ *then it is measurable and its Dixmier trace is given by*

$$\mathrm{Tr}_{Dix}(T_P) = \frac{1}{n!(2\pi)^n} \int_{\partial\Omega} \widetilde{\sigma}(T_Q)(z,1)\,\nu_z\,. \tag{1.22}$$

Remarks 1.2.19.

i) *(P1) is used in (P2) to show that GTOs form an algebra:* $T_{P'}T_P = \Pi P'\Pi P = \Pi P' Q = T_{P'Q}$*, with* $T_Q = T_P$*,* $[\,Q,\,\Pi\,] = 0$.

ii) *From (P8),* $\widetilde{\mathrm{ord}}\big([\,T_P,\,T_Q\,]\big) \leq \widetilde{\mathrm{ord}}(T_P) + \widetilde{\mathrm{ord}}(T_Q) - 1$.

iii) *Note that* (1.21) *differs from the original Weyl law* (3.7) *by replacing the volume of the unit ball with the one of* Σ_{T_P}.

iv) *(P13) is the analogous of [40, Theorem 1] we mentioned above. Note also that from* (A.5) *and since* $\widetilde{\sigma}(T_P)$ *is homogeneous of degree* $-n$*,* (1.22) *is independent of the defining function.*

Note also that $T_P \in \mathrm{GTO}^m$ maps continuously holomorphic Sobolev spaces, namely

$$T_P : W^{s+m}_{hol}(\partial\Omega) \to W^s_{hol}(\partial\Omega),\ \text{for any } s \in \mathbb{R}\,,$$

because Π is (or rather extends to) a continuous map from $W^s(\partial\Omega)$ onto $W^s_{hol}(\partial\Omega)$ for any real number s.

1.2.2.5 GTOs with log-polyhomogeneous symbols

In [59], the theory of GTOs constructed with log-polyhomogeneous pseudodifferential operators has been established as an extension of the previous usual GTOs (see Definition C.1.10).

Definition 1.2.20. *Let* $m \in \mathbb{C}$*. An operator of the form* $T_P = \Pi P \Pi$*, where* Π *is again the Szegö projector and* $P \in \Psi\mathrm{DO}^m_{\log}(\partial\Omega)$ *is called a GTO of log type, and the corresponding space is denoted* $\mathrm{GTO}^m_{\log}$*. The set* $\mathrm{GTO}^{m,0}_{\log} \subset \mathrm{GTO}^m_{\log}$ *is the subspace of GTOs whose principal symbol is classical (i.e. with no logarithmic term).*

They enjoy similar properties as usual GTOs and we refer to [59, Proposition 3] for the details.

1.2.2.6 GTOs involving differential operators on Ω

We can generalize the definition of Λ_m (1.8) and construct the operator

$$\Lambda_{m,\mathbf{P}} := K_m^* \mathbf{P} K_m = K^* w_m \mathbf{P} K \,, \tag{1.23}$$

acting on the boundary $\partial\Omega$, where $\mathbf{P}$ is a differential operator on $\mathbb{C}^n$ of order $d \in \mathbb{N}$ of the form

$$\mathbf{P} = \sum_{|\alpha| \le d} a_\alpha(z)\, r^j(z)\, \partial_z^\alpha + \sum_{|\alpha'| \le d'} b_{\alpha'}(z)\, r(z)^{j'}\, \partial_{\bar{z}}^{\alpha'} \tag{1.24}$$

for some $d, d' \in \mathbb{N}$, $j, j' \in \mathbb{R}^+$, $\alpha, \alpha' \in \mathbb{N}^n$ and some functions $a_\alpha, b_{\alpha'} \in C^\infty(\overline{\Omega})$. Similarly, for a function $f \in C^\infty(\overline{\Omega})$, we denote the operator

$$\Lambda_{w_m f} := K^* \,(w_m f)\, K_m \,.$$

Proposition 1.2.21. *The operator $T_{\Lambda_{m,\mathbf{P}}}$ is a GTO of order less or equal to $d - (m + j + 1)$. When the order is exactly $d - (m+j+1)$, the symbol of $T_{\Lambda_{m,\mathbf{P}}}$ is given in local coordinates $(z,t) \in \partial\Omega \times \mathbb{R}_{>0}$ by*

$$\widetilde{\sigma}(T_{\Lambda_{m,\mathbf{P}}})(z,t) = \frac{\Gamma(m+j+1)}{2|\eta_z|}\, \chi(z,0) \Big(\sum_{|\alpha|=d} a_\alpha \prod_{k=1}^{n} (\partial_{z_k} r)^{\alpha_k} \Big)(z)\, t^{d-(m+j+1)} \,. \tag{1.25}$$

Proof. By Boutet de Monvel's theory [19], $\Lambda_{m,\mathbf{P}}$ is again a pseudodifferential operator, so $T_{\Lambda_{m,\mathbf{P}}}$ makes sense as a GTO. For $k \in \{1, \cdots, n\}$, define the operators Z_k and $\bar{Z}_k$ on $\partial\Omega$ by

$$Z_k := \gamma\, \partial_{z_k} K, \quad \bar{Z}_k := \gamma\, \partial_{\bar{z}_k} K \,,$$

whose symbol at $(z,t) \in \Sigma$ is $\sigma(Z_k)(z,t) := i\langle t\, \eta_z \,,\, Z_k \rangle = t\, \partial_{z_k} r(z)$. Using (1.12), we have

$$\begin{aligned} K^* w \mathbf{P} K &= \sum_{|\alpha| \le d} K^*(w_m a_\alpha r^j) \partial_z^\alpha K + \sum_{|\alpha'| \le d'} K^* w_m b_{\alpha'} r^{j'} \partial_{\bar{z}}^{\alpha'} K \\ &= \sum_{|\alpha| \le d} \Lambda_{w_m a_\alpha r^j} Z^\alpha + \sum_{|\alpha'| \le d'} \Lambda_{w_m b_{\alpha'} r^{j'}} \bar{Z}^{\alpha'} \,, \end{aligned}$$

with $Z^\alpha := \prod_{k=1}^n Z_k^{\alpha_k}$ and the same for $\bar{Z}$. Note that $H^2 \subset \mathrm{Ker} \bar{Z}$ so the second term on the right hand side disappears in $T_{\Lambda_{w\mathbf{P}}} = \Pi\, \Lambda_{m,\mathbf{P}}\, \Pi$. So at $(z,t) \in \partial\Omega \times \mathbb{R}_{>0}$, we get from (1.9)

$$\sigma\Big(\sum_{|\alpha| \le d} \Lambda_{w_m a_\alpha r^j} Z^\alpha \Big)(z, t\,\eta_z) = \frac{\Gamma(m+j+1)}{2|\eta_z|}\, \chi(z) \Big(\sum_{|\alpha|=d} a_\alpha \prod_{k=1}^{n} (\partial_{z_k} r)^{\alpha_k} \Big)(z)\, t^{d-(m+j+1)} \,.$$

and the result follows from Corollary 1.2.16. $\square$

1.3 Relations between Toeplitz operators

1.3.1 Link between Toeplitz on Bergman and GTOs

From Example 1.2.17, the operator T_{Λ_m} exists as a positive, elliptic and compact GTO of order $-(m+1)$ on $L^2(\partial\Omega)$ and maps continuously $W^s_{hol}(\partial\Omega)$ into $W^{s+m+1}_{hol}(\partial\Omega)$, for any $s \in \mathbb{R}$.
If $u \in \mathrm{Ker}(T_{\Lambda_m}) \subset W^s_{hol}(\partial\Omega)$ for a certain $s \in \mathbb{R}$, then

$$0 = \langle T_{\Lambda_m} u \,,\, u \rangle_{W^s_{hol}(\partial\Omega)} = \langle \Pi \Lambda_m u \,,\, u \rangle_{W^s_{hol}(\partial\Omega)} = \langle \Lambda_m u \,,\, \Pi u \rangle_{W^s_{hol}(\partial\Omega)} \,.$$

Since $\Pi u = u$ we get, using the injectivity of Λ_m,

$$0 = \langle \Lambda_m u \, , \, u \rangle_{W^s_{hol}(\partial\Omega)} = \| \Lambda_w^{1/2} u \|^2 \Rightarrow u = 0 \, .$$

Thus, for any $s \in \mathbb{R}$, the inverse operator $T^{-1}_{\Lambda_m}$ exists from $\mathrm{Ran}(T_{\Lambda_m}) = W^{s+m+1}_{hol}(\partial\Omega)$ onto $W^s_{hol}(\partial\Omega)$. For completeness, we give the proof of the following result.

Proposition 1.3.1. *[59, Theorem 4] Let T be a positive selfadjoint operator on $H^2(\partial\Omega)$ such that $T \sim T_P$, where P is an elliptic pseudodifferential operator of some order and such that $\widetilde{\sigma}(T_P) > 0$.*
Let $W^T_{hol}(\partial\Omega)$ be the completion of $C^\infty_{hol}(\partial\Omega)$ with respect to the norm $\| u \|^2_T := \langle Tu \, , \, u \rangle_{H^2}$. Then, we have

$$W^T_{hol}(\partial\Omega) = W^{p/2}_{hol}(\partial\Omega) \, ,$$

where $p := \widetilde{\mathrm{ord}}(T_P)$.

Proof. We may assume that P commutes with Π. From (P11) of Proposition 1.2.18, the equivalence between T and T_P induces $T^{1/2} \sim T_P^{1/2} \sim T_{P^{1/2}} = \Pi \, P^{1/2}|_{H^2} = P^{1/2}|_{H^2}$. Since $P^{1/2}$ is elliptic, the GTO $T_{P^{1/2}}$ admits a parametrix, so is Fredholm. As a consequence $T^{1/2}$ is a positive, so an injective Fredholm operator, thus an isomorphism from $W^{p/2}(\partial\Omega)$ onto $H^2(\partial\Omega)$. Now if $u \in W^{p/2}_{hol}(\partial\Omega)$, $T^{1/2}u$ belongs to $H^2(\partial\Omega)$. Thus we have the finite quantities

$$\|T^{1/2}u\|^2_{H^2} = \langle Tu \, , \, u \rangle_{H^2} = \| u \|^2_{W^T_{hol}} ,$$

which proves the equality. □

Proposition 1.3.2. *[59, Proof of Theorem 1] The operator K_m maps bijectively the space $W^{T_{\Lambda_m}}_{hol}(\partial\Omega)$ onto $A^2_m(\Omega)$.*

Proof. Let $f \in C^\infty_{hol}(\overline{\Omega}) \subset A^2_m(\Omega)$ and $u = \gamma_m(f) \in C^\infty(\partial\Omega)$. Then

$$\| f \|^2_{A^2_m} = \langle K_m u \, , \, K_m u \rangle_{L^2(\Omega)} = \langle \Lambda_m u \, , \, u \rangle_{L^2(\partial\Omega)} = \langle \Pi\Lambda_m u \, , \, u \rangle_{L^2(\partial\Omega)} = \langle T_{\Lambda_m} u \, , \, u \rangle_{L^2(\partial\Omega)} \, .$$

Thus K_m is an isometry of $W^{T_{\Lambda_m}}_{hol}(\partial\Omega)$ onto the completion of $C^\infty_{hol}(\overline{\Omega})$ in A^2_m.
As for (1.11) (see [62] for details), we get

$$\mathbf{\Pi}_m = K_m \, \Pi \, T^{-1}_{\Lambda_m} \, \Pi \, K^*_m. \tag{1.26}$$

Note that $C^\infty(\overline{\Omega})$ is dense in $L^2(\Omega, w_m)$, while the projection $\mathbf{\Pi}_m$ maps each $W^s_{hol}(\Omega)$, and, hence, $C^\infty(\overline{\Omega})$ into itself. So $C^\infty_{hol}(\overline{\Omega})$ is dense in A^2_m and the claim follows. □

From the fact that K_m is an isomorphism of $W^s_{hol}(\partial\Omega)$ onto $W^{s+1/2}_{hol}(\Omega)$ for any $s \in \mathbb{R}$, we also see that $A^2_m(\Omega) = K_m W^{-(m+1)/2}(\partial\Omega) = W^{-m/2}_{hol}(\Omega)$ and that γ_m is an isomorphism of A^2_m onto the Sobolev $W^{-(m+1)/2}(\partial\Omega)$.
As we already said, $T^{1/2}_{\Lambda_m}$ is an isomorphism of $W^s_{hol}(\partial\Omega)$ onto $W^{s+(m+1)/2}(\partial\Omega)$ for all $s \in \mathbb{R}$, with equivalent norms. As a consequence,

Lemma 1.3.3. *[61, Lemma 2.13] The operator*

$$V_m := K_m \, T^{-1/2}_{\Lambda_m} \text{ is a unitary map from } H^2(\partial\Omega) \text{ onto } A^2_m(\Omega). \tag{1.27}$$

Proof. We have $V_m^* V_m = T_{\Lambda_m}^{-1/2} K_m^* K_m T_{\Lambda_m}^{-1/2} = T_{\Lambda_m}^{-1/2} T_{\Lambda_m} T_{\Lambda_m}^{-1/2} = I_{H^2}$. Similarly, we get from (1.26), $V_m V_m^* = I_{A_m^2(\Omega)}$. □

We now identify a Toeplitz operator $\mathbf{T}_f$ on $A_m^2(\Omega)$ with generalized Toeplitz operators acting on $H^2(\partial\Omega)$ via γ_m, K_m and V_m:

Proposition 1.3.4 ([61, Proposition 1.3.4])**.** *For $f \in C^\infty(\overline{\Omega})$, we have*

$$\begin{aligned} &\gamma_w \, \mathbf{T}_f \, K_m = T_{\Lambda_m}^{-1} \, T_{\Lambda_{w_m f}} \quad \textit{on } W_{hol}^{-(m+1)/2}(\partial\Omega)\,, \\ &\mathbf{T}_f = V_m \, T_{\Lambda_m}^{-1/2} \, T_{\Lambda_{w_m f}} \, T_{\Lambda_m}^{-1/2} \, V_m^* \quad \textit{on } A_m^2(\Omega)\,. \end{aligned} \tag{1.28}$$

Proof. For any u an v in $W_{hol}^{-(m+1)/2}(\partial\Omega)$, we get

$$\begin{aligned} \langle \mathbf{T}_f \, K_m u \,,\, K_m v \rangle_{A_m^2(\Omega)} &= \langle \mathbf{\Pi}_m \, f \, K_m u \,,\, K_m v \rangle_{A_m^2(\Omega)} = \langle \, f \, K_m u \,,\, \mathbf{\Pi}_m \, K_m v \rangle_{A_m^2(\Omega)} \\ &= \langle \, f \, K_m u \,,\, K_m v \rangle_{A_m^2(\Omega)} = \langle \, w_m f \, K u \,,\, K v \rangle_{L^2(\Omega)} \\ &= \langle \, (K^* \, w_m f \, K) u \,,\, v \rangle_{H^2(\partial\Omega)} = \langle \, \Lambda_{w_m f} u \,,\, \Pi v \rangle_{H^2(\partial\Omega)} \\ &= \langle \, T_{\Lambda_{w_m f}} u \,,\, v \rangle_{H^2(\partial\Omega)} = \langle \, K_m \, T_{\Lambda_m}^{-1} \, T_{\Lambda_{w_m f}} \, u \,,\, K_m v \rangle_{H^2(\partial\Omega)} \,. \end{aligned}$$

Thus $\mathbf{T}_f \, K_m = K_m \, T_{\Lambda_m}^{-1} \, T_{\Lambda_{w_m f}}$ on $W_{hol}^{-(m+1)/2}(\partial\Omega)$, hence $\gamma_m \mathbf{T}_f K_m = T_{\Lambda_m}^{-1} \, T_{\Lambda_{w_m f}}$.
Finally, we get $V_m^* \, \mathbf{T}_f \, V_m = V_m^* \, (K_m \gamma_m) \, \mathbf{T}_f \, (K_m \gamma_m) \, V_m = T_{\Lambda_m}^{-1/2} \, T_{\Lambda_{w_m f}} \, T_{\Lambda_m}^{-1/2}$. □

From the GTO's theory and the mapping properties of K_m and γ_m, we see that the right-hand side in (1.28) extends to a bounded operator on any $W_{hol}^s(\Omega)$, hence the left-hand side enjoys the same property.

Let $\mathcal{T}^{-\infty}$ denote the ideal in $\mathrm{GTO}^{\leq 0}$ of GTO's of order $-\infty$, i.e. of (smoothing) generalized Toeplitz operators with Schwartz kernel in $C^\infty(\partial\Omega \times \partial\Omega)$.

Lemma 1.3.5. *[61, Proposition 5.5] For $m > -1$, let $\mathcal{A}_{B,m}$ be the $*$-algebra generated by Toeplitz operators $\mathbf{T}_f$ acting on $A_m^2(\Omega)$, with $f \in C^\infty(\overline{\Omega})$. The map $\Theta_m : a \in \mathcal{A}_{B,m} \mapsto \Theta_m(a) := V_m^* \, a \, V_m \in \mathrm{GTO}^{\leq 0}$ is a *-isomorphism of $\mathcal{A}_{B,m}$ onto a subalgebra $\Theta_m(\mathcal{A}_{B,m}) \subset \mathrm{GTO}^{\leq 0}$. Moreover,*

$$\mathrm{GTO}^{\leq 0} = \Theta_m(\mathcal{A}_{B,m}) + \mathcal{T}^{-\infty}\,.$$

Proof. Thanks to (1.28), $\Theta_m(\mathbf{T}_f) = V_m^* \, \mathbf{T}_f \, V_m = T_{\Lambda_m}^{-1/2} \, T_{\Lambda_{w_m f}} \, T_{\Lambda_m}^{-1/2} \in \mathrm{GTO}^{\leq 0}$. Since $\mathcal{A}_{B,m}$ is generated by the $\mathbf{T}_f$, the map defines an isomorphism from $\mathcal{A}_{B,m}$ into $\Theta_m(\mathcal{A}_{B,m})$ which preserves the adjoint.

We now prove the equality: let $T_P \in \mathrm{GTO}^{-s}$, $s \geq 0$, with symbol $\widetilde{\sigma}(T_P)(z,t) =: t^{-s} u_P(z)$, where $(z,t) \in \partial\Omega \times \mathbb{R}_{>0}$ and $s_P \in C^\infty(\partial\Omega)$ (see (P7)). Then

$$f_0(w) := \tfrac{\Gamma(m+1)}{\Gamma(m+s+1)} \, K(u_P)(w)\,, \quad w \in \overline{\Omega}\,,$$

is in $C^\infty(\overline{\Omega})$ with $v_{\partial\Omega}(f_0) = 0$, and by Proposition 1.3.4, (1.9) and (1.25), the operator $\Theta_m(\mathbf{T}_{r^s f_0})$ is a GTO also of order $-s$ and with the same principal symbol as T_P. Thus $T_1 := T_P - \Theta_m(\mathbf{T}_{r^s f_0})$ is a GTO of order $-s-1$. Applying the same reasoning to T_1 in the place of T_P yields $f_1 \in C^\infty(\overline{\Omega})$ such that $\Theta_m(\mathbf{T}_{r^{s+1} f_1})$ has the same order and principal symbol as T_1, hence $T_2 := T_P - \Theta_m(\mathbf{T}_{r^s f_0 + r^{s+1} f_1})$

is a GTO of order $-s-2$. By iteration, this yields a sequence $f_2, f_3, \dots$. Finally, let $f \in C^\infty(\overline{\Omega})$ be a function which has the same boundary jet as the formal sum $\sum_{j=0}^\infty r^j f_j$, i.e. such that for any $N \in \mathbb{N}\backslash\{0\}$

$$f - \sum_{j=0}^{N-1} r^j f_j = O(r^N)$$

vanishes to order N at the boundary (such an f can be obtained in a completely standard manner along the lines of the classical Borel theorem). Set $g := r^s f$. Then by Proposition 1.3.4 and (1.9) again, for any $N \in \mathbb{N}\backslash\{0\}$, the difference

$$R := T_P - \Theta_m(\mathbf{T}_g) = T_P - \Theta_m(\mathbf{T}_{\sum_{j=0}^{N-1} r^{s+j} f_j}) - \Theta_m(\mathbf{T}_{r^s(f - \sum_{j=0}^{N-1} r^j f_j)}) = T_N - \Theta_m(\mathbf{T}_{O(r^{N+s})})$$

is a GTO of order (at most) $-s-N$. Since N is arbitrary, R is a GTO of order $-\infty$, i.e. $R \in \mathcal{T}^{-\infty}$, and the proof is complete. □

Similarly as for the computation of (1.9), it can be shown that the order of $\Theta_m(\mathbf{T}_f)$ is precisely the vanishing order of f on the boundary $\partial\Omega$, and if f has compact support inside Ω, then $\Theta_m(\mathbf{T}_f)$ is smoothing.

Remark 1.3.6. *It remains an open question to know whether the inclusion $\Theta_m(\mathcal{A}_{B,m}) \subset \mathrm{GTO}^{\leq 0}$ is strict or not.*

1.3.2 Toeplitz operators over $\mathbb{B}^n$ as elements of $\mathfrak{h}^n$

Lemma 1.3.7. *[61, Remark 2.16] For a general strictly pseudoconvex domain Ω with smooth boundary $\partial\Omega$ and defining function r, we have on the unweighted Bergman space $A^2(\Omega)$:*

$$\mathbf{T}_{\partial_{z_j}} = V\, T_\Lambda^{-1/2} T_{(\partial_{z_j} r)/(2\|\partial r\|)} T_\Lambda^{-1/2}\, V^*. \tag{1.29}$$

Proof. Using Stokes' formula, we get for $f, g \in C^\infty_{hol}(\overline{\Omega})$

$$\begin{aligned}\langle \mathbf{T}_{\partial_{z_j}} f\,,\, g\rangle_{A^2(\Omega)} = \langle \partial_{z_j} f\,,\, g\rangle_{A^2(\Omega)} &= \int_\Omega d\mu\, (\partial_{z_j} f)\, \bar g = \int_{\partial\Omega} d\sigma\, f\, \bar g\, \frac{\partial_{z_j} r}{2\|\partial r\|} - \int_\Omega d\mu\, f\, \partial_{z_j} \bar g \\ &= \int_{\partial\Omega} d\sigma\, f\, \bar g\, \frac{\partial_{z_j} r}{2\|\partial r\|}\,.\end{aligned}$$

Thus $K^* \mathbf{T}_{\partial_{z_j}} K = T_{(\partial_{z_j} r)/(2\|\partial r\|)}$ and the result follows by density of $C^\infty_{hol}(\overline{\Omega})$ in $A^2(\Omega)$ (see the proof of Proposition 1.3.2) and from the definition of V (1.27) and also (1.12). □

In this section, we restrict to the case $\Omega = \mathbb{B}^n$, without weight, and denote as usual $A^2 := A^2(\mathbb{B}^n)$. Since $z \mapsto z^\alpha$ is holomorphic, $\mathbf{T}_{z^\alpha}$ is the multiplication by z^α on A^2, and $\partial^\alpha = \mathbf{\Pi}\partial^\alpha =: \mathbf{T}_{\partial^\alpha}$. Strictly speaking, the latter is not a classical Toeplitz operator as in Definition 1.2.1 since its " symbol " is a differential operator, but the notation still makes sense. We will extend the definition in (1.36).
The following result shows that the operators $\mathbf{T}_{z^\alpha}$ and $\mathbf{T}_{\partial^\alpha}$ acting on A^2 can be expressed as representations of elements in the enveloping algebra $\mathrm{Env}(\mathfrak{h}^n)$. We follow [95, Chapter 4.2] for the first equality.

Proposition 1.3.8. *[61, Proposition 4.1] For $\alpha \in \mathbb{N}^n$, let*

$$(a^{\pm})^{\alpha} := \prod_{j=1}^{n} (a_j^{\pm})^{\alpha_j} \quad \text{and} \quad d_\alpha^{\pm} := \prod_{k=1}^{|\alpha|} \big(\mathfrak{N} - i(\tfrac{n}{2} \pm k)\,\mathbb{T}\big)$$

be elements of $\mathrm{Env}(\mathfrak{h}^n)$*. The operators $\mathbf{T}_{z^\alpha}$ and $\mathbf{T}_{\partial^\alpha}$ from A^2 to A^2 can be written as*

$$\mathbf{T}_{z^\alpha} = \pi_A\big((a^+)^\alpha\big)\,\big(\pi_A(d_\alpha^+)\big)^{-1/2} \quad \text{and} \quad \mathbf{T}_{\partial^\alpha} = \pi_A\big((a^-)^\alpha\big)\,\big(\pi_A(d_\alpha^-)\big)^{-1/2}.$$

Proof. We have, on the basis (1.6)

$$\begin{aligned} \mathbf{T}_{z^\alpha}\, v_\beta &= z^\alpha\, v_\beta = \tfrac{b_\alpha}{b_{\beta+\alpha}}\, v_{\beta+\alpha} = \big(\tfrac{(\beta+\alpha)!}{\beta!}\,\tfrac{(|\beta|+n)!}{(|\beta|+|\alpha|+n)!}\big)^{1/2}\, v_{\beta+\alpha}\,, \quad \forall \alpha,\beta\,,\\ \mathbf{T}_{\partial^\alpha}\, v_\beta &= \partial^\alpha\, v_\beta = \tfrac{\beta!}{(\beta-\alpha)!}\,\tfrac{b_\alpha}{b_{\beta-\alpha}}\, v_{\beta-\alpha} = \big(\tfrac{\beta!}{(\beta-\alpha)!}\,\tfrac{(|\beta|+n)!}{(|\beta|-|\alpha|+n)!}\big)^{1/2}\, v_{\beta-\alpha}\,, \quad \beta \geq \alpha\,, \end{aligned} \tag{1.30}$$

where for the second equality, the case $\alpha > \beta$ corresponds to the null operator. As a consequence, we only consider $\mathbf{T}_{\partial^\alpha}$ on the domain $\mathrm{span}_{\beta\geq\alpha}\{v_\beta\}$. From Proposition 1.1.16, we deduce

$$\begin{aligned} \pi_A\big((a^+)^\alpha\big)\, v_\beta &= \big(\tfrac{(\beta+\alpha)!}{\beta!}\big)^{1/2}\, v_{\beta+\alpha}\,, \quad \forall \alpha,\beta,\\ \pi_A\big((a^-)^\alpha\big)\, v_\beta &= \big(\tfrac{\beta!}{(\beta-\alpha)!}\big)^{1/2}\, v_{\beta-\alpha}\,, \qquad \beta \geq \alpha\,. \end{aligned}$$

Moreover, the elements $d_\alpha^{\pm}$ act on A^2 as

$$\pi_A(d_\alpha^{\pm})\, v_\beta = \prod_{k=1}^{|\alpha|} \big(\pi_A(\mathfrak{N}) - i(\tfrac{n}{2} \pm k)\pi_A(\mathbb{T})\,\big)\, v_\beta = \prod_{k=1}^{|\alpha|} (|\beta| + n \pm k)\, v_\beta = \tfrac{(|\beta|\pm|\alpha|+n)!}{(|\beta|+n)!}\, v_\beta\,.$$

Thus the operators $\pi_A(d_\alpha^{\pm})$ are invertible on A^2 and we get the claimed formulæ. □

The following result links GTOs over the unit sphere $\mathbb{S}^{2n-1}$ of $\mathbb{R}^{2n}$ and elements of $\mathfrak{h}^n$.

Lemma 1.3.9. *[61, Lemma 4.2] The operator $\mathbf{R} := \sum_{j=1}^n \mathbf{T}_{z_j}\mathbf{T}_{\partial_{z_j}} = \sum_{j=1}^n \mathbf{T}_{z_j\partial_{z_j}}$ on A^2 is positive and*

$$\pi_A(\mathbb{P}_j) = -\tfrac{i}{\sqrt{2}}\,\big(\mathbf{T}_{\partial_{z_j}}\,(\mathbf{R}+n)^{-1/2} - (\mathbf{T}_{\partial_{z_j}}\,(\mathbf{R}+n)^{-1/2})^*\big) \tag{1.31}$$

(showing again that $\pi_A(\mathbb{P}_j)$ is selfadjoint).
The operator $V^*\,\pi_A(\mathbb{P}_j)\,V$ belongs to $\mathrm{GTO}^{1/2}(\mathbb{S}^{2n-1})$ *with principal symbol*

$$\widetilde{\sigma}\big(V^*\pi_A(\mathbb{P}_j)V\big)(z,t) = 2^{1/4}\,\tfrac{\mathrm{Im}(\partial_{z_j} r)}{\sqrt{\mathbf{R}r}}(z)\; t^{1/2}\,. \tag{1.32}$$

Proof. By (1.30), $\mathbf{R}\, v_\beta = |\beta|\, v_\beta$ for any $\beta \in \mathbb{N}^n$, so $\mathbf{R}$ is positive on A^2. Since

$$\mathbf{T}_{\partial_{z_j}}\,(\mathbf{R}+n)^{-1/2}\, v_\alpha = \alpha_j^{1/2}\, v_{\alpha-1_j}\,, \quad (\mathbf{R}+n)^{1/2}\mathbf{T}_{z_j}\, v_\alpha = (\alpha_j+1)^{1/2}\, v_{\alpha+1_j}\,, \quad \text{and}$$
$$(\mathbf{T}_{z_j})^*\, v_\alpha = \big(\tfrac{\alpha_j}{|\alpha|+n}\big)^{1/2}\, v_{\alpha-1_j}\,,$$

we get on A^2

$$\mathbf{T}_{z_j}^* = (\mathbf{R}+n+1)^{-1}\mathbf{T}_{\partial_{z_j}} = \mathbf{T}_{\partial_{z_j}}\,(\mathbf{R}+n)^{-1}. \tag{1.33}$$

By Proposition 1.1.16, this yields

$$\pi_A(\mathbb{P}_j)\, v_\alpha = -\tfrac{i}{\sqrt{2}}\Big((|\alpha|+n)^{-1/2}\,\mathbf{T}_{\partial_{z_j}} v_\alpha - (|\alpha|+n+1)^{1/2}\,\mathbf{T}_{z_j} v_\alpha\Big),$$

and $\pi_A(\mathbb{P}_j) = -\tfrac{i}{\sqrt{2}}\left[\mathbf{T}_{\partial_{z_j}}\,(\mathbf{R}+n)^{-1/2} - \mathbf{T}_{z_j}(\mathbf{R}+n+1)^{1/2}\right]$. Thus, (1.33) implies (1.31).

As in the proof of Lemma 1.3.7, we get for $f,\ g \in A^2$:

$$\begin{aligned}\langle\, (\mathbf{R}+n) f\,,\, g\,\rangle_{A^2} &= \int_{\mathbb{B}^n} d\mu \sum_{j=1}^{n} \partial_{z_j}(z_j f)\,\bar g = \int_{\mathbb{B}^n} d\mu \sum_{j=1}^{n} \partial_{z_j}(z_j f \bar g) = \int_{\mathbb{S}^{2n-1}} d\sigma\, f\,\bar g \sum_{j=1}^{n} \tfrac{z_j \partial_{z_j} r}{2\|\,\partial r\,\|} \\ &= \int_{\mathbb{S}^{2n-1}} d\sigma\, f\,\bar g\, \tfrac{\mathbf{R} r}{2\|\,\partial r\,\|}\,.\end{aligned}$$

The strict pseudoconvexity of the unit ball implies that the function $\mathbf{R}r$ is positive on the boundary $\mathbb{S}^{2n-1}$. So with $u := \big(\tfrac{\mathbf{R}r}{2\|\,\partial r\,\|}\big)|_{\partial\Omega}$, we have on $W^{-1/2}_{hol}(\mathbb{S}^{2n-1})$

$$K^*(\mathbf{R}+n)\,K = T_u\,,$$

hence $V^*\,(\mathbf{R}+n)^{-1/2}\,V = \big(T_\Lambda^{-1/2} T_u T_\Lambda^{-1/2}\big)^{-1/2}$ is a GTO. Since $\sigma(\Lambda)$ and u are not identically 0 on $\mathbb{S}^{2n-1}$, we get from Corollary 1.2.16

$$\widetilde{\sigma}\big(V^*\,(\mathbf{R}+n)^{-1/2}\,V\big) = \sigma(\Lambda)^{1/2}|_\Sigma\, u^{-1/2}|_{\partial\Omega}\,.$$

Now with $T := (V^*\mathbf{T}_{\partial_{z_j}} V)\,(V^*(\mathbf{R}+n)^{-1/2} V) = V^*\,\mathbf{T}_{\partial_{z_j}}\,(\mathbf{R}+n)^{-1/2}\,V$, we have from Lemma 1.3.7 and (1.31)

$$\begin{aligned}\widetilde{\sigma}\big(V^*\pi_A(\mathbb{P}_j)V\big) &= -\tfrac{i}{\sqrt{2}}\,\big(\widetilde{\sigma}(T) - \widetilde{\sigma}(T^*)\big) = \sqrt{2}\,\mathrm{Im}\big(\widetilde{\sigma}(T)\big) \\ &= \sqrt{2}\,\widetilde{\sigma}\big(V^*(\mathbf{R}+n)^{-1/2}V\big)\,\mathrm{Im}\big(\widetilde{\sigma}(V^*\mathbf{T}_{\partial_{z_j}} V)\big) \\ &= \sqrt{2}\,\Big(\sigma(\Lambda)^{1/2}|_\Sigma\,\rho^{-1/2}\Big)\,\Big(\sigma(\Lambda)^{-1}|_\Sigma\,\mathrm{Im}\big(\tfrac{\partial_{z_j} r}{2\|\,\partial r\,\|}\big)|_{\partial\Omega}\Big) \\ &= \sqrt{2}\,u^{-1/2}\,\sigma(\Lambda)^{-1/2}|_\Sigma\,\Big(\tfrac{\mathrm{Im}(\partial_{z_j} r)}{2\|\,\partial r\,\|}\Big)|_{\mathbb{S}^{2n-1}}\end{aligned}$$

From the definition of u, (1.9) and (A.9), we get for any $(z,t) \in \partial\Omega \times \mathbb{R}_{>0}$

$$\widetilde{\sigma}\big(V^*\pi_A(\mathbb{P}_j)V\big)(z,t) = \sqrt{2}\,\frac{\sqrt{2\|\,\partial r\,\|(z)}}{\sqrt{(\mathbf{R}r)(z)}}\,2^{1/4}\sqrt{\|\,\partial r\,\|}\,t^{1/2}\,\frac{\mathrm{Im}(\partial_{z_j} r)(z)}{2\|\,\partial r\,\|(z)} = 2^{1/4}\,\frac{\mathrm{Im}(\partial_{z_j} r)}{\sqrt{\mathbf{R}r}}(z)\;t^{1/2}\,. \qquad \square$$

Remark 1.3.10. *In the previous lemma, the fact that $V^*\pi_A(\mathbb{P}_j)V$ is a GTO of order $1/2$ is actually a consequence of a more general result introduced in [95], described in [82, Section 9] and rigorously proved in [141, Appendix B]. It states that for any $m \in \mathbb{R}$, the set of operators on $\mathbb{R}^n$ whose symbol p lies in $\mathscr{S}^m(\mathbb{R}^n)$ (see Definition C.1.11) with asymptotical expansion*

$$p(x,\xi) \approx \sum_{j\in\mathbb{N}} p_{m-2j}(x,\xi)\,, \qquad \textit{as } |x|,\ |\xi| \to \infty\,,$$

(this set, denoted $\mathcal{H}^m_b$ in [141, (B.38)]), is isomorphic to $\mathrm{GTO}^{m/2}(\mathbb{S}^{2n-1})$.

1.3.3 Link between Toeplitz operators on the Fock spaces and Weyl operators

Here, we restrict to the case $n = 1$ since the results from [24, 25] we are interested in are established for the one dimensional Fock space.

Let $\rho_{\mathfrak{m}}$ be a weight on $\mathbb{C}$ as in Definition 1.1.8. From [25, Proposition 9] (3), the Toeplitz operator $\mathscr{T}_{\rho_{\mathfrak{m}}}$

(3) It concerns only $\mathbb{C}$ but the results are actually valid for $\mathbb{C}^n$.

is densely defined, positive, selfadjoint and $\mathscr{T}_{\rho_{\mathfrak{m}}}^{1/2}$ extends to a unitary isomorphism from $\mathscr{F}_{\mathfrak{m}}$ onto $\mathscr{F}$. Moreover, denoting $\mathscr{T}^{(\mathfrak{m})}$ and $\mathscr{T}$ the Toeplitz operators acting on $\mathscr{F}_{\mathfrak{m}}$ and $\mathscr{F}$ respectively, we have the unitary equivalence

$$\mathscr{T}_f^{(\mathfrak{m})} \approx \mathscr{T}_{\rho_{\mathfrak{m}}}^{-1/2} \mathscr{T}_{\rho_{\mathfrak{m}} f} \mathscr{T}_{\rho_{\mathfrak{m}}}^{-1/2}, \tag{1.34}$$

for any $f \in L^\infty(\mathbb{C})$. Equation [24, (16)] gives also a relation between Toeplitz operators on $\mathscr{F}$ and Weyl operators on $L^2(\mathbb{C})$ (see Definition 1.1.10):

$$\mathscr{T}_f = \mathscr{W}_{\mathcal{E} f}, \tag{1.35}$$

where $\mathcal{E} = e^{\Delta/8}$ is the heat operator at time 1/8. We remark also that for any $f \in \mathscr{S}^m$, $m \in \mathbb{R}$, $\mathcal{E} f$ is also in $\mathscr{S}^m$.

1.3.4 A diagram as a summary

Figure 1.1 illustrates the unitaries between the different spaces (simple arrows), the corresponding orthogonal projectors (dashed arrows), isomorphisms (dotted arrows) and the representations of the Heisenberg Lie algebra (double arrows).

The diagram must be seen in three dimensions, and presents three levels: the top level corresponds to $L^2(\mathbb{S}^{2n-1})$, $L^2(\mathbb{B}^n)$ and $L^2(\mathbb{C}^n)$, which are placed above their corresponding Hilbert spaces, the middle one contains the latter together with $L^2(\mathbb{R}^n)$, while the lowest level concerns only the Heisenberg Lie algebra $\mathfrak{h}^n$. The space $L^2(\mathbb{R}^n)$ has been placed in the middle level since it does not have any underlying Hilbert space like the other L^2 spaces over complex domains. Finally, simple light-gray edges complete the diagram in order to link spaces belonging to the same level.
The diagram presents the particular case $\Omega = \mathbb{B}^n$ for two reasons: first, the notations are simpler and secondly this example will be often treated in the following. Moreover, the representations π_{A_m} and π_A of $\mathfrak{h}^n$ on the (un)weighted Bergman spaces are defined for the unit ball.

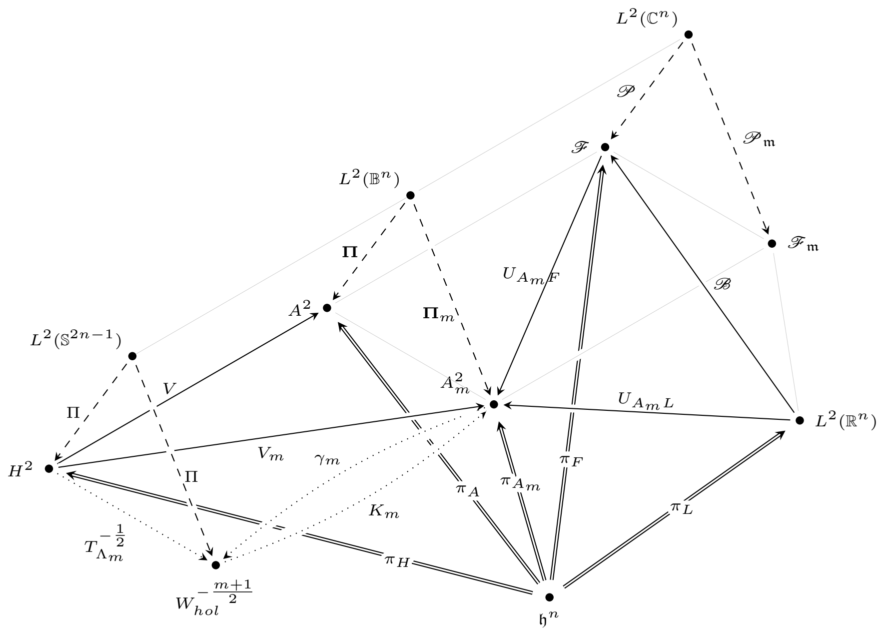

Figure 1.1: The case of $\mathbb{B}^n$: projectors, unitaries and representations involved between the Hilbert spaces and the Heisenberg Lie algebra.

1.4 Examples

We investigate in this section two different classes of Toeplitz operators on the weighted Bergman spaces, which will be needed in Section 4 to build spectral triples.

1.4.1 GTO-like operators on the Bergman space

In Section 1.3.2, the operator $\mathbf{T}_{\partial_{z_j}}$ was considered as a Toeplitz operator acting on the Bergman space. Similarly to GTOs, we can think of Toeplitz operator $\mathbf{T}_{\mathbf{P}}$ acting on $A^2_m(\Omega)$, where the differential operator $\mathbf{P}$ on $\mathbb{C}^n$ replaces the function $f \in C^\infty(\overline{\Omega})$ as is Definition 1.2.1. Thus, for $\mathbf{P}$ of the form (1.24), we define

$$\mathbf{T}_{\mathbf{P}} := \mathbf{\Pi}_m \mathbf{P}|_{A^2_m(\Omega)} \,. \tag{1.36}$$

We can generalize the result of (1.3.7).

Lemma 1.4.1. *[61, Lemma 2.15] For $\mathbf{P}$ of the form (1.24), we have*

$$\mathbf{T}_{\mathbf{P}} = V_m \, T_{\Lambda_m}^{-1/2} \, T_{\Lambda_{m,\mathbf{P}}} \, T_{\Lambda_m}^{-1/2} \, V_m^* \qquad \text{on } A^2_m(\Omega)\,.$$

Moreover, $\mathbf{T}_{\mathbf{P}}$ is selfadjoint on $A^2_m(\Omega)$ when $\mathbf{P}$ has a selfadjoint extension on $L^2(\Omega, w_m)$.

Proof. A similar calculation as in the proof of (1.28) shows the equalities. We only need to prove that $T_{\Lambda_{m,\mathbf{P}}}$ is selfadjoint, which follows from (1.23). Indeed for $u,\, v \in H^2(\partial\Omega)$, we get

$$\begin{aligned}\langle\, (T_{\Lambda_{m,\mathbf{P}}})^* \, u\,,\, v\,\rangle_{H^2} &= \langle\, u\,,\, K^* w_m \mathbf{P} K\, v\,\rangle_{H^2} = \langle\, Ku\,,\, \mathbf{P}Kv\,\rangle_{L^2(\Omega,w_m)} = \langle\, \mathbf{P}Ku\,,\, Kv\,\rangle_{L^2(\Omega,w_m)} \\ &= \langle\, w_m \mathbf{P}Ku\,,\, Kv\,\rangle_{L^2(\Omega)} = \langle\, K^* w_m \mathbf{P} K\, u\,,\, v\,\rangle_{H^2} = \langle\, T_{\Lambda_{m,\mathbf{P}}}\, u\,,\, v\,\rangle_{H^2}\,. \end{aligned}$$ □

An interesting example of selfadjoint operator $\mathbf{T}_{\mathbf{P}}$, where $\mathbf{P}$ is not selfadjoint on $L^2(\Omega, w)$ is given by the “ weighted normal derivative ” operator

$$\mathbf{P}_{w_m} := \tfrac{1}{w_m} \sum_{j=1}^n (\partial_{\bar z_j} w_{m+1})\, \partial_{z_j}\,, \tag{1.37}$$

(note that $\partial_{\bar z_j}(w_{m+1})/w_m$ is smooth up to the boundary). For f and g in $A^2_m(\Omega)$, we get

$$\int_\Omega d\mu\, \partial_{z_j}(w_{m+1}\, f\bar g) = \int_{\partial\Omega} d\sigma\, w_{m+1}\, f\bar g\, \tfrac{\partial_{z_j} r}{2\|\,\partial r\,\|} = 0\,,$$

since $w_{m+1} = 0$ on $\partial\Omega$. Applying Leibniz rule to the left hand side gives

$$\int_\Omega w_m d\mu\, \tfrac{\partial_{z_j}(w_{m+1})}{w_m}\, f\bar g = \int_\Omega w_m d\mu\, (r\partial_{z_j}) f\, \bar g\,,$$

hence $\mathbf{T}_{(\partial_{z_j} w_{m+1})/w_m} = \mathbf{T}_{r\partial_{z_j}}$, as operators acting on $A^2_m(\Omega)$. From the relations $\mathbf{T}^*_h = \mathbf{T}_{\bar h}$ and $\mathbf{T}_{h\partial_{z_j}} = \mathbf{T}_h \mathbf{T}_{\partial_{z_j}}$, we get

$$\mathbf{T}_{\mathbf{P}_{w_m}} = \sum_{j=1}^n \big(\mathbf{T}_{(\partial_{z_j} w_{m+1})/w_m}\big)^* \, \mathbf{T}_{\partial_{z_j}} = \sum_{j=1}^n \big(\mathbf{T}_{r\partial_{z_j}}\big)^* \, \mathbf{T}_{\partial z_j} = \sum_{j=1}^n \mathbf{T}^*_{\partial_{z_j}} \, \mathbf{T}_r \, \mathbf{T}_{\partial_{z_j}}\,.$$

So not only is $\mathbf{T}_{\mathbf{P}_{w_m}}$ selfadjoint but it is also negative on $A^2_m(\Omega)$.

In the case of the unweighted Bergman space,

$$\mathbf{P}_0 := \mathbf{P}_{w_0=1} = -\sum_{j=1}^{n} (\partial_{\bar{z}_j} r)\, \partial_{z_j} \,. \tag{1.38}$$

From Lemma 1.4.1, the operator $V^*\, \mathbf{T}_{\mathbf{P}_0}\, V$ is an elliptic GTO of order 1, whose symbol is given by (1.9) and (1.25):

$$\begin{aligned} \widetilde{\sigma}(V^*\, \mathbf{T}_{\mathbf{P}_0}\, V)(z,t) &= \widetilde{\sigma}(T_\Lambda)^{-1}\, \widetilde{\sigma}(T_{\Lambda_{0,\mathbf{P}_0}}) = -2t|\eta_z|\, \tfrac{\Gamma(1)}{2|\eta_z|} \sum_{j=1}^{n} (\partial_{\bar{z}_j} r)(\partial_{z_j} r)(z) = -2\|\, \eta_z \,\|^2\, t \\ &= -\|\, \partial r \,\|^2(z)\, t \end{aligned} \tag{1.39}$$

(the last equality uses (A.9)).

Remark 1.4.2. *The hypothesis $\mathbf{P} = \mathbf{P}^*$ in Lemma 1.4.1 is quite strong since $\mathbf{T}^*_{\mathbf{P}} \neq \mathbf{T}_{\mathbf{P}^*}$ in general. For instance, when $w_m = 1$, one deduces from (1.29) that*

$$\mathbf{T}^*_{\partial_{z_j}} = V\, T_\Lambda^{-1/2} T_{(\partial_{\bar{z}_j} r)/(2\|\, \partial r \,\|)} T_\Lambda^{-1/2}\, V^* \neq 0 = \mathbf{T}_{-\partial_{\bar{z}_j}}$$

while $-\partial_{\bar{z}_j}$ is the formal adjoint of ∂_{z_j} on the domain of smooth compactly supported functions on Ω. Of course, any selfadjoint extension of a differential operator $\mathbf{P}$ needs to take care of the boundary conditions on $\partial\Omega$.

1.4.2 Unitary operators

We now give examples of unitary GTOs and also unitaries $\mathbf{U}$ on $A^2_w(\Omega)$ such that $V^*_m\, \mathbf{U}\, V_m$ is a GTO. These classes of operators will be used later on in Section 4.1 and Section 4.2.

A natural class of unitary GTOs is given by operators of the form

$$T_{\exp(iP)}\,, \qquad \text{with } P \text{ a pseudodifferential operator on } \partial\Omega \text{ such that } [\,\Pi,\, P\,] = 0\,.$$

However, this expression makes sense only if $\exp(iP)$ is itself a pseudodifferential operator. For instance, $P := -i\partial_\theta$ on $L^2(\mathbb{S}^1)$ belongs to $\Psi\mathrm{DO}^1(\mathbb{S}^1)$ with principal symbol $\sigma(P)(\theta,t) = t$, while e^{iP} is not a pseudodifferential operator since its principal symbol e^{it} does not belong to any Hörmander class S^m. The idea to bypass this problem is to control the behaviour of P by means of a function which behaves nicely at infinity. The following lemma gives a sufficient condition on this function and on P for $\exp(iP)$ to be a (unitary) classical pseudodifferential operator.

Lemma 1.4.3. *[61, Lemma 2.18] Let $\varphi \in C^\infty(\mathbb{R})$ be real valued function in $S^0(\mathbb{R})$ (see Definition C.1.1), i.e. verifying*

$$\forall k \in \mathbb{N} \text{ there are } c_k > 0 \text{ such that for all } x \in \mathbb{R}, \quad |\partial_x^k \varphi|(x) \le c_k\, (1+|x|)^{-k}\,, \tag{1.40}$$

and let Q be an elliptic selfadjoint pseudodifferential operator of order 1 on a compact manifold M. Then the operator $\exp(i\varphi(Q)) \in \Psi\mathrm{DO}^0(M)$.

Proof. From Faà di Bruno's formula, any smooth functions f and g on $\mathbb{R}$ verify

$$\partial_x^k (f \circ g)(x) = \sum_{\alpha \in E_k} \tfrac{k!}{\alpha!} \big((\partial^{|\alpha|} f) \circ g\big)(x) \prod_{j=1}^{k} (\tfrac{1}{j!}\, \partial^j g)^{\alpha_j}(x)\,, \quad \forall\, k \in \mathbb{N}\,,$$

where $E_k := \{\alpha \in \mathbb{N}^k, \alpha_1 + 2\alpha_2 + \cdots + k\alpha_k = k\}$. Thus with $e : x \in \mathbb{R} \mapsto \exp(i\,x)$, we get for any $k \in \mathbb{N}$

$$|\partial_x^k (e \circ \varphi)|(x) \leq \sum_{\alpha \in E_k} \tfrac{k!}{\alpha!} \,\big[\prod_{j=1}^{k} (\tfrac{c_j}{j!})^{\alpha_j}\big]\, (1+|x|)^{-k} =: c'_k\, (1+|x|)^{-k}\,,$$

so $x \in \mathbb{R} \mapsto \exp\big(i\varphi(x)\big)$ verifies also (1.40), thus from [137, Theorem 1] (or [140, Theorem 1.2]), $\exp(i\varphi(Q)) \in \Psi\mathrm{DO}^0(M)$. □

Corollary 1.4.4. *[61, Corollary 1.4.4] Let φ verify (1.40) and $Q \in \Psi\mathrm{DO}^1(\partial\Omega)$ such that and $T_Q = (T_{\Lambda_m})^{-1/(m+1)}$ and $[\,Q,\, \Pi\,] = 0$. Then $T_{\exp(i\varphi(Q))}$ is a unitary GTO.*

Proof. From (P1), such operator Q exists and by previous lemma, $\exp(i\varphi(Q)) \in \Psi\mathrm{DO}^0(\partial\Omega)$ is a unitary which commutes with Π, thus $T_{\exp(i\varphi(Q))} = \exp\big(i\varphi(T_{\Lambda_m}^{-1/(m+1)})\big)$ is a unitary GTO. □

In Section 4.2, we will need to find unitary operators on $A^2_m(\Omega)$ of the form $V_m T V_m^*$, with T a GTO, to deduce non-positive Dirac-like operators from positive ones. So we can take

$$\mathbf{U} := V_m T_{\exp\big(i\varphi(Q)\big)} V_m^*$$

as in Corollary 1.4.4.

Another class of unitary operators on $A^2_m(\Omega)$ can be obtained as follows. Take any GTO T_P which is invertible (as an operator on H^2) and not a constant multiple of a positive operator. Take for instance, $T_P = T_f$ with f a nonconstant zero-free holomorphic function: the zero-free condition ensures $T_f = M_f$ is invertible, while $T_f = cA$ with A a positive operator, would mean that multiplication by the nonconstant holomorphic function f/c is a positive operator, which is quickly seen to lead to contradiction. From (P1), we know there exists another pseudodifferential operator Q such that $T_Q = T_P$ and $[\,\Pi,\, Q\,] = 0$. Hence also $T_P^* T_P = T_Q^* T_Q = T_{Q^*Q} = T_{|Q|}^2$, implying that $U := T_P T_{|Q|}^{-1}$ is a unitary GTO. From (1.29), the operator $\mathbf{U} := V_m\, U\, V_m^*$ is unitary from $A^2_m(\Omega)$ onto $A^2_m(\Omega)$. Furthermore, $\mathbf{U}$ is not a multiple of the identity; for, if it were, then so would be U, hence $T_P = UT_{|Q|}$ would be a constant multiple of the positive operator $T_{|Q|}$, contrary to the hypothesis.

Chapter 2

Quantization

We will focus on two different (but related) programmes of quantization: the geometric quantization and the deformation quantization.

2.1 Geometric quantization

This section recalls very briefly the main concepts of geometric quantization and we refer to [156, 135] and particularly to [51] for deeper details.

2.1.1 The different steps

2.1.1.1 Prequantization

Here (Ω, ω) denotes a symplectic manifold of real dimension $2n$, representing the phase space of a classical system. The proposed *prequantization* is the following: to any L^2 measurable smooth function f over Ω is associated the operator

$$Q_f^\theta := -i\hbar(X_f - \tfrac{i}{\hbar}\theta(X_f)) + f\,, \tag{2.1}$$

where X_f is the Hamiltonian vector field associated to f, and θ is a local symplectic potential, i.e. a one form verifying locally $\omega = d\theta$ (see Appendix A). Actually, (2.1) is the result of successive corrections for Q^θ to respect at least points i) and iii) of Conditions 0 from the introduction. Indeed, first $f \mapsto -i\hbar\, X_f$ is a non injective map since two functions differing by a constant give the same operator. Then the map $Q : f \mapsto -i\hbar\, X_f + f$ is not enough since it verifies the linearity but not the canonical commutation relations. Indeed for any smooth function ψ over Ω,

$$\begin{aligned}[]
[\,Q_f,\, Q_g\,] &= (-i\hbar\, X_f + f)(-i\hbar\, X_g + g)\,\psi - (-i\hbar\, X_g + g)(-i\hbar\, X_f + f)\,\psi \\
&= -\hbar^2[\,X_f,\, X_g\,]\,\psi - i\hbar\big(X_f(g\psi) + fX_g(\psi) - X_g(f\psi) - gX_f(\psi)\big) \\
&= -i\hbar\big(-i\hbar X_{\{f,g\}} + (X_f - X_g)\big)\,\psi = -i\hbar\big(-i\hbar X_{\{f,g\}} + 2\{\,f\,,\,g\,\}\big)\,\psi \neq Q_{\{f,g\}}\,\psi\,.
\end{aligned}$$

The last correction is given by introducing a one form θ and by setting the previous quantization map Q^θ (2.1). Switching locally from θ to $\theta' := \theta + d\alpha$, with some smooth function α, we get for any smooth square summable function ψ over Ω:

$$\begin{aligned} Q_f^{\theta'}(e^{i\alpha/\hbar}\psi) &= -i\hbar\big(X_f(e^{i\alpha/\hbar}\psi) - \tfrac{i}{\hbar}\theta(X_f)(e^{i\alpha/\hbar}\psi) - \tfrac{i}{\hbar}d\alpha(X_f)(e^{i\alpha/\hbar}\psi)\big) + f e^{i\alpha/\hbar}\psi \\ &= -i\hbar\big(\tfrac{i}{\hbar}d\alpha(X_f)(e^{i\alpha/\hbar}\psi) + e^{i\alpha/\hbar}X_f\psi - \tfrac{i}{\hbar}\theta(X_f)(e^{i\alpha/\hbar}\psi) - \tfrac{i}{\hbar}d\alpha(X_f)(e^{i\alpha/\hbar}\psi)\big) \\ &\qquad + f e^{i\alpha/\hbar}\psi \\ &= e^{i\alpha/\hbar}Q_f^{\theta}\psi\,. \end{aligned}$$

In other words, the change of the local symplectic potential is compensated by a change of the phase of the function ψ. As a consequence, the quantum states are not functions but belong instead to the Hilbert space $\mathcal{H}_{pre}$ (for prequantum Hilbert space) of square summable compactly supported sections on a hermitian complex line bundle $L \to \Omega$, with $U(1)$ as structural group. The hermitian metric h on $\Gamma(L)$ gives rise to the following inner product on $\mathcal{H}_{pre}$

$$\langle\, \psi_1\,,\, \psi_2 \,\rangle_L := (2\pi\hbar)^n \int_\Omega h(\psi_1,\psi_2)(z)\,\tfrac{1}{n!}\omega^n(z) =: \int_\Omega h(\psi_1,\psi_2)(z)\,\tilde\omega(z)\,. \tag{2.2}$$

Now considering a connection ∇ in $L \to \Omega$, we can rewrite (2.1) independently of θ, as a prequantization map from $C^\infty(\Omega)$ to $\mathrm{End}(\mathcal{H}_{pre})$:

$$Q_f^{pre}\psi := (-i\hbar\,\nabla_{X_f} + f)\,\psi\,. \tag{2.3}$$

Comparing (2.3) with (2.1), the connection ∇ is of the form $\nabla_X = X - \frac{i}{\hbar}\,\theta(X)$, for any (non necessarily Hamiltonian) vector field $X \in T\Omega$. The computation of the curvature of ∇ gives for any $X, Y \in T\Omega$

$$\begin{aligned} \mathrm{curv}(\nabla)(X,Y) &= [\,\nabla_X,\,\nabla_Y\,] - \nabla_{[X,Y]} = [\,X,\,Y\,] - \tfrac{i}{\hbar}\,\big(X\theta(Y) - Y\theta(X)\big) + \tfrac{i}{\hbar}\,\theta([\,X,\,Y\,]) \\ &= \tfrac{1}{i\hbar}\,d\theta(X,Y) = \tfrac{1}{i\hbar}\,\omega(X,Y)\,. \end{aligned} \tag{2.4}$$

In other words, the prequantization map (2.3) makes sense if and only if the curvature of the connection is a multiple of the symplectic form.

Let us check that this prequantization map verifies the selfadjointness for real valued function and also the canonical commutation relations iii) of Conditions 0 (the proof can also be found in [51, Theorem 8]). First, when f is real valued and $\psi_1, \psi_2 \in \mathcal{H}_{pre}$, we have from (2.2)

$$\begin{aligned} \langle\, Q_f^{pre}\psi_1\,,\, \psi_2 \,\rangle_L &= \int_\Omega h\big((-i\hbar\,\nabla_{X_f} + f)\,\psi_1, \psi_2\big)(z)\,\tilde\omega(z) \\ &= -i\hbar\int_\Omega X_f\,\big(h(\psi_1,\psi_2)\big)(z)\,\tilde\omega(z) + \int_\Omega h\big(\psi_1, (-i\hbar\nabla_{X_f} + f)\psi_2\big)(z)\,\tilde\omega(z) \\ &= -i\hbar\int_\Omega X_f\,\big(h(\psi_1,\psi_2)\big)(z)\,\tilde\omega(z) + \langle\, \psi_1\,,\, Q_f^{pre}\,\psi_2 \,\rangle_L\,. \end{aligned} \tag{2.5}$$

Now denoting $\mathcal{L}_X$ the Lie derivative of a vector field X and $g := h(\psi_1,\psi_2) \in C^\infty(\Omega)$ which is compactly supported (recall that ψ_1, ψ_2 have compact support), we have

$$X_f(g)\,\tilde\omega = (\mathcal{L}_{X_f}g)\,\tilde\omega = (\mathcal{L}_{X_f}g)\,\tilde\omega + g\,(\mathcal{L}_{X_f}\,\tilde\omega) = \mathcal{L}_{X_f}(g\,\tilde\omega) = d\big(i_{X_f}(g\,\tilde\omega)\big)\,,$$

so the first term in (2.5) vanishes as the integration of a compactly supported exact form of maximal degree.

Secondly, for any $f, g \in C^\infty(\Omega)$ and $\psi \in \Gamma(L)$, we have

$$\begin{aligned}
[\, Q^{pre}_f ,\, Q^{pre}_g \,]\,\psi &= (-i\hbar\,\nabla_{X_f} + f)(-i\hbar\,\nabla_{X_g} + g)\,\psi - (-i\hbar\,\nabla_{X_g} + g)(-i\hbar\,\nabla_{X_f} + f)\,\psi \\
&= -\hbar^2[\,\nabla_{X_f},\, \nabla_{X_g}\,]\,\psi - i\hbar\big(\nabla_{X_f}(g\psi) + f\,\nabla_{X_g}(\psi) - \nabla_{X_g}(f\psi) - g\,\nabla_{X_f}(\psi)\big) \\
&= -\hbar^2\,\big(\nabla_{[\,X_f, X_g\,]} + \mathrm{curv}(\nabla)(X_f, X_g)\big)\,\psi - i\hbar\big(X_f(g) - X_g(f)\big)\,\psi \\
&= -i\hbar\,\big(-i\hbar\nabla_{X_{\{\,f\,,\,g\,\}}} - \omega(X_f, X_g) + 2\{\,f\,,\,g\,\}\big)\,\psi \qquad \text{from (2.4) and (A.3)} \\
&= -i\hbar\,\big(-i\hbar\nabla_{X_{\{\,f\,,\,g\,\}}} + \{\,f\,,\,g\,\}\big)\,\psi = -i\hbar\,Q^{pre}_{\{\,f\,,\,g\,\}}\,.
\end{aligned}$$

When Ω is symplectic, the existence of such a connection on a complex line bundle $L \to \Omega$, which is compatible with the existing hermitian metric h, i.e. such that for any $\psi_1, \psi_2 \in \Gamma(L)$ and $X \in T\Omega$,

$$X(h(\psi_1, \psi_2)) = h(\nabla_X \psi_1, \psi_2) + h(\psi_1, \nabla_X \psi_2)\,,$$

is the result of the Chern–Weil theorem [150] (a suitable statement can also be found in [155, Theorem 13]). When the previous objects L and ∇ make sense and verify the conditions, the manifold Ω is called *prequantizable*. Note that the denomination " prequantum Hilbert space " comes from the fact that $\mathcal{H}_{pre}$, or rather the projective Hilbert space $P\mathcal{H}_{pre}$, does not really correspond to the physical quantum states. The latter are actually sections of the line bundle which are covariantly constant in the direction of a given *polarization*.

2.1.1.2 Polarization

The second step consists in reducing the number of variables by restricting the prequantum Hilbert space and also the space of observables which can be quantized. The n variables are selected by means of selecting an additional geometric structure: a *complex polarization* $\mathcal{P}$. This is a distribution [(1)] verifying the following properties:

i) $\dim_{\mathbb{R}} \mathcal{P}_p = n$ for any $p \in \Omega$, and $\omega(\mathcal{P}, \mathcal{P}) = 0$ (Lagrangian),

ii) $[\,\mathcal{P},\, \mathcal{P}\,] \subset \mathcal{P}$ (involutive),

iii) $\dim_{\mathbb{R}}(\mathcal{P}_p \cap \overline{\mathcal{P}}_p \cap T_p\Omega)$ is constant for any $p \in \Omega$.

Remark 2.1.1. *In the particular case when Ω is a Kähler manifold (Definition A.4.1), the antiholomorphic tangent space $\mathcal{T}''\Omega$ (Definition A.3.2) defines the so-called* Kähler polarization $\mathcal{P}$. *Kähler polarizations have the property to be compatible with the connection ∇, in the sense that around any point $p \in \Omega$, there is a local symplectic potential θ such that the connection one form is θ (see for instance [87, Proposition 23.6]), or equivalently, $\theta(\mathcal{P}) = 0$.*

The space of *polarized quantum states* $\Gamma_{\mathcal{P}}(L)$ consists of the set of smooth sections in $\Gamma(L)$ being constant along $\mathcal{P}$, which means that for any section $\psi \in \Gamma_{\mathcal{P}}(L)$ and any $X \in \mathcal{P}$, $\nabla_X \psi = 0$. However, given a classical observable $f \in C^\infty(\Omega)$, the range of the corresponding operator Q^{pre}_f acting on $\Gamma_{\mathcal{P}}(L)$ is not necessarily again in $\Gamma_{\mathcal{P}}(L)$.

The idea is to define the set of *quantizable observables* as the functions f in $C^\infty(\Omega)$ such that for any states $\psi \in \Gamma_{\mathcal{P}}(L)$, $Q^{pre}_f \psi$ stays in $\Gamma_{\mathcal{P}}(L)$. Then we can show [51, Proposition 24] that an observable f

[(1)] In this context, this is a subbundle of $T\Omega \otimes \mathbb{C}$.

is quantizable if and only if one of the following relations is verified:

$$[\,Q_f^{pre},\,\nabla_X\,]=0 \text{ on } \Gamma_{\mathcal{P}}(L)\,,\qquad \text{or}\qquad [\,X_f,\,X\,]\subset\mathcal{P}\,,\qquad \text{for any } X\in\mathcal{P}\,. \tag{2.6}$$

Note that this constraint reduces considerably the space of classical observables: for instance, in the case $\Omega=\mathbb{R}^{2n}$, with the polarization spanned by the vector fields $(\partial_{x_j})_{j=1\ldots n}$, the set of quantizable observables is formed by functions at most linear in the dual variables $(\xi_j)_{j=1\ldots n}$ (see [1, Section 3.7]).

2.1.1.3 Metaplectic correction

At this point, the process of quantization is not achieved yet since we want to define a measure on the space of quantum states. Indeed, in general, elements of $\Gamma_{\mathcal{P}}(L)$ are not square summable, and the scalar product is not even well defined. The last step, called the *metaplectic correction*, aims to bring a solution to this problem.

Note that this step is not necessary for our purposes since we will work with strictly pseudoconvex manifolds, which are Kähler. In this case, the space of quantum states $\Gamma_{\mathcal{P}}$, where $\mathcal{P}$ is the Kähler polarization (see Remark 2.1.1), is already endowed by a hermitian inner product. The final quantization map is given by $Q_f := Q_f^{pre}$ of (2.3) and the space of square summable holomorphic sections in $\Gamma(L)$ is denoted $\Gamma_{hol}(L)$, whose completion with respect to the natural norm defines the quantum Hilbert space $\mathcal{H}$. Also, for compact Kähler manifolds, the space of holomorphic sections $\Gamma_{hol}(L)$ is finite dimensional and $\mathcal{B}(\mathcal{H})$ can be identified with the space of finite dimensional matrices.

For a more general manifold Ω , the idea is to replace sections of $\Gamma_{\mathcal{P}}(L)$ by elements called half-forms, and then define an inner product on such element. Denote $\kappa_{\mathcal{P}}$ the canonical bundle of $\mathcal{P}$, i.e. the complex line bundle for which sections φ are n-forms verifying $i_X\varphi=0$ for any $X\in\overline{\mathcal{P}}$. A section φ in $\Gamma(\kappa_{\mathcal{P}})$ is said to be polarized if $d\varphi=0$, so that polarized sections are exactly holomorphic $(n,0)$-forms over Ω. Suppose there exists a square root of $\kappa_{\mathcal{P}}$, i.e. a complex line bundle $\delta_{\mathcal{P}}$ over Ω such that $\delta_{\mathcal{P}}\otimes\delta_{\mathcal{P}}$ is isomorphic to $\kappa_{\mathcal{P}}$. Thus for $\phi_1,\phi_2\in\Gamma(\delta_{\mathcal{P}})$, the tensor product $\phi_1\otimes\phi_2$ is identified with a section in $\Gamma(\kappa_{\mathcal{P}})$. As a consequence, $\delta_{\mathcal{P}}$ is endowed with a natural hermitian inner product [87, Proposition], which, combined with the one on L, gives rise to the half-form Hilbert space $\mathcal{H}$ (or the quantum Hilbert space), defined as the set of square integrable polarized sections on $L\otimes\delta_{\mathcal{P}}$. As before, physical quantum states correspond to elements of the corresponding Hilbert space $P\mathcal{H}$. Finally, the quantum observables are modified accordingly to this correction: to any smooth function f on Ω verifying (2.6) is associated the following operator

$$Q_f s := (Q_f^{pre}\psi)\otimes\phi - i\hbar\,\psi\otimes X_f(\phi)\,, \tag{2.7}$$

where locally $s=\psi\otimes\phi\in\Gamma(L)\otimes\Gamma(\delta_{\mathcal{P}})$.

2.1.2 Link with Toeplitz operators and an example of quantization

Toeplitz operators naturally arise in the context of geometric quantization of a compact Kähler manifold Ω. Indeed, instead of restricting the set of quantizable classical observables with (2.6), another approach is to force the operators Q_f to stay in the Hilbert space of quantum states $\mathcal{H}$ (or $P\mathcal{H}$). This procedure is obtained by considering the orthogonal projection $\mathbf{\Pi}$ from $L^2(L)$ to $\Gamma_{hol}(L)=\mathcal{H}$ and modifying the quantization map (2.7) as

$$\mathbf{Q}'_f := \mathbf{\Pi} Q_f$$

(since we are working on the whole domain Ω, the bold notations has been chosen in analogy to the ones of the respective objects on the Bergman space in Definition 1.2.1). When Ω is compact and Kähler, the Tuynman relation [143] (or [27, Proposition 4.1] for a coordinate free proof) gives

$$\mathbf{Q}'_f = \mathbf{\Pi}\big(- i\hbar\nabla_{X_f} + f\big) = \mathbf{T}_{\frac{\hbar}{2}\Delta f + f} \,. \tag{2.8}$$

Remark 2.1.2. *Different factors are used in the literature. Here, we chose to introduce the reduced Planck constant $\hbar$ so it will reappear in the derived formulæ.*

Again, from this relation, it becomes obvious that the quantum observable $\mathbf{Q}_f$ corresponding to a classical one f (real valued) is selfadjoint since $\mathbf{T}_f^* = \mathbf{T}_{\bar{f}} = \mathbf{T}_f$.

Let us illustrate the previous notions on a simple example. Consider $\mathbb{C}^n$, equipped with the hermitian form $\omega := i\sum_{j=1}^n dz_j \wedge d\bar{z}_j$, for which we take the holomorphic polarization, i.e. $\mathcal{P} = \mathcal{T}''\Omega$, which is locally spanned by the $(\partial_{\bar{z}_j})_{j=1,\dots,n}$. A local symplectic potential is given by $\theta = -\frac{i}{2}(\partial\rho - \bar{\partial}\rho)$, where $\rho : z \mapsto \rho(z) := |z|^2$. For any $j = 1, \dots, n$, we have

$$\nabla_{\partial_{\bar{z}_j}} = \partial_{\bar{z}_j} - \tfrac{i}{\hbar}\theta(\partial_{\bar{z}_j}) = \partial_{\bar{z}_j} + \tfrac{1}{2\hbar} z_j \,,$$

and any quantum sections ψ verify $\nabla_X \psi = 0$ for $X \in \mathcal{P}$. The solutions of this equation are given by

$$\psi(z) = \phi(z) e^{-\frac{|z|^2}{2\hbar}} = \phi(z) e^{-\frac{\rho}{2\hbar}} \,,$$

where ϕ is holomorphic in $\mathbb{C}^n$. As a consequence, quantum states are exactly (normalized) vectors of the Fock space $\mathscr{F}(\mathbb{C}^n)$ in Definition 1.1.8, by using the measure $e^{-|z|^2/(2\hbar)} d\mu(z)$ instead of the original one.

2.1.3 Approximating the classical observables

The idea proposed in [28] is to introduce a parameter $m \in \mathbb{N}$ in the context of geometric quantization in order to see the Lie algebra of classical observables as the semi-classical limit of the Lie algebras of quantum operators, when m goes to infinity. As a consequence, this limit has to be defined so that the product of quantum observables and commutators correspond to the pointwise product of classical observables and the Poisson bracket respectively. This leads to the notion of *Lie algebra quasilimit*.

We suppose we are given a compact Kähler manifold Ω with complex line bundle L, a fibre metric h and compatible connection ∇. To make the geometric quantization depends on some parameter $m \in \mathbb{N}$, consider the m^{th} tensor product of L, denoted $L^{(m)} := \bigotimes_{j=1}^m L$, endowed with the hermitian inner product $h^{(m)} := \bigotimes_{j=1}^m h$ and $\nabla^{(m)} := \sum_{j=1}^m 1 \otimes \cdots \otimes \nabla \otimes \cdots \otimes 1$ (where ∇ is at the j^{th} position). For $f \in C^\infty(\Omega)$, we denote also $Q_f := Q_f^{pre}$ of (2.3) (recall that in the Kähler case, the prequantization map Q_f^{pre} is actually the final quantization map since no metaplectic correction is needed) and $\mathcal{H}^{(m)}$ the Hilbert space of holomorphic sections in $\Gamma_{hol}(L^{(m)})$ which are square summable with respect to the inner product induced by $h^{(m)}$. The dependence on m of the other quantities corresponding to the m^{th} level are given by

$$\operatorname{curv}(\nabla^{(m)}) = m\operatorname{curv}(\nabla) = \tfrac{m}{i\hbar}\omega\,, \quad X_f^{(m)} = \tfrac{1}{m}X_f\,, f \in C^\infty(\Omega)\,, \quad \{\, f\,,\, g\,\}_m = \tfrac{1}{m}\{\, f\,,\, g\,\}\,,$$
$$\|\,.\,\|_m := \tfrac{1}{m}\|\,.\,\| \text{ on } \operatorname{End}(\mathcal{H}^{(m)})\,, \quad Q_f^{(m)} := m\big(- i\hbar\nabla^{(m)}_{X_f^{(m)}} + f\big)\,.$$

Note that the expression of $Q_f^{(m)}$ has been rescaled by a factor m to recover

$$[\, Q_f^{(m)},\, Q_g^{(m)}\,] = -i\hbar\, Q_{\{\,f\,,g\,\}}^{(m)}\,,$$

indeed, for each level $m \in \mathbb{N}$, we want a representation of the Poisson algebra $(C^\infty(\Omega), \{\,.\,,\,.\,\})$, and not $(C^\infty(\Omega), \{\,.\,,\,.\,\}_m)$. As a consequence, the rescaled norm $\|\,.\,\|_m$ still makes sense for $m = 0$. The Tuynman's relation (2.8) remains valid for every level $m \in \mathbb{N}$ and denoting the corresponding Toeplitz operators

$$\mathbf{T}_f^{(m)} := \mathbf{\Pi}^{(m)}\mathbf{M}_f : \Gamma_{hol}(L^{(m)}) \to \Gamma_{hol}(L^{(m)})\,, \tag{2.9}$$

for $f \in C^\infty(\Omega)$, we have

$$Q_f^{(m)} = \mathbf{T}^{(m)}_{\frac{\hbar}{2}\Delta f + mf} = m\,\mathbf{T}^{(m)}_{\frac{\hbar}{2m}\Delta f + f}\,.$$

This kind of Toeplitz operators is fundamentally different from the one of Definition 1.2.1. Indeed, for any $m \in \mathbb{N}$, the Toeplitz quantization map $\mathbf{T}^{(m)} : f \in C^\infty(\Omega) \to \mathrm{End}(\mathcal{H}^{(m)})$ is surjective, i.e. every bounded linear operator over $\mathcal{H}^{(m)}$ is actually a Toeplitz operator (see [28, Proposition 4.2]). In particular, for any $m \in \mathbb{N}$, the previous Toeplitz operators $\mathbf{T}_f^{(m)}$, $f \in C^\infty(\Omega)$, form a noncommutative Lie algebra $\mathcal{A}_m$, naturally endowed with the usual commutator and the norm $\|\,.\,\|_m$.
In this context, a " semi-classical version " of Toeplitz operators is used (see for instance in [36]): a Toeplitz operator is defined as a sequence $(\mathbf{T}^{(m)})_{m\in\mathbb{N}}$ such that for any $m \in \mathbb{N}$, $\mathbf{T}^{(m)}$ is of the form

$$\mathbf{T}^{(m)} := \mathbf{\Pi}^{(m)}\,\mathbf{M}_{f(.,m)}\,\mathbf{\Pi}^{(m)} + \mathbf{r}^{(m)} : L^2(L^{(m)}) \to L^2(L^{(m)})\,,$$

where $(f(.,m))_{m\in\mathbb{N}}$ is a sequence of smooth functions on Ω admitting an asymptotic expansion of the form $\sum_{j=0}^\infty m^{-j}\, f_j$ for the C^∞ topology, with smooth coefficients f_j, and $(\mathbf{r}^{(m)})_{m\in\mathbb{N}}$ is a sequence of operators on $L^2(L^{(m)})$ such that $\mathbf{r}^{(m)} = \mathbf{\Pi}^{(m)}\mathbf{r}^{(m)}\mathbf{\Pi}^{(m)}$ and $\|\,\mathbf{r}^{(m)}\,\| = O(m^{-\infty})$.

From a deformation point of view, we want this family of Lie algebras, representing the quantum observables, $\hbar$-converge to the Lie algebra of classical observables $(C^\infty(\Omega), \{\ ,\ \}, \|\,.\,\|_\infty)$. Here, $\hbar/m$, in which $\hbar$ is fixed, plays the role of the varying reduced Planck constant [(2)] and the classical limit is obtained as m tends to infinity. A natural way to define a convergence of Lie algebras which respects the compatibility between the Lie structures, is given by the notion of *quasilimit*:

Definition 2.1.3 ([27, Axioms 3.1 and 3.2]). *Let $(\mathcal{L}_m, [\,.,\,.\,]_m, \|\,.\,\|_m)_{m\in\mathbb{N}}$ be a family of Lie algebras $\mathcal{L}_m$ with brackets $[\,.,\,.\,]_m$ and endowed with a norm $\|\,.\,\|_m$. A Lie algebra $(\mathcal{L}, [\,.,\,.\,])$ with bracket $[\,.,\,.\,]$ is a $\mathcal{L}_m$-quasilimit with respect to the family of surjective maps $(p_m : \mathcal{L} \to \mathcal{L}_m)_{m\in\mathbb{N}}$ if for any $x, y \in \mathcal{L}$,*

i) $\|\,p_m(x) - p_m(y)\,\|_m \underset{m\to\infty}{\longrightarrow} 0 \Rightarrow x = y,$

ii) $\|\,[\,p_m(x),\, p_m(y)\,]_m - p_m([\,x,\, y\,])\,\|_m \underset{m\to\infty}{\longrightarrow} 0.$

The family $(\mathcal{L}_m, [\,.,\,.\,]_m, \|\,.\,\|_m)_{m\in\mathbb{N}}$ is then called an approximating sequence of $(\mathcal{L}, [\,.,\,.\,])$ with respect to $(p_m)_{m\in\mathbb{N}}$.

Remark 2.1.4. *The original definition uses a family of distances $(d_m(.,.))_{m\in\mathbb{N}}$ on the Lie algebras instead of norms $(\|\,.\,\|_m)_{m\in\mathbb{N}}$.*

[(2)] If this quantity still can be called a constant.

The principal result for us is that Toeplitz operators form an approximating sequence of classical observables, as the following result states.

Theorem 2.1.5 ([28]). *With respect to the family of maps* $(\mathbf{T}^{(m)} : f \to \frac{m}{\hbar}\,\mathbf{T}_f^{(m)})_{m\in\mathbb{N}}$, *the Poisson algebra* $(C^\infty(\Omega), \{\,.\,,\,.\,\})$ *is a* $\mathrm{End}\big(\Gamma_{hol}(L^{(m)})\big)$*-quasilimit. Moreover*

$$\|\,\mathbf{T}_f^{(m)}\,\| \underset{m\to\infty}{\longrightarrow} \|\,f\,\|_\infty\,.$$

Remark 2.1.6. *Here, we have assumed that the complex line bundle is* very ample*, which means that the Kähler manifold* Ω *can be embedded in some analytic projective space* $P\mathbb{C}^N$ *[38, 99]. As a consequence, there are " enough " global holomorphic sections and we have the following asymptotic estimation [28, (2-13)]*

$$\dim_{\mathbb{R}}\big(\Gamma_{hol}(L^{(m)})\big) = \tfrac{m^n}{(2\pi)^n}\,\mathrm{vol}(\Omega) + O(m^{n-1})\,,$$

where $\mathrm{vol}(\Omega) := \int_\Omega \omega^n$.

As immediate consequences, we get

$$\|\,[\,\mathbf{T}_f^{(m)},\,\mathbf{T}_g^{(m)}\,]\,\| \underset{m\to\infty}{\longrightarrow} 0\,, \tag{2.10}$$

$$\|\,\tfrac{m}{-i\hbar}[\,\mathbf{T}_f^{(m)},\,\mathbf{T}_f^{(m)}\,] - \mathbf{T}_{\{\,f\,,g\,\}}^{(m)}\,\| \underset{m\to\infty}{\longrightarrow} 0\,. \tag{2.11}$$

The first relation (2.10), which means that the product of two Toeplitz operators tends to commute as m tends to infinity, has to be related to (2.17). Moreover, the term $\hbar$ in (2.11), which does not appear in the original statement, has been introduced for the quasilimit to verify (2.18) (here $\frac{m}{\hbar}\to\infty$ plays the role of $\hbar \to 0$ in (2.18)), and the minus sign comes from Remark A.1.2.

Although we do not give the technical details of the proof of this theorem, let us just explain how the theory of generalized Toeplitz operators appears in this context. First, consider the unit circle bundle L^+ of the dual of the line bundle L. Since L^+ is the boundary of a unit disk bundle which is a strictly pseudoconvex domain in $\mathbb{C}^n$, we can consider the Hardy space $\mathcal{H}^+ := H^2(L^+)$ (see Section 1.1.1). Using the corresponding Szegö projector $\Pi^+ : L^2(L^+) \to \mathcal{H}^+$ we obtain GTOs

$$T_P^+ := \Pi^+ P : \mathcal{H}^+ \to \mathcal{H}^+\,, \qquad \text{where } P \in \Psi\mathrm{DO}(Q)\,. \tag{2.12}$$

Now the link with the family of Toeplitz operators $\mathbf{T}_f^{(m)}$ is the following: since $\mathcal{H}^+$ is invariant by the unit circle action, we get the decomposition $\mathcal{H}^+ = \bigoplus_{m\in\mathbb{N}} \mathcal{H}_m^+$, where $\mathcal{H}_m^+$ consists of functions over L, subject to the relation $f(e^{i\theta}\,\lambda) = e^{im\theta}\,f(\lambda)$, for any $\theta \in [0, 2\pi)$, $\lambda \in L$ and $m \in \mathbb{N}$. From the one-to-one correspondence between functions in $\mathcal{H}_m^+$ and elements of $\mathcal{H}^{(m)} := \Gamma_{hol}(L^m)$, we have the identification $\mathcal{H}^+ \approx \bigoplus_{m\in\mathbb{N}} \mathcal{H}^{(m)}$. Moreover, for any $f \in C^\infty(L)$, the GTO T_f is invariant under the circle action and thus can be identified with the direct sum $T_f^+ \approx \bigoplus_{m\in\mathbb{N}} \mathbf{T}_f^{(m)}$.
As a consequence, to any sequence of Toeplitz operators $(\mathbf{T}_f^{(m)})_{m\in\mathbb{N}}$ is associated a GTO T_f^+ acting on $\mathcal{H}^+$. This correspondence brings all the known properties of GTOs (symbolic calculus, notion of orders, etc.) back to the Toeplitz operators of geometric quantization. We will use a similar construction in Section 4.5 for strictly pseudoconvex domains to build a spectral triple from Theorem 2.1.5. As we will see in Section 2.2.2, this construction also leads to a deformation quantization.

2.2 Deformation quantization

Here, Ω denotes a smooth manifold. The idea of formal deformation quantization is to consider that the algebra of quantum observables is a formal deformation of the algebra of classical ones. It means that the composition of the quantum observables Q_f and Q_g, corresponding to the classical observables f and g on the phase space Ω, can be written a formal power series in $\hbar$:

$$Q_f Q_g = Q_{fg} + O(\hbar)\,, \tag{2.13}$$

This reflects the fact that the quantum system is obtained by adding to the classical one some quantum perturbations controlled by the asymptotical behaviour of the Planck constant. Since quantum observables form an algebra, there should exist some function depending on f and g, denoted $f \star_\hbar g$, which verifies $Q_f Q_g = Q_{f\star_\hbar g}$. To fulfil (2.13) and also i) to iv) of Conditions 0 from the introduction, the function $f \star_\hbar g$ should be of the form

$$f \star_\hbar g = \sum_{j\in N} \hbar^j\, C_j(f,g)$$

where $C_j(f,g)$ are smooth functions verifying

$$C_0(f,g) = fg\,, \quad C_1(f,g) - C_1(g,f) = -i\,\{\, f\,,\, g\,\}\,, \quad C_j(f,1) = C_j(1,f) = 0\,, \quad \forall j \geq 1 \tag{2.14}$$

(we could also require that for any symplectomorphism ϕ, $(f\circ\phi)\star_\hbar (g\circ\phi) = (f\star_\hbar g)\circ\phi$, but we will not develop this constraint).
The first two conditions mean that we recover the Poisson structure of the classical observables as $\hbar$ tends to 0 and the third one is added to keep the property $Q_f Q_1 = Q_{f\cdot 1} = Q_f$.
Thus the induced new object $\star_\hbar$ can be made mathematically precise as an associative and noncommutative product on $C^\infty(\Omega)$; which leads to the following definition:

Definition 2.2.1. *Let $\mathcal{P} := (C^\infty(\Omega), \{\,.\,,\,.\,\})$ be a Poisson algebra. A star product on $\mathcal{P}$ is an associative product*

$$\begin{array}{rcccc} \star_\hbar & : & C^\infty(\Omega)\times C^\infty(\Omega) & \to & C^\infty(\Omega)[[\hbar]] \\ & & (f,g) & \mapsto & f\star_\hbar g := \sum_{j=0}^\infty \hbar^j\, C_j(f,g)\,, \end{array} \tag{2.15}$$

where C_j are bidifferential operators which are linear in each argument, such that for any f, g in $C^\infty(\Omega)$,

i) $f\star_\hbar g \mod \hbar = f_0\, g_0\,,$

ii) $\frac{1}{\hbar}(f\star_\hbar g - g\star_\hbar f) \mod \hbar = -i\,\{\, f_0\,,\, g_0\,\}$

Note that the conditions i) and ii) are equivalent to (2.14). Also, (2.13) and (2.15) involve formal power series in $\hbar$, and this is the reason why we speak of *formal deformation quantization*.
A star product extends to $C^\infty(\Omega)[[\hbar]]$ by $\mathbb{C}[[\hbar]]$ linearity :

$$\big(\sum_{j\in\mathbb{N}} \hbar^j\, f_j\big)\star\big(\sum_{j\in\mathbb{N}} \hbar^j\, g_j\big) = \sum_{j\in\mathbb{N}} \hbar^j\,(f\star g)_j\,, \quad \text{with } (f\star g)_j := \sum_{k+l+m=j} C_m(f_k,g_l)\,.$$

Formal star products, whose notion was introduced in [12], exist for arbitrary symplectic [66] and even for Poisson manifolds [100]. In the next section, we will investigate the case of Kähler manifolds. Also, when C_j are bidifferential operators, the star product is said to be *local*.

Given a star product $\star$, defined by its differential operators C_j, there is a recipe to construct another $\star'$ with coefficients C'_j. The set of all maps $D : C^\infty(\Omega)[[\hbar]] \to C^\infty(\Omega)[[\hbar]]$ of the form

$$D := \sum_{j\in\mathbb{N}} \hbar^j\, D_j\,, \qquad \text{where } D_j \text{ are differential operators on } C^\infty(\Omega)\text{, and } D_0 = I,$$

form a group G (sometimes called the group of *gauge actions*), so for any such $D \in G$, one can check that

$$f \star' g := D\big(D^{-1}(f) \star D^{-1}(g)\big)\,, \quad f, g \in C^\infty(\Omega)[[\hbar]]\,, \tag{2.16}$$

defines another star product.

Definition 2.2.2. *Two star products are called formally equivalent if they can be related by an element of G as in (2.16).*

The reformulation in terms of star product brings a fresh perspective to the quantization process: at the quantum level, we can work either with the set of operators $\{Q_f\,, f \in C^\infty(\Omega)[[\hbar]]\}$, or equivalently with the noncommutative algebra $(C^\infty(\Omega)[[\hbar]], \star_\hbar)$. The map $Q : f \mapsto Q_f$ is thus seen as an intermediate step in the construction of a noncommutative structure on the initial phase space: this is the *phase space formulation* of quantum mechanics.

Example 2.2.3. *The archetypal example of deformation quantization concerns the flat phase space $\mathbb{R}^{2n}$ endowed with the usual Poisson structure. It is defined as the Weyl quantization [151] which assigns to $f \in \mathcal{S}(\mathbb{R}^{2n})$ the operator $W_f : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$*

$$(W_f\, u)(x) := \tfrac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}(x-y).\xi}\, f(\tfrac{x+y}{2}, \xi)\, u(y)\, dy d\xi\,.$$

Then, for any $f, g \in \mathcal{S}(\mathbb{R}^{2n})$, there is another distribution $f \star_W g \in \mathcal{S}'(\mathbb{R}^{2n})$ such that

$$W_f W_g = W_{f\star_W g}\,,$$

inducing the so-called Moyal star product $\star_W$ [10]. Denoting $\omega := \left[\begin{smallmatrix} 0 & I \\ -I & 0\end{smallmatrix}\right]$ the canonical symplectic form on $\mathbb{R}^{2n}$ we get

$$\begin{aligned} f \star_W g =& f\, g - \tfrac{i\hbar}{2} \sum_{j=1}^n (\partial_{x_j} f)(\partial_{\xi_j} g) - (\partial_{\xi_j} f)(\partial_{x_j} g) + \tfrac{(-i\hbar)^2}{8} \sum_{j,k=1}^n (\partial^2_{x_j} f)(\partial^2_{\xi_k} g) - (\partial^2_{\xi_j} f)(\partial^2_{x_k} g) \\ &+ O(\hbar^3)\,, \end{aligned}$$

which verifies $f \star_W g \mod \hbar = f\, g$ and $\frac{1}{-i\hbar}(f \star_W g - g \star_W f) \mod \hbar = \{\, f\,,\, g\,\}$.

We can also avoid the formal aspect of deformation quantization by considering the notion of *strict quantization*.

Let $\mathcal{A}$ (with norm $\|\,.\,\|$) be a dense $*$-subalgebra of $C^\infty(\Omega)$ stable by the action of the Poisson bracket, $\mathcal{I}_0 \subset \mathbb{R}$ a collection of points which has $0 \notin \mathcal{I}_0$ as an accumulation point, and $\mathcal{I} := \mathcal{I}_0 \cup \{0\}$.

Definition 2.2.4 ([103, Definition II.1.1.1])**.** *A strict quantization of $\mathcal{A}$ is given by a family of C^*-algebras $(\mathcal{A}_\hbar)_{\hbar\in\mathcal{I}}$ (with corresponding norms $\|\,.\,\|_\hbar$), with $\mathcal{A}_0 = \mathcal{A}$ and a family of linear maps $(Q^{(\hbar)} : \mathcal{A} \to \mathcal{A}_\hbar)_{\hbar\in\mathcal{I}}$, such that* *i)* $Q^{(0)}$ *is the identity,*

ii) *for any* $f \in \mathcal{A}$*, the function* $\hbar \mapsto \| Q_f^{(\hbar)} \|_\hbar$ *is continuous. In particular,* $\| Q_f^{(\hbar)} \|_\hbar \underset{\hbar \to 0}{\longrightarrow} \| f \|$*,*

iii) *for any* $f, g \in \mathcal{A}$*,*

$$\left\| Q_f^{(\hbar)} Q_g^{(\hbar)} - Q_{fg}^{(\hbar)} \right\|_\hbar \underset{\hbar \to 0}{\longrightarrow} 0\,, \tag{2.17}$$

$$\left\| \tfrac{1}{-i\hbar} \big(Q_f^{(\hbar)} Q_g^{(\hbar)} - Q_g^{(\hbar)} Q_f^{(\hbar)} \big) - Q_{\{f,g\}}^{(\hbar)} \right\|_\hbar \underset{\hbar \to 0}{\longrightarrow} 0\,. \tag{2.18}$$

Note that the morphisms $f \mapsto Q_f^{(\hbar)}$ are quantization maps, since from Theorem 3.1.2, $Q_f^{(\hbar)} \in \mathcal{A}_\hbar$ can be represented as bounded operators on some Hilbert space.

As an example of deformation quantization, we present now the *Berezin quantization*, which uses the Berezin transform in the context of reproducing Hilbert spaces. *Berezin–Toeplitz quantization*, described in Section 2.2.2, is an example of strict quantization of the Poisson algebra of smooth functions which uses (classical) Toeplitz operators as a quantization map.

Note that strict quantization is less restrictive than the so-called *strict deformation quantization*, for which $Q_f^{(\hbar)} Q_g^{(\hbar)}$ necessarily lies in $\mathcal{A}_\hbar$ (see [122, Definition 1] and [123, Chapter 9]).

From now on, Ω is an open bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary $\partial\Omega$, as in Section 1.1. On Ω, consider a family of general weights $(w_\hbar)_{\hbar \in \mathcal{I}}$, where $\mathcal{I} \subset \mathbb{R}$ is as above. Here, the weights $w_\hbar$ are not necessarily of the form (1.4) (but they will usually be that with $\hbar = m^{-1}$), and we only need $w_\hbar > 0$ on Ω.

2.2.1 Berezin quantization

For any $\hbar \in \mathcal{I}$, the weighted Bergman space $A_\hbar^2$ are reproducing kernel Hilbert spaces, which means that for any $x \in \Omega$, there is an element $K_{\hbar,x} \in A_\hbar^2$ [(3)] such that for any function $f \in A_\hbar^2$,

$$f(x) = \langle\, f\,,\, K_{\hbar,x} \,\rangle_\hbar = \int_{y \in \Omega} f(y)\, \overline{K_{\hbar,x}(y)}\, w_\hbar(y) d\mu(y)\,.$$

Then the reproducing kernel of $A_\hbar^2$ is given by

$$(x, y) \mapsto K_\hbar(x, y) := \langle\, K_{\hbar,x}\,,\, K_{\hbar,y} \,\rangle_\hbar\,.$$

Note that for any $x \in \Omega$, $\| K_{\hbar,x} \| \neq 0$ since $0 < 1(x) = \langle\, K_{\hbar,x}\,,\, 1 \,\rangle_\hbar \leq \| K_{\hbar,x} \|$.

Definition 2.2.5. *The Berezin symbol (or covariant symbol) of a bounded linear operator A on $A_\hbar^2$ is the function*

$$\mathrm{Ber}_\hbar(A)(x) := \tfrac{\langle\, K_{\hbar,x}\,,\, A K_{\hbar,x} \,\rangle_\hbar}{\langle\, K_{\hbar,x}\,,\, K_{\hbar,x} \,\rangle_\hbar}\,,$$

and the Berezin transform [15] of $f \in L^\infty(\Omega)$ is defined as

$$\mathrm{ber}_\hbar(f)(x) := \mathrm{Ber}_\hbar(\mathbf{T}_f)(x) = \tfrac{\langle\, K_{\hbar,x}\,,\, f K_{\hbar,x} \,\rangle_\hbar}{\langle\, K_{\hbar,x}\,,\, K_{\hbar,x} \,\rangle_\hbar}\,,$$

where $\mathbf{T}_f$ is the Toeplitz operator on $A_\hbar^2$ associated to f (see Definition 1.2.1).

(3) It must not be confused with the Poisson extension operator introduced in Definition 1.1.4.

It is easy to check that for any $\hbar \in \mathcal{I}$, the map $\mathrm{Ber}_\hbar$ is linear, $\mathrm{Ber}_\hbar(I) = 1$, $\mathrm{Ber}_\hbar(A^*) = \overline{\mathrm{Ber}_\hbar(A)}$ and $\|\, \mathrm{Ber}_\hbar(A) \,\|_\infty \leq \|\, A \,\|$. Moreover, there is a one-to-one correspondence between $A \in \mathcal{B}(A^2_\hbar)$ and $\mathrm{Ber}_\hbar(A)$ (this is true when considering holomorphic functions on Ω). As a consequence, for any $\hbar \in \mathcal{I}$, we get on $\mathcal{A}_\hbar := \mathrm{Ber}_\hbar(\mathcal{B}(A^2_\hbar))$ the noncommutative product $\star_\hbar$ defined as

$$f \star_\hbar g := \mathrm{Ber}_\hbar\big(\mathrm{Ber}_\hbar^{-1}(f)\, \mathrm{Ber}_\hbar^{-1}(g)\big)\,,$$

for any $f, g \in \mathcal{A}_\hbar$. However in general, this does not verify the conditions (2.15) and (2.14) of a star product.

The procedure of *Berezin quantization* consists in choosing the weights $(w_\hbar)_{\hbar\in\mathcal{I}}$ such that the Berezin transform $\mathrm{ber}_\hbar$ has the expansion [(4)]

$$\mathrm{ber}_\hbar \underset{\hbar\to 0}{=} \sum_{j=0}^{\infty} \hbar^j\, Q_j\,, \tag{2.19}$$

where $Q_j := \sum_{\alpha,\beta} c_{j\alpha\beta}\partial^\alpha\bar{\partial}^\beta$ are differential operators of order j verifying

$$Q_0 = I\,, \quad \text{and} \quad \sum_{\alpha,\beta} c_{1\alpha\beta}(\bar{\partial}^\beta f \partial^\alpha g - \bar{\partial}^\beta g \partial^\alpha f) = i\,\{\, f\,,\, g\,\}\,, \quad f, g \in \mathcal{A}\,.$$

Then, we get the so-called *Berezin star product* $\star_B$:

$$f \star_B g := \sum_{j=0}^{\infty} \hbar^j C_j^B(f, g)\,,$$

where $C_j^B(f,g) := \sum_{\alpha,\beta} c_{j\alpha\beta}\bar{\partial}^\beta f \partial^\alpha g$.
In our context, the existence of the expansion (2.19) is ensured by [54, Theorem B], where it is also shown that Q_1 is nothing else but (some multiple of) the Laplace–Beltrami operator Δ.

Example 2.2.6. *When $\Omega = \mathbb{B}^n$ with weights of the form $w_m(z) := (1 - |z|^2)^m$, $m > -1$, the Berezin transform is*

$$(\mathrm{ber}_m f)(x) = \int_{\mathbb{B}^n} f(y) \tfrac{(1-|x|^2)^{m+n+1}}{|1-x\bar{y}|^{2(m+n+1)}}\,(1 - |y|^2)^m\, d\mu(y)\,.$$

The use of the stationary phase method [93, Chapter 7.7] leads to the expansion

$$\mathrm{ber}_m f \underset{m\to\infty}{=} f + \tfrac{1}{4m}\widetilde{\Delta} + O(m^{-2})\,,$$

where $\widetilde{\Delta} f(z) := (1 - |z|^2)^{2m}\Delta f(z)$.

2.2.2 Berezin–Toeplitz quantization

Following Definition 2.2.4, we denote here $\mathbf{T}_f^{(\hbar)}$ the Toeplitz operator acting on the weighted Bergman space $A^2_\hbar$, where $f \in C^\infty(\Omega)$. The idea of Berezin–Toeplitz quantization is to choose the weights $(w_\hbar)_{\hbar\in\mathcal{I}}$ such that the product of two Toeplitz operators admits the following expansion

$$\mathbf{T}_f^{(\hbar)}\mathbf{T}_g^{(\hbar)} \underset{\hbar\to 0}{=} \sum_{j=0}^{\infty} \hbar^j\, \mathbf{T}^{(\hbar)}_{C_j^{BT}(f,g)}\,,$$

[(4)] The meaning of " $\underset{\hbar\to 0}{=}$ " is defined below.

where the equality is understood in the strong sense:

$$\Big\| \mathbf{T}_f^{(\hbar)} \mathbf{T}_g^{(\hbar)} - \sum_{j=0}^{N-1} \hbar^j \, \mathbf{T}^{(\hbar)}_{C_j^{BT}(f,g)} \Big\|_\hbar \underset{\hbar \to 0}{=} O(\hbar^N)\,, \qquad \text{for any } N \in \mathbb{N}\,. \tag{2.20}$$

Then, if the C_j^{BT} are bidifferential operators on $C^\infty(\Omega)$ and verify (2.14), we get the Berezin–Toeplitz star product on

$$f \star_{BT} g := \sum_{j=0}^{\infty} \hbar^j \, C_j^{BT}(f,g)\,, \qquad \text{for } f, g \in C^\infty(\Omega)\,,$$

which allows us to write symbolically

$$\mathbf{T}_f \mathbf{T}_g = \mathbf{T}_{f \star_{BT} g}\,.$$

In the case of strictly pseudoconvex manifolds, the result of Berezin–Toeplitz quantization has been established in [57]:

Theorem 2.2.7 ([57, Theorem 3])**.** *Let Ω be a smoothly bounded strictly pseudoconvex domain of $\mathbb{C}^n$ with a defining function r such that $\log(-1/r)$ is strictly plurisubharmonic (see Appendix A.4). Consider on Ω weights of the form*

$$w_m := (-r)^{m+1}\, \mathcal{J}[r]\,, \quad m \in \mathbb{N}\,, \tag{2.21}$$

where $\mathcal{J}[r]$ is the Monge–Ampère determinant of r, see Appendix (B.2). With $\mathbf{T}_f^{(m)} := \mathbf{T}_f|_{A_m^2}$, we have

i) for any $f \in C^\infty(\overline{\Omega})$, $\| \mathbf{T}_f^{(m)} \| \underset{m \to \infty}{\longrightarrow} \| f \|_\infty$,

ii) there are bidifferential operators C_j^{BT}, $j \in \mathbb{N}$, verifying (2.14) such that for any f and g in $C^\infty(\overline{\Omega})$ and any $N \in \mathbb{N}$,

$$\Big\| \mathbf{T}_f^{(m)} \mathbf{T}_g^{(m)} - \sum_{j=0}^{N-1} m^{-j} \, \mathbf{T}^{(m)}_{C_j^{BT}(f,g)} \Big\| \underset{m \to \infty}{=} O(m^{-N})\,. \tag{2.22}$$

Remark 2.2.8. *The weights w_m here are not exactly the same as in the original statement: a shift has been chosen between the index of the weight and the power of r in order to avoid irrelevant technical difficulties. The strict positivity of $\mathcal{J}[r]$ on Ω and $\partial\Omega$ follows from the hypothesis on r and from the strict pseudoconvexity of Ω respectively.*

As a consequence, the Berezin–Toeplitz quantization defines a strict quantization and gives rise to a local star product. Since we will need elements of the proof of this theorem in Section 4.5, let us describe the main idea behind and some technical constructions, which by the way are very similar to the ones presented at the end of Section 2.1.3. We refer to [57, p. 235] for details in this setting (the idea however goes back to Forelli and Rudin).

In the following, we add a superscript " $+$ " sign to the notations of objects related to the following disc bundle over Ω:

$$\Omega^+ := \{(z,s) \in \Omega \times \mathbb{C} : |s|^2 < -r(z)\}\,,$$

with boundary $\partial\Omega^+ := \{(z,s) \in \Omega \times \mathbb{C} : |s|^2 = -r(z)\}$ ($\partial\Omega^+$ is the unit circle bundle L^+ of Section 2.1.3). As a subset of $\mathbb{C}^{n+1}$, Ω^+ is a smoothly bounded strictly pseudoconvex domain with defining function $\rho(z,s) := |s|^2 + r(z)$ (see iii) of Example A.5.3). On $\partial\Omega^+$, it is more convenient to use the change of coordinates

$$(z,s) \mapsto (z, e^{i\theta}\sqrt{-r(z)})\,. \tag{2.23}$$

Replacing r by ρ in (1.2) and (1.3), we get the corresponding contact form

$$\eta^+(z,\theta) = \tfrac{1}{2i}\Big(\sum_{j=1}^{n} \partial_{z_j} r(z)\, dz_j - \partial_{\bar z_j} r(z)\, d\bar z_j\Big) - r(z)\, d\theta\,,$$

the volume form ν^+ on $\partial\Omega^+$ and the symplectic cone Σ^+. From now on, we choose ν^+ for the measure on $\partial\Omega^+$. We consider the Hardy space $H^2(\partial\Omega^+)$ and the corresponding Szegö orthogonal projector $\Pi^+ : L^2(\partial\Omega^+, \nu^+) \to H^2(\partial\Omega^+)$, which induces the theory of GTOs $T_P^+ := \Pi^+ P$, where P are now pseudodifferential operators on $\partial\Omega^+$.
Now consider the Taylor expansion of a function in Ω^+ in the fibre variable

$$f(z,s) = \sum_{m=0}^{\infty} f_m(z)\, s^m\,.$$

Denote by $H^{(m)}$, $m \in \mathbb{N}$, the subspace in $H^2(\partial\Omega^+)$ of functions of the form $f_m(z)\, s^m$ (i.e. for which all the Taylor coefficients vanish except for the m^{th}). Alternatively, $H^{(m)}$ is the subspace of functions f in $H^2(\partial\Omega^+)$ satisfying

$$f(z,\lambda s) = \lambda^m f(z,s), \quad \forall \lambda \in \mathbb{S}^1\,. \tag{2.24}$$

The generator of the circle action $D^+ := -i\partial_\theta$ acts as the multiplication by m on each $H^{(m)}$ and is actually a GTO of order 1 acting on $H^2(\partial\Omega^+)$ with symbol (seen as a function in the variables $(z,\theta,t) \in \Omega \times [0,2\pi) \times \mathbb{R}_{>0} \approx \Sigma^+$)

$$\widetilde{\sigma}(D^+)(z,\theta,t) := i\langle\, t\,\eta^+_{(z,\theta)}\,,\, -i\partial_\theta\,\rangle = -r(z)\, t\,.$$

Using (2.23), it follows from [121, Lemma VII.3.9] and also [57, Section 5] that for any function (2.24), we have the relation

$$\begin{aligned}\|\, f\,\|^2_{H^{(m)}} &= \int_{(z,s)\in\Omega^+} |f_m(z)s^m|^2\, \nu^+(z,s) = n! \int_{(z,s)\in\Omega^+} |f_m(z)s^m|^2\, \frac{\mathcal{J}[\rho]}{\|\,\partial\rho\,\|}(z,s)\, d\mu(z,s)\\ &= n! \int_{(z,\theta)\in\Omega\times[0,2\pi)} |f_m(z)|^2\, \big((-r)^m \mathcal{J}[\rho]\big)(z,\theta)\, d\bar z dz d\theta = 2\pi n!\, \|\, f_m\,\|^2_{A^2_m}\,,\end{aligned}$$

for any $m \in \mathbb{N}$. The last equality comes from the fact that we have on the boundary $\mathcal{J}[\rho] = \mathcal{J}[r]$, and we also understand at this point the presence of the Monge–Ampère determinant in the weight. Consequently, the correspondence $f_m(z)\, s^m \longleftrightarrow f_m(z)$ is, up to the constant factor $2\pi n!$, an isometric isomorphism of $H^{(m)}$ onto the weighted Bergman space A^2_m with weight (2.21). Thus

$$H^2(\partial\Omega^+) = \bigoplus_{m=0}^{\infty} H^{(m)} \approx \bigoplus_{m=0}^{\infty} A^2_m =: H^{\oplus}\,. \tag{2.25}$$

Furthermore, since any $f \in C^\infty(\overline{\Omega})$ generates the function

$$\tilde f(z,s) := f(z)\,, \quad (z,s) \in \partial\Omega^+\,,$$

which is constant along the fibres, we have, under the above isomorphism, the following relation between GTOs and Toeplitz on $H^{\oplus}$:

$$T^+_{\tilde{f}} \approx \bigoplus_{m\in\mathbb{N}} \mathbf{T}^{(m)}_f =: \mathbf{T}^{\oplus}_f \,. \tag{2.26}$$

The coefficients C^{BT}_j in (2.22) are obtained using the fact that any GTO T^+ of order 0 on $H^2(\partial\Omega^+)$ commuting with D^+ can be written as $T^+ = T^+_{\tilde{f}} + (D^+)^{-1}R^+$, where $f \in C^\infty(\overline{\Omega})$ is uniquely determined and R^+ is some GTO of order 0 which also commutes with D^+. Then, recursively, for any $N \in \mathbb{N}$ we obtain some functions $(f_j)_{j=0,\ldots,N-1}$ and $R^+_{N-1} \in \mathrm{GTO}^0$ verifying

$$T^+ = \sum_{j=0}^{N-1}(D^+)^{-j}\,T^+_{\tilde{f}_j} + (D^+)^N\,R^+_{N-1}\,, \qquad \text{i.e.} \qquad (D^+)^N\big(T^+ - \sum_{j=0}^{N-1}(D^+)^{-j}\,T^+_{\tilde{f}_j}\big) = R^+_{N-1}\,.$$

Evaluating the norm of the two sides of the last relation, as bounded operators on the Hilbert space $H^2(\partial\Omega^+) = \bigoplus_{m\in\mathbb{N}} H^{(m)}$, we obtain, using (2.26)

$$m^N\left\|T^+|_{H^{(m)}} - \sum_{j=0}^{N-1} m^{-j}\,T^+_{\tilde{f}_j}|_{H^{(m)}}\right\| = \|\,R^+_{N-1}\,\|\,,\ \text{hence}\ \left\|T^+|_{H^{(m)}} - \sum_{j=0}^{N-1} m^{-j}\,\mathbf{T}^{(m)}_{f_j}\right\| = O(m^{-N})\,,$$

which leads to (2.22). The uniqueness of the coefficients $C^{BT}_j(f,g)$ can be deduced from i) of Theorem 2.2.7.

For compact Kähler manifolds, there is a naturally defined star product in the context of geometric quantization [28, 126, 127]. The proof is the same as above and involves the Toeplitz operators (2.9) and the GTOs (2.12).

Chapter 3

Noncommutative geometry

We underline the fact that the work of this thesis focuses on the study of spectral triples. Therefore, we only present in this section the related notions we handle and we refer to [41, 44, 76] for a complete overview about the subject.

3.1 Duality between topology and algebra

In the Introduction we mentioned the fact that considering the commutative algebra of functions over a space rather than the space itself, is a different (but more interesting) approach to describe the same object. Let us first recall two major results which illustrate how topology and algebra are closely related: the celebrated Gelfand–Naimark theorems [70].

Recall that for a commutative Banach algebra $\mathcal{A}$, a (continuous) character on $\mathcal{A}$ is a multiplicative linear functionals on $\mathcal{A}$, i.e. a non-zero (continuous) morphism from $\mathcal{A}$ to $\mathbb{C}$, and the set $\mathcal{M}_{\mathcal{A}}$ of characters on $\mathcal{A}$ is called the spectrum of $\mathcal{A}$.

Let us start with an example. Consider the commutative algebra $\mathcal{A} = C(X)$ of continuous functions from X to $\mathbb{C}$ on a compact Hausdorff space X, with the usual pointwise addition and multiplication. Endowed with the involution $f \mapsto \bar{f}$ and the norm $\| f \|_\infty := \sup_{x\in X} |f(x)|$, $\mathcal{A}$ is a C^*–algebra. To any $x \in X$, we can associate the character on $\mathcal{A}$ with $\varphi_x : f \mapsto \varphi_x(f) := f(x)$, and reciprocally, one can show that any character on $\mathcal{A}$ is of this form. As a consequence, via the map $x \mapsto \varphi_x$, the space X is homeomorphic to $\mathcal{M}_{\mathcal{A}}$ (a short proof can be found for instance in [26, Example 10]).

First, we can see that we recover (trivially) a famous result which states the spectrum of a unital (resp. non unital) commutative Banach algebra $\mathcal{A}$ is a compact (resp. locally compact) Hausdorff space (in this case, precisely homeomorphic to X). Secondly, we get that $\mathcal{A}$ is isomorphic to $C(\mathcal{M}_{\mathcal{A}})$: the initial algebra can be described without any loss of information as the space of continuous functions on some compact Hausdorff space (here, X itself). This induces a topological characterization of a purely algebraic object.

Generalizing this result, the first Gelfand–Naimark theorem describes how a commutative unital (resp. non unital) C^*–algebra can be seen as the space of continuous functions that vanish at infinity on some compact (resp. locally compact) Hausdorff space.

Theorem 3.1.1 (Gelfand–Naimark theorem)**.** *If $\mathcal{A}$ is a commutative C^*–algebra, the Gelfand transform,* *defined as*

$$\mathscr{G} : a \in \mathcal{A} \mapsto \mathscr{G}(a) \in C_0(\mathcal{M}_{\mathcal{A}}) \quad \text{where for any } \varphi \in \mathcal{M}_{\mathcal{A}},\ \mathscr{G}(a)[\varphi] := \varphi(a),$$

is an isometric $*$–isomorphism.

The second theorem concerns the representation of a not necessarily commutative C^*–algebra:

Theorem 3.1.2 (Gelfand–Naimark–Segal theorem)**.** *Any C^*–algebra has an isometric representation as a closed subalgebra of bounded operators on some Hilbert space.*

The main difference between the two theorems lies in the fact that the algebra is not necessarily commutative in the latter. This result has an immediate application for quantum physics (see Chapter 2). For instance, if the phase space X of a dynamical system happens to be compact (resp. locally compact) Hausdorff, then to any classical observable, represented as an element in $C_0(X)$, we can associate a quantum operator, i.e. a bounded operator on some Hilbert space. In other words, this second theorem ensures the existence of an abstract quantization procedure [(1)].

Hausdorff spaces are interesting but turn out to be insufficient in general to describe physical problems. Indeed, in addition to the topology of the space, we also have to take into account the notions of metric and smoothness in order to define local invariants or differentiation of functions. From a physicist's point of view and in the spirit of noncommutative geometry, it is interesting to look for an algebraic characterization of (compact) Riemannian manifolds which carry a spin structure: those spaces are the essential ingredients to describe fermionic dynamical systems. We will see that this spin structure allows to define the so-called *Dirac operator*, which generalizes the one Dirac was looking for, and carries a lot of information about the manifold. Before introducing its equivalent in the context of noncommutative geometry, let us briefly recall some facts about the Dirac operator.

3.2 The Dirac operator and its properties

3.2.1 Construction

The origins of the Dirac operator go back to the establishment of the quantum version of the Klein–Gordon equation [(2)]

$$\left(\tfrac{1}{c^2} \tfrac{\partial^2}{\partial t^2} - \Delta + \tfrac{m^2 c^2}{\hbar^2} \right) \psi = 0\,, \tag{3.1}$$

valid in the Minkowski space time. In [48], Dirac emphasizes the fact that the equation must be linear in both time and space variables[(3)], hence he considers the equation

$$(p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta)\psi = 0\,, \tag{3.2}$$

[(1)] This is abstract: we do not take into account the conditions we have seen in Chapter 2 to get a " reasonable " quantization.

[(2)] Proposed by Gordon[75] and Klein [98] independently to give a relativistic version of the Schrödinger equation, which describes the motion of a spinless massive free particle.

[(3)] This argument comes from the quantum mechanics which describes evolutions of a particle in terms of linear transformations.

where the four-momentum $(p_0, \vec{p})$ has been quantized via the canonical quantization scheme [(4)]

$$p_0 = \frac{i\hbar}{c}\frac{\partial}{\partial t}\,, \quad \vec{p}_\mu = -i\hbar\frac{\partial}{\partial x_\mu}\,, \quad \mu = 1,2,3\,.$$

The square of (3.2) must be (3.1) and the terms α_i and β can be represented using the Pauli matrices [113] σ_μ, $\mu = 1,2,3$, and by defining the 4×4 matrices

$$\alpha_0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\,, \quad \alpha_\mu = \begin{bmatrix} 0 & \sigma_\mu \\ \sigma_\mu & 0 \end{bmatrix}\,, \quad \mu = 1,2,3\,.$$

Setting $\beta = mc\,\alpha_0$, we get

$$(p_0\, I + \vec{\alpha}.\vec{p} + mc\,\alpha_0)\psi = 0\,.$$

The covariant formulation (with respect to the signature (+ - - -)) is given by

$$(\gamma_\mu p^\mu - mc\,\gamma_0)\psi = 0\,, \tag{3.3}$$

where $\gamma_0 = \gamma^0 = \alpha_0$ and $\gamma_\mu = -\gamma^\mu = -\alpha_0\alpha_\mu$, $\mu = 1,2,3$. This equation describes the evolution of a massive free particle with spin $\frac{1}{2}$ [(5)]. Denoting $g_{\mu\nu}$ the metric tensor of the Minkowski space time with the previous signature convention, the γ_μ matrices then verify

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}\, I\,. \tag{3.4}$$

The operator

$$\not{D} := \gamma_\mu p^\mu = -i\gamma_\mu\partial^\mu \tag{3.5}$$

is called *the Dirac operator* on the Minkowski space (the factor $-i$ is part of the definition to make the operator symmetric). Being the square root of the Laplacian operator relative to the Minkowski space time, hence a differential operator of order one, it fulfils the conditions Dirac was looking for [(6)] [(7)].

A generalization of the Dirac operator to an n-dimensional Riemannian manifold M is possible if we assume that M carries a spinor bundle. We denote here $P_G(X)$ a G-principal bundle over a space X. Recall that a spin structure on M is a couple $(P_{\mathrm{Spin}_n}(TM), \eta)$, where η is a two-fold covering map from $P_{\mathrm{Spin}_n}(TM)$ to $P_{\mathrm{SO}_n}(TM)$, such that the following diagram commutes

$$\begin{tikzcd} P_{\mathrm{Spin}_n}(TM) \times \mathrm{Spin}_n \arrow[r] \arrow[d, "\eta \times \xi"'] & P_{\mathrm{Spin}_n}(TM) \arrow[d, "\eta"'] \arrow[dr] & \\ P_{\mathrm{SO}_n}(TM) \times \mathrm{SO}_n \arrow[r] & P_{\mathrm{SO}_n}(TM) \arrow[r] & M \end{tikzcd}$$

(4)See Chapter 2.

(5)The notion of spin, introduced by Pauli in 1924 [112] for the electron to give an interpretation of experimental results, was not understood at that time and will not be theorized until Pauli in 1927 [113], Dirac in 1928 [48] and Wigner in 1939 [152].

(6)Actually, Hamilton already discovered in 1843 the generators $\{1, i, j, k\}$ of the quaternion group. These elements are nothing else than the Pauli matrices via the identification $1 \mapsto I$, $i \mapsto -i\sigma_1$, $j \mapsto -i\sigma_2$, $k \mapsto -i\sigma_3$.

(7)The field ψ is no longer scalar but consists of a 4 dimensional vector: such fields are called *bispinors* and they actually represent an element of the Lorentz group in the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation: this reflects the fact that ψ is invariant under Lorentz transformations, as needed.

It is also denoted $\mathrm{Spin}(TM)$. A spin manifold M is a manifold admitting a spin structure [(8)].
The spinor bundle $\$$ of M is the vector bundle associated to $\mathrm{Spin}(TM)$ via the spinor representation $\rho : \mathrm{Spin}_n \longrightarrow \mathrm{Mat}_{2^k}(\mathbb{C}) \approx \mathbb{C}^{2^k}$, with $k := \lfloor \frac{n}{2} \rfloor$, i.e.

$$\$:= \mathrm{Spin}(TM) \times_\rho \mathbb{C}^{2^k} .$$

In other words, a spin bundle is a principal bundle that carries a spin representation on each fibre, and from a physicist point of view, square integrable sections represent the wave function of a fermion, which leads to consider the Hilbert space of spinors

$$L^2(\$) := \{\psi \in \Gamma^\infty(M, \$) \, , \int_M \langle \psi \, , \, \psi \rangle \sqrt{g} dx < \infty\}$$

($\langle \cdot \, , \cdot \rangle$ is a $C^\infty(M)$-valued hermitian scalar product on $\$$). Up to orientations on M, the *spin connection* $\nabla^{\$} : \Gamma(M, \$) \to \Gamma(M, \$) \otimes \Gamma(M, T^*M)$ is the unique connection on $\$$ verifying

$$[\, \nabla^{\$} , \, c(\cdot) \,] = c(\nabla^{LC} \cdot) \, ,$$

where $c : \Gamma\big(M, C\ell(T^*M)\big) \longrightarrow \mathrm{End}\big(\Gamma(M, \$)\big)$ is the Clifford action on $\$$.

Definition 3.2.1. *The Dirac operator on M is the map $\not{D} : \Gamma(M, \$) \to \Gamma(M, \$)$ defined by*

$$\not{D} := -i \, \hat{c} \circ \nabla^{\$} \, , \tag{3.6}$$

where

$$\begin{array}{rrcl} \hat{c} : & \Gamma(M, \$) \otimes \Gamma(C\ell(T^*M)) & \longrightarrow & \Gamma(M, \$) \\ & \psi \otimes a & \longmapsto & \hat{c}(\psi \otimes a) := c(a)\psi \, . \end{array}$$

In local coordinates, it is given by $\not{D} = -i \, \sum_{j=1}^n c(dx_i) \, \nabla^{\$}_{\partial_{x_i}}$ (the previous definition is independent of the basis).

The relation with the differentiation is the following: for $f \in C^\infty(M)$, and $\psi \in \Gamma(M, \$)$, we have

$$\begin{aligned} [\, \not{D}, \, f \,]\psi &= -i \left(\hat{c}(\nabla^{\$}(f\psi)) - \hat{c}(f \nabla^{\$}(\psi)) \right) = -i\hat{c} \left(\nabla^{\$}(f\psi) - \nabla^{\$}(\psi) \right) \\ &= -i\hat{c}(\psi \otimes df) = -ic(df)\psi \, , \end{aligned}$$

or equivalently $[\, \not{D}, \, f \,] = -ic(df)$, and as an operator on $L^2(M, \$)$, the norm of $[\, \not{D}, \, f \,]$ is $\| \, df \, \|_\infty$. In other words, the map $f \in C^\infty(M) \mapsto [\, \not{D}, \, f \,] \in \mathrm{End}\big(\Gamma(M, \$)\big)$, is analogous to the differentiation on the commutative algebra $C^\infty(M)$.

3.2.2 Hearing the shape of the manifold

Besides the fact that the Dirac operator (3.6) generalizes (3.5), its spectrum contains topological information about the manifold it is defined on. It means that the geometry of a such a manifold is of *spectral* origin: this is the reason why noncommutative geometry is seen as a *spectral geometry*. Indeed, some information about the manifold (dimension, volume, curvature, etc.) is encoded in the spectrum of $\not{D}$.

[(8)] The existence of spin structure depends on some topological conditions on M.

The original motivation of this approach leads back to the famous problem of describing the geometry of a bounded domain by studying the spectrum of the corresponding Laplacian [96].

When M is an oriented Riemannian spin manifold without boundary, the closed extension $\not{D}^{**}$ to $L^2(S)$ is a selfadjoint elliptic differential operator of order one [76, Theorem 9.15][73, Proposition 1.3.4]. In the case of compact manifold, the spectrum of $\not{D}$ consists of a discrete and unbounded (from both sides) sequence of real eigenvalues whose asymptotic behaviour obeys Weyl's law. Indeed, the counting function $N_{\not{D}} : \lambda \in \mathbb{R} \mapsto \mathrm{card}\{\mu \in \mathrm{Spec}(\not{D})\,, \mu \leq |\lambda|\}$ verifies

$$N_{\not{D}}(\lambda) \underset{\lambda\to\infty}{\sim} \tfrac{\mathrm{vol}(\mathbb{B}^n)}{(2\pi)^n}\, \mathrm{vol}(M)\, \lambda^n \,, \tag{3.7}$$

where the volumes of the unit ball and the one of the manifold appear. The computation of $\not{D}^2$ gives rise to the Schrödinger–Lichnerowicz formula [128, 107]

$$\not{D}^2 = \Delta + \tfrac{1}{4} s\, I$$

and involves the scalar curvature s of M.

Another way to derive topological invariants from the Dirac operator is the use of the heat kernel method [71, 72, 145]. Recall that on the Euclidean space $\mathbb{R}^n$, the heat kernel

$$K_0(t,x,y) = \tfrac{1}{(4\pi t)^{n/2}}\, e^{-(x-y)^2/(4t)}$$

solves the heat equation

$$(\partial_t + \Delta_x)\, K_0(t,x,y) = 0 \text{ for } t > 0\,, \quad \text{and} \quad \lim_{t\to 0} K_0(t,x,y) = \delta(x-y)\,.$$

On compact Riemannian manifold M without boundary and of dimension n, the kernel K of $e^{-t\Delta}$, where Δ is the corresponding Laplace–Beltrami operator, admits the following asymptotics on the diagonal of $M \times M$ [110]:

$$K(t,x,x) \underset{t\to 0^+}{\sim} \sum_{k=0}^{\infty} a_k(x)\, t^{(k-n)/2}\,,$$

where the coefficients a_k are smooth functions over M. We then get an asymptotic expression for the trace of $e^{-t\Delta}$:

$$\mathrm{Tr}(e^{-t\Delta}) = \int_{x\in M} \sqrt{g} dx\, K(t,x,x) \underset{t\to 0^+}{\sim} \sum_{k=0}^{\infty} A_k\, t^{(k-n)/2}\,,$$

where A_k is obtained after integration of a_k on M. These coefficients can actually be expressed in terms of topological invariants of M.

We consider more generally operators of " Laplace type " on a vector bundle V over M, i.e. operators of the form $P = -(g^{\mu\nu}\partial_\mu\partial_\nu + A^\mu\partial_\mu + B)$, where $g^{\mu\nu}$ is the inverse metric tensor on M, A and B are matrix valued functions on M. Let

$$\omega_\mu := \tfrac{1}{2} g_{\nu\mu}(A^\nu + g^{\delta\rho}\,\Gamma_{\delta\rho}{}^{\nu} I)\,, \quad \text{and} \quad E := B - g^{\nu\mu}(\partial_\mu\omega_\nu + \omega_\nu\omega_\mu - \omega_\rho\,\Gamma_{\nu\mu}{}^{\rho})\,,$$

where $\Gamma_{\nu\mu}{}^{\rho} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$ denotes the Christoffel symbols. Recall also that the Riemann curvature tensor and the curvature of the connection ω are defined respectively as

$$R^\mu{}_{\nu\rho\sigma} := \partial_\sigma\Gamma^\mu{}_{\nu\rho} - \partial_\rho\Gamma^\mu{}_{\nu\sigma} + \Gamma^\lambda{}_{\nu\rho}\Gamma^\mu{}_{\lambda\sigma} - \Gamma^\lambda{}_{\nu\sigma}\Gamma^\mu{}_{\lambda\rho}\,, \qquad \Omega_{\mu\nu} := \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + \omega_\mu\omega_\nu - \omega_\nu\omega_\mu\,.$$

The main theorem states that for any function φ in $C^\infty(\mathbb{R})$, we have the following asymptotic expansion

$$\mathrm{Tr}(\varphi(e^{-tP})) \underset{t\to 0^+}{\sim} \sum_{k=0}^{\infty} a_k(\varphi, P)\, t^{(k-n)/2}\,. \tag{3.8}$$

The coefficients $a_k(\varphi, P)$ vanish when k is odd, and otherwise can be expressed as an integral of linear combinations of local invariants $\mathcal{I}_{k,i}(P)$ of M [(9)]:

$$a_k(\varphi, P) = \int_M \sqrt{g}dx\ \mathrm{tr}_V\big(\varphi(x)\sum_i \alpha_{k,i}\,\mathcal{I}_{k,i}(P)(x)\big)\,, \quad (\alpha_{k,i} \text{ are coefficients}). \tag{3.9}$$

The first two non null coefficients are

$$a_0(\varphi, P) = \tfrac{1}{(4\pi)^{n/2}} \int_M \sqrt{g}dx\ \mathrm{tr}_V\big(\varphi(x)\, I_x\big)$$
$$a_2(\varphi, P) = \tfrac{1}{(4\pi)^{n/2}}\tfrac{1}{6} \int_M \sqrt{g}dx\ \mathrm{tr}_V\big(f(x)(6E_x + 4R_x)\big)$$

(see [85, 4, 144] for computations to higher order). When M has a boundary, additional terms appears in (3.9), involving boundary local invariants and normal derivatives of φ and depend on the type of conditions put on the boundary. See [145, Section 5] for more details.

The notion of distance can also be formulated algebraically using the Dirac operator. To illustrate this, consider two points x and y on the real line $\mathbb{R}$. The distance $d(x, y)$ is given by the supremum of all $|f(x) - f(y)|$, when the function $f \in C^1(\mathbb{R})$ verifies $\|\, f' \,\|_\infty \leq 1$. In this case, it is satisfied by the identity function.
Now, using the previous identification between points of a general compact Hausdorff space X and elements of $\mathcal{M}_{C(X)}$, and replacing the derivation of a function by commutation with $\not{D}$, we get the Connes' notion of distance between two states φ_i:

$$d(\varphi_1, \varphi_2) = \sup\{\, |\varphi_1(f) - \varphi_2(f)|\,,\ \|\,[\,\not{D},\, f\,]\,\| \leq 1\,\}\,,$$

which coincides with the previous one. This shows how the Dirac operator encodes the metrics on a space, and more generally on a manifold. Note that this formulation also makes sense when the algebra is no longer commutative.

3.3 Main tools in noncommutative geometry

We describe in this section the objects of noncommutative geometry we will use later on: the notions of spectral triples, spectral dimension, regularity and the spectral action.

3.3.1 Spectral triples

We have seen that information about a compact Riemannian spin manifold (M, g) can be recovered from three principal objects: the commutative algebra $\mathcal{A} := C^\infty(M)$, whose spectrum encodes the points of M, the Hilbert space $\mathcal{H} := L^2(M, \not{S})$ the algebra acts on, related to the spin structure, and finally the Dirac operator $\not{D}$, acting on $\mathcal{H}$, which contains metric and also topological invariants. The generalization of this example considers more general (not necessarily commutative) algebras, which leads to the concept of spectral triple.

(9) Local invariants are built from R, E, Ω and their derivatives.

Definition 3.3.1. *A (unital) spectral triple is defined by the data $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ with*

i) an involutive unital *-algebra $\mathcal{A}$,

ii) a faithful representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$,

iii) a selfadjoint operator $\mathcal{D}$ acting on $\mathcal{H}$ with compact resolvent such that for any $a \in \mathcal{A}$, the extended operator of $[\,\mathcal{D},\, \pi(a)\,]$ is bounded.

Remarks 3.3.2.

i) When $\mathcal{A}$ is not unital, consider its unitization $\widetilde{\mathcal{A}}$ and replace the compactness of the resolvent by the compactness of $\pi(a)(\mathcal{D}-\lambda)^{-1}$ for any $a \in \mathcal{A}$ and $\lambda \notin \operatorname{Spec}(\mathcal{D})$.

ii) An extension of the notion of spectral triple involving von Neumann algebras has been investigated in [14].

The operator $\mathcal{D}$ is assumed to be selfadjoint and with compact resolvent, that is, for any λ not in $\operatorname{Spec}(\mathcal{D})$, the operator $(\mathcal{D}-\lambda)^{-1}$ is compact. This condition is equivalent to the compactness of $(\mathcal{D}^2+1)^{-1/2}$. Indeed, for any $\lambda \notin \operatorname{Spec}(\mathcal{D})$, from the well known resolvent formula

$$(\mathcal{D}-\lambda)^{-1} = (\mathcal{D}-\mu)^{-1} + (\lambda-\mu)(\mathcal{D}-\lambda)^{-1}(\mathcal{D}-\mu)^{-1}\,,$$

we see that $(\mathcal{D}-\lambda)^{-1}$ is compact if and only if $(\mathcal{D}-\mu)^{-1}$ is. Since $\mathcal{D}$ is selfadjoint, it means that $(\mathcal{D}+i)^{-1}$ is compact, or equivalently $\big(((\mathcal{D}+i)^{-1})^*(\mathcal{D}+i)^{-1}\big)^{-1/2} = (\mathcal{D}^2+1)^{-1/2}$ is compact.

In the context of noncommutative geometry, terms such as $\sum_j \pi(a_j)[\,\mathcal{D},\, \pi(b_j)\,]$, $a_j, b_j \in \mathcal{A}$ are called one-forms, by analogy with the usual ones $\sum_j f_j\, dg_j$, with f_j, g_j smooth functions.

Example 3.3.3. *If (M,g) is a compact oriented Riemannian manifold without boundary, admitting a spinor bundle $\$$, then $(C^\infty(M), L^2(M,\$), \not{D})$, as defined in the previous section, is a spectral triple, sometimes called the standard commutative spectral triple. All conditions have already been proved above, and the known properties of the Laplacian operator induce the compactness of $(\not{D}^2+1)^{-1/2}$.*

A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ alone is not sufficient to give a complete algebraic description of a Riemannian spin manifold, and additional data and other conditions are needed. We put as definitions the ones we are interested in and we refer to [41, 43, 146] for the complete list.

Definition 3.3.4. *The spectral dimension of a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is*

$$d := \inf\{s \in \mathbb{R}\,,\, \operatorname{Tr}|\mathcal{D}|^{-s} < +\infty\}\,.$$

When d is even, there is a selfadjoint unitary operator $\Gamma : \mathcal{H} \to \mathcal{H}$, called chirality, such that

$$\Gamma(\operatorname{dom}(\mathcal{D})) = \operatorname{dom}(\mathcal{D})\,, \quad [\,\Gamma,\, \pi(\mathcal{A})\,] = 0\,, \quad \text{and} \quad \Gamma\mathcal{D} = -\mathcal{D}\Gamma\,.$$

In the even dimensional case, denoting $\mathcal{H}^+$ (resp. $\mathcal{H}^-$) the eigenspace of Γ with respect to the eigenvalue $+1$ (resp. -1), $\mathcal{D}$ sends $\mathcal{H}^\pm$ to $\mathcal{H}^\mp$, hence can be decomposed on $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ as

$$\mathcal{D} = \begin{bmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{bmatrix}$$

where $\mathcal{D}^+ := \frac{1}{4}(I-\Gamma)\mathcal{D}(I-\Gamma) = (\mathcal{D}^-)^*$.

Definition 3.3.5. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple and define δ on $\mathcal{B}(\mathcal{H})$ as $\delta(T) := [\,|\mathcal{D}|,\, T\,]$, with domain $\mathrm{dom}(\delta) := \{T \in \mathcal{B}(\mathcal{H}), [\,|\mathcal{D}|,\, T\,] \in \mathcal{B}(\mathcal{H})\}$. Then the spectral triple is said to be regular if $\pi(\mathcal{A})$ and $[\,\mathcal{D},\, \pi(\mathcal{A})\,]$ lie in $\cap_{k\in\mathbb{N}} \mathrm{dom}(\delta^k)$.*

The definition of the map δ, which encodes the notion of smoothness of the corresponding noncommutative manifold, involves the absolute value of $\mathcal{D}$ instead of $\mathcal{D}$ itself, which is a priori more natural. Indeed in the standard commutative case (Example 3.3.3), one can show that $[\,|\not{\mathcal{D}}|,\, [\not{\mathcal{D}},\, f\,]]$, $f \in C^\infty(M)$, is always bounded, whereas $[\,\not{\mathcal{D}},\, [\not{\mathcal{D}},\, f\,]\,]$ is not in general [147, p. 69]. Also, the spaces $\mathrm{dom}(|\mathcal{D}|^{-s})$ can be seen as the noncommutative analogues of usual Sobolev spaces in functional analysis.

Definition 3.3.6. *The triple is said to be real with KO-dimension $d \in \mathbb{Z}/8$, if there is an antiunitary operator $J : \mathcal{H} \to \mathcal{H}$, also called reality or real structure, such that $J^2 = \varepsilon I$, $J\mathcal{D} = \varepsilon' \mathcal{D} J$ and when the spectral dimension is even, $J\Gamma = \varepsilon'' \Gamma J$. The signs ε, ε and ε'' depend on the spectral dimension d modulo 8 [42, (1.46)]:*

d mod 8	0	1	2	3	4	5	6	7
ε	+	+	-	-	-	-	+	+
ε'	+	-	+	+	+	-	+	+
ε''	+		-		+		-	

Moreover, J must satisfy

$$\begin{aligned} &[\,\pi(\mathcal{A}),\, J\pi(\mathcal{A}^*)J^{-1}\,] = 0, \ \ \textit{and} \\ &[\,[\,\mathcal{D},\, \pi(\mathcal{A})\,],\, J\pi(\mathcal{A}^*)J^{-1}\,] = 0 \quad \textit{(first order condition).} \end{aligned} \tag{3.10}$$

The reality operator J encodes the representation of the opposite algebra $\mathcal{A}^\circ$ of $\mathcal{A}$, via the application $b^\circ \mapsto Jb^*J^{-1}$.

3.3.2 The spectral action

We end this section by introducing the notion of spectral action. A spectral version of the action functional (10), from which we can derive the dynamics of relativistic elementary particles, is proposed in [35]:

Definition 3.3.7. *Given a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, the spectral action is defined as*

$$\mathcal{S}(\mathcal{D}, f, \Lambda) := \mathrm{Tr}\big(f(\mathcal{D}^2/\Lambda^2)\big)\,, \tag{3.11}$$

where f is an even positive function on $\mathbb{R}$ (which plays the role of a cut-off necessary to assume the traceability of the operator) and Λ a real parameter, assumed to be sufficiently large.

Note that this quantity depends entirely on the spectrum of $\mathcal{D}$. When the function f is taken to be the characteristic function on the interval $[-1, 1]$, then $f(\mathcal{D}^2/\Lambda^2)$ is just the cardinal of $\mathrm{Spec}(\mathcal{D}^2) \cap [-\Lambda^2, \Lambda^2]$.

This spectral action (11) succeeded in describing the Standard Model from a spectral point of view [35], where the parameter Λ plays the role of the energy scale we choose to look at.

(10)Given by the sum of the Einstein–Hilbert action and the one for the Standard Model.
(11)Or rather the full one, which includes a term $\langle\, \psi\,,\, \mathcal{D}\psi\,\rangle$, $\psi \in \mathcal{H}$.

The relation between the spectral action and the heat kernel is the following: if f is the Laplace transform of some function ϕ [(12)], then from (3.8) (with $\varphi = 1$),

$$\begin{aligned}\mathcal{S}(\mathcal{D}, f, \Lambda) &= \int_0^{+\infty} \phi(t)\, \mathrm{Tr}(e^{-t\mathcal{D}^2/\Lambda^2})\, dt = \int_0^{+\infty} \phi(\Lambda^2\, t)\, \mathrm{Tr}(e^{-t\mathcal{D}^2})\, \Lambda^2 dt \\ &\underset{\Lambda\to+\infty}{\sim} \sum_{k=0}^{\infty} a_k(1, \mathcal{D}^2) \int_0^{\infty} \phi(\Lambda^2\, t)\, t^{(k-n)/2}\, \Lambda^2 dt = \sum_{k=0}^{\infty} a_k(1, \mathcal{D}^2)\, \phi_k\, \Lambda^{n-k}\,,\end{aligned} \tag{3.12}$$

where $\phi_k := \int_0^{+\infty} \phi(t)\, t^{(k-n)/2}\, dt$. Thus the computation of the spectral action reduces to the study of the small-time asymptotics of the heat kernel.

In practice, the spectral action can be computed by means of the Wodzicki residue (noncommutative integral) or the Dixmier trace so we recall briefly their definition and basic properties.
For a compact manifold M of dimension n without boundary and an operator $P \in \Psi\mathrm{DO}^m(M)$ with total symbol $\sigma \sim \sum_j \sigma_{m-j}$ (given on a local chart), the quantity

$$c(P)(x) := (2\pi)^{-n} \int_{S_x^* M} \mathrm{Tr}(\sigma_{-n})(x, \xi)\, d\xi\,,$$

is equal to the coefficient of the logarithmic divergence [(13)] of the Schwartz kernel of P on the diagonal of $M \times M$, and $c(P)(x)|dx|$ [(14)] defines a density on M. The main result is the following:

Theorem 3.3.8 ([153, 84, 154])**.** *Let M be a compact manifold of dimension n without boundary. For any $D \in \Psi\mathrm{DO}^1(M)$ elliptic and $P \in \Psi\mathrm{DO}^m(M)$, $m \in \mathbb{Z}$,*

$$\mathrm{Wres}(P) := \underset{s=0}{\mathrm{res}}\, \zeta_D^P(s) = -\!\!\!\!\!\!\int_M c(P)(x)|dx|\,, \tag{3.13}$$

where $\zeta_D^P(s) := \mathrm{Tr}(P|D|^{-s})$. $\mathrm{Wres}(P)$ is independent of the operator D and when M is connected and $n > 2$, it is the only trace on $\Psi\mathrm{DO}^{-\mathbb{N}}(M)$ up to a multiplication constant.

We refer to [63, 146, 114] for an overview of the Wodzicki residue and its applications in the framework of noncommutative geometry. In this context, the quantity $\mathrm{Wres}(P)$ is called the noncommutative integral $-\!\!\!\!\!\int P$, and the operator $\mathcal{D}$ in a spectral triple plays the role of D in (3.13) (when $\mathcal{D}$ is not invertible, consider the operator $\mathcal{D} + \Pi_{\mathrm{Ker}(\mathcal{D})}$, where $\Pi_{\mathrm{Ker}(\mathcal{D})}$ is the projector on $\mathrm{Ker}(\mathcal{D})$), which is always invertible. Denote also $\zeta_D : s \in \mathbb{C} \mapsto \mathrm{Tr}(|D|^{-s})$.

The asymptotics (3.12) can be expressed in terms of noncommutative integrals.

Theorem 3.3.9. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple of spectral dimension n such that (3.12) is valid, then*

$$\mathcal{S}(\mathcal{D}, f, \Lambda) \underset{\Lambda\to+\infty}{\sim} \sum_{k=1,\dots,n} f_k\, a_k\, (1, \mathcal{D}^2)\, \Lambda^k + f(0)\, a_n + \dots,$$

with $f_k := \big(\Gamma(\tfrac{n-k}{2})\big)^{-1} \int_0^{+\infty} f(s)\, s^{(n-k)/2-1}\, ds$ and

$$\begin{aligned} a_k &= \tfrac{1}{2}\Gamma(\tfrac{n-k}{2}) -\!\!\!\!\!\!\int |\mathcal{D}|^{k-n}\,, \quad k = 0, \dots, n-1\,,\\ a_n &= \dim(\mathrm{Ker}\mathcal{D}) + \zeta_{\mathcal{D}}(0)\,. \end{aligned}$$

[(12)]This condition is discussed in [63].
[(13)]For instance, the renormalization in QED aims to give an interpretation of these divergences, which are common in physics.
[(14)]$|dx| := |dx_1 \wedge \cdots \wedge dx_n|$.

The Dixmier trace was the first example of a positive singular trace (i.e. vanishes on the ideal of finite rank operators). Its domain is the Macaev ideal $\mathcal{L}^{1,+}$, i.e. the set of compact operators P whose singular values $\mu_j(P)$, arranged in decreasing order (with multiplicity), verify $\sup_{N>1}\log(N)^{-1}\sum_{j=0}^{N}\mu_j(P) < +\infty$, and, for short, measures the logarithmic divergence of the spectrum of an operator T

$$\mathrm{Tr}_{Dix}(P) = \lim_{N\to\infty}\frac{1}{\log(N)}\sum_{j=0}^{N}\mu_j(P)$$

(see [49] for details). The relation with the Wodzicki residue is due to Connes [40, Theorem 1]: when $E \to M$ is a complex vector bundle over a compact Riemannian manifold M of dimension n and P a pseudodifferential operator in $\Psi\mathrm{DO}^{-n}(M,E)$, then

$$\mathrm{Tr}_{Dix}(P) = \tfrac{1}{n}\,\mathrm{Wres}(P)\,.$$

Finally, if the eigenvalues of $\mathcal{D}$ and their multiplicity are known, the spectral action can also be directly computed by means of the Poisson summation or the Euler–Maclaurin formulae.

Chapter 4

Applications in noncommutative geometry

The purpose of this chapter is to use the properties of Toeplitz operators we have established in Chapter 1 in order to build spectral triples, check their regularity and compute their spectral dimension. The first results concern spectral triples over the Hardy space and also weighted Bergman spaces over Ω (both are actually closely related). Enjoying the fruitful properties of GTOs, we first present a generic spectral triple on the boundary and propose a natural operator $\mathcal{D}$ for any bounded strictly pseudoconvex domain Ω with smooth boundary and also when Ω is the unit ball of $\mathbb{C}^n$. Then, using the relations established in Section 1.3, we shift this spectral triple on the whole domain, using algebras generated by Toeplitz operators over weighted Bergman spaces, and present different examples of $\mathcal{D}$. We also show how to add a phase to an only positive or negative operator $\mathcal{D}$ and how to modify the spectral triple accordingly. Finally, we present an example of commutative spectral triple together with a real structure J. The third class of spectral triples concerns the complex plane $\mathbb{C}$ and uses the relations between Toeplitz operators on Fock spaces and Weyl operators, which possess similar properties as the classical pseudodifferential operators.

4.1 Hardy space and spectral triples

4.1.1 A generic result

Proposition 4.1.1. *[61, Proposition 5.2] Let $\Omega \in \mathbb{C}^n$ be a strictly pseudoconvex manifold with smooth boundary $\partial\Omega$. Let $\mathcal{A}_H := \mathrm{GTO}^{\leq 0}$, with the identity representation π on $\mathcal{H} := H^2(\partial\Omega)$, and $\mathcal{D} \in \mathrm{GTO}^1$ be selfadjoint and elliptic.*
Then $(\mathcal{A}_H, \mathcal{H}, \mathcal{D})$ is a regular spectral triple of dimension $n = dim_{\mathbb{C}}\, \Omega$.

Proof. Clearly $\mathcal{A}_H$ is an algebra with unit $T_1 = I$ and involution $T_P^* = T_{P^*}$, where P^* is the adjoint of P in $L^2(\partial\Omega)$, and trivially π is faithful. From (P10) of Proposition 1.2.18, since $\mathcal{D}$ is elliptic of order 1, it has a parametrix of order -1, hence compact, so $\mathcal{D}$ has compact resolvent. Moreover, for any $T_P \in \mathcal{A}_H$, the commutator $[\,\mathcal{D},\, T_P\,]$ is bounded since, from (P8)

$$\widetilde{\mathrm{ord}}([\,\mathcal{D},\, T_P\,]) \leq \widetilde{\mathrm{ord}}(\mathcal{D}) + \widetilde{\mathrm{ord}}(T_P) - 1 \leq 1 + 0 - 1 = 0\,.$$

So $(\mathcal{A}_H, \mathcal{H}, \mathcal{D})$ is a spectral triple.

Since from (P11), $|\mathcal{D}| = (\mathcal{D}^*\mathcal{D})^{1/2}$ is of order 1, one can check recursively that for all $k \in \mathbb{N}$ and $T_P \in \mathcal{A}_H$, $\delta^k(T_P) = \delta(T_k)$, where T_k is a GTO of order 0 (see Definition 3.3.5). From the previous result, $\delta^k(T_P)$ is bounded and the same is true for elements of the form $T_P = [\mathcal{D}, T_Q]$, $T_Q \in \mathcal{A}_H$, so the regularity follows.

For the dimension computation, we follow [62, Theorem 3]. We order the points λ_j of the spectrum of $|\mathcal{D}|$ counting multiplicities as $0 < \lambda_1 \leq \lambda_2 \leq \cdots$. Denoting $N(\lambda) := N_{\mathcal{D}}(\lambda)$ the number of eigenvalues λ_j less than λ, we apply (P12) to $|\mathcal{D}|$:

$$N(\lambda) \underset{\lambda\to\infty}{=} c\,\lambda^n + \mathcal{O}(\lambda^{n-1}),$$

where $c := \frac{\mathrm{vol}(\Sigma_{\mathcal{D}})}{(2\pi)^n}$. So we get for large λ:

$$\lambda^n = \tfrac{N(\lambda)}{c} + \mathcal{O}(\lambda^{n-1}) = \tfrac{N(\lambda)}{c} + \mathcal{O}\big(\lambda^{-1}N(\lambda)\big).$$

Since $N(\lambda)^{-1/n} \sim \mathcal{O}(\lambda^{-1})$, we have $\lambda^n = \frac{N(\lambda)}{c} + \mathcal{O}(N(\lambda)^{1-1/n}) = \frac{N(\lambda)}{c}[1 + O(N(\lambda)^{-1/n})]$ as $\lambda \to \infty$, so given $s \in \mathbb{R}$,

$$\lambda^{-s} = \tfrac{c^{s/n}(1+O(N(\lambda)^{-1/n}))}{N(\lambda)^{s/n}} = \tfrac{c^{s/n}}{N(\lambda)^{s/n}} + \mathcal{O}\big(\tfrac{1}{N(\lambda)^{(s+1)/n}}\big).$$

Thus

$$\begin{aligned}\mathrm{Tr}|\mathcal{D}|^{-s} &= \sum_{j=1}^{\infty}\lambda_j^{-s} = \int_{\lambda_1}^{\infty}\lambda^{-s}dN(\lambda) = \int_{\lambda_1}^{\infty}\big(\tfrac{c^{s/n}}{N(\lambda)^{s/n}} + \mathcal{O}\big(\tfrac{1}{N(\lambda)^{(s+1)/n}}\big)\big)\,dN(\lambda)\\ &= \int_1^{\infty}\big(\tfrac{c^{s/n}}{N^{s/n}} + \mathcal{O}\big(\tfrac{1}{N^{(s+1)/n}}\big)\big)\,dN\end{aligned}$$

is finite if and only if $s > n$. □

Remarks 4.1.2.

i) If we assume in this proposition that $\mathcal{D}$ is of order $a < 1$, then the commutators with T_P will be GTOs of order $a-1$, hence not only bounded but even compact.

ii) As a subalgebra of $\mathcal{A}_H$, we can consider the algebra generated by classical Toeplitz operators T_u on H^2, where $u \in C^\infty(\partial\Omega)$.

iii) It would be more accurate to say that the spectral dimension derived here corresponds to the quantity $\dim_{\mathbb{R}}(\Sigma)/2$ (see [22, Theorems 12.9 and 13.1]). In the context of pseudoconvex manifolds however, the natural symplectic cone Σ characterizing the contact structure on the boundary has real dimension $2\dim_{\mathbb{C}}\Omega$.

A possible extension of Proposition 4.1.1 consists in replacing the usual GTOs by the ones with log-polyhomogeneous symbols (see Definition 1.2.20):

Proposition 4.1.3. *If $\mathcal{A} := \mathrm{GTO}^{0,0}_{\log}$, with identity representation on $\mathcal{H} := H^2(\partial\Omega)$ and the operator $\mathcal{D} \in \mathrm{GTO}^{1,0}_{\log}$ is pure elliptic and selfadjoint, then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a regular spectral triple of dimension $n = \dim_{\mathbb{C}}\Omega$.*

Proof. First, $\mathcal{D}$ is pure elliptic so has a parametrix $\mathcal{D}^{-1} = \Pi Q$, with $Q \in \Psi\mathrm{DO}^{-1}_{\log} \subset \Psi\mathrm{DO}^{-1+\varepsilon}$, for any $\varepsilon > 0$, thus is compact and $\mathcal{D}$ has compact resolvent.
Now we check the boundedness of $[\,\mathcal{D},\, T_P\,]$, for $T_P \in \mathcal{A}$. There exist classical pseudodifferential operators D_1 and P_0 of order 1 and 0 respectively, and $D_0 \in \Psi\mathrm{DO}^0_{\log}$ and $P_{-1} \in \Psi\mathrm{DO}^{-1}_{\log}$, all commuting with Π and such that $\mathcal{D} = T_{D_1+D_0}$, and $T_P = T_{P_0+P_{-1}}$. Hence, we get

$$[\,\mathcal{D},\, T_P\,] = \Pi\,[\,D_1 + D_0,\, P_0 + P_{-1}\,] = \Pi\,\big([\,D_1,\, P_0\,] + [\,D_1,\, P_{-1}\,] + [\,D_0,\, P_0\,] + [\,D_0,\, P_{-1}\,]\big)\,.$$

These four commutators terms belong respectively to $\Psi\mathrm{DO}^0$, $\Psi\mathrm{DO}^{-1}_{\log}$, $\Psi\mathrm{DO}^{-1}_{\log}$ and $\Psi\mathrm{DO}^{-2}_{\log}$, so are bounded operators.
The regularity and dimension are shown similarly as in the proof of Proposition 4.1.1. □

4.1.2 Examples of operators $\mathcal{D}$

4.1.2.1 Some natural examples for general strictly pseudoconvex manifolds

– The first example of operator $\mathcal{D}$ for Proposition 4.1.1 is given by $\mathcal{D} := T_\Lambda^{-1}$, which is in a sense natural since $\Lambda = K^*K$ depends only on the domain Ω and the symbol $\sigma(\Lambda)^{-1}(x,\xi) = 2|\xi|$ is, up to a factor, the one of the positive square root of the Laplacian. However, the usual Dirac operator is only selfadjoint and not positive in general, and the K-homology class of the spectral triple induced by a positive operator $\mathcal{D}$, like for instance T_Λ^{-1}, is trivial. As we will discuss in the next section, this drawback can be bypassed by adding a phase to the operator and by doubling the Hilbert space.

– Similarly, an example of natural operator $\mathcal{D}$ for Proposition 4.1.3 is given by $T_{\Lambda_f}^{-1}$, where f is the solution of the Monge–Ampère equation (B.1).
Let a_j and f_j be the functions of the expansion (B.3). Using similar arguments as in [21], we know that for each $j = 1, 2, \dots$, the operator $\Lambda_{a_j r^s}$ is a classical pseudodifferential one of order $-(s + v_j + 1)$ on $\partial\Omega$, for all $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > -1$ and where $v_j := v_{\partial\Omega}(a_j)$ denotes the vanishing order of a_j on $\partial\Omega$ (replace the cone $\mathcal{C}$ by $\partial\Omega$ in Definition 1.2.7). Moreover, denoting t the inward normal coordinate, a similar computation as in the proof of Lemma 1.1.5 shows that its principal symbol is

$$\sigma(\Lambda_{a_j r^s})(x,\xi) = \tfrac{\Gamma(s+v_j+1)}{2^{v_j+1} v_j!}\,(\partial_t^{v_j} a_j)(x)\,(\tfrac{|\partial_t r|(x)}{\sqrt{2}})^s |\xi|^{-(s+v_j+1)}\,.$$

Differentiating this expression with respect to s and evaluating at $s = j(n+1)$, we get, for $j = 1, 2, \dots$, $\partial_s^j(r^s a_j) = \big(r^{n+1}\log r\big)^j a_j = f_j$, hence Λ_{f_j} belongs to $\Psi\mathrm{DO}^{-(j(n+1)+v_j+1),j}_{\log}$ and has principal symbol

$$\sigma(\Lambda_{f_j})(x,\xi) = \tfrac{(\partial_t^{v_j} a_j)(x)}{2^{v_j+1} v_j!}\Big(\sum_{k_1+k_2+k_3=j} \tfrac{j!}{k_1!k_2!k_3!}\Gamma_{k_1}\log(\tfrac{|\partial_t r|}{\sqrt{2}})^{k_2}\log(\tfrac{1}{|\xi|})^{k_3}\Big)(\tfrac{|\partial_t r|}{\sqrt{2}})^{j(n+1)}\,|\xi|^{-(j(n+1)+v_j+1)},$$

where $\Gamma_{k_1} := \partial_s^{k_1}\Gamma(s + v_j + 1)|_{s=j(n+1)}$. Finally, the operator Λ_f belongs to $\Psi\mathrm{DO}^{-1,0}_{\log}$, and is pure elliptic and selfadjoint. Its total symbol in a local chart is of the form (C.3) with $k_j = 0$ for $j = 0, \dots, n$, $k_j = 1$ for $j = n+1, \dots, 2(n+1) - 1$, etc.
Since $\sigma(\Lambda_{f_0})$ does not vanish on Σ, $T_{\Lambda_f}^{-1} \in \mathrm{GTO}^{1,0}_{\log}$.

– Another interesting example is given by the operator

$$\mathcal{D} := T_{E'}\,,\qquad \text{where}\qquad E' := \tfrac{1}{i\,\eta(E)}\,E\,,$$

E being the complex normal direction defined in (A.12).
The particularity of this GTO is that its symbol is simply $\widetilde{\sigma}(\mathcal{D})(z,t) = i\langle t\,\eta_z\,,\, \frac{1}{i\,\eta_z(E)}E\,\rangle = t$, in the chosen parametrization of Σ (A.11), which is invariant under change of defining function (considering another defining function changes also E' so its symbol on Σ remains the same). Since η and E' are dual, up to the factor $-i$, this operator $\mathcal{D}$ is the one which is the most related to the contact structure on $\partial\Omega$. Moreover, the Levi form can be expressed naturally by means of its symbol. First, we know that for any $f, g \in C^\infty(\partial\Omega)$ such that $\{\,\tau(f)\,,\,\tau(g)\,\}_\Sigma \neq 0$, where τ is the extension map from (A.13), $\mathcal{D}[\,T_f,\,T_g\,]$ is a GTO of order exactly 0 with symbol $-i\widetilde{\sigma}(\mathcal{D})\{\,\tau(f)\,,\,\tau(g)\,\}_\Sigma$. Now using $e_k : z \mapsto z_k$, $j = 1,\dots,n$, in (A.14), we have $\overline{\bar\partial_b \bar e_j} = dz_j$, $\overline{\bar\partial_b \bar e_k} = dz_k$ (the other two combinations vanish) and

$$\begin{aligned}\widetilde{\sigma}\big(\mathcal{D}\,[\,T^*_{e_j},\,T_{e_k}\,]\big)(z,t) &= -i\,\widetilde{\sigma}(\mathcal{D})(z,t)\;\{\,\tau(\bar e_j)\,,\,\tau(e_k)\,\}_\Sigma(z,t)\\ &= \langle\,dz_j\,,\,(\mathrm{Hess}^*_r)^{-1}(z)\,dz_k\,\rangle = \big[(\mathrm{Hess}^*_r)^{-1}(z)\big]_{j,k}\,.\end{aligned}$$

One could also take for $\mathcal{D}$ the GTOs associated to the other normal derivatives $D_{m,j}$ and $\bar D_{m,j}$, $j \neq m$, $m = 1,\dots,n$, but they are only defined on the open sets X_m of $\partial\Omega$, hence not well globally defined.

4.1.2.2 Transporting $\not{D}$ from $\mathbb{R}^n$ to $H^2(\mathbb{S}^{2n-1})$

In the case when $\partial\Omega = \mathbb{S}^{2n-1}$, another idea is to bring the usual Dirac operator $\not{D} = -i\gamma_\mu\partial^\mu$ acting on $L^2(\mathbb{R}^n)$ back to $H^2(\mathbb{S}^{2n-1})$ using the unitaries presented in Section 1.1. Since $\not{D}$ can be written as $\sum_{j=1}^n \gamma_j \pi_L(\mathbb{P}_j)$, we consider

$$\mathcal{D}_{\mathbb{S}^{2n-1}} := \sum_{j=1}^n \gamma_j\, V^*\,\pi_A(\mathbb{P}_j)\,V\,, \tag{4.1}$$

acting on $H^2(\mathbb{S}^{2n-1}) \otimes C\ell(\mathbb{R}^n) \approx H^2(\mathbb{S}^{2n-1}) \otimes \mathbb{C}^{2^{\lfloor n/2\rfloor}}$ where V is the unweighted version of (1.27).

Proposition 4.1.4. *[61, Proposition 5.9] As in Proposition 4.1.1, let $\mathcal{A}_H$ be the algebra of GTOs of order less or equal to 0, acting on the Hardy space $\mathcal{H} = H^2(\mathbb{S}^{2n-1}) \otimes C\ell(\mathbb{R}^n)$ via $\pi(T_Q) := T_Q \otimes I$. Then $(\mathcal{A}, \mathcal{H}, \mathcal{D}_{\mathbb{S}^{2n-1}})$ is a regular spectral triple of dimension $2n$.*

Proof. The requirement of compact resolvent is automatically fulfilled, since it is satisfied in the case of the standard Dirac operator on $\mathbb{R}^n$, from which the operator $\mathcal{D}_{\mathbb{S}^{2n-1}}$ was obtained by transfer via various *-isomorphisms, which also shows it is selfadjoint. From Lemma 1.3.9, $\mathcal{D}_{\mathbb{S}^{2n-1}}$ is a GTO of order $1/2$ so the remainder of the proof is similar to the one of Proposition 4.1.1. $\square$

Remark 4.1.5. *Note that a spectral triple using algebra generated by Toeplitz operators on the Hardy space over the unit circle $\mathbb{S}^1$ has already be established in [39, Section 4.2]. More precisely, they use the algebra generated by*

$$\mathcal{A} := \{T_u\,,\, u \in C(\mathbb{S}^1)\} \cap \Psi\mathrm{DO}^0(\mathbb{S}^1)\,,$$

acting diagonally on $\mathcal{H} = H^2(\mathbb{S}^1) \oplus H^2(\mathbb{S}^1)$, whose orthonormal basis is given by $(e^{ij\theta})_{j\in\mathbb{N}}$, $\theta \in [0, 2\pi)$. For any $k \in \mathbb{N}$, they consider the operator $\mathcal{D}_k := \left[\begin{smallmatrix} 0 & \mathcal{D}_k^- \\ \mathcal{D}_k^+ & 0\end{smallmatrix}\right]$ where $\mathcal{D}_k^- := N^{1/2}S^k$ and $\mathcal{D}_k^+ := (\mathcal{D}_k^-)^*$, with the number and shift operators

$$N : e^{ij\theta} \mapsto je^{ij\theta}\,, \quad S : e^{ij\theta} \mapsto e^{i(j+1)\theta}\,.$$

They obtain a spectral triple with spectral dimension 2. This can be related to Proposition 4.1.4 for the case $n = 1$: since V maps unitarily $H^2(\mathbb{S}^1)$ onto $A^2(\mathbb{B}^1)$, $\mathbb{P}_1 \in \mathfrak{h}^1$ acts on $H^2(\mathbb{S}^1)$ via a representation π_H as

$$\pi_H(\mathbb{P}_1)\, e^{ij\theta} := -\tfrac{i}{\sqrt{2}} \big(j^{1/2} e^{i(j-1)\theta} - (j+1)^{1/2} e^{i(j+1)\theta}\big) = -\tfrac{i}{\sqrt{2}} \big((N^{1/2}S)^* - (N^{1/2}S)\big)\, e^{ij\theta}\,,$$

using the previous notations. The power in S^k and the matrix formulation does not affect the result. Finally, since $\mathbb{S}^1$ is the boundary of the complex one dimensional domain $\mathbb{B}^1$, the obtained dimension corresponds to $2\dim_{\mathbb{C}}\mathbb{B}^1 = 2$.

4.2 Bergman spaces and spectral triples

4.2.1 Over a general strictly pseudoconvex domain

In the Bergman case, we have a similar result as Proposition 4.1.1:

Proposition 4.2.1. *[61, Proposition 5.4] Let $\mathcal{A}_{B,m}$ be the algebra generated by the Toeplitz operators $\mathbf{T}_f$, where the functions f are in $C^\infty(\overline{\Omega})$, π be the identity representation on $\mathcal{H} := A^2_m(\Omega)$ and $\mathcal{D} := V_m\, T_Q\, V_m^*$, where T_Q in GTO^1 is selfadjoint and elliptic with V_m as in (1.27).*
Then $(\mathcal{A}_{B,m}, \mathcal{H}, \mathcal{D})$ is a regular spectral triple of dimension $n = \dim_{\mathbb{C}}(\Omega)$.

Proof. As in the Hardy case, clearly $\mathcal{A}_{B,m}$ is a unital involutive algebra with a faithful representation on $\mathcal{H}$. Since T_Q has a parametrix of order -1, hence compact, $\mathcal{D}$ has compact resolvent by unitary equivalence.
To see that $[\,\mathcal{D},\, \mathbf{T}_f\,]$ is bounded for all $\mathbf{T}_f$ in $\mathcal{A}_{B,m}$, we use (1.28) and remark that

$$[\,\mathcal{D},\, \mathbf{T}_f\,] = V_m\, [\, T_Q,\, T_{\Lambda_m}^{-1/2} T_{\Lambda_{w_m f}} T_{\Lambda_m}^{-1/2}\,]\, V_m^*\,. \tag{4.2}$$

From (P8) of Proposition 1.2.18, since the orders of the GTOs T_Q and $T_{\Lambda_m}^{-1/2} T_{\Lambda_{w_m f}} T_{\Lambda_m}^{-1/2}$ are respectively 1 and less or equal to 0, the commutator on the right hand side has order less or equal to 0, hence is bounded in particular on $H^2(\partial\Omega)$.

Since $|\mathcal{D}| = V_m\, |T_Q|\, V_m^*$ and $|\mathcal{D}|^{-s} = V_m\, |T_Q|^{-s}\, V_m^*$, for $s \in \mathbb{R}$, the regularity and dimension computation are shown by using the same arguments as for Proposition 4.1.1. □

We have seen in Section 4.1.1 that GTOs of log type generate a spectral triple. The idea now is to shift Proposition 4.1.3 from Hardy to the Bergman spaces just as we did in the previous result. This leads to consider a larger class of symbols f for Toeplitz operators on Bergman spaces, which involve also logarithmic terms near the boundary of Ω.
Let $p \in \mathbb{N}\backslash\{0\}$. Consider the class $\mathrm{MA}^p(\Omega)$ of functions f over Ω verifying:

$$f \approx \sum_{j=0}^{\infty} \big(r^p \log r\big)^j\, a_j\,, \quad f|_{\partial\Omega} = 0,\ a_0 \text{ not identically } 0 \text{ and } a_j \in C^\infty(\overline{\Omega})$$

where " $\approx$ " has the same meaning as in (B.3). Note that this class does not depend on r. Indeed, if ρ is another defining function of Ω, there is a function $\phi \in C^\infty(\overline{\Omega})$ such that $r = \phi\rho$ (see Remark A.3.5). So we have

$$f \approx \sum_{j=0}^{\infty} \big((\rho\phi)^p \log(\rho\phi)\big)^j \, a_j \approx \sum_{j=0}^{\infty} \phi^{pj} a_j \sum_{k=0}^{j} \tbinom{j}{k} \rho^{pk} \rho^{p(j-k)} (\log\rho)^k (\log\phi)^{j-k} \approx \sum_{j=0}^{\infty} \big(\rho^p \log\rho\big)^j \, \widetilde{a}_j \,,$$

where the $\widetilde{a}_j$ consist in a sum of products of the a_k and powers of ϕ and $\log(\phi)$, which are in $C^\infty(\overline{\Omega})$.

The two constraints on $f \in \mathrm{MA}^p(\Omega)$ make the operator $\Lambda_{w_m f}$ belong to $\Psi\mathrm{DO}_{\log}^{-(m+1),0}$. Moreover, the relation (1.28) remains valid with this class of functions, so we get a unitary equivalence between $\mathbf{T}_f$ and GTOs of log type. Consequently, the following result is proved using similar arguments as for Proposition 4.2.1 by replacing usual GTOs by the ones relative to pure polyhomogeneous pseudodifferential operators, and using Proposition 4.1.3 instead of Proposition 4.1.1:

Proposition 4.2.2. *Let $\mathcal{A}_p$ be the algebra generated by the Toeplitz operators of the form $\mathbf{T}_f$ where $f \in \mathrm{MA}^p(\Omega)$, acting on $\mathcal{H} := A^2_m(\Omega)$. Let $\mathcal{D} := V_m \, T \, V_m^*$, with T a selfadjoint elliptic operator in $\mathrm{GTO}^{1,0}_{\log}$.*
Then $(\mathcal{A}_p, \mathcal{H}, \mathcal{D})$ is a regular spectral triple of dimension n.

4.2.2 Examples of operators $\mathcal{D}$

Using Proposition 1.2.21, we may take $\mathcal{D} = \mathbf{T}_{\mathbf{P}}$, with any differential operator $\mathbf{P}$ on $\overline{\Omega}$ such that $T_{\Lambda_{w_m \mathbf{P}}}$ is of order 1, $\mathbf{T}_{\mathbf{P}} = \mathbf{T}^*_{\mathbf{P}}$ and also (1.25) is nonzero on Σ. So a first example is given by the weighted normal derivative (1.37).

As a second example, we can take $\mathcal{D} = \mathbf{T}_r^{-1}$. Indeed, we know that $\mathbf{T}_r^{-1}$ exists on $\mathrm{Ran}(\mathbf{T}_r)$ which is dense in A^2_m, and since r vanishes to order 1 on the boundary, we deduce from (1.28) that $\mathbf{T}_r^{-1}$ corresponds to a GTO of order 1. So we get the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ with the same $\mathcal{A}$ and $\mathcal{H}$ as in of Proposition 4.2.1. Again, this operator depends only on the defining function so seems quite natural to consider. However, the fact that $\mathcal{D} = \mathbf{T}_r^{-1}$ is a negative operator induces a trivial K-homology class for the spectral triple. We now get around this triviality:

Proposition 4.2.3. *[61, Proposition 5.7] Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be the spectral triple of Proposition 4.2.1 with $\mathcal{D} = \mathbf{T}_r^{-1}$. Define $\mathcal{A}'$ as the algebra of all $\mathbf{T}_f$ acting diagonally on $\mathcal{H}' := \mathcal{H} \oplus \mathcal{H}$ and let $\mathcal{D}'$ be the operator*

$$\mathcal{D}' := \begin{bmatrix} 0 & \mathbf{U}\,\mathbf{T}_r^{-1} \\ \mathbf{T}_r^{-1}\,\mathbf{U}^* & 0 \end{bmatrix}$$

where $\mathbf{U}$ is a unitary operator on $A^2_m(\Omega)$. If $\mathbf{U}$ is such that

$$V_m^* \, \mathbf{U} \, V_m \text{ is a unitary GTO,} \tag{4.3}$$

then $(\mathcal{A}', \mathcal{H}', \mathcal{D}')$ is a regular spectral triple.
The triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and $(\mathcal{A}', \mathcal{H}', \mathcal{D}')$ have the same dimension.

Proof. For any $\mathbf{T}'_f \in \mathcal{A}'$, $[\,\mathcal{D}', \mathbf{T}'_f\,] = \left[\begin{smallmatrix} 0 & \mathcal{D}_1 \\ \mathcal{D}_2 & 0 \end{smallmatrix}\right]$ with $\mathcal{D}_1 := [\,\mathbf{U}\,\mathbf{T}_r^{-1}, \mathbf{T}_f\,]$ and $\mathcal{D}_2 := [\,\mathbf{T}_r^{-1}\mathbf{U}^*, \mathbf{T}_f\,]$.

From Proposition 1.3.4, we have (1.28) and $\mathbf{T}_r^{-1} = V_m\, T_{\Lambda_m}^{1/2} T_{\Lambda_{w_m r}}^{-1} T_{\Lambda_m}^{1/2}\, V_m^*$, so we get

$$\begin{aligned}
\mathcal{D}_1 &= \mathbf{U}\mathbf{T}_r^{-1}\,\mathbf{T}_f - \mathbf{T}_f\,\mathbf{U}\mathbf{T}_r^{-1} \\
&= \mathbf{U}\,(V_m T_{\Lambda_m}^{1/2} T_{\Lambda_{w_m r}}^{-1} T_{\Lambda_m}^{1/2} V_m^*)\,(V_m T_{\Lambda_m}^{-1/2} T_{\Lambda_{w_m f}} T_{\Lambda_m}^{-1/2} V_m^*) \\
&\qquad - (V_m T_{\Lambda_m}^{-1/2} T_{\Lambda_{w_m f}} T_{\Lambda_m}^{-1/2} V_m^*)\,\mathbf{U}\,(V_m T_{\Lambda_m}^{1/2} T_{\Lambda_{w_m r}}^{-1} T_{\Lambda_m}^{1/2} V_m^*) \\
&= \mathbf{U}\,(V_m T_{\Lambda_m}^{1/2} T_{\Lambda_{w_m r}}^{-1} T_{\Lambda_{w_m f}} T_{\Lambda_m}^{-1/2} V_m^*) - V_m T_{\Lambda_m}^{-1/2} T_{\Lambda_{w_m f}} T_{\Lambda_m}^{-1/2}\,(V_m^*\mathbf{U}V_m)\,T_{\Lambda_m}^{1/2} T_{\Lambda_{wr}}^{-1} T_{\Lambda_m}^{1/2} V_m^* \\
&= (V_m V_m^*)\,\mathbf{U}\,V_m T_{\Lambda_m}^{1/2} T_{\Lambda_{w_m r}}^{-1}\,(T_{\Lambda_m}^{1/2} T_{\Lambda_m}^{-1/2})\,T_{\Lambda_{w_m f}} T_{\Lambda_m}^{-1/2} V_m^* \\
&\qquad - V_m T_{\Lambda_m}^{-1/2} T_{\Lambda_{w_m f}} T_{\Lambda_m}^{-1/2}\,(V_m^*\mathbf{U}V_m)\,T_{\Lambda_m}^{1/2} T_{\Lambda_{w_m r}}^{-1} T_{\Lambda_m}^{1/2} V_m^* \\
&= V_m\,[\,(V_m^*\mathbf{U}V_m)\,T_{\Lambda_m}^{1/2} T_{\Lambda_{w_m r}}^{-1} T_{\Lambda_m}^{1/2},\ T_{\Lambda_m}^{-1/2} T_{\Lambda_{w_m f}} T_{\Lambda_m}^{-1/2}\,]\,V_m^*\,.
\end{aligned}$$

From the hypothesis, $V_m\,\mathbf{U}\,V_m^*$ is a bounded GTO, $T_{\Lambda_{w_m r}}^{-1} T_{\Lambda_m}$ is a GTO of order 1 and $T_{\Lambda_m}^{-1} T_{\Lambda_{wf}}$ is a GTO of order less or equal to 0, so the commutator is a GTO of order less or equal to 0, thus is a bounded operator on A_m^2. Similar arguments show that

$$D_2 = V_m\,[\,T_{\Lambda_m}^{1/2} T_{\Lambda_{w_m r}}^{-1} T_{\Lambda_m}^{1/2}\,(V_m^*\mathbf{U}^*V_m),\ T_{\Lambda_m}^{-1/2} T_{\Lambda_{w_m f}} T_{\Lambda_m}^{-1/2}\,]\,V_m^*$$

is also bounded on A_m^2, which makes $[\,\mathcal{D}',\ \mathbf{T}'_f\,]$ bounded on the direct sum $\mathcal{H}'$.
We remark that the expressions of $\mathcal{D}_1$ and $\mathcal{D}_2$ differ from (4.2) by the term $V_m^*\mathbf{U}V_m$ which is a GTO of order 0. So the regularity of the spectral triple is shown as in Proposition 4.2.1.
Finally $\mathcal{D}'$ has compact resolvent since $\mathcal{D}'^{-1} = \left[\begin{smallmatrix} 0 & \mathbf{U}\mathbf{T}_r \\ \mathbf{T}_r\mathbf{U}^* & 0 \end{smallmatrix}\right]$ is compact because the operators $\mathbf{U}\mathbf{T}_r$ and $\mathbf{T}_r\mathbf{U}^*$ are compact.
Since $\mathcal{D}'^2 = \left[\begin{smallmatrix} \mathbf{U}\mathbf{T}_r^{-2}\mathbf{U}^* & 0 \\ 0 & \mathbf{T}_r^{-2} \end{smallmatrix}\right]$, we deduce that the unitary $\mathbf{U}$ does not influence the computation of eigenvalues so the spectral dimension is not altered. □

Remark 4.2.4. *The two classes of unitaries* $\mathbf{U}$ *defined in Section 1.4.2 satisfy* (4.3)*, so provide examples of spectral triples* $(\mathcal{A}', \mathcal{H}', \mathcal{D}')$ *on (the sum of two copies of) the Bergman space with non-negative* $\mathcal{D}'$ *when* $\mathcal{D} = \mathbf{T}_r^{-1}$*.*

4.2.3 The case of the unit ball

4.2.3.1 Radial symbols and commutative algebra

In the case $\Omega = \mathbb{B}^n$, the Proposition 4.2.1 can be made much more explicit. Indeed, if f is a radial function in $C^\infty(\overline{\mathbb{B}^n})$ and $w_m = (-r)^m\,\chi$, $m > -1$, is a weight with χ and r also radial, the family $(v_\alpha)_{\alpha\in\mathbb{N}^n}$ defined in (1.5) diagonalizes $\mathbf{T}_f : A_m^2(\mathbb{B}^n) \to A_m^2(\mathbb{B}^n)$ and the eigenvalues only depend on $|\alpha|$. Namely,

$$\langle\, \mathbf{T}_f\, v_\alpha\,,\ v_\beta\,\rangle_{A_m^2} = \tfrac{\delta_{\alpha\beta}}{\int_0^1 t^{2n+2|\alpha|-1}\,w_m(t)\,dt} \int_0^1 t^{2n+2|\alpha|-1}\,f(t)\,w_m(t)\,dt\,, \tag{4.4}$$

as it is easily seen by passing to the polar coordinates. As a consequence, Toeplitz operators on $A_m^2(\mathbb{B}^n)$ with radial symbols commute.
To give an example of a computation of the spectral dimension, consider the weight $w_m := (-r)^m$ with the radial defining function $r : z \in \mathbb{B}^n \mapsto |z|^2 - 1$, and $\mathcal{D} := \mathbf{T}_r^{-1}$. A direct calculation shows that

$$\begin{aligned}
\lambda_{|\alpha|}(\mathbf{T}_r^{-1}) &= -\Big(\int_0^1 t^{2n+2|\alpha|-1}\,(t^2-1)^{m+1}\,dt\Big)^{-1} \int_0^1 t^{2n+2|\alpha|-1}\,(t^2-1)^m\,dt \\
&= -\tfrac{1}{m+1}(|\alpha| + n + m + 1)\,,
\end{aligned} \tag{4.5}$$

with multiplicity $M_{|\alpha|} = \binom{n-1+|\alpha|}{n-1}$. Since

$$\mathrm{Tr}\big(|\mathcal{D}|^{-s}\big) = (m+1)^s \sum_{|\alpha|=0}^{\infty} \binom{n-1+|\alpha|}{n-1} \big(|\alpha|+n+m+1\big)^{-s}$$

and $\binom{n-1+|\alpha|}{n-1} \underset{|\alpha|\to\infty}{\sim} \frac{|\alpha|^{n-1}}{(n-1)!}$, we have $\mathrm{Tr}\big(|\mathcal{D}|^{-s}\big) < \infty$ if and only if $\sum_{|\alpha|=0}^{\infty} |\alpha|^{n-1-s} < \infty$, so for each $s > n$, which gives the result of Proposition 4.2.1.

Finally, we can transport unitarily the Dirac operator $\not{\mathcal{D}}$ from $L^2(\mathbb{R}^n)$ to $A^2(\mathbb{B}^n)$ as it is done in Proposition 4.1.4, by setting

$$\mathcal{D}_{\mathbb{B}^n} := \sum_{j=1}^{n} \gamma_j \, \pi_A(\mathbb{P}_j) = V \, \mathcal{D}_{\mathbb{S}^{2n-1}} \, V^* \,. \tag{4.6}$$

Of course, the corresponding spectral triple build as previous from the algebra generated by Toeplitz operators on the unweighted Bergman space over the unit ball of $\mathbb{C}^n$ is also regular and has spectral dimension $2n$.

4.2.3.2 Example of spectral triple with a real structure

The main difficulty in finding a compatible real structure J (see Definition 3.3.6) using algebras of Toeplitz operators comes from the first order condition in (3.10). Indeed, there is very little known about commutants in Toeplitz algebras and commutation properties are rare. The following example uses the commutation property of Toeplitz operators with radial symbols on Bergman spaces over the unit ball of $\mathbb{C}^n$. Despite the drastic simplifications induced by the framework considered here, the result is not without interest: we see that the introduction of a real structure, which aims to describes the relations between the algebra and the opposite algebra, naturally leads to consider the antiholomorphic version of Toeplitz operators.

Let $r(z) = r(|z|)$ be a radial defining function on $\mathbb{B}^n$ and consider the radial weight $w_m = (-r)^m$. Denote by $\mathbf{C}$ the complex conjugation operator and $\bar{A}^2_m := \mathbf{C}(A^2_m)$ the space of antiholomorphic functions in $L^2(\mathbb{B}^n, w_m\, d\mu)$. For $f \in C^\infty(\overline{\mathbb{B}^n})$, denote respectively by $\mathbf{T}_f$ the Toeplitz operators acting on A^2_m (here the dependence on m in the weight is not very important so we lighten the notations). We can define naturally its mirrored version $\bar{\mathbf{T}}_f := \overline{\mathbf{\Pi}}_m \mathbf{M}_f$ acting on $\bar{A}^2_m$, where $\overline{\mathbf{\Pi}}_m : L^2(\mathbb{B}^n) \to \bar{A}^2_m$ is the orthogonal projector. The corresponding Toeplitz operators are intertwined via the relation

$$\bar{\mathbf{T}}_f = \mathbf{C}\, \mathbf{T}_{\bar{f}}\, \mathbf{C} \,. \tag{4.7}$$

Let $\mathcal{A}_{rad}$ (resp. $\bar{\mathcal{A}}_{rad}$) be the algebra generated by Toeplitz operators $\mathbf{T}_f$ (resp. $\bar{\mathbf{T}}_f$), with radial symbols $f \in C^\infty(\overline{\mathbb{B}^n})$. The fact that this algebra is commutative can be deduced from (4.4).
For $f \in C^\infty(\overline{\mathbb{B}^n})$, the antiholomorphic version of (1.28) is given by

$$\bar{\mathbf{T}}_{\bar{f}} = \mathbf{C}\, V_m \, T_{\Lambda_m}^{-1/2} T_{\Lambda_{w_m f}} T_{\Lambda_w}^{-1/2}\, V_m^* \, \mathbf{C} \,.$$

Proposition 4.2.5. *Let $\mathcal{H}' := A^2_{0,m} \oplus \mathbb{C} \oplus \bar{A}^2_{0,m}$, where $A^2_{0,m} := \{\phi \in A^2_m(\mathbb{B}^n)\,,\, \phi(0) = 0\}$ and $\bar{A}^2_{0,m} := \mathbf{C}(A^2_{0,m})$. Let π' be the representation of $\mathcal{A}' := \mathcal{A}_{rad} \times \bar{\mathcal{A}}_{rad}$ on $\mathcal{H}'$ defined as*

$$\pi'(\mathbf{T}_f, \bar{\mathbf{T}}_g) := \begin{bmatrix} \mathbf{T}_f & 0 & 0 \\ 0 & \mathbf{T}_f & 0 \\ 0 & 0 & \bar{\mathbf{T}}_g \end{bmatrix} ,$$

and set

$$\mathcal{D}' := \begin{bmatrix} \mathbf{T}_r^{-1} & 0 & 0 \\ 0 & \mathbf{T}_r^{-1} & 0 \\ 0 & 0 & \bar{\mathbf{T}}_r^{-1} \end{bmatrix}, \quad \text{and} \quad J' := \tilde{\varepsilon} \begin{bmatrix} 0 & 0 & -i\mathbf{C} \\ 0 & -1 & 0 \\ i\mathbf{C} & 0 & 0 \end{bmatrix}, \quad \text{where } \tilde{\varepsilon} := \begin{cases} 1 & \text{if } \varepsilon = 1\,, \\ i & \text{if } \varepsilon = -1 \end{cases}$$

(see Definition 3.3.6 for the definition of ε).
Then J' defines a real structure for the spectral triple $(\mathcal{A}', \mathcal{H}', \mathcal{D}')$.

Proof. From previous results, $(\mathcal{A}', \mathcal{H}', \mathcal{D}')$ is a regular spectral triple with spectral dimension n, and we get directly $J'^2 = \varepsilon I$. Then for $\begin{bmatrix} \phi \\ \lambda \\ \bar{\psi} \end{bmatrix}$ in $\mathcal{H}'$, where $\phi, \psi \in A^2_{0,m}$, we get

$$[\,\mathcal{D}',\, J'\,] \begin{bmatrix} \phi \\ \lambda \\ \bar{\psi} \end{bmatrix} = \mathcal{D}' \begin{bmatrix} -i\psi \\ -\lambda \\ i\bar{\phi} \end{bmatrix} - J' \begin{bmatrix} \mathbf{T}_r^{-1}\phi \\ \lambda\mathbf{T}_r^{-1} \\ \bar{\mathbf{T}}_r^{-1}\bar{\psi} \end{bmatrix} = \begin{bmatrix} -i\mathbf{T}_r^{-1}\psi \\ -\lambda\mathbf{T}_r^{-1} \\ \bar{\mathbf{T}}_r^{-1}\bar{\psi} \end{bmatrix} - \begin{bmatrix} -i\overline{\bar{\mathbf{T}}_r^{-1}\bar{\psi}} \\ -\lambda\mathbf{T}_r^{-1} \\ i\overline{\mathbf{T}_r^{-1}\phi} \end{bmatrix} = 0\,,$$

(we used (4.7) for the last equality). The first order condition is trivially fulfilled since for any $(\mathbf{T}_f, \bar{\mathbf{T}}_g)$ in $\mathcal{A}'$, $[\,\mathcal{D}',\, \pi'\big((\mathbf{T}_f, \bar{\mathbf{T}}_g)\big)\,] = 0$. □

Remark 4.2.6. *The choice of the middle terms of $\mathcal{D}'$ and J' imply $[\,\mathcal{D},\, J\,] = 0$ since $\mathbf{T}_r^{-1}$ commutes trivially with $\mathbb{C}$. As a consequence this result is valid only when $\varepsilon' = 1$ in Definition 3.3.6, i.e. for a dimension n such that $n \mod 8 = 0, 2, 3, 4, 6$ or 7.*

4.2.3.3 Computation of the spectral action

We fix here the radial defining function for $\mathbb{B}^n$ as $r(z) := |z|^2 - 1$ and the weight $w_m := (-r)^m$.
The following proposition gives the expression of the coefficients $a_k(1, \mathcal{D}^2)$ appearing in the asymptotic expansion of the spectral action (3.12) in the context of the spectral triples Proposition 4.2.1 for the case $\mathcal{D} := \mathbf{T}_r^{-1}$.

Proposition 4.2.7. *The spectral action relative to the spectral triple $(\mathcal{A}_{B,m}, A^2_m(\mathbb{B}^n), \mathbf{T}_r^{-1})$ has the following asymptotics*

$$S(\mathbf{T}_r^{-1}, f, \Lambda) \underset{\Lambda\to\infty}{\sim} \sum_{k=0}^{\infty} f_k\, a_k\, \Lambda^{n-k}\,, \qquad \text{with}$$
$$a_k = \tfrac{m+1}{2(n-1)!}\, \Gamma(\tfrac{n-k}{2})\, c_{n-k-1}\,, \qquad k = 0, \dots, n-1\,,$$

where

$$f_k := \big(\Gamma(\tfrac{n-k}{2})\big)^{-1} \int_0^{+\infty} f(s)\, s^{(n-k)/2-1}\, ds\,,$$
$$c_l := (m+1)^l \sum_{i=l}^{n-1} \tbinom{i}{l}\, s(n-1, i)\, (-m-1)^{i-l}, \text{ for } l = 0, \dots, n-1\,.$$

Proof. We denote $p := |\alpha| \in \mathbb{N}$ from (4.5), the eigenvalues of $\mathbf{T}_r^{-1}$ are given by

$$\lambda_p = -\tfrac{1}{m+1}(p + n + m + 1) =: -\alpha(p + n - \beta)\,,$$

where $\alpha := 1/(m+1)$, $\beta := -(m+1)$, and the corresponding multiplicities $M_p := \binom{n-1+p}{n-1}$. If $n \geq 2$, we can express M_p as a polynomial in $|\lambda_p|$ (when $n = 1$, then $M_p = 1$ and the following remains valid):

$$\begin{aligned} M_p &= \tfrac{1}{(n-1)!}(p+n-1)(p+n-2)\dots(p+1) = \tfrac{1}{(n-1)!}\,(\tfrac{|\lambda_p|}{\alpha}+\beta)_{(n-1)} \\ &= \tfrac{1}{(n-1)!}\sum_{j=1}^{n-1} s(n-1,j)\,(\tfrac{|\lambda_p|}{\alpha}+\beta)^j =: \tfrac{1}{(n-1)!}\sum_{l=0}^{n-1} c_l\,|\lambda_p|^l\,. \end{aligned}$$

From Theorem 3.3.9, for any $k = 0,\dots,n-1$,

$$a_k = \tfrac{1}{2}\Gamma(\tfrac{n-k}{2}) ⨍ |\mathcal{D}|^{k-n} = \tfrac{1}{2}\Gamma(\tfrac{n-k}{2})\,\underset{s=0}{\operatorname{res}}\,(\operatorname{Tr}|\mathcal{D}|^{k-n-s})\,,$$

hence, setting $\beta' := \beta + n$, we obtain

$$\begin{aligned} \operatorname{Tr}|\mathcal{D}|^{k-n-s} &= \sum_{p\in\mathbb{N}} M_p|\lambda_p|^{k-n-s} = \tfrac{1}{(n-1)!}\sum_{p\in\mathbb{N}}\sum_{l=0}^{n-1} c_l|\lambda_p|^{l+k-n-s} \\ &= \tfrac{1}{(n-1)!}\sum_{l=0}^{n-1} c_l\,\alpha^{l+k-n-s}\sum_{p\in\mathbb{N}}\tfrac{1}{(p+\beta')^{s+n-k-l}} \\ &= \tfrac{1}{(n-1)!}\sum_{l=0}^{n-1} c_l\,\alpha^{l+k-n-s}\big[\zeta(s+n-k-l) - \sum_{p=1}^{\beta'-1}\tfrac{1}{(p+\beta')^{s+n-k-l}}\big]\,. \end{aligned}$$

Hence for any $k = 0,\dots,n-1$, $\underset{s=0}{\operatorname{res}}\,(\operatorname{Tr}|\mathcal{D}|^{k-n-s}) = \frac{m+1}{(n-1)!}\,c_{n-k-1}$ which gives the result. $\square$

4.3 Remarks on Dixmier traces

In all above examples of spectral triples $(\mathcal{A},\mathcal{H},\mathcal{D})$, one can also give a formula for the Dixmier traces $\operatorname{Tr}_{Dix}(a|\mathcal{D}|^{-n})$ where $a \in \mathcal{A}$ and n is the spectral dimension.

First, from (P13) of Proposition 1.2.18, if T_P is a GTO on $\partial\Omega$ of order $-n$, then T_P is measurable and Dixmier traceable:

$$\operatorname{Tr}_{Dix}(T_P) = \tfrac{1}{n!\,(2\pi)^n}\int_{\partial\Omega}\widetilde{\sigma}(T_P)(z,1)\,\nu_z\,.$$

That trace is independent of the defining function, see Remark 1.2.19 iv).
In the context of the Hardy space spectral triple from Section 4.1, we thus have for $T_Q \in GTO^0$ and $\mathcal{D}$ as in Proposition 4.1.1

$$\operatorname{Tr}_{Dix}(T_Q|\mathcal{D}|^{-n}) = \tfrac{1}{n!\,(2\pi)^n}\int_{\partial\Omega}\widetilde{\sigma}(T_Q)(z,1)\,|\sigma(\mathcal{D})(z,1)|^{-n}\,\nu_z\,.$$

In particular, if $T_Q = T_u$, with $u \in C^\infty(\partial\Omega)$, replace the symbol of T_Q in the integrand by u itself.

For the Bergman case, the Dirac operator in Proposition 4.2.1 is of the form $\mathcal{D} = V_m\,T\,V_m^*$, where T is a selfadjoint elliptic GTO of order 1. So from (1.28), we have for any $f \in C^\infty(\overline{\Omega})$

$$\begin{aligned} \operatorname{Tr}_{Dix}(T_f|\mathcal{D}|^{-n}) &= \operatorname{Tr}_{Dix}(V_m\,T_{\Lambda_m}^{-1/2}\,T_{\Lambda_{w_m f}}\,T_{\Lambda_m}^{-1/2}\,V_m^*\,V_m\,|T|^{-n}\,V_m^*) \\ &= \operatorname{Tr}_{Dix}(T_{\Lambda_m}^{-1/2}\,T_{\Lambda_{w_m f}}\,T_{\Lambda_m}^{-1/2}\,|T|^{-n})\,, \end{aligned}$$

which is treated as above.

For $\mathcal{D} = \mathbf{T}_{\mathbf{P}_0}$, with $\mathbf{P}_0$ as in (1.38), we use a similar trick to compute

$$\begin{aligned}\mathrm{Tr}_{Dix}(\mathbf{T}_f\,|\mathcal{D}|^{-n}) &= \mathrm{Tr}_{Dix}\big(V\,(T_\Lambda^{-1/2}T_{\Lambda_f}T_\Lambda^{-1/2})\,V^*\,\mathbf{T}_{\mathbf{P}_0}^{-n}VV^*\big)\\ &= \mathrm{Tr}_{Dix}\big(T_\Lambda^{-1/2}T_{\Lambda_f}T_\Lambda^{-1/2}\,(V^*\mathbf{T}_{\mathbf{P}_0}V)^{-n}\big)\,,\end{aligned}$$

and we get the result from (1.39).

For the triple $(\mathcal{A}', \mathcal{H}', \mathcal{D}')$ from Proposition 4.2.3, the Dixmier traces get multiplied by 2 due to the appearance of 2×2 block matrices. Similarly, a factor n appears in the computation of Dixmier traces for spectral triples involving the Dirac operators $\mathcal{D}_{\mathbb{S}^{2n-1}}$ (4.1) and $\mathcal{D}_{\mathbb{B}^n}$ (4.6), since both contain gamma matrices. From Lemma 1.3.9, we remark that $\mathcal{D}_{\mathbb{S}^{2n-1}} = \sum_{j=1}^n \gamma_j T_{Q_j}$, where T_{Q_j} are GTOs of order 1/2 whose symbols are given by (1.32). Hence $\mathrm{Tr}_{Dix}(|\mathcal{D}_{\mathbb{S}^{2n-1}}|^{-2n}) = \mathrm{Tr}_{Dix}(|\mathcal{D}_{\mathbb{B}^n}|^{-2n})$ is finite and can be computed by using the previous identity.

Finally, note that the operator $\mathcal{D}_{S^{2n}}$, defined over a compact manifold $\partial\Omega$ of complex dimension n, corresponds to the usual Dirac operator $\not{D}$ acting on all $\mathbb{R}^n$ of real dimension n.

As we already mentioned in Remark 1.3.10, an operator of Weyl type of order k over $\mathbb{R}^n$ is unitarily equivalent to a GTO of order $k/2$. According to [149, Theorem 2.7.1], the right order for a Weyl operator over $\mathbb{R}^n$ to be Dixmier-traceable is precisely $-2n$, and the corresponding unitarily equivalent GTO is of order $-n$, as (P13) states.

Actually, this can be recast in the context of noncommutative geometry in the following way: as already said in i) of Remarks i, for a non unital spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, the axiom " $\mathcal{D}$ has a compact resolvent " is replaced by " $\pi(a)(I+\mathcal{D}^2)^{-1}$ is a compact operator for any $a \in \mathcal{A}$ ". For instance, for a spectral triple on $\mathbb{R}^n$ like $\big(\mathcal{S}(\mathbb{R}^n), L^2(\mathbb{R}^n)\otimes C\ell(\mathbb{R}^n), \not{D}\big)$, we get that the operator $(I+\mathcal{D}^2)^{-n/2} = \big((I-\Delta)\otimes I_{C\ell(\mathbb{R}^n)}\big)^{-n/2}$ is not Dixmier-traceable, whereas $\pi(f)(I+\mathcal{D}^2)^{-n/2}$, $f \in \mathcal{S}(\mathbb{R}^n)$ is. Here, the dimension n appears twice: one in the power of $|\mathcal{D}|$ and the other through the algebra of Schwartz functions, via the n variables of f.

4.4 A spectral triple for the Fock space over $\mathbb{C}$

Proposition 4.4.1. *Let $\rho_{\mathfrak{m}} \in \mathscr{S}^{\mathfrak{m}}(\mathbb{C})$, with $\mathfrak{m} > -1$.*
Define $\mathcal{A} := \{\mathscr{T}_f^{(\mathfrak{m})}, f \in \bigcup_{s\leq 0}\mathscr{S}^s\}$, $\mathcal{H} := \mathscr{F}_{\mathfrak{m}}$ and $\mathcal{D} := \mathscr{T}_g^{(\mathfrak{m})}$ where g is a strictly positive measurable function in $\mathscr{S}^1(\mathbb{C}) \subset \mathrm{GLS}^1(\mathbb{C})$.
Then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a regular spectral triple of dimension 2.

Proof. Again, $\mathcal{A}$ is a unital algebra with unit $\mathscr{T}_1^{(\mathfrak{m})} = I$, with involution $* : \mathscr{T}_f^{(\mathfrak{m})} \mapsto \mathscr{T}_{\bar f}^{(\mathfrak{m})}$, and its representation is trivially faithful. From Section 1.3.3, $\mathscr{T}_g^{(\mathfrak{m})}$ is densely defined, self-adjoint and positive on $\mathscr{F}_{\mathfrak{m}}$, hence has an inverse $\mathcal{D}^{-1}$ with the same properties. Combining (1.34) and (1.35), we get

$$\mathcal{D} \approx \big(T_{\rho_{\mathfrak{m}}}^{(\mathfrak{m})}\big)^{-1/2}\,T_{\rho_{\mathfrak{m}}g}^{(\mathfrak{m})}\,\big(T_{\rho_{\mathfrak{m}}}^{(\mathfrak{m})}\big)^{-1/2} = \mathscr{W}_{\mathcal{E}\rho_{\mathfrak{m}}}^{-1/2}\,\mathscr{W}_{\mathcal{E}(\rho_{\mathfrak{m}}g)}\,\mathscr{W}_{\mathcal{E}\rho_{\mathfrak{m}}}^{-1/2} = \mathscr{W}_c\,,$$

where $\mathscr{W}_c$ has order $-\mathfrak{m}/2 + \mathfrak{m} + 1 - \mathfrak{m}/2 = 1$. So there exists a parametrix $\mathscr{W}_{c'}$ of $\mathscr{W}_c$, with c' in $\mathrm{GLS}^{-1}(\mathbb{C})$ such that $\mathcal{D}^{-1} = \mathscr{W}_{c'}$ modulo smoothing operators, which proves that $\mathcal{D}^{-1}$ is compact, hence $\mathcal{D}$ has compact resolvent.

Same argument shows that the commutator $[\,\mathcal{D},\ \mathscr{T}_f^{(\mathfrak{m})}\,]$ is bounded for all $\mathscr{T}_f^{(\mathfrak{m})}$ in $\mathcal{A}$: let f in a certain $\mathscr{S}^s$, $s \leq 0$, we have

$$[\,\mathscr{T}_g^{(\mathfrak{m})},\ \mathscr{T}_f^{(\mathfrak{m})}\,] \approx [\,\mathscr{W}_{\mathcal{E}\rho_{\mathfrak{m}}}^{-1/2} \mathscr{W}_{\mathcal{E}(\rho_{\mathfrak{m}} g)} \mathscr{W}_{\mathcal{E}\rho_{\mathfrak{m}}}^{-1/2},\ \mathscr{W}_{\mathcal{E}\rho_{\mathfrak{m}}}^{-1/2} \mathscr{W}_{\mathcal{E}(\rho_{\mathfrak{m}} f)} \mathscr{W}_{\mathcal{E}\rho_{\mathfrak{m}}}^{-1/2}\,]\,.$$

The left part of the commutator is a complex Weyl operator of order $-\mathfrak{m}/2 + \mathfrak{m} + 1 - \mathfrak{m}/2 = 1$ while the right part has order $-\mathfrak{m}/2 + \mathfrak{m} + s - \mathfrak{m}/2 = s \leq 0$, so the commutator has order less or equal to $1 + s - 1 = s \leq 0$, hence is a bounded operator.
Since $|\mathcal{D}| = \mathcal{D} = \mathscr{T}_g^{(\mathfrak{m})}$, one can verify recursively that for all $k \in \mathbb{N}$ and $\mathscr{T}_f^{(\mathfrak{m})} \in \mathcal{A}$, we get

$$\delta^k(\mathscr{T}_f^{(\mathfrak{m})}) \approx [\,\mathscr{W}_{\mathcal{E}\rho_{\mathfrak{m}}}^{-1/2} \mathscr{W}_{\mathcal{E}(\rho_{\mathfrak{m}} g)} \mathscr{W}_{\mathcal{E}\rho_{\mathfrak{m}}}^{-1/2},\ \mathscr{W}_{\varphi_k}\,]\,,$$

where $\varphi_k \in \mathrm{GLS}^0(\mathbb{C})$, so the commutators are bounded, and the same is true for elements in $[\,\mathcal{D},\ \mathcal{A}\,]$, so the regularity follows.
Since $\mathcal{D}$ is unitarily equivalent to the complex Weyl operator $\mathscr{W}_c$ of order 1 as above, a general result on Weyl operators ([149, Theorem 2.7.1], for $l = 1$) implies that $\mathrm{Tr}\big(|\mathcal{D}|^{-s}\big) = \mathrm{Tr}\big(|\mathscr{W}_c|^{-s}\big)$ is finite for $s > 2$, which leads to the result. □

4.5 Links with quantization

4.5.1 A spectral triple from the Berezin–Toeplitz quantization

It has been shown in [68] that from a noncommutative point of view, the plane $\mathbb{R}^{2n}$ endowed with the Moyal star product $\star_W$ described in Example 2.2.3 is nothing but a non compact noncommutative space, described by the following spectral triple:

$$\big(\mathcal{A} := (\mathcal{S}(\mathbb{R}^{2n}), \star_W), \mathcal{H} := L^2(\mathbb{R}^{2n}) \otimes \mathbb{C}^{2^n}, \mathcal{D} := \not{\partial} = -i\gamma_\mu \partial^\mu\big)\,,$$

with the representation $\pi(f)$ given by the left multiplication $M_f \otimes I_{2^n}$, where $M_f : g \mapsto f \star_W g$ (see [68] for details). Note that the algebra is nonunital and the chosen unitization consists of smooth functions bounded together with all their derivatives. The spectral triple is regular and its spectral dimension is exactly $2n$.

We show in this section that we can similarly build a spectral triple associated to the Berezin–Toeplitz quantization. On one hand, we presented Theorem 2.2.7, which establishes the expression of the star product $\star_{BT}$ by using sequences of Toeplitz operators acting at each level on some weighted Bergman space over the strictly pseudoconvex domain Ω. On the other hand, Proposition 4.2.1 shows that a spectral triple can be constructed at every level. Thus, gluing together this family of spectral triples yields a single " composed " one directly related to the standard Berezin–Toeplitz star product on Ω. Here is the detailed construction.

We assume here that the hypothesis of Theorem 2.2.7 are verified, so that the theory of GTOs over $\partial\Omega^+$ exists. Notations with a superscript " $\oplus$ " concern objects related to the following orthogonal direct sum

$$H^\oplus := \bigoplus_{m=0}^{\infty} A_m^2\,,$$

where A^2_m are the weighted Bergman spaces over Ω with previous weight. Let $\mathbf{\Pi}_m : L^2(\Omega) \to A^2_m$ be the orthogonal projection and for any $f \in C^\infty(\overline{\Omega})$, define the corresponding Toeplitz operator acting on $H^\oplus$

$$\mathbf{T}^\oplus_f := \bigoplus_{m\in\mathbb{N}} \mathbf{T}^{(m)}_f = \bigoplus_{m\in\mathbb{N}} \mathbf{T}_f|_{A^2_m}\,.$$

Clearly each $\mathbf{T}^\oplus_f$ is again bounded with $\|\,\mathbf{T}^\oplus_f\,\| \le \|\,f\,\|_\infty$, $(\mathbf{T}^\oplus_f)^* = \mathbf{T}^\oplus_{\bar f}$, and $[\,\mathbf{T}^\oplus_f, \mathbf{\Pi}_m\,] = 0$ for any $m \in \mathbb{N}$. Denote also the number operator on $H^\oplus$

$$\mathbf{N}^\oplus := \bigoplus_{m\in\mathbb{N}} (m+1)\,\mathbf{\Pi}_m\,.$$

Finally, let $B^\oplus$ be the set of bounded operators $\mathbf{A}^\oplus$ on $H^\oplus$ commuting with $\mathbf{\Pi}_m$ for any $m \in \mathbb{N}$ and such that there exist $f_j \in C^\infty(\overline{\Omega})$ for which

$$\Big\|\mathbf{\Pi}_m\big(\mathbf{A}^\oplus - \sum_{j=0}^{N-1} (\mathbf{N}^\oplus)^{-j}\,\mathbf{T}^\oplus_{f_j}\big)\,\mathbf{\Pi}_m\Big\| \underset{m\to\infty}{=} O(m^{-N})\,, \qquad \text{for any } N \in \mathbb{N}\,, \tag{4.8}$$

or symbolically $\mathbf{A}^\oplus \approx \sum_{j\in\mathbb{N}} (\mathbf{N}^\oplus)^{-j}\,\mathbf{T}^\oplus_{f_j}$. It is the main result of the Berezin–Toeplitz quantization on Ω (Theorem 2.2.7) that finite products of $\mathbf{T}^\oplus_f$ belong to $B^\oplus$. More specifically, one has

$$\mathbf{T}^\oplus_f \mathbf{T}^\oplus_g \approx \sum_{j=0}^{\infty} (\mathbf{N}^\oplus)^{-j}\,\mathbf{T}^\oplus_{C^{BT}_j(f,g)}\,, \tag{4.9}$$

written symbolically $\mathbf{T}^\oplus_f \mathbf{T}^\oplus_g = \mathbf{T}^\oplus_{f\star_{BT} g}$.

Since we handle similar objects as in the end of Section 2.1.3, we keep the same notations. Let K^+ be the Poisson extension operator for Ω^+, and define as before the elliptic selfadjoint operator $\Lambda^+ := (K^+)^* K^+$ in $\Psi\mathrm{DO}^{-1}(\partial\Omega^+)$, acting on $H^2(\partial\Omega^+)$. Since the fibre rotations $(z, s) \mapsto (z, e^{i\theta} s)$, $\theta \in [0, 2\pi)$, preserve holomorphy and harmonicity of functions, the operators K^+, Λ^+ and the Szegö projection Π^+ commute with them. The GTO $T^+_{\Lambda^+}$ on $H^2(\partial\Omega^+)$ therefore likewise commutes with these rotations, and hence commutes also with the projections in $H^2(\partial\Omega^+)$ onto $H^{(m)}$, i.e. is diagonal in the decomposition $H^+(\partial\Omega^+) = \bigoplus_{m\in\mathbb{N}} H^{(m)}$. Let $\mathbf{L}^\oplus := \bigoplus_{m\in\mathbb{N}} \mathbf{L}_m$ be the operator corresponding to $T^+_{\Lambda^+}$ under the isomorphism (2.25).

Proposition 4.5.1. *Let $\mathcal{A}^\oplus$ be the algebra (no closures taken) generated by the $\mathbf{T}^\oplus_f$, where $f \in C^\infty(\overline{\Omega})$, acting (via identity representation) on $\mathcal{H}^\oplus := H^\oplus$ and $\mathcal{D}^\oplus := (\mathbf{L}^\oplus)^{-1}$.*
Then $(\mathcal{A}^\oplus, \mathcal{H}^\oplus, \mathcal{D}^\oplus)$ is a regular spectral triple of dimension $n+1$.

Proof. Using the above isomorphisms, we can actually switch from $H^\oplus$ to the space $H^2(\partial\Omega^+)$, from $\mathcal{A}^\oplus$ to the algebra generated by $T^+_{\tilde f} \in \mathrm{GTO}(\partial\Omega^+)$, $f \in C^\infty(\overline{\Omega})$, and from $\mathcal{D}$ to $(T^+_{\Lambda^+})^{-1}$. Everything then follows in exactly the same way as in Section 4.1, noting that the complex dimension of Ω^+ is $n+1$. $\square$

So we have constructed a spectral triple using the operators $\mathbf{T}^\oplus_f$, which are known to induce the Berezin–Toeplitz star product (4.9) over Ω. In the spirit of deformation quantization and the phase space formulation of quantum mechanics, we are now looking for a spectral triple whose algebra is (some subalgebra of) $(C^\infty(\overline{\Omega})[[\hbar]], \star_{BT})$, taking for the representation $f \mapsto \mathbf{T}^\oplus_f$.

More specifically, consider the linear map κ from $B^{\oplus}$ to $C^{\infty}(\overline{\Omega})[[m^{-1}]]$ (the latter algebra equipped with the usual involution $\big(\sum_{j\in\mathbb{N}} m^{-j}\, f_j(z)\big)^* := \sum_{j\in\mathbb{N}} m^{-j}\, \overline{f_j(z)}$) defined by

$$\kappa : \mathbf{A}^{\oplus} \in B^{\oplus} \mapsto \sum_{j=0}^{\infty} m^{-j} f_j(z) \ \text{ for } \mathbf{A}^{\oplus} \text{ as in (4.8).}$$

As noted previously, κ is well defined owing to the convergence $\|\,\mathbf{\Pi}_m \mathbf{T}_f^{(m)}\,\| \to \|\,f\,\|_\infty$ as m tends to infinity, although it is not injective. Extending as usual $\star_{BT}$ from $C^{\infty}(\overline{\Omega})$ to all of $C^{\infty}(\overline{\Omega})[[m^{-1}]]$ by $\mathbb{C}[[m^{-1}]]$-linearity, we get

$$\kappa(\mathbf{A}_1^{\oplus}\mathbf{A}_2^{\oplus}) = \kappa(\mathbf{A}_1^{\oplus}) \star_{BT} \kappa(\mathbf{A}_2^{\oplus})\,, \ \text{ and } \quad \kappa[(\mathbf{A}^{\oplus})^*] = [\kappa(\mathbf{A}^{\oplus}]^*\,,$$

which make $\kappa : (B^{\oplus}, \circ) \to (C^{\infty}(\overline{\Omega})[[m^{-1}]], \star_{BT})$ a $*$-algebra homomorphism.
Then we have the following:

Theorem 4.5.2. *Let $\mathcal{A}^{\oplus}$ be the polynomial subalgebra over $\mathbb{C}[[m^{-1}]]$ of $(C^{\infty}(\overline{\Omega})[[m^{-1}]], \star_{BT})$ generated by $\kappa(\mathbf{T}_f^{\oplus})$, $f \in C^{\infty}(\overline{\Omega})$ endowed with the representation $\pi^{\oplus}$ on $\mathcal{H}^{\oplus} := H^{\oplus}$ defined as*

$$\pi^{\oplus}(m^{-j} f) := (\mathbf{N}^{\oplus})^{-j}\, \mathbf{T}_f^{\oplus}\,, \quad f \in C^{\infty}(\overline{\Omega}), \quad j \in \mathbb{N}\,, \tag{4.10}$$

which is well-defined from $\mathcal{A}^{\oplus}$ into $B^{\oplus}$, and $\mathcal{D}^{\oplus} := \bigoplus_{m\in\mathbb{N}} \mathbf{L}_m^{-1}$ on $\mathcal{H}^{\oplus}$.
Then $(\mathcal{A}^{\oplus}, \mathcal{H}^{\oplus}, \mathcal{D}^{\oplus})$ is a regular spectral triple of dimension $n+1$.

Proof. In view of previous results, the only thing we need to check is that $\pi^{\oplus}$ is well-defined and faithful. The former is immediate from (4.10) and the fact that κ is a $*$-algebra homomorphism. For the faithfulness, note that $\kappa \circ \pi^{\oplus} = 1$ on $\mathcal{A}^{\oplus}$; thus $\pi^{\oplus}(a) = 0$ implies $a = \kappa\big(\pi^{\oplus}(a)\big) = 0$. □

Again, proceeding as in Proposition 4.2.3, one can adjoin to the last construction an appropriate unitary GTO on $\partial\Omega^+$ to obtain non-positive operators $\widetilde{\mathcal{D}}$ (see Remark 4.2.4).

Some interesting remarks on the spectral dimension can be made. First, notice that in the noncommutative Moyal plane mentioned in Example 2.2.3, the spectral dimension matches the real dimension of the domain $\mathbb{R}^{2n}$, while in our result, the corresponding noncommutative space has an extra complex dimension compared to the initial domain Ω. This “ $n+1$ ” phenomenon is mathematically clear by paying attention to the proof: the considered algebra corresponds to a direct sum of Toeplitz operators, which is unitarily equivalent to a subalgebra of GTOs acting on the boundary of the strictly pseudoconvex manifold Ω^+ of complex dimension $n+1$. In other words, the direct sum in Proposition 4.5.1, which does not appear in the Moyal plane, is responsible for the extra dimension, and reflects the presence of the underlying disk bundle.

Let us also remark that we have a spectral triple at each level $m \in \mathbb{N}$ of $(\mathcal{A}^{\oplus}, \mathcal{H}^{\oplus}, \mathcal{D}^{\oplus})$. Indeed, the algebra $\mathcal{A}^{\oplus}$ is generated by a sequence of algebras $(\mathcal{A}_m \ni \mathbf{T}_f^{(m)})_{m\in\mathbb{N}}$, while $\mathcal{H}^{\oplus}$ and $\mathcal{D}^{\oplus}$ are obtained after direct summation of $(\mathcal{H}_m := A_m^2)_{m\in\mathbb{N}}$ and $(\mathcal{D}_m := \mathbf{L}_m^{-1})_{m\in\mathbb{N}}$ respectively. Since $\mathcal{D}^{\oplus} \approx (T_{\Lambda^+}^{+})^{-1}$ is a compact operator, $\mathcal{D}_m^{-1}$ is compact on $\mathcal{H}_m$ so $\mathcal{D}_m$ has compact resolvent for any $m \in \mathbb{N}$. Moreover for any $m \in \mathbb{N}$, $[\,\mathcal{D}_m^{-1},\ \mathbf{T}_f^{(m)}\,]$ is bounded since $[\,\mathcal{D}^{\oplus},\ \mathbf{T}_f^{\oplus}\,]$ is bounded, which makes $(\mathcal{A}_m, \mathcal{H}_m, \mathcal{D}_m)_{m\in\mathbb{N}}$ a family of spectral triples.

Remark 4.5.3. *Parametrized spectral triples involving Toeplitz algebras have also been investigated in [33], but the approach are different. An operator $\mathcal{D}_{\alpha,\beta}$, depending on two real positive parameters α*

and β, induces a family of spectral triples. As $\alpha \to 0$ (with $\beta = 1$), the spectral triple tends to reveal the infinitesimal structure of the underlying noncommutative space (i.e. sees only compact operators). On the other hand, in the case of a commutative algebra, the limit $\beta \to 0$ (with $\alpha = 1$) can be understood as a " passage from a noncommutative compact metric space into a commutative compact metric space ". This can be seen as a formulation of the semi-classical limit discussed in Section 2.1.3 in the context of noncommutative geometry. Compared to our construction, the difference lies in the fact that we do not study any convergence of the family of spectral triples $(\mathcal{A}_m, \mathcal{H}_m, \mathcal{D}_m)_{m\in\mathbb{N}}$ when $m \to \infty$, but concatenate all of them in order to get a new one.

4.5.2 (Des)Integration of spectral triples

4.5.2.1 Conditions for integrability

The spectral triple of Proposition 4.5.1 can be *desintegrated* into a family of spectral triples $(\mathcal{A}_m, \mathcal{H}_m, \mathcal{D}_m)_{m\in\mathbb{N}}$. More generally if an abstract spectral triple $(\mathcal{A}^\oplus, \mathcal{H}^\oplus, \mathcal{D}^\oplus)$ [(1)] can be decomposed in this way, the induced triplets $(\mathcal{A}_m, \mathcal{H}_m, \mathcal{D}_m)_{m\in\mathbb{N}}$ are necessarily spectral triples. If this implication is easy to verify, the converse is slightly more subtle: as the following result states, a family of spectral triples must verify some (quite restrictive) conditions if we want the sum to be a spectral triple again. We now present these *integrability* conditions.

Proposition 4.5.4. *Let $(\mathcal{A}_m, \mathcal{H}_m, \mathcal{D}_m)_{m\in\mathbb{N}}$ be a family of (not necessarily unital) spectral triples, with corresponding representations $(\pi_m)_{m\in\mathbb{N}}$, and denote $\|\,.\,\|_m$ the norm on $\pi_m(\mathcal{A}_m)$.*
Let $(\beta_m)_{m\in\mathbb{N}}$ be a sequence of non-zero real numbers such that

$$\| (1+\beta_m^2\mathcal{D}_m^2)^{-1/2} \|_m \underset{m\to+\infty}{\longrightarrow} 0\,. \tag{4.11}$$

Define the following objects:

- $\mathcal{H}^\oplus := \bigoplus_{m\in\mathbb{N}} \mathcal{H}_m$,
- $\mathcal{D}^\oplus := \bigoplus_{m\in\mathbb{N}} \beta_m \mathcal{D}_m$, *acting on* $\mathcal{H}^\oplus$,
- $\mathcal{A}^\oplus := \{(a_m)_{m\in\mathbb{N}} \in \prod_{m\in\mathbb{N}} \mathcal{A}_m : \sup_{m\in\mathbb{N}} \| \pi_m(a_m) \|_m < +\infty,$ *and* $\sup_{m\in\mathbb{N}} \| [\beta_m\mathcal{D}_m, \pi_m(a_m)] \|_m < +\infty\}$,
- $\pi^\oplus(a^\oplus) := \bigoplus_{m\in\mathbb{N}} \pi_m(a_m)$ *for* $a^\oplus \in \mathcal{A}^\oplus$.

Then $(\mathcal{A}^\oplus, \mathcal{H}^\oplus, \mathcal{D}^\oplus)$ is a (not necessarily unital) spectral triple.

Remark 4.5.5.

i) Equation (4.11) is equivalent to the compactness of the resolvent of $\mathcal{D}^\oplus$ (recall that an operator $\bigoplus_{m\in\mathbb{N}} A_m$ is compact if and only if A_m is compact for any $m \in \mathbb{N}$ and $\| A_m \|_m \to 0$ as $m \to \infty$; see [34, Exercice II.4.13]). As a consequence, the sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is such that

$$\sum_{m\in\mathbb{N}} \dim\left(\operatorname{Ker} \mathcal{D}_m\right) < \infty\,.$$

In particular, if we take the same $\mathcal{D}_m = \mathcal{D}_0$ at each level $m \in \mathbb{N}$, the latter must be invertible.

[(1)]We keep this notation to underline the fact that the triple can be desintegrated.

ii) The two conditions in the definition of $\mathcal{A}^\oplus$ correspond to the boundedness of both the representation $\pi^\oplus$ and the commutator $[\,\mathcal{D}^\oplus,\, \pi^\oplus(\mathcal{A}^\oplus)\,]$ for the norm $\|\,.\,\|^\oplus := \sup_{m\in\mathbb{N}} \|\,.\,\|_m$ on $\pi^\oplus(\mathcal{A}^\oplus)$.

iii) We chose to add the additional parameter $(\beta_m)_{m\in\mathbb{N}}$ in order to control the behaviour of the sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ as m tends to infinity. This can be avoided by putting some restrictions directly on the operators $\mathcal{D}_m$, but this restricts the set of summable families of spectral triples. Indeed, in the case when $\mathcal{D}^\oplus := \bigoplus_{m\in\mathbb{N}} \mathcal{D}_0$, with $\mathcal{D}_0$ invertible, then the resolvent of $\mathcal{D}^\oplus$ is not compact and the use of a sequence $(\beta_m)_{m\in\mathbb{N}}$ is necessary for the integration.
An alternative would consist in rescaling the norm $\|\,.\,\|_m$ at each level by multiplying it by the term β_m and set $\mathcal{D}^\oplus$ as the simple direct sum of all $\mathcal{D}_m$. This rescaling mechanism of the norm has already been encountered in Section 2.1.3.

Proof. For two elements $a^\oplus = (a_m)_{m\in\mathbb{N}}$ and $b^\oplus = (b_m)_{m\in\mathbb{N}}$ in $\mathcal{A}^\oplus$, we have:

$$\sup_{m\in\mathbb{N}} \|\,\pi_m(a_m b_m)\,\|_m \leq \sup_{m\in\mathbb{N}} \|\,\pi_m(a_m)\,\|_m \,\sup_{m\in\mathbb{N}} \|\,\pi_m(b_m)\,\|_m < +\infty\,, \qquad \text{and}$$

$$\begin{aligned}\sup_{m\in\mathbb{N}} \|\,[\,\beta_m\mathcal{D}_m,\, \pi_m(a_m b_m)\,]\,\|_m \leq{}& \sup_{m\in\mathbb{N}} \|\,\pi_m(a_m)\,\|_m \,\sup_{m\in\mathbb{N}} \|\,[\,\beta_m\mathcal{D}_m,\, \pi_m(b_m)\,]\,\|_m \\ &+ \sup_{m\in\mathbb{N}} \|\,[\,\beta_m\mathcal{D}_m,\, \pi_m(a_m)\,]\,\|_m \,\sup_{m\in\mathbb{N}} \|\,\pi_m(b_m)\,\|_m < +\infty\,,\end{aligned}$$

hence $\mathcal{A}^\oplus$ is an algebra with involution $* : a^\oplus = (a_m)_{m\in\mathbb{N}} \mapsto (a^\oplus)^* := (a_m^*)_{m\in\mathbb{N}}$.
The operator $\mathcal{D}^\oplus$ is selfadjoint and we have for $a^\oplus \in \mathcal{A}^\oplus$

$$\pi^\oplus(a^\oplus)\,\big(1 + (\mathcal{D}^\oplus)^2\big)^{-1/2} = \bigoplus_{m\in\mathbb{N}} \pi_m(a_m)\,(1 + \beta_m^2\,\mathcal{D}_m^2)^{-1/2}.$$

For any $m \in \mathbb{N}$, the summand $\pi_m(a_m)\,(1 + \beta_m^2\,\mathcal{D}_m^2)^{-1/2}$ is compact. From (4.11) and the fact that $\pi^\oplus$ is a bounded representation, $\|\,\pi_m(a_m)\,(1 + \beta_m^2\,\mathcal{D}_m^2)^{-1/2}\,\|_m$ tends to 0 as $m \to +\infty$. As a consequence, $\pi^\oplus(a^\oplus)\,\big(1 + (\mathcal{D}^\oplus)^2\big)^{-1/2}$ is compact.
By definition of $\mathcal{A}^\oplus$, the commutator $[\,\mathcal{D}^\oplus,\, \pi^\oplus(a^\oplus)\,] = \bigoplus_{m\in\mathbb{N}}[\,\beta_m\mathcal{D}_m,\, \pi_m(a_m)\,]$ is bounded. □

4.5.2.2 Example

The following result gives an explicit example of integration of spectral triples.
Consider the case $\Omega = \mathbb{B}^n$ with defining function $r(z) = |z|^2 - 1$ and a weight of the form (1.4) with $\chi = 1$.
Denote the operators $\boldsymbol{\mathcal{R}} := \sum_{j=1}^n \boldsymbol{\mathcal{R}}_j$ and $\overline{\boldsymbol{\mathcal{R}}} := \sum_{j=1}^n \overline{\boldsymbol{\mathcal{R}}}_j$ with $\boldsymbol{\mathcal{R}}_j := z_j\partial_{z_j}$ and $\overline{\boldsymbol{\mathcal{R}}}_j := \bar{z}_j\,\partial_{\bar{z}_j}$, acting on $C^\infty(\overline{\mathbb{B}^n})$ ($\boldsymbol{\mathcal{R}}$ differ from the $\mathbf{R}$ from Lemma 1.3.9 which is defined only on the Bergman spaces). Let $\mathrm{Pol}(\mathbb{B}^n)$ be the set of polynomials on $\mathbb{B}^n$ in z and $\bar{z}$.

Theorem 4.5.6. *For $m \in \mathbb{N}$, let*

- $\mathcal{H}_m := A_m^2(\mathbb{B}^n)$,
- $\mathcal{D}_m := (\mathbf{T}_r^{-1})^{(m)}$,
- *$\mathcal{A}_m$ be the *-algebra generated by Toeplitz operators $\mathbf{T}_p^{(m)}$ acting on $\mathcal{H}_m$, with $p \in \mathrm{Pol}(\mathbb{B}^n)$,*
- *π_m be the identity representation on $\mathcal{H}_m$ and $\|\,.\,\|_m$ as the usual norm of operators,*

- $\beta_m := m + 1$, $m \in \mathbb{N}$.

If we define $\mathcal{H}^\oplus$, $\mathcal{D}^\oplus$, $\pi^\oplus$ as in Proposition 4.5.4 and $\mathcal{A}'^\oplus$ as the algebra generated by elements of the form $(\mathbf{T}_p^{(m)})_{m\in\mathbb{N}}$, with $p \in \mathrm{Pol}(\mathbb{B}^n)$ (i.e. keeping the same polynomial at all level $m \in \mathbb{N}$), then $(\mathcal{A}'^\oplus, \mathcal{H}^\oplus, \mathcal{D}^\oplus)$ is a spectral triple of dimension $n+1$.

Proof. First, we know from Section 4.2 that for any $m \in \mathbb{N}$, $(\mathcal{A}_m, \mathcal{H}_m, \mathcal{D}_m)$ defines a spectral triple. Moreover,

$$\begin{aligned}\| \,(1+\beta_m^2\,\mathcal{D}_m^2)^{-1/2}\,\|_m = \|\,(1+\beta_m^2\,(\mathbf{T}_r^{(m)})^{-2})^{-1/2}\,\|_m \leq |\beta_m|^{-1} \|\,\mathbf{T}_r^{(m)}\,\|_m \\ \leq |\beta_m|^{-1}\|\,r\,\|_\infty \underset{m\to+\infty}{\longrightarrow} 0\,.\end{aligned}$$

Let us show that $\mathcal{A}'^\oplus$ is a subalgebra of $\mathcal{A}^\oplus$ of Proposition 4.5.4: if $(a_m)_{m\in\mathbb{N}} = (\mathbf{T}_p^{(m)})_{m\in\mathbb{N}}$ of $\mathcal{A}'^\oplus$, $p \in \mathrm{Pol}(\mathbb{B}^n)$, is a generator, the conditions are satisfied since

$$\sup_{m\in\mathbb{N}} \|\,\pi_m(a_m)\,\|_m \leq \|\,p\,\|_\infty < +\infty \quad \text{and from the next Proposition 4.5.7,}$$

$$\sup_{m\in\mathbb{N}} \|\,[\,\beta_m\mathcal{D}_m,\,\pi_m(a_m)\,]\,\|_m = \sup_{m\in\mathbb{N}} \tfrac{m+1}{m+1}\|\,\mathbf{T}^{(m)}_{(\boldsymbol{\mathcal{R}}-\overline{\boldsymbol{\mathcal{R}}})p}\,\| \leq \|\,(\boldsymbol{\mathcal{R}}-\overline{\boldsymbol{\mathcal{R}}})\,p\,\|_\infty < +\infty\,.$$

These inequalities remain valid for a general element of $\mathcal{A}'^\oplus$, which is composed, at each level $m \in \mathbb{N}$, by the same finite sum of finite products of Toeplitz operators acting on A_m^2. Since $\mathcal{A}'^\oplus$ form a *-algebra, we conclude that it is a *-subalgebra of $\mathcal{A}^\oplus$ and from Proposition 4.5.4, $(\mathcal{A}'^\oplus, \mathcal{H}^\oplus, \mathcal{D}^\oplus)$ is a spectral triple. We now compute its dimension. For $s \in \mathbb{R}$, we have

$$\begin{aligned}\mathrm{Tr}\,|\mathcal{D}^\oplus|^{-s} &= \sum_{m\in\mathbb{N}} \beta_m^{-s}\,\mathrm{Tr}\,(\mathbf{T}_r^{(m)})^s = \sum_{m\in\mathbb{N}} (\tfrac{\beta_m}{m+1})^{-s}\,\mathrm{Tr}\,(\boldsymbol{\mathcal{R}}+m+n+1)^{-s} \\ &= \sum_{m\in\mathbb{N}} \mathrm{Tr}\,(\boldsymbol{\mathcal{R}}+m+n+1)^{-s}\,.\end{aligned}$$

Denoting $\lambda_k(m) := k+m+n+1$ the eigenvalues of $\boldsymbol{\mathcal{R}}+m+n+1$, and $M_k := \binom{k+n-1}{n-1}$ the corresponding multiplicities, we get

$$\mathrm{Tr}\,\,(\boldsymbol{\mathcal{R}}+m+n+1)^{-s} = \sum_{k=0}^{\infty} M_k\,\lambda_k(m)^{-s} =: I_m(s)\,. \tag{4.12}$$

We know from Section 4.2 that $I_m(s)$ is finite for $s > n$. So when $s = n+\varepsilon$, $\varepsilon > 0$, we can estimate the asymptotic behaviour of this quantity as $m \to \infty$ using Lemma 4.5.9, and so $\mathrm{Tr}\,|\mathcal{D}^\oplus|^{-(n+\varepsilon)}$ is finite when

$$\sum_{m\in\mathbb{N}} (\tfrac{m+1}{\beta_m})^{n+\varepsilon}\,m^{-\varepsilon} < +\infty\,, \tag{4.13}$$

i.e. when $\varepsilon > 1$, which leads to the result. □

Proposition 4.5.7. *For $p(z) = \sum_{|\alpha|\leq d, |\beta|\leq d'} p_{\alpha\beta}\,z^\alpha\,\bar{z}^\beta \in \mathrm{Pol}(\mathbb{B}^n)$, and denoting briefly $\mathbf{T}_p = \mathbf{T}_p^{(m)}$, we get*

$$[\,\mathbf{T}_r^{-1},\,\mathbf{T}_p\,] = \tfrac{1}{m+1}\,\mathbf{T}_{(\boldsymbol{\mathcal{R}}-\overline{\boldsymbol{\mathcal{R}}})\,p}\,, \qquad \text{on } A_m^2(\mathbb{B}^n)\,.$$

Proof. By [157, (2.9)], a standard orthonormal basis of $A^2_m(\mathbb{B}^n)$ is given by

$$v_\alpha(z) = \big(\tfrac{\Gamma(|\alpha|+m+n+1)}{\Gamma(m+n+1)\,\alpha!}\big)^{1/2}\, z^\alpha =: b_\alpha z^\alpha.$$

Using the shift operators $\boldsymbol{S}_j : v_\alpha \mapsto v_{\alpha+1_j}$, $j = 1,\dots,n$, we have the relations

$$\mathbf{T}_{z_j} = \boldsymbol{S}_j\,\big(\tfrac{\boldsymbol{\mathcal{R}}_j+1}{\boldsymbol{\mathcal{R}}+m+n+1}\big)^{1/2}, \quad \boldsymbol{\mathcal{R}}_j\boldsymbol{S}_j = \boldsymbol{S}_j(\boldsymbol{\mathcal{R}}_j+1)\,, \quad \boldsymbol{S}_j^*\boldsymbol{S}_j = 1\,, \quad \text{for } j=1,\dots,n, \quad \text{and}$$

$$\begin{aligned}\mathbf{T}_r^{-1} &= (1-\sum_{j=1}^n \mathbf{T}_{|z_j|^2})^{-1} = (1-\sum_{j=1}^n (\mathbf{T}_{z_j})^*\mathbf{T}_{z_j})^{-1} = (1-\sum_{j=1}^n \tfrac{\boldsymbol{\mathcal{R}}_j+1}{\boldsymbol{\mathcal{R}}+m+n+1})^{-1}\\ &= \tfrac{1}{m+1}(\boldsymbol{\mathcal{R}}+m+n+1)\,.\end{aligned}$$

Hence we get

$$\begin{aligned}[\,\mathbf{T}_r^{-1},\,\mathbf{T}_{z_j}\,] &= \tfrac{1}{m+1}\,\big((\boldsymbol{\mathcal{R}}+m+n+1)\,\boldsymbol{S}_j\,\big(\tfrac{\boldsymbol{\mathcal{R}}_j+1}{\boldsymbol{\mathcal{R}}+m+n+1}\big)^{1/2} - \boldsymbol{S}_j\,\big(\tfrac{\boldsymbol{\mathcal{R}}_j+1}{\boldsymbol{\mathcal{R}}+m+n+1}\big)^{1/2}(\boldsymbol{\mathcal{R}}+m+n+1)\big)\\ &= \tfrac{1}{m+1}\,\boldsymbol{S}_j\,\big(\tfrac{\boldsymbol{\mathcal{R}}_j+1}{\boldsymbol{\mathcal{R}}+m+n+1}\big)^{1/2}\,\big(\boldsymbol{\mathcal{R}}+m+n+2-(\boldsymbol{\mathcal{R}}+m+n+1)\big)\\ &= \tfrac{1}{m+1}\,\mathbf{T}_{z_j}\,.\end{aligned}$$

From this last equality, the fact that $[\,\mathbf{T}_{z_j},\,\mathbf{T}_{z_k}\,] = 0$, for $j,k = 1,\dots,n$, we get by iteration of the formula $[\,A,\,BC\,] = B[\,A,\,C\,] + [\,A,\,B\,]C$ for $\alpha,\beta\in\mathbb{N}^n$

$$[\,\mathbf{T}_r^{-1},\,\prod_{j=1}^n \mathbf{T}_{z_j}^{\alpha_j}\,] = \tfrac{|\alpha|}{m+1}\,\prod_{j=1}^n \mathbf{T}_{z_j}^{\alpha_j}\,, \quad \text{and similarly} \quad [\,\mathbf{T}_r^{-1},\,\prod_{j=1}^n (\mathbf{T}_{z_j}^*)^{\beta_j}\,] = -\tfrac{|\beta|}{m+1}\,\prod_{j=1}^n (\mathbf{T}_{z_j}^*)^{\beta_j}\,.$$

Hence, the relation $\mathbf{T}_{z^\alpha\bar z^\beta} = \big(\prod_{j=1}^n (\mathbf{T}_{z_j}^*)^{\beta_j}\big)\big(\prod_{j=1}^n \mathbf{T}_{z_j}^{\alpha_j}\big)$ yields to

$$\begin{aligned}[\,\mathbf{T}_r^{-1},\,\mathbf{T}_p\,] &= \sum_{|\alpha|\le d,|\beta|\le d'} [\,\mathbf{T}_r^{-1},\,\mathbf{T}_{z^\alpha\bar z^\beta}\,] = \sum_{|\alpha|\le d,|\beta|\le d'} [\,\mathbf{T}_r^{-1},\,\big(\prod_{j=1}^n (\mathbf{T}_{z_j}^*)^{\beta_j}\big)\big(\prod_{j=1}^n \mathbf{T}_{z_j}^{\alpha_j}\big)\,]\\ &= \sum_{|\alpha|\le d,|\beta|\le d'} p_{\alpha\beta}\Big(\big(\prod_{j=1}^n (\mathbf{T}_{z_j}^*)^{\beta_j}\big)[\,\mathbf{T}_r^{-1},\,\prod_{j=1}^n \mathbf{T}_{z_j}^{\alpha_j}\,] + [\,\mathbf{T}_r^{-1},\,\prod_{j=1}^n (\mathbf{T}_{z_j}^*)^{\beta_j}\,]\prod_{j=1}^n \mathbf{T}_{z_j}^{\alpha_j}\Big)\\ &= \tfrac{1}{m+1}\sum_{|\alpha|\le d,|\beta|\le d'} p_{\alpha\beta}\,(|\alpha|-|\beta|)\big(\prod_{j=1}^n (\mathbf{T}_{z_j}^*)^{\beta_j}\big)\big(\prod_{j=1}^n \mathbf{T}_{z_j}^{\alpha_j}\big)\\ &= \tfrac{1}{m+1}\sum_{|\alpha|\le d,|\beta|\le d'} p_{\alpha\beta}\,(|\alpha|-|\beta|)\,\mathbf{T}_{z^\alpha\bar z^\beta}\\ &= \tfrac{1}{m+1}\,\mathbf{T}_{(\boldsymbol{\mathcal{R}}-\overline{\boldsymbol{\mathcal{R}}})\,p}\,.\end{aligned}$$

□

Remark 4.5.8. *The previous result is restricted to polynomials only. Indeed, we cannot apply the Stone–Weierstrass theorem to extend the result for general smooth functions over* $\overline{\mathbb{B}^n}$ *since the map* $f\mapsto[\,\mathbf{T}_r^{-1},\,\mathbf{T}_f\,]$ *is not continuous on* A^2_m *for the norm* $\|\,.\,\|_\infty$.

Lemma 4.5.9. *With the notations of* (4.12), *we have* $I_m(n+\varepsilon) \underset{m\to\infty}{\sim} (n-1)!\,\frac{\Gamma(\varepsilon)}{\Gamma(\varepsilon+n)}\,m^{-\varepsilon}$.

Proof. First, if $n\ge 2$, we can express M_k as a polynomial in $\lambda_k(m)$:

$$\begin{aligned}M_k &= \tfrac{1}{(n-1)!}\,(k+n-1)(k+n-2)\cdots(k+1) = \tfrac{1}{(n-1)!}\,\big(\lambda_k(m)-(m+2)\big)_{(n-1)}\\ &= \tfrac{1}{(n-1)!}\sum_{j=1}^{n-1} s(n-1,j)\,\big(\lambda_k(m)-(m+2)\big)^j = \tfrac{1}{(n-1)!}\sum_{l=0}^{n-1} c_l(m)\,\lambda_k(m)^l,\end{aligned}$$

where we used the Pochhammer symbols and $s(a,b)$ are the Stirling numbers of first kind and

$$c_l(m) := \sum_{i=l}^{n-1} \tbinom{i}{l} s(n-1,i)\big(-(m+2)\big)^{i-l}, \quad l = 0, \cdots, n-1\,.$$

So we have $I_m(n+\varepsilon) = \frac{1}{(n-1)!}\sum_{k=0}^{\infty} g_m(k)$, where

$$g_m(k) := \sum_{l=0}^{n-1} c_l(m)\,\lambda_k(m)^{l-(n+\varepsilon)}.$$

We use the Euler–Maclaurin formula on g_m:

$$\begin{aligned}\sum_{k\in\mathbb{N}} g_m(k) &= \int_0^{+\infty} g_m(x)\,dx + \tfrac{1}{2}\big(g_m(0) + \lim_{x\to+\infty} g_m(x)\big)\\ &\qquad + \sum_{j=2}^{N} \tfrac{B_j}{j!} \lim_{x\to+\infty} \big(\partial_x^{j-1} g_m(x) - \partial_x^{j-1} g_m(0)\big) + R(m)\\ &=: T_1(m) \qquad + T_2(m) \qquad + T_3(m) \qquad + R(m),\end{aligned}$$

where B_j is the j^{th} Bernoulli number and $R(m) := \frac{(-1)^{N+1}}{N!}\int_0^\infty \partial_x^N g_m(x)\, b_N(x - \lfloor x\rfloor)\,dx$, and b_N being the N^{th}-Bernouilli polynomial. We get

$$\begin{aligned}T_1(m) &= \int_0^\infty g_m(x)\,dx = \sum_{l=0}^{n-1} c_l(m)\int_0^\infty (x+m+n+1)^{l-(n+\varepsilon)}\,dx\\ &= -\sum_{l=0}^{n-1} c_l(m)\,\tfrac{1}{l-(n+\varepsilon)+1}\,(m+n+1)^{l-(n+\varepsilon)+1}\\ &= -\sum_{l=0}^{n-1}\sum_{i=l}^{n-1} \tbinom{i}{l} s(n-1,i)\big(-(m+2)\big)^{i-l}\tfrac{1}{l-(n+\varepsilon)+1}(m+n+1)^{l-(n+\varepsilon)+1}\\ &\underset{m\to\infty}{\sim} \sum_{l=0}^{n-1}\tbinom{n-1}{l}\,\tfrac{(-1)^{n-l}}{l-(n+\varepsilon)+1}\,m^{-\varepsilon} = (n-1)!\,\tfrac{\Gamma(\varepsilon)}{\Gamma(\varepsilon+n)}\,m^{-\varepsilon}\,, \qquad \text{and}\\ T_2(m) &= \tfrac{1}{2}g(0) = \tfrac{1}{2}\sum_{l=0}^{n-1}\sum_{i=l}^{n-1} s(n-1,i)\big(-(m+2)\big)^{i-l}\,(m+n+1)^{l-(n+\varepsilon)} = O(m^{-(\varepsilon+1)})\,.\end{aligned}$$

Since, for $j \geq 2$,

$$\partial_x^{j-1} g_m(x) = \sum_{l=0}^{n-1} c_l(m)(l-(n+\varepsilon))_{(j-1)}\,(x+m+n-1)^{l-(n+\varepsilon)-(j-1)}, \tag{4.14}$$

we get, for $N \geq 2$, $T_3(m) = O(m^{-(\varepsilon+2)})$. We have the following upper bound for the rest (obtained by computing the Fourier series of the Bernouilli polynomial b_N)

$$|R(m)| \leq \tfrac{2}{(2\pi)^N}\,\zeta(N)\int_0^{+\infty} |g_m^{(N)}|(x)\,dx\,,$$

which gives, using (4.14), after integration over x for $N \geq 2$,

$$\begin{aligned}|R(m)| &\leq \tfrac{2}{(2\pi)^N}\,\zeta(N)\sum_{l=0}^{n-1} |c_l(m)||(l-n+\varepsilon)_{(N)}|(N-1+n+\varepsilon-l)^{-1}\,(m+n+1)^{l-(n+\varepsilon)-N+1}\\ &= O(m^{-(\varepsilon+N-1)}) = O(m^{-(\varepsilon+2)})\,.\end{aligned}$$

□

Remarks 4.5.10.

i) *A possible extension of Theorem 4.5.6, in which any $(a_m)_{m\in\mathbb{N}} \in \mathcal{A}'^{\oplus}$ is defined as the copy of the same element on each level $m \in \mathbb{N}$, consists of replacing a finite number of a_m by arbitrary elements of $\mathcal{A}_m$.*
Thus the representation of an element $a^{\oplus}$ of this new algebra $\mathcal{A}''^{\oplus}$ is of the form

$$\pi''^{\oplus}(a^{\oplus}) = \bigoplus_{m\leq N} \pi_m(a_m) \oplus \bigoplus_{m>N} \sum_{i=1}^{p} \prod_{j=1}^{q_i} \mathbf{T}^{(m)}_{p_{ij}}\,,$$

for some integer N, some arbitrary $a_m \in \mathcal{A}_m$, $m \leq N$, and fixed family of polynomials p_{ij} in $\mathrm{Pol}(\mathbb{B}^n)$, $i = 1, \dots, p$, $j = 1, \dots, q_i$.

ii) *In Theorem 4.5.6, we can consider a more general sequence $(\beta_m)_{m\in\mathbb{N}}$ such that $\beta_m \sim m^{\lambda}$, as $m \to \infty$, for $0 < \lambda \leq 1$ (the upper bound comes from the boundedness of the commutator between the representation of an element of the algebra and $\mathcal{D}^{\oplus}$). Then, the conclusions of Proposition 4.5.4 remain valid but the dimension changes: in this case, (4.13) is true when $\varepsilon > \big(1 + n(1-\lambda)\big)/\lambda$. As a consequence, making λ varying in $(0, 1]$ leads to a dimension lying in $[n + 1, +\infty)$.*

Conclusion and perspectives

The results presented in this thesis show that we have met the initial goals.

Firstly, various examples of spectral triples using algebras of Toeplitz operators have been exhibited and the underlying geometry explored. After studying the principal microlocal features of GTOs, we have tested their remarkable efficiency to derive results not only in both geometric and Berezin–Toeplitz quantizations, but also in the field of noncommutative geometry. The spectral triples built on the boundary and inside a smoothly bounded strictly pseudoconvex domain Ω involve GTOs by means of the powerful underlying machinery. For the Fock case, a similar construction has been used with operators of Weyl type admitting similar properties. Using the more abstract definition of GTOs given in the initial theory, an interesting problem would be to investigate similar constructions of spectral triples over general smooth compact manifolds admitting a Toeplitz structure and to study the induced geometry.

Secondly, we showed that a natural spectral triple is associated to the Berezin–Toeplitz quantization, as in the case of the Moyal quantization. More generally, a star product derived from a deformation quantization endows the algebra of smooth functions with a noncommutative structure, which makes it a good candidate to describe a noncommutative space. I plan to study whether a spectral triple can be built from other quantization schemes.

From a geometrical point of view, we obtained smooth noncommutative spaces with the same dimension as the corresponding manifold the Hilbert spaces were defined on, except for the spectral triple based on the Berezin–Toeplitz quantization and for the example of integration of spectral triples. In these cases, the extra dimension we obtained has been interpreted as the additional degree of freedom induced by the summation. A possible extension of this result would be to consider continuous fields of algebras and Hilbert spaces and to study the corresponding dimension. Moreover, it would also be interesting to get a converse of the result about integration of spectral triples, that is to determine the conditions under which a spectral triple can be decomposed as a summation of an infinite number of spectral triples. A possible link with deformation quantization cannot be excluded.

Fefferman proposed in [65] a programme to find all the local invariants of strictly pseudoconvex domains (and CR manifolds more generally). In the context of spectral triples on the boundary of Ω, we exhibited an operator $\mathcal{D}$ based on the complex normal derivative from which we were able to compute the coefficients of the matrix of the Levi form. This is a first result on the characterization of a contact manifold, and a possible perspective would consist in capturing additional local invariants with the help of GTOs. Similar works has been done for CR and conformal manifolds in [8, 89, 119] and more recently in [120].

Appendix A

Geometric framework

This section is dedicated to recalling some definitions and properties of the different classes of manifolds we consider and we refer to [94, 17, 131, 102] for further details.

A.1 Symplectic manifolds

Recall that a symplectic manifold is given by a pair (Ω, ω), where Ω is a smooth even-dimensional manifold and ω is a symplectic form on Ω, i.e. a closed non-degenerate 2-form.

Example A.1.1.

i) $(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$, *where* $\omega_{\mathbb{R}^{2n}} := \sum_{j=1}^{n} dx_j \wedge dy_j$, *is symplectic.*

ii) If Ω *is an* n*-dimensional differentiable manifold, denote* $(x_1, \ldots, x_n)$ *and* $(\xi_1, \ldots, \xi_n)$ *the local and dual variables respectively. Then denoting (in local coordinates)* $\beta := \sum_{j=1}^{n} \xi_j \, dx_j$ *the Liouville form,* $(T^*\Omega, -d\beta)$ *is a symplectic manifold.*

One of the main differences between Riemannian and symplectic geometries is that symplectic manifolds do not admit local invariants, thanks to the Darboux theorem which states that around any point of Ω, the symplectic form ω can be written as $\omega_{\mathbb{R}^{2n}}$ of the previous example. In other words, symplectic manifolds are locally flat. Global invariants however exists, like for instance the (Liouville's) volume form $\frac{1}{n!}\omega^n = \frac{1}{n!}\,\omega \wedge \cdots \wedge \omega$ (which vanishes nowhere).
Given a differentiable function $f : \Omega \to \mathbb{R}$ we can associate a unique [(1)] vector field $X_f \in T\Omega$, called the Hamiltonian vector field of f, verifying for any $Y \in T\Omega$

$$\begin{aligned} &\omega(X_f, Y) := -df(Y)\,, \qquad \text{given locally by} \\ &X_f = \sum_{j=1}^{n} (-\partial_{y_j} f)\, \partial_{x_j} + (\partial_{x_j} f)\, \partial_{y_j}\,. \end{aligned} \tag{A.1}$$

Remark A.1.2. *The Hamiltonian vector field is sometimes defined as* $\omega(X_f, Y) := df(Y)$ *(with a positive sign).*

[(1)]The unicity comes from the non-degeneracy of ω.

Any $2n$-dimensional symplectic manifold (Ω, ω) admits a Poisson bracket, i.e. a skew-symmetric bilinear map $\{\,.\,,\,.\,\}$ defined on $C^\infty(\Omega) \times C^\infty(\Omega)$ as

$$\{\, f \,,\, g \,\} := \omega(X_f, X_g) = -df(X_g) = -X_g(f) = X_f(g)\,, \tag{A.2}$$

which in canonical coordinates writes $\{\, f \,,\, g \,\} = \sum_{j=1}^n \partial_{x_j} f\, \partial_{y_j} g - \partial_{x_j} g\, \partial_{y_j} f$. Using the Poisson bracket, (A.1) can be also expressed as

$$X_f = \{\, f \,,\, .\, \}\,.$$

We have the following properties

$$\begin{aligned} &\{\, f \,,\, \{\, g \,,\, h \,\}\,\} + \{\, g \,,\, \{\, h \,,\, f \,\}\,\} + \{\, h \,,\, \{\, f \,,\, g \,\}\,\} = 0 \quad \text{(Jacobi's identity)}, \\ &\{\, fg \,,\, h \,\} = f\{\, g \,,\, h \,\} + g\{\, f \,,\, h \,\}, \qquad \text{and} \qquad [\, X_f,\, X_g\,] = X_{\{\, f \,,g \,\}}\,. \end{aligned} \tag{A.3}$$

The last expression uses the Jacobi's identity:

$$\begin{aligned} [\, X_f,\, X_g\,](h) &= X_f(X_g(h)) - X_g(X_f(h)) = \{\, f \,,\, \{\, g \,,\, h \,\}\,\} - \{\, g \,,\, \{\, f \,,\, h \,\}\,\} \\ &= \{\, f \,,\, \{\, g \,,\, h \,\}\,\} + \{\, g \,,\, \{\, h \,,\, f \,\}\,\} = -\{\, h \,,\, \{\, f \,,\, g \,\}\,\} = \{\,\{\, f \,,\, g \,\}\,,\, h \,\} = X_{\{\, f \,,g \,\}}\,. \end{aligned}$$

A smooth manifold endowed with a Poisson bracket is called a Poisson manifold.

A.2 Contact manifolds

The " odd-dimensional analogue " of symplectic geometry is called contact geometry. It plays an important role in geometric quantization. See [2, 69] for a good overview and [81] for deeper details.

Definition A.2.1. *Let Ω be a differentiable manifold of dimension $2n-1$. A contact manifold is the pair $(\Omega, \mathcal{C})$, where $\mathcal{C} \subset T\Omega$ is defined as*

$$\mathcal{C} := \ker(\eta)\,, \qquad \textit{with } \eta \in T^*\Omega \textit{ such that } \eta \wedge (d\eta)^{n-1} \textit{ vanishes nowhere on } \Omega.$$

We call $\mathcal{C}$ the contact structure and η a contact form [(2)].

Example A.2.2. *On $\mathbb{R}^{2n-1}$, with variables $(x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}, t)$, the natural contact structure is given by the kernel of $\eta := dt + \sum_{j=1}^{n-1} x_j dy_j - y_j dx_j$.*

Note that the form

$$\nu := \eta \wedge (d\eta)^{n-1} \tag{A.4}$$

defines a volume form on Ω, which implies Ω is orientable. Moreover, the 1-form $\theta := g\eta$, where g is a smooth function from Ω to $\mathbb{R}\backslash\{0\}$, defines the same contact structure, and the relation on the volume form is the following

$$\theta \wedge (d\theta)^{n-1} = g\,\eta \wedge (dg\,\eta + g\,d\eta)^{n-1} = g^n\,\eta \wedge (d\eta)^{n-1}\,. \tag{A.5}$$

Remark A.2.3. *The condition in the definition of a contact form $\eta \wedge (d\eta)^{n-1} \neq 0$ is equivalent to say that $(d\eta)^n|_{\mathcal{C}} \neq 0$. In other words, $(\mathcal{C}_p, d\eta|_{\mathcal{C}_p})$ is a symplectic vector space for any $p \in \Omega$.*

[(2)]The contact structure is often denoted ξ in the literature but we chose to change the notations to avoid confusions with an element of the cotangent space.

A.3 Complex manifolds

Recall that a complex manifold is a manifold together with an atlas of charts to $\mathbb{C}^n$ such that the transition functions are biholomorphisms [(3)]. For instance, $\mathbb{C}^n$ and $\mathbb{CP}^n$ are complex manifolds. Let V be an even-dimensional real vector space. A linear map $J : V \to V$ is a complex structure if $J^2 = -I$. Its extension to $V \otimes \mathbb{C}$ is given by $J(\lambda v) := \lambda J(v)$, for $\lambda \in \mathbb{C}$ and $v \in V$. For instance, the canonical complex structure on $T_p(\mathbb{R}^{2n})$, $p \in \mathbb{R}^{2n}$, is given by $J(\partial_{x_j}) := \partial_{y_j}$ and $J(\partial_{y_j}) := -\partial_{x_j}$. The induced complex structure J^* on $T_p^*(\mathbb{R}^{2n})$ is defined by duality: $J^*(dx_j) := -dy_j$ and $J^*(dy_j) := dx_j$.
An integrable almost complex structure on a manifold is a smoothly varying complex structure on each fibre of its tangent space. Note that any complex manifold admits a complex structure by pushforwarding the one from $\mathbb{C}^n$ to its tangent space via a coordinate chart.

The complexification $V \otimes \mathbb{C}$ admits the decomposition $V = V^{1,0} \oplus V^{0,1}$, where $J|_{V^{1,0}} = iI$ and $J|_{V^{0,1}} = -iI$, and since for any $v \in V \otimes \mathbb{C}$, $J\bar{v} = \overline{Jv}$, we get $V^{1,0} = \overline{V^{0,1}}$. When $V = T_p\Omega$ or $T_p^*\Omega$, with Ω an n-dimensional complex manifold and $p \in \Omega$, define the spaces

$$\begin{aligned}
T_p^{1,0}\Omega &:= \operatorname*{span}_{j=1,\dots,n} \left\{\partial_{z_j} := \tfrac{1}{2}(\partial_{x_j} - i\partial_{y_j})\right\}, & T_p^{0,1}\Omega &:= \operatorname*{span}_{j=1,\dots,n} \left\{\partial_{\bar{z}_j} := \tfrac{1}{2}(\partial_{x_j} + i\partial_{y_j})\right\}, \\
T_p^{*^{1,0}}\Omega &:= \operatorname*{span}_{j=1,\dots,n} \left\{dz_j := dx_j + idy_j)\right\}, & T_p^{*^{0,1}}\Omega &:= \operatorname*{span}_{j=1,\dots,n} \left\{d\bar{z}_j := dx_j - idy_j)\right\}, \\
\Lambda^{a,b}T_p\Omega &:= \operatorname*{span}_{|\alpha|=a,|\beta|=b} \left\{\partial_z^\alpha \wedge \partial_{\bar{z}}^\beta\right\}, & \Lambda^{a,b}T_p^*\Omega &:= \operatorname*{span}_{|\alpha|=a,|\beta|=b} \left\{dz^\alpha \wedge d\bar{z}^\beta\right\},
\end{aligned}$$

where $a, b \in \mathbb{N}$, $\alpha, \beta \in \mathbb{N}^n$, $\partial_z^\alpha := \partial_{z_1}^{\alpha_1} \wedge \cdots \wedge \partial_{z_n}^{\alpha_n}$ and similarly for $\partial_{\bar{z}}^\beta$, dz^α and $d\bar{z}^\beta$. The set of smooth differential forms of bi-degree (a, b) (i.e. smooth section of $\Lambda^{a,b}T\Omega$) is denoted $\Gamma^{a,b}\Omega$.

Definition A.3.1. *The Dolbeault operators ∂ and $\bar{\partial}$ [(4)] are defined as follows.*
For $f \in \Gamma^{0,0}\Omega$,

$$\partial f := \sum_{j=1}^n \partial_{z_j} f \, dz_j \,, \quad \bar{\partial} f := \sum_{j=1}^n \partial_{\bar{z}_j} f \, d\bar{z}_j \,,$$

and for any $a, b \in \mathbb{N}$, $|\alpha| = a$, $|\beta| = b$,

$$\partial : f \in \Gamma^{a,b}\Omega \mapsto \partial f \wedge dz^\alpha \wedge d\bar{z}^\beta \in \Gamma^{a+1,b}\Omega \,, \quad \bar{\partial} : f \in \Gamma^{a,b}\Omega \mapsto \bar{\partial} f \wedge dz^\alpha \wedge d\bar{z}^\beta \in \Gamma^{a,b+1}\Omega \,.$$

One checks easily that $d = \partial + \bar{\partial}$, $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$.

Definition A.3.2. *The complex tangent space of a complex manifold Ω at $p \in \Omega$ is*

$$\mathcal{T}_p\Omega := T_p\Omega \cap JT_p\Omega \,.$$

Define respectively the holomorphic and antiholomorphic tangent spaces as

$$\mathcal{T}_p'\Omega := T_p^{1,0}\Omega \cap (T_p\Omega \otimes \mathbb{C}) \,, \quad \text{and} \quad \mathcal{T}_p''\Omega := T_p^{0,1}\Omega \cap (T_p\Omega \otimes \mathbb{C}) \,.$$

Define also the bundles $\mathcal{T}\Omega := \bigcup_{p\in\Omega} \mathcal{T}_p\Omega$, $\mathcal{T}'\Omega := \bigcup_{p\in\Omega} \mathcal{T}_p'\Omega$ and $\mathcal{T}''\Omega := \bigcup_{p\in\Omega} \mathcal{T}_p''\Omega$.

[(3)] This means that they are holomorphic with holomorphic inverse.
[(4)] $\bar{\partial}$ is also called the Cauchy–Riemann operator.

First, since $J \circ J|_{\mathcal{T}_z\Omega} = -I$ for any $z \in \Omega$, then $\dim_{\mathbb{R}} \mathcal{T}_z\Omega$ is even. Moreover, the dimension of $\mathcal{T}_p\Omega$ (hence $\mathcal{T}'_p\Omega$ and $\mathcal{T}''_p\Omega$) depends in general on the point p and more precisely, if $\Omega \subset \mathbb{C}^n$ is a real manifold which has real dimension $2n - d$, then [17, Section 7, Lemma 1]

$$2n - 2d \leq \dim_{\mathbb{R}} \mathcal{T}_p\Omega \leq 2n - d\,. \tag{A.6}$$

Definition A.3.3. *The manifolds for which the dimension of its complex tangent space is the same everywhere is called a CR-manifold (or Cauchy–Riemann manifold).*

This very large class of manifolds contains all complex manifolds, but also from (A.6) hypersurfaces of $\mathbb{C}^n$ and consequently contact manifolds.

Definition A.3.4. *Let $\Omega \subset \mathbb{C}^n$ be an open bounded complex manifold with smooth boundary $\partial\Omega$. The boundary Cauchy–Riemann operator $\bar\partial_b : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega, \mathcal{T}''^*)$ is defined as*

$$\bar\partial_b u := d\tilde u|_{\mathcal{T}''}\,,$$

where $\tilde u$ is any smooth extension of u in a neighborhood of $\partial\Omega$ in $\mathbb{C}^n$ [5].

When Ω is a bounded open set of $\mathbb{C}^n$ with smooth boundary $\partial\Omega$, we use in practice a defining function r, i.e. a smooth function $r : \overline{\Omega} \to \mathbb{R}$ verifying

$$r|_\Omega < 0\,, \quad r|_{\partial\Omega} = 0 \quad \text{and} \quad dr|_{\partial\Omega} \neq 0.$$

Remark A.3.5. *Note that for two defining functions r and ρ relative to the same manifold Ω, there is a smooth function ϕ strictly positive on Ω such that $r = \phi\,\rho$ [131, Section 24 Theorem 4]*

Most of the results concern this kind of domains and we work almost systematically with a defining function r. In this case, a vector field $X' := \sum_{j=1}^n X'_j\,\partial_{z_j} \in T_p(\mathbb{C}^n)$ is a holomorphic tangent vector field if

$$\langle\, \partial r\,,\, X'\,\rangle_p = \sum_{j=1}^n X'_j\,\partial_{z_j} r(p) = 0\,,$$

and $X'' := \sum_{j=1}^n X''_j\,\partial_{\bar z_j} \in T_p(\mathbb{C}^n)$ is an antiholomorphic tangent vector field if

$$\langle\, \bar\partial r\,,\, X''\,\rangle_p = \sum_{j=1}^n X''_j\,\partial_{\bar z_j} r(p) = 0\,.$$

Such CR-manifolds inherit the following additional structure:

Definition A.3.6. *Let Ω be a CR-manifold with defining function r. The Levi form at $p \in \Omega$ is given by*

$$L' := \partial\bar\partial r|_{\mathcal{T}'_p\Omega}\,.$$

In local holomorphic coordinates, setting $X' = \sum_{j=1}^n X'_j\,\partial_{z_j}$, $Y' = \sum_{k=1}^n Y'_k\,\partial_{z_k}$ in $\mathcal{T}'_p\Omega$, and also $X'' = \sum_{j=1}^n X''_j\,\partial_{\bar z_j}$, $Y'' = \sum_{k=1}^n Y''_k\,\partial_{\bar z_k}$ in $\mathcal{T}''_p\Omega$, we get

$$\begin{aligned} L'_r(X', Y')(p) &= \sum_{j,k=1}^n \partial_{z_j}\partial_{\bar z_k} r(p)\, X'_j\, \overline{Y'_k} = \langle\, Y'\,,\, \mathrm{Hess}_r(p)^t X'\,\rangle \\ L''_r(X'', Y'')(p) &= \sum_{j,k=1}^n \partial_{z_k}\partial_{\bar z_j} r(p) X''_j\, \overline{Y''}_k = \langle\, Y''\,,\, \mathrm{Hess}_r(p) X''\,\rangle\,, \end{aligned} \tag{A.7}$$

[5] $\bar\partial_b$ is independent of the choice of such extension.

(we follow the notations of [62]). The notation with the scalar product in $\mathbb{C}^n$ uses implicitly the identification between an element of $\mathcal{T}'$ or $\mathcal{T}''$ with a vector of $\mathbb{C}^n$. Note that for another defining function ρ such that $r = \phi\rho$, we have on $\partial\Omega$

$$\mathrm{Hess}_r = \phi\,\mathrm{Hess}_\rho + A(\rho,\phi) + A(\phi,\rho)\,, \qquad \text{where } \big[A(u,v)\big]_{k,j} := \partial_{z_k} u\,\partial_{\bar{z}_j} v\,.$$

Since $\mathcal{T}'' \subset \mathrm{Ker}(A(\phi,\rho)) \cap \mathrm{Ker}(A^t(\rho,\phi))$, we have on $\partial\Omega$, $L'_r = \phi L'_\rho$ and $L''_r = \phi L''_\rho$ (see [17, Chapter 10.3]). For any X', Y' in $\mathcal{T}'$ and denoting $\overline{X'} := \sum_{j=1}^n \overline{X'}_j\,\partial_{\bar{z}_j}$, we also get $L'(X',Y') = L''(\overline{Y'},\overline{X'})$.

A.4 Kähler manifolds

Definition A.4.1. *A complex manifold Ω is hermitian if there is a positive hermitian form h_p on each fibre $T_p\Omega$. Locally, we have*

$$h_p(X,Y) = \sum_{j,k=1}^n h_{jk}(p)\,dz_j(X)d\bar{z}_k(Y)\,, \quad X,Y \in T_p\Omega\,,$$

with $h_{jk} = \overline{h_{kj}}$.
Kähler manifolds are those for which the corresponding differential form (called Kähler form)

$$\omega := \tfrac{i}{2}\sum_{j,k=1}^n h_{jk}\,dz_j \wedge d\bar{z}_k$$

is closed.

Since the corresponding hermitian metric $h := \sum_{j,k=1}^n h_{jk}\,dz_j \otimes d\bar{z}_k$ induces a Riemannian metric on Ω by setting $g := \frac{1}{2}(h + \bar{h})$, hermitian manifolds can be seen as the " complex analogues of Riemannian manifolds ". In particular, Kähler manifolds are particularly interesting since they are both complex and symplectic. From a symplectic point of view, they consist in a symplectic manifold (Ω,ω) endowed with a compatible integrable almost complex structure J. The compatibility condition means that for any vector fields X and Y, $\omega(X,Y) = \omega(JX,JY)$, which induces that the previous h and g are also compatible. The relation between h, ω and g is given by

$$h(X,Y) = g(X,Y) - i\omega(X,Y)\,,$$

for any $X, Y \in T\Omega$.

Locally, there is an analytic characterization of Kähler manifolds. For any point $p \in \Omega$, there is a neighborhood U and a plurisubharmonic function $f : U \to \mathbb{R}$ (i.e. such that its complex Hessian $\mathrm{Hess}_f = [\partial_{z_j}\partial_{\bar{z}_k} f]_{j,k}$ is positive definite on U) such that $\omega|_U = \frac{i}{2}\partial\bar{\partial}f$. This function f is called a *local Kähler potential*.

A.5 Pseudoconvex manifolds

Definition A.5.1. *An open bounded CR-manifold $\Omega \subset \mathbb{C}^n$ is (strictly) pseudoconvex if its Levi form is positive (definite) on its boundary $\partial\Omega$.*

Remark A.5.2. *Equivalently, [131, Section 39, Theorem 3] states that Ω is strictly pseudoconvex if there is a strictly plurisubharmonic defining function. Also, the strict pseudoconvexity is independent of the choice of the defining function.*

The notion of pseudoconvexity arises naturally by analogy with the usual convexity in real geometry. Let $D := \{x \in \mathbb{R}^n \,, r(x) < 0\}$ be a domain with smooth boundary [6], and assume without loss of generality that the origin belongs to its boundary. Then the Taylor expansion of r around 0 reads

$$r(x) = 0 + A(x) + \tfrac{1}{2}H(x) + o(|x|^2)\,,$$

where $A(x) := \sum_{j=1}^n \partial_{x_j} r(0)\, x_j$ is such that for any $X \in T_0\partial D$, $\langle\, dA \,,\, X \,\rangle = 0$ and where we set $H(x) := \sum_{j,k=1}^n \partial_{x_j}\partial_{x_k} r(0)\, x_j x_k$. If D is convex, a small neighborhood of the origin only intersects the tangent space $T_0\partial D$ at the origin, hence the quadratic form induced by $H(x)$ and restricted to $T_0\partial D$ is positive definite.
Now for a complex domain $\Omega := \{z \in \mathbb{C}^n \,, r(z) < 0\}$ with smooth boundary $\partial\Omega$, the Taylor expansion of r at $0 \in \partial\Omega$, with respect to the variables $z_j, \bar{z}_k$, gives

$$r(z) = 0 + 2\mathrm{Re}(A)(z) + 2\mathrm{Re}(B)(z) + \tfrac{1}{2}H(z) + o(|z|^2)\,,$$

with

$$A(z) := \sum_{j=1}^n \partial_{z_j} r(0)\, z_j\,, \quad B(z) := \sum_{j,k=1}^n \partial_{z_j}\partial_{z_k} r(0)\, z_j z_k\,, \text{ and}$$
$$H(z) := \sum_{j,k=1}^n \partial_{z_j}\partial_{\bar{z}_k} r(0)\, z_j \bar{z}_k$$

(A corresponds to $\mathbf{R}(r)|_{z=0}$ in Section 1.3.2). This time, $\langle\, \partial A \,,\, X' \,\rangle = 0$ for any $X' \in \mathcal{T}_0'\partial\Omega$ and the term $B(z)$ can always be cancelled by a suitable biholomorphic mapping [131, Section 37]. Similarly as in the real case, the strict pseudoconvexity is defined by assuming the Levi form is positive definite. Note also that $H(z)$ is invariant by biholomorphic mapping.

Example A.5.3.

i) In the particular case $n = 1$, all simply-connected open set of $\mathbb{C}$ are pseudoconvex (one first check that the unit open disk $\mathbb{D}$ of $\mathbb{C}$ is pseudoconvex and applies the Riemann mapping theorem).

ii) The unit open ball $\mathbb{B}^n$ of $\mathbb{C}^n$ is strictly pseudoconvex: the defining function $r(z) = |z|^2 - 1$ defines a positive definite Levi form on its boundary.

iii) If Ω is a strictly pseudoconvex domain with a defining function r such that $\log(-1/r)$ is strictly plurisubharmonic, then $\Omega^+ := \{(z, w) \in \Omega \times \mathbb{C} \,, |w|^2 < -r(z)\}$ is also strictly pseudoconvex.

A bounded strictly pseudoconvex manifold Ω with smooth boundary $\partial\Omega$ and defining function r admits a contact structure on its boundary, given by the contact form

$$\eta_p := \tfrac{1}{2i}(\partial r - \bar{\partial} r)|_p = \tfrac{1}{2i}\sum_{j=1}^n (\partial_{z_j} r(p)\, dz_j - \partial_{\bar{z}_j} r(p)\, d\bar{z}_j)\,, \quad p \in \partial\Omega\,, \tag{A.8}$$

[6] C^2 regularity is enough.

thus we have

$$\| \partial r \| = \sqrt{2}\, |\eta| \,. \tag{A.9}$$

Note that the kernel of η_p is exactly $\mathcal{T}'\partial\Omega \oplus \mathcal{T}''\partial\Omega$. Moreover, from Remark A.2.3 and the equality

$$d\eta_p = i \sum_{j,k=1}^{n} \tfrac{\partial^2 r}{\partial_{z_j}\partial_{\bar{z}_k}}(p)\, dz_j d\bar{z}_k \,,$$

we see that the strict pseudoconvexity of Ω is equivalent to saying that $(\partial\Omega, \mathcal{T}'\partial\Omega \oplus \mathcal{T}''\partial\Omega)$ is a contact manifold. Also, the set of all positive multiples of the contact form

$$\Sigma := \{t\,\eta\,, t \in \mathbb{R}_{>0}\} \subset T^*\partial\Omega \tag{A.10}$$
$$= \{(z, t\,\eta_z)\,, z \in \partial\Omega\,, t \in \mathbb{R}_{>0}\}\,, \tag{A.11}$$

is a symplectic subset of $T^*\partial\Omega\backslash\{0\}$. Following [23, 62], let X_m, $m = 1, \dots, n$, be the open set of $\partial\Omega$ where $\partial_{\bar{z}_m} r(z) \neq 0$; they also verify $\cup_{m=1}^{n} X_m = \partial\Omega$. For any $z \in X_m$, the complex tangent space $\mathcal{T}_z\partial\Omega$ is spanned by the vector fields

$$\begin{aligned} &\bar{D}_{m,j} := \partial_{\bar{z}_j} - \tfrac{\partial_{\bar{z}_j} r}{\partial_{\bar{z}_m} r}\partial_{\bar{z}_m}\,, \quad D_{m,j} := \partial_{z_j} - \tfrac{\partial_{z_j} r}{\partial_{z_m} r}\partial_{z_m}\,, \quad j \neq m\,, \quad \text{and} \\ &E := \sum_{j=1}^{n} \partial_{z_j} r\, \partial_{\bar{z}_j} - \partial_{\bar{z}_j} r\, \partial_{z_j}\,, \end{aligned} \tag{A.12}$$

(all coefficients are evaluated at $z \in X_m$), while $\mathcal{T}_{(z,t)}\Sigma \approx \mathcal{T}_z\partial\Omega \times \mathbb{R}$ is spanned by (A.12) together with ∂_t.

In the parametrization (A.11), from [62, Lemma 5], the corresponding symplectic form ω_Σ and volume form vol_Σ on Σ are given at $(z, t) \in \partial\Omega \times \mathbb{R}_{>0}$ by

$$\omega_\Sigma = dt \wedge \eta_z + t\, d\eta_z\,, \quad \mathrm{vol}_\Sigma = \tfrac{1}{(n-1)!}\,\omega_\Sigma^n = \tfrac{t^{n-1}}{(n-1)!}\, dt \wedge \eta_z \wedge (d\eta_z)^{n-1} = \tfrac{t^{n-1}}{(n-1)!}\, dt \wedge \nu_z\,,$$

where ν is the volume form on $\partial\Omega$ given in (A.4). The induced Poisson bracket on Σ is denoted $\{\,.\,,\,.\,\}_\Sigma$. The parametrization (A.11) depends on the defining function. Indeed, considering another defining function ρ such that $r = \phi\,\rho$, for some strictly positive $\phi \in C^\infty(\overline{\Omega})$, induces the change of coordinates $(x, \xi) \leftarrow (x, \phi(x)^{-1}\,\xi)$. To underline the dependence of the parametrization on the defining function, the cone (or rather its parametrization) is also denoted Σ_r.

The form L''_r from on $\mathcal{T}''$ (A.7) induces the dual form $\mathcal{L}''_r$ on $\mathcal{T}''^*$. Since the matrix Hess_r is positive definite on $\partial\Omega$, for any $\alpha \in \mathcal{T}''^*$ there is a unique element $Z''_\alpha \in \mathcal{T}''$ verifying

$$L''_r(X, Z''_\alpha) = \alpha(X) \quad \text{for any } X \in \mathcal{T}''.$$

Writing the elements as vectors of $\mathbb{C}^n$, we get

$$Z''_\alpha = \mathrm{Hess}_r^{*-1}\overline{\alpha}\,,$$

where * stands for conjugate transposition and $\overline{\alpha} := (\overline{\alpha_1}, \overline{\alpha_2}, \dots) \in \mathbb{C}^n$. Again, Z''_α depends on r. Then, the dual form $\mathcal{L}''_r$ is defined for any $\alpha, \beta \in \mathcal{T}''^*$ as

$$\begin{aligned} \mathcal{L}''_r(\alpha, \beta) &:= L''_r(Z''_\beta, Z''_\alpha) = \langle\, Z''_\alpha\,,\, \mathrm{Hess}_r^{-1} Z''_\beta\,\rangle = \langle\, \mathrm{Hess}_r^{*-1}\overline{\alpha}\,,\, \mathrm{Hess}_r \mathrm{Hess}_r^{*-1}\overline{\beta}\,\rangle = \langle\, \overline{\alpha}\,,\, \mathrm{Hess}_r^{*-1}\overline{\beta}\,\rangle \\ &= \langle\, \beta\,,\, \big(\overline{\mathrm{Hess}_r}\big)^{-1}\alpha\,\rangle. \end{aligned}$$

The relation between these objects and the bracket $\{\,.\,,\,.\,\}_\Sigma$ relative to the bundle Σ is given explicitly in [62, Corollary 8]: let

$$\begin{array}{rcccc} \tau & : & C^\infty(\partial\Omega) & \to & C^\infty(\Sigma) \\ & & f & \mapsto & \tau(f) \end{array} \tag{A.13}$$

where $\tau(f)$ is the extension of f to the bundle Σ, constant along each fibre.
In the previous parametrization of Σ, we get $\tau(f)(z,t) := f(z)$ for any $(z,t) \in \partial\Omega \times \mathbb{R}_{>0}$. Thus for $f, g \in C^\infty(\partial\Omega)$, we get

$$\begin{aligned} \{\,\tau(f)\,,\,\tau(g)\,\}_{\Sigma_r}(z,t) &= \tfrac{i}{t}\big(\mathcal{L}''_r(\bar\partial_b f, \bar\partial_b \bar g) - \mathcal{L}''_r(\bar\partial_b g, \bar\partial_b \bar f)\big)(z)\,, \\ &= \tfrac{i}{t}\big(\langle\,\overline{\bar\partial_b\, f}\,,\,(\mathrm{Hess}^*_r)^{-1}\overline{\bar\partial_b\, \bar g}\,\rangle - \langle\,\overline{\bar\partial_b\, g}\,,\,(\mathrm{Hess}^*_r)^{-1}\overline{\bar\partial_b\, \bar f}\,\rangle\big)(z)\,. \end{aligned} \tag{A.14}$$

A.6 Diagram

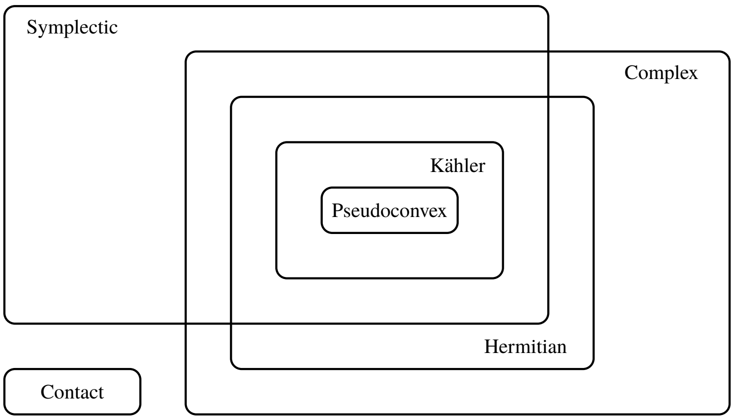

Figure A.1: Diagram of all types of considered manifolds

Appendix B

Biholomorphically invariant defining functions and logarithmic divergences

Recall that a biholomorphism is a holomorphic map whose inverse is also holomorphic. The Riemann mapping theorem states that in the complex plane $\mathbb{C}$, any non-empty simply connected open domain, except $\mathbb{C}$ itself, is biholomorphic to the unit disk $\mathbb{D}$. This is actually a particularity of the dimension 1 and in general this statement is false for higher dimensions [77] (1). Biholomorphic transformations preserve angles and are closely related to conformal mappings.
It seems natural in our context to look for a defining function which would be invariant under biholomorphic coordinate changes. A good candidate verifying this constraint is given by the solution to the complex Monge–Ampère equation [64, 105], i.e. a negative function f of class C^2 on Ω verifying

$$\begin{aligned} &\mathcal{J}[f] = 1 \qquad \text{on } \Omega\,, \\ &f = 0\,, \quad df \neq 0 \qquad \text{on } \partial\Omega\,, \end{aligned} \tag{B.1}$$

where $\mathcal{J}[f]$ is the Monge–Ampère determinant defined by

$$\mathcal{J}[f] := \det \left[\begin{array}{c|ccc} f & \partial_{z_1} f & \dots & \partial_{z_n} f \\ \hline \partial_{\bar z_1} f & \partial_{\bar z_1}\partial_{z_1} f & \dots & \partial_{\bar z_1}\partial_{z_n} f \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{\bar z_n} f & \partial_{\bar z_n}\partial_{z_1} f & \dots & \partial_{\bar z_n}\partial_{z_n} f \end{array}\right] = f^{n+1} \det \left[\partial_{z_j}\partial_{\bar z_k} \log(-1/f)\right]_{j,k} . \tag{B.2}$$

The last equality is obtained by considering the following columns manipulations:

$$\mathcal{J}[f] = \det \left[\begin{array}{c|ccc} f & 0 & \dots & 0 \\ \hline \partial_{\bar z_1} f & \partial_{\bar z_1}\partial_{z_1} f - \frac{1}{f}(\partial_{\bar z_1} f)(\partial_{z_1} f) & \dots & \partial_{\bar z_1}\partial_{z_n} f - \frac{1}{f}(\partial_{\bar z_1} f)(\partial_{z_n} f) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{\bar z_n} f & \partial_{\bar z_n}\partial_{z_1} f - \frac{1}{f}(\partial_{\bar z_n} f)(\partial_{z_1} f) & \dots & \partial_{\bar z_n}\partial_{z_n} f - \frac{1}{f}(\partial_{\bar z_n} f)(\partial_{z_n} f) \end{array}\right]$$

$$= f^{n+1} \det \left[\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline \frac{1}{f}\partial_{\bar z_1} f & \frac{1}{f^2}\big(f\partial_{\bar z_1}\partial_{z_1} f - (\partial_{\bar z_1} f)(\partial_{z_1} f)\big) & \dots & \frac{1}{f^2}\big(f\,\partial_{\bar z_1}\partial_{z_n} f - (\partial_{\bar z_1} f)(\partial_{z_n} f)\big) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{f}\partial_{\bar z_n} f & \frac{1}{f^2}\big(f\,\partial_{\bar z_n}\partial_{z_1} f - (\partial_{\bar z_n} f)(\partial_{z_1} f)\big) & \dots & \frac{1}{f^2}\big(f\,\partial_{\bar z_n}\partial_{z_n} f - (\partial_{\bar z_n} f)(\partial_{z_n} f)\big) \end{array}\right]$$

and noting that the lower right block matrix is exactly the complex Hessian of $\log(-1/f)$.

(1) Poincaré showed in 1907 that the unit ball of $\mathbb{C}^n$ is not biholomorphic to the unit polydisk [118].

Example B.1.1. *For the case of the unit ball of $\mathbb{C}^n$, the function $f : z \mapsto |z|^2 - 1$ is solution to the complex Monge–Ampère equation.*

It is known [105] that for Ω strictly pseudoconvex, such a solution exists and is unique; however, it is in general not smooth at the boundary, and has a singularity

$$f \approx \sum_{j=0}^{\infty} \left(r^{n+1} \log r\right)^j a_j =: \sum_{j=0}^{\infty} f_j \,, \tag{B.3}$$

with $a_j \in C^\infty(\overline{\Omega})$, r a defining function of Ω and " $\approx$ " means that the difference of u and the partial sum $\sum_{j=0}^{N-1}$ on the right-hand side belongs to $C^{N(n+1)-1}(\overline{\Omega})$ and vanishes on $\partial\Omega$ to order $N(n+1)-1$ for any integer $N > 0$.

The Taylor expansion of a solution f near a boundary point in the inward normal direction $t > 0$ gives

$$\begin{aligned} f|_{\partial\Omega} &\approx \sum_{k=0}^{\infty} \tfrac{t^k}{k!} \left(\partial_t^k \sum_{j=0}^{\infty} (r^{n+1} \log r)^j a_j\right)|_{\partial\Omega} \approx \sum_{k=0}^{n} \tfrac{t^k}{k!}\, \partial_t^k a_0|_{\partial\Omega} + \tfrac{t^{n+1}}{(n+1)!}\, \partial_t^{n+1} \left(a_1 r^{n+1} \log r\right)|_{\partial\Omega} + \ldots \\ &\approx \sum_{k=0}^{n} \tfrac{t^k}{k!}\, \partial_t^k a_0|_{\partial\Omega} + t^{n+1} \left(a_1 \left(\partial_t r\right)^{n+1} \log r\right)|_{\partial\Omega} + \ldots \end{aligned}$$

One can also compute the next terms of the rest " $\ldots$ " which depend on the normal derivatives of the a_j evaluated on the boundary and some increasing powers of $\log r$ terms. This makes the boundary values of functions a_j completely determined. Of course, within Ω, the a_j are not unique (replace for instance a_0 and a_1 by $a_0 + g\, r^{n+1} \log r$ and $a_1 - g$, for any smooth function g with compact support in Ω).

Appendix C

Pseudodifferential operators

Some references for this section are [90, 91, 139, 80, 133, 81]. If U is a smoothly bounded open set of $\mathbb{R}^n$ and $s \in \mathbb{R}$, we denote $W^s(U)$ the usual Sobolev space on U.

Definition C.1.1. *A smooth function $p : (x,\xi) \in U \times \mathbb{R}^n \to \mathbb{C}$ is a symbol of order $m \in \mathbb{C}$ if for any compact set $K \subset U$ and any multiindices $\alpha, \beta \in \mathbb{N}^n$,*

$$|\partial_x^\alpha \partial_\xi^\beta p|(x,\xi) \leq C_{K\alpha\beta}(1+|\xi|^2)^{\frac{1}{2}(\mathrm{Re}(m)-|\beta|)}\,, \quad x \in K \text{ and } |\xi| \geq 1\,.$$

The class is denoted $S^m(U \times \mathbb{R}^n)$ or just S^m. Denote also $S^{-\infty} := \bigcap_{m\in\mathbb{R}} S^m$.

Remark C.1.2. *S^m corresponds to the Hörmander class $S^m_{\rho,\delta}$ for $\rho = 1$ and $\delta = 0$ [93, Definition 7.8.1].*

Example C.1.3.

i) Smooth functions on $U \times \mathbb{R}^n$ homogeneous of degree m in the second variable, belong to S^m.

ii) If $p = p(x,\xi)$ has compact support in the second variable, then $p \in S^{-\infty}$.

Definition C.1.4. *A symbol $p \in S^m$ is polyhomogeneous if it admits the asymptotic expansion*

$$p(x,\xi) \approx \sum_{j=0}^{\infty} p_{m-j}(x,\xi)\,, \tag{C.1}$$

where p_{m-j} is ξ-homogeneous of degree $m-j$ [(1)] and " $\approx$ " means that for any $N \in \mathbb{N}\backslash\{0\}$, $p - \sum_{j=0}^{N-1} p_{m-j}$ lies in $S^{\mathrm{Re}(m)-N}$.

A polyhomogeneous symbol $p \in S^m$ gives rise to the operator $p(x,D) : C_0^\infty(U) \to C^\infty(U)$ by setting

$$Pf(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi}\, p(x,\xi)\widehat{f}(\xi)\, d\xi\,. \tag{C.2}$$

An operator of the form (C.2) whose symbol is in $S^{-\infty}$ is called smoothing, and we denote $\Psi\mathrm{DO}^{-\infty}$ the set of smoothing operators. If two operators A and B differ from a smoothing operator, then we write $A \sim B$.

[(1)] It means that $p_{m-j}(x,\lambda\xi) = \lambda^{m-j} p_{m-j}(x,\xi)$, for any $x \in U, \xi \neq 0$ and $\lambda \in \mathbb{C}$.

Example C.1.5. *For $d \in \mathbb{N}$, the operator with symbol $p(x,\xi) = \sum_{|\alpha|\le d} p_\alpha(x)\,\xi^\alpha$, where p_α lies in $C^\infty(U)$, is $p(x,D) = \sum_{|\alpha|\le d} p_\alpha(x)\,(-i\partial_x)^\alpha$.*

Definition C.1.6. *A pseudodifferential operator $P : C_0^\infty(U) \to C^\infty(U)$ of degree $m \in \mathbb{C}$ is an operator of the form*

$$P = p(x,D) + R,$$

with $p \in S^m$ and $R \in \Psi\mathrm{DO}^{-\infty}$. The set of pseudodifferential operators of degree m (resp. less or equal to m) is denoted $\Psi\mathrm{DO}^m$ (resp. $\Psi\mathrm{DO}^{\le m}$) or OPS^m (resp. $OPS^{\le m}$).
The function $(x,\xi) \mapsto p(x,\xi)$ is called the total symbol of P and p_m its principal symbol.
Define also the maps

$$\begin{aligned}
&\sigma_{tot} : P \in \Psi\mathrm{DO}^m \mapsto \sigma_{tot}(P) := p\,,\ \textit{the total symbol of } P\,,\\
&\sigma : P \in \Psi\mathrm{DO}^m \mapsto \sigma(P) := p_m \in S^m\,,\\
&\mathrm{ord} : P \in \Psi\mathrm{DO}^m \mapsto \mathrm{ord}(P) := m \in \mathbb{C}\,.
\end{aligned}$$

$P \in \Psi\mathrm{DO}^m$ is elliptic if $\sigma(P)$ does not vanish on $U \times \mathbb{R}^n \backslash \{0\}$.

Remark C.1.7. *More generally, if S is a set of symbols, we denote by OPS the set of operators of the form $p(x,D) + R$, where $p \in S$ and R is a smoothing operator.*

We recall the main properties of pseudodifferential operators.

Proposition C.1.8.

- *The Schwartz kernel is smooth outside the diagonal of $U \times U$,*
- *$\cup_{m\in\mathbb{C}} \Psi\mathrm{DO}^m$ is an algebra,*
- *for any $s \in \mathbb{R}$, $P \in \Psi\mathrm{DO}^m(\mathbb{R}^n)$, $m \in \mathbb{R}$, is continuous from $W^s(\mathbb{R}^n)$ to $W^{s-m}(\mathbb{R}^n)$,*
- *$P \in \mathcal{B}(L^2(\mathbb{R}^n)) \Leftrightarrow \mathrm{Re}(\mathrm{ord}(P)) \le 0$, and $P \in \mathcal{K}(L^2(\mathbb{R}^n)) \Leftrightarrow \mathrm{Re}(\mathrm{ord}(P)) < 0$,*
- *$\mathrm{ord}(P_1P_2) = \mathrm{ord}(P_1) + \mathrm{ord}(P_2)$,*
- *$\mathrm{ord}([\,P_1,\,P_2\,]) \le \mathrm{ord}(P_1) + \mathrm{ord}(P_2) - 1$,*
- *$\sigma(P_1P_2) = \sigma(P_1)\sigma(P_2)$,*
- *$\sigma([\,P_1,\,P_2\,]) = -i\{\,\sigma(P_1)\,,\,\sigma(P_2)\,\}_{\mathbb{R}^n} = -i\sum_{j=1}^n \partial_{\xi_j}\sigma(P_1)\partial_{x_j}\sigma(P_2) - \partial_{x_j}\sigma(P_1)\partial_{\xi_j}\sigma(P_2)$,*
- *if $P \in \Psi\mathrm{DO}^m$ is elliptic, there is $Q \in \Psi\mathrm{DO}^{-m}$ (called parametrix) such that PQ and QP are smoothing,*
- *the principal symbol is covariant under action of diffeomorphisms (the total symbol is not in general).*

The previous definitions extend to general manifolds M.

Definition C.1.9. *Let U and V be open sets of M and $\mathbb{R}^n$ respectively. An operator P from $C^\infty(M)$ to $C^\infty(M)$ is a pseudodifferential operator of order $m \in \mathbb{C}$ if its Schwartz kernel is smooth outside the diagonal of $M \times M$ and for any diffeomorphism $\phi : U \to V$ and $f \in C_0^\infty(V)$*

$$P_U f := P(f \circ \phi) \circ \phi^{-1}\,.$$

is a pseudodifferential operator of order m on V.
We then denote $P \in \Psi\mathrm{DO}^m(M)$.

The principal symbol of $P \in \Psi\mathrm{DO}^m(M)$ is globally defined as a function on the bundle $T^*M \to M$:

$$\sigma(P)(x,\xi) := \lim_{t\to\infty} t^m\, e_t^-(x)\big(P\, e_t^+\big)(x)\,, \quad x \in M\,, \xi \in T_x^*M\,,$$

with $e_t^{\pm} : x \mapsto e^{\pm itf(x)}$, where $df|_x = \xi$. Thus Proposition C.1.8 can be extended in the case of manifolds. Definition C.1.9 can also be extended by considering a vector bundle $E \to M$, and linear operators from $\Gamma_0^\infty(M,E)$ to $\Gamma^\infty(M,E)$, whose principal symbol is a matrix-valued function. The set of such operators is denoted $\Psi\mathrm{DO}(M,E)$.
A theory of pseudodifferential operators on manifold with boundary can be found in [18, 80].
We have seen that the solution to de Monge–Ampère equation (B.1) admits an asymptotic expansion at the boundary which involves logarithmic terms (B.3). Actually, symbols admitting logarithmic divergences give rise to a larger class of pseudodifferential operators which admit similar properties as above.

Definition C.1.10. *A symbol p is log-polyhomogeneous if it admits the asymptotic expansion*

$$p(x,\xi) \approx \sum_{j=0}^{\infty} p_{m-j}(x,\xi), \ \ \textit{where } p_{m-j}(x,\xi) = \|\,\xi\,\|^{m-j} \sum_{k=0}^{k_j} p_{m-j,k}\big(x, \tfrac{\xi}{\|\,\xi\,\|}\big)\,\big(\log\|\,\xi\,\|\big)^k \tag{C.3}$$

for $\|\,\xi\,\| > 2$, $m \in \mathbb{C}$ and finite $k_j \in \mathbb{N}$, where "$\approx$" means that $p - \sum_{j=0}^{N-1} p_{m-j} \in S^{\mathrm{Re}(s)-N+\varepsilon}$ for any $\varepsilon > 0$ and $N \in \mathbb{N}$.

The set of pseudodifferential operators whose symbol satisfies (C.3) is denoted $\Psi\mathrm{DO}^m_{\log}$, and we set $\Psi\mathrm{DO}_{\log} := \bigcup_{m\in\mathbb{C}} \Psi\mathrm{DO}^m_{\log}$. When $k_0 = k$, we denote it $\Psi\mathrm{DO}^{m,k}_{\log}$. For all $\varepsilon > 0$, we have the inclusion $\Psi\mathrm{DO}^m_{\log} \subset \Psi\mathrm{DO}^{\mathrm{Re(m)}+\varepsilon}$.
Following [59], an operator $P \in \Psi\mathrm{DO}_{\log}$ is said to be pure if $k_0 = 0$, so belongs to $\Psi\mathrm{DO}^{m,0}_{\log}$. It is pure elliptic if $k_0 = 0$ and $p_m(x,\xi) \neq 0$ for all $\|\,\xi\,\| \neq 0$. As a consequence, one can decompose any $P \in \Psi\mathrm{DO}^{m,0}_{\log}$ in the form $P = P_0 + P_1$, where $P_0 \in \Psi\mathrm{DO}^m$ and $P_1 \in \Psi\mathrm{DO}^{m-1}_{\log}$. For $P \in \Psi\mathrm{DO}^m_{\log}$ such that p_m does not vanish identically, its order is defined as $\mathrm{ord}(P) := m \in \mathbb{C}$, and its principal symbol as $\sigma(P) := p_m$.
See [129, 106] for more details about log-polyhomogeneous pseudodifferential operators.

Definition C.1.11. *For $m \in \mathbb{R}$, denote $\mathrm{GLS}^m(\mathbb{R}^n \times \mathbb{R}^n)$ [(2)] the set of functions $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ verifying*

$$|\partial_x^\alpha \partial_\xi^\beta p|(x,\xi) \leq c_{\alpha\beta}\,(1+|x|^2+|\xi|^2)^{\frac{1}{2}(m-|\alpha|-|\beta|)}\,, \quad \textit{as } |x|,\, |\xi| \to \infty\,,$$

for some constants $c_{\alpha\beta}$.
A Weyl operator of order m is of the form

$$W_p f(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\, p(\tfrac{x+y}{2},\xi)\, f(y)\, dy d\xi\,, \tag{C.4}$$

where p in some $\mathrm{GLS}^m(\mathbb{R}^n \times \mathbb{R}^n)$ is called its symbol.

Weyl operators enjoy similar properties to pseudodifferential ones concerning their orders and the induced symbolic calculus.

(2) For Grossman–Loupias–Stein. See also [149, 133].

Definition C.1.12. *The space $\mathscr{S}^m(\mathbb{R}^n \times \mathbb{R}^n)$ consists of functions $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ admitting the asymptotical expansion*

$$p(x,\xi) \approx \sum_{j \in \mathbb{N}} p_{m-j}(x,\xi)\,,$$

with $p_{m-j} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ are homogeneous of degree $m-j$ and where " $\approx$ " means that for any $N \in \mathbb{N}$, $p - \sum_{j=0}^{N-1} p_j(x,\xi) \in \mathrm{GLS}^N(\mathbb{R}^n \times \mathbb{R}^n)$.

Note that the set $\mathscr{S}^m(\mathbb{R}^n \times \mathbb{R}^n)$ corresponds to the symbol class $\Gamma_1^m(\mathbb{R}^n \times \mathbb{R}^n)$ in [133, Definitions 23.1, 23.2].

Notations and symbols

The integer n is reserved for the dimension of the considered domain or space. The canonical coordinates are denoted by $(x_j, y_j) \in \mathbb{R}^{2n} \approx \mathbb{C}^n \ni z_j := x_j + iy_j$, $j = 1, \dots, n$. For any $x, y \in \mathbb{R}^n$, $z, z' \in \mathbb{C}^n$, $\alpha \in \mathbb{N}^n$:

$$xy := \sum_{j=1}^n x_j\, y_j\,, \quad z^2 = \sum_{j=1}^n z_j^2\,, \quad |z|^2 := \sum_{j=1}^n |z_j|^2\,, \quad \langle\, z\,,\, z'\,\rangle := \sum_{j=1}^n z_j \bar{z}'_j$$

$$z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}\,, \quad |\alpha| := \sum_{j=1}^n \alpha_j\,, \quad \alpha! := \prod_{j=1}^n \alpha_j!\,,$$

$$1_j := (0, \dots, 1, \dots, 0) \in \mathbb{N}^n \text{ (1 at the } n^{\text{th}} \text{ position)}\,.$$

For $x \in \mathbb{R}$ and $j, k \in \mathbb{N}$, the Pochhammer symbol $(x)_{(k)}$ and Stirling number of first kind $s(k, j)$ are defined as follows

$$(x)_{(k)} := x(x-1)\dots(x-k+1) =: \sum_{j=0}^k s(k,j)\, x^j\,.$$

$\lfloor x \rfloor$	integer value of $x \in \mathbb{R}$	
$\widehat{f}$	Fourier transform of f	
$\{\, f\,,\, g\,\}$	Poisson bracket between f and g	
$\sim$	equality between operators modulo the smoothings	p. 101
$\underset{\hbar\to 0}{=}$	asymptotic expansion	p. 56
a_j^+, a_j^-	creation and annihilation elements of $\mathfrak{h}^n$	p. 24
$\mathcal{A}_{B,m}$, $\mathcal{A}_{\mathcal{H}}$	algebras related to Toeplitz operators	pp. 37, 69, 73
A^2, $A^2_w(\Omega)$	Bergman spaces	p. 20
$\mathbb{B}^n$	unit ball of $\mathbb{C}^n$	
$\mathcal{B}(\mathcal{H})$	space of bounded linear operators on $\mathcal{H}$	
$\mathscr{B}$	Bargmann transform	p. 23
$\mathbf{C}$	complex conjugation operator	p. 76
$C_0(X)$	continuous functions on X vanishing at infinity	p. 59
$C^\infty(\Omega)[[\hbar]]$	formal power series in $\hbar$ with coefficients in $C^\infty(\Omega)$	p. 52
$\mathbb{CP}^n$	complex projective space of complex dimension n	p. 93
∇^{LC}	the Levi-Civita connexion	
$\not{D}$	the Dirac operator	pp. 62, 72, 76
$d\mu$, $d\sigma$	Lebesgue measures on Ω (or $\mathbb{C}^n$) and $\partial\Omega$	pp. 38, 20

$\mathrm{Env}(X)$	enveloping algebra of X	p. 24
$\mathrm{End}(\mathcal{H})$	space of endomorphisms on $\mathcal{H}$	
$\mathscr{F}_{\mathfrak{m}}$	$\mathfrak{m}^{\text{th}}$ Fock space	p. 23
γ, γ_m	trace operator	p. 22
γ_μ	gamma matrices	p. 61
$\Gamma^\infty(X, E)$	space of smooth sections of $E \longrightarrow X$	
GTO^m	set of GTOs of order m	p. 27
$\mathscr{H}^m$		p. 28
$H^2(\partial\Omega)$	Hardy space	p. 23
Herm^m	Hermite operator	p. 28
Hess_f	complex Hessian matrix of f	p. 94
$\mathcal{J}[f]$	Monge–Ampère determinant of f	p. 99
$\Lambda_m := K_m^* K_m$		p. 21
$\mathcal{K}(\mathcal{H})$	space of compact linear operators on $\mathcal{H}$	
K, K_m	Poisson operator	p. 21
$\mathcal{L}_X$	Lie derivative along X	p. 46
$\mathcal{L}^{1,\infty}$	Macaev ideal	
$\mathcal{M}_{\mathcal{A}}$	spectrum of a commutative Banach algebra $\mathcal{A}$	p. 59
MA^p	Monge–Ampère class of functions	p. 73
$\tilde{\omega}$	rescaled Liouville form	p. 46
$OP\mathscr{H}^m$		p. 28
$OPS^{m,k}$		p. 28
$\pi_{A_m}, \pi_F, \pi_H, \pi_L$	different representations of $\mathfrak{h}^n$	p. 25
$\mathbb{P}_j, \mathbb{Q}_j, \mathbb{T}$	basis of $\mathfrak{h}^n$	p. 24
$\Psi\mathrm{DO}^m$	pseudodifferential operators of order m	p. 102
$\Psi\mathrm{DO}^m_{\log}$	log-polyhomogeneous pseudodifferential operators	p. 103
$P_G(X)$	principal G-bundle over X	
$\mathrm{Pol}(X)$	set of polynomials on X	p. 84
$\mathbb{R}_{>0}$	$\mathbb{R}^+ \backslash \{0\}$	
$\mathbf{R}$		p. 39
$\boldsymbol{\mathcal{R}}, \overline{\boldsymbol{\mathcal{R}}}$		p. 84
$\sigma_{tot}(Q), \underline{\sigma}(Q), \mathrm{ord}(Q)$	total and principal symbol, order of $Q \in \Psi\mathrm{DO}$	p. 102
$\widetilde{\sigma}(T_Q), \widetilde{\mathrm{ord}} T_Q$	(principal) symbol and order of $T_Q \in \mathrm{GTO}$	p. 31
$\$$	spinor bundle	p. 62
$\mathbb{S}^{n-1}$	unit sphere in $\mathbb{R}^n$	
$\mathscr{S}^m$		p. 103
$\mathcal{S}, \mathcal{S}'$	Schwartz space and its dual (tempered distributions)	
S^*M	unit sphere in T^*M, with M a manifold	
$S^m, S^{m,k}$		pp. 101,28
$SO(TM)$	SO_n-frame bundle on TM	
$\mathscr{T}_\psi, T_u, \mathbf{T}_f$	Toeplitz on Fock, Hardy and Bergman spaces	p. 26
u_α, v_α	orthonormal basis of the Fock and Bergman spaces	pp. 20, 23
$v_X(f)$	vanishing order of f on a subset $X \subset \mathrm{dom}(f)$	p. 29
$W^s, W^s_{hol}, W^s_{harm}$	Sobolev spaces	pp. 21, 101

$\mathscr{W}_\sigma, W_\sigma$	Weyl operators on $\mathscr{F}$ and $L^2(\mathbb{R}^n)$ with symbol σ	pp. 24, 103
χ	smooth strictly positive function over $\overline{\Omega}$	p. 20

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