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Équations de Stokes et de Navier-Stokes avec des conditions aux limites de Navier

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Introduction générale

Dans cette thèse on s'intéresse à l'étude théorique des écoulements de fluides visqueux incompressibles qui peuvent intervenir dans plusieurs domaines physiques, tels que la météorologie, l'aérodynamique, océangraphie et de nombreux autres domaines. Ils sont souvent modélisés par des équations de Stokes :

$$\frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \qquad \text{dans } \Omega, \tag{0.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{dans } \Omega, \tag{0.2}$$

ou de Navier-Stokes

$$\frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \qquad \text{dans } \Omega, \tag{0.3}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{dans } \Omega, \tag{0.4}$$

où $\boldsymbol{f}$, $\boldsymbol{u}$ et π désignent respectivement les forces extérieures, le champ de vitesses et le champ de pression et Ω est un ouvert borné connexe de $\mathbb{R}^3$.

À ce jour la plupart des travaux mathématiques portant sur les équations de Stokes et de Navier-Stokes dans un domaine borné considèrent des conditions classiques de non-glissement, dites aussi conditions de Dirichlet introduites par G. G. Stokes en 1845 [60] :

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{sur } \partial\Omega, \tag{0.5}$$

qui signifie que le fluide adhère à la paroi.

Néanmoins, dans les applications on peut se trouver face à des situations physiques où il est nécessaire de faire intervenir d'autres types de condition au bord pour bien décrire le comportement du fluide près de la paroi. En effet, comme l'a indiqué Serrin dans [56], les conditions aux limites classiques de non-glissement ne sont pas toujours réalistes et conduisent à des phénomènes de couche limite près de la surface.

En 1827, Navier [53] a proposé une condition dite condition de glissement avec friction à la paroi qui permet de prendre en compte l'effet de glissement du fluide près du bord et de mesurer la friction en considérant la composante tangentielle du tenseur des contraintes proportionnelle à la composante tangentielle, désigneée par τ, du champ de vitesses :

$$\boldsymbol{u}\cdot\boldsymbol{n}=g \qquad \text{et} \qquad 2\left[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\tau}+\alpha\boldsymbol{u}_{\tau}=\boldsymbol{h} \qquad \text{sur } \partial\Omega \tag{0.6}$$

où α est un coefficient de friction et $\mathbf{D}(\boldsymbol{u})$ est le tenseur de déformations défini par

$$\mathbf{D}(\boldsymbol{u})_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \qquad 1\leq i,j\leq 3.$$

Remarquons formellement, si $\alpha\longrightarrow\infty$, $g=0$ et $\boldsymbol{h}=\boldsymbol{0}$ sur $\partial\Omega$ alors les composantes normale et tangentielle de la vitesse sont nulles et on retrouve la condition classique de Dirichlet ; physiquement on dit que la paroi rigide freine le fluide. Il est important de noter que le cas $\alpha=0$ est aussi intéressant et qui correspond à la condition de glissement de Navier sans frottement, dite aussi condition de Navier parfaite.

Notons que ces conditions sont souvent utilisées dans les simulations numériques d'écou- lement en présence de parois perforées et aussi en présence de parois visqueuses. Dans ce contexte on peut citer les travaux de Amirat et al [2] qui ont étudié les équations de Navier-Stokes considérées dans une couche horizontale dont une surface horizontale rugueuse, les travaux de D. Bucur et al [23, 24] sur une étude de l'effet de la rugosité près de bord. Citons aussi le travail de W.Jagen et Miklic [41, 42]. Ces conditions sont également utilisées pour les parois perforées [13]. Observons que dans le cas d'un bord plat, si $\alpha=0$, $g=0$ et $\boldsymbol{h}=\mathbf{0}$ sur $\partial\Omega$, alors les conditions de Navier (0.6) coïncident avec les conditions suivantes :

$$\boldsymbol{u}\cdot\boldsymbol{n}=0 \qquad \text{et} \qquad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=\mathbf{0} \quad \text{sur } \partial\Omega \tag{0.7}$$

. Dans le cas d'un bord régulier, de classe $C^{1,1}$ les deux conditions diffèrent d'un terme d'ordre 0 :

$$\left[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\right]_{\tau}=\nabla_{\tau}\left(\boldsymbol{v}\cdot\boldsymbol{n}\right)+\left(\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}}\right)_{\tau}-\sum_{k=1}^{2}\left(\boldsymbol{v}_{\tau}\cdot\frac{\partial\boldsymbol{v}}{\partial s_k}\right)\boldsymbol{\tau}_k \quad \text{sur}\,\Gamma.$$

Dans certaines situations, d'autres types de conditions aux limites peuvent intervenir. Par exemple les conditions d'imperméabilités sur le champ de vitesses et le champ de vorticité

$$\boldsymbol{u}\cdot\boldsymbol{n}=0, \quad \mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{n}=\mathbf{0} \quad \text{et} \quad \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{n}=0 \quad \text{sur } \partial\Omega. \tag{0.8}$$

Concernant l'analyse mathématique portant sur ce type de conditions aux limites, on peut citer les travaux de Bellout, Neustupa et Penel [16] où les auteurs étudient les équations de Stokes et de Navier Stokes. Dans [52], les auteurs démontrent l'existence de solutions partiellement fortes pour les équations de Navier-Stokes stationnaires descriptives de fluides visqueux barotropiques compressibles. Nous pouvons également citer le travail de J. Neustupa et P. Penel [54].

Pour certaine écoulements, il est important de prescrire la valeur de la pression tout au moins sur une partie du bord, on peut par exemple donner les exemples suivants :

$$\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \qquad \text{et} \qquad \pi + \frac{1}{2}|\boldsymbol{u}|^2 = 0 \qquad \text{sur } \partial\Omega. \tag{0.9}$$

Ce dernier type de conditions aux limites et les conditions (0.7) ont fait l'objet de plusieurs travaux. Bernard [18, 19], Kozono et Yanagisawa [44] et récemment Amrouche et Seloula [8, 9] ont développé des propriétés sur les potentiels vecteurs, ils ont établi quelques inégalitées de Sobolev et ils ont donné des résultats d'existence de solutions faible, forte et très faibles dans la théorie $\boldsymbol{L}^p$. Dans [14, 15, 27], les auteurs ont étudié les équations de Stokes et de Navier-Stokes avec les conditions aux limites (0.9).

En outre, on peut donner la condition de Neumann suivante :

$$\Big((\nabla \boldsymbol{u})^\top + \gamma \nabla \boldsymbol{u}\Big)\,\boldsymbol{n} - \pi \boldsymbol{n} = \boldsymbol{0} \quad \text{sur } \partial\Omega \tag{0.10}$$

avec $-1 < \gamma \leq 1$. Physiquement, ces conditions expliquent l'absence de contraintes sur l'interface séparant deux milieux. De nombreux auteurs se sont intéréssés à ce type de Conditions. Dans [48], les auteurs ont défini l'opérateur de Stokes et ils ont montré l'existence de solution "mild" pour le problème de Stokes avec les conditions aux limites (0.10). Solonnikov [58] a prouvé l'existence localement en temps de la solution classique pour les équations de Navier-Stokes non-stationnaires avec les conditions aux limites (0.10) avec $\gamma = 1$.

Cette thèse est consacrée à l'étude mathématique des équations de Stokes et de Navier-Stokes stationnaires et aussi le problème d'évolution de Stokes avec des conditions aux limites de Navier. Bien que moins vaste que celles portant sur des conditions aux limites de type Dirichlet, il existe cependant une littérature importante qui s'intéresse à l'étude mathématiques des équations de Stokes et de Navier-Stokes avec des conditions aux bords de Navier (0.6).

Le premier travail a été effectué par V.A Solonnikov et V.E Schadilov en 1973 [59]. Dans ce papier les auteurs ont étudié le problème de Navier-Stokes avec la condition de Navier sur une partie du bord et une condition de Dirichlet sur le reste du bord, ils ont obtenu des résultats d'existence de la solution faible dans $\boldsymbol{H}^1(\Omega) \times L^2(\Omega)$ et aussi la solution forte dans $\boldsymbol{H}^2(\Omega) \times H^1(\Omega)$.

En 1985, G. Mulon et F. Salemi [50, 51] ont donné des résultats d'existence, d'unicité et de régularité pour les problèmes de Navier-Stokes stationnaires et d'évolution avec des conditions aux limites de Navier dans le cas d'un domaine borné ou d'un domaine extérieur.

H. Beirao da Veiga [31] a prouvé l'existence de la solution faible pour l'équation de Navier-Stokes avec des conditions aux limites de Navier dans le demi-espace. L'équation de Navier-Stokes dans le cas d'un bord plat a été étudiée par Berselli [20], dans ce travail il a prouvé l'existence de la solution généralisée, de la solution forte et de la solution très faible dans $\boldsymbol{L}^3(\Omega)$. Nous pouvons citer également H. Beirao da Veiga et F. Crispo [28, 30] qui ont établi des résultats de convergence pour le problème de Navier-Stokes quand la viscosité tend vers 0. Dans le cas de problème d'évolution, il est important aussi de citer le travail de H. Beirao da Veiga [29] où il démontre que l'opérateur de Stokes avec la condition de type Navier est maximal, monotone et auto-adjoint et par suite il engendre un semi-groupe analytique de contractions. Et par conséquent, il a établi l'existence de la solution forte pour le problème de Stokes.

En 2**D**, T. Clopeau, A. Milkelić et R. Robert [26] ont obtenu des résultats d'existence de la solution faible et de la solution forte. Des résultats similaires ont été donné par Mucha [49] pour les équations de Navier-Stokes dans un tuyau infini. Busuioc and Ratiu [25] démontrent l'existence de la solution globale dans $\boldsymbol{H}^3$ pour les équations de Navier-Stokes. D'un point de vue analyse numériques du problème, on peut se référer aux travaux de R. Verfürth [63] et A. Liakos [45, 46].

Ce manuscrit est composé de trois chapitres:

Dans le premier chapitre, nous considérons les équations de Stokes stationnaires avec des conditions aux limites de Navier dans un domaine borné connexe de $\mathbb{R}^3$:

$$(S_T)\begin{cases} -\Delta \boldsymbol{u} + \nabla\pi &= \boldsymbol{f} \qquad \text{dans } \Omega \\ \operatorname{div} \boldsymbol{u} &= \chi \qquad \text{dans } \Omega \\ \boldsymbol{u}\cdot\boldsymbol{n} &= g \qquad \text{sur } \partial\Omega \\ 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau &= \boldsymbol{h} \qquad \text{sur } \partial\Omega \end{cases}$$

L'objet de ce chapitre est d'étudier quelques résultats d'existence et d'unicité pour des différents type de solutions, notamment la solution généralisée, la solution forte et la solution très faible. Nous commençons par prouver quelques relations sur lesquelles nous nous appuyons dans la suite : pour un champ de vecteurs $\boldsymbol{v} \in \mathfrak{D}(\overline{\Omega})$, nous montrons les relations suivantes:

$$\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n} = \nabla_\tau(\boldsymbol{v}\cdot\boldsymbol{n}) - \left(\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}}\right)_\tau - \sum_{k=1}^{2}\left(\boldsymbol{v}_\tau\cdot\frac{\partial\boldsymbol{v}}{\partial s_k}\right)\tau_k \quad \text{sur } \Gamma \tag{0.11}$$

et

$$[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_\tau = \nabla_\tau(\boldsymbol{v}\cdot\boldsymbol{n}) + \left(\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}}\right)_\tau - \sum_{k=1}^{2}\left(\boldsymbol{v}_\tau\cdot\frac{\partial\boldsymbol{v}}{\partial s_k}\right)\tau_k \quad \text{sur } \Gamma. \tag{0.12}$$

Par conséquent, nous obtenons : pour $\boldsymbol{v} \in \mathfrak{D}(\overline{\Omega})$ avec $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ sur Γ

$$[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_\tau = -\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n} - 2\sum_{k=1}^{2}\left(\boldsymbol{v}_\tau\cdot\frac{\partial\boldsymbol{v}}{\partial s_k}\right)\tau_k \quad \text{sur } \Gamma \tag{0.13}$$

ou bien aussi

$$[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_\tau = \mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n} - 2\left(\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}}\right)_\tau \quad \text{sur } \Gamma. \tag{0.14}$$

Pour simplifier les notations nous introduisons l'opérateur d'ordre 0 : $\boldsymbol{\Lambda}\boldsymbol{w} = \sum_{k=1}^{2}(\boldsymbol{w}_{\boldsymbol{\tau}}\cdot\dfrac{\partial\boldsymbol{n}}{\partial s_k})\boldsymbol{\tau}_k$.

En procédant comme dans [55] et en se servant de le densité de $\mathfrak{D}(\overline{\Omega})$ dans $\boldsymbol{E}^p(\Omega)$, nous montrons que pour $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$ on a bien $[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_\tau \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ et nous avons la formule de Green suivante : pour $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$ et $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega)$,

$$-\langle\Delta\boldsymbol{v},\boldsymbol{\varphi}\rangle_\Omega = 2\int_\Omega \mathbf{D}(\boldsymbol{v}):\mathbf{D}(\boldsymbol{\varphi})\,d\boldsymbol{x} - 2\langle[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}},\boldsymbol{\varphi}\rangle_\Gamma \tag{0.15}$$

avec

$$\boldsymbol{V}^{p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\, \operatorname{div} \boldsymbol{v} = 0 \,\text{in}\, \Omega,\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \,\text{on}\, \Gamma\},$$

$$\boldsymbol{E}^{p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\, \Delta \boldsymbol{v} \in [\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'\},$$

$\langle .,.\rangle_\Gamma = \langle .,.\rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}$ et $\langle .,.\rangle_\Omega = \langle .,.\rangle_{[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'\times \boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)}$.

Supposons que $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'$, $\chi = 0$ dans Ω, $g = 0$ sur $\partial\Omega$ et $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ qui satisfait

$$\boldsymbol{h}\cdot\boldsymbol{n} = 0 \quad \text{sur} \quad \Gamma. \tag{0.16}$$

Nous utilisons la formule de Green (I.15), nous montrons que trouver $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ solution du problème $(\mathcal{S}_T)$ est équivalent à

$$\begin{cases} \text{Trouver} \quad \boldsymbol{u} \in \boldsymbol{V}^{p}(\Omega) \quad \text{tel que,} \\ \\ \forall \boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega), \quad 2\displaystyle\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_\Omega + \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_\Gamma . \end{cases} \tag{0.17}$$

Comme indiqué par V. A. Solonnikov and V. E. Ščadilov dans [59] il est facile de voir que

1) Si l'ouvert Ω est obtenu par rotation autour d'un vecteur $\boldsymbol{b}$ fixé de $\mathbb{R}^3$, alors le noyau correspondant au problem (I.1) est une droite vectorielle engendrée par le champ de vecteurs $\boldsymbol{\beta}$ défini par :

$$\boldsymbol{\beta}(\boldsymbol{x}) = \boldsymbol{b}\times\boldsymbol{x}, \quad \text{pour}\, \boldsymbol{x} \in \Omega.$$

Ce noyau, ne dépendant pas de p, sera noté par $\boldsymbol{\mathcal{T}}(\Omega)$.

2) Si le domaine Ω n'est pas obtenu par rotation autour d'un vecteur, alors le noyau se réduit à zero.

Il est important de signaler que pour assurer l'existence dans le cas d'un ouvert obtenu par rotation autour d'un vecteur $\boldsymbol{b}$, la condition de compatibilité suivante est nécessaire :

$$\langle \boldsymbol{f}, \boldsymbol{\beta}\rangle_\Omega + \langle \boldsymbol{h}, \boldsymbol{\beta}\rangle_\Gamma = 0, \tag{0.18}$$

Dans de récents travaux, C. Amrouche et Nour Seloula [8] ont considéré le problème de Stokes avec des conditions aux limites portant sur la composante tangentielle de la tourbillon :

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi &= \boldsymbol{f} & \text{dans } \Omega \\ \operatorname{div} \boldsymbol{u} &= \chi & \text{dans } \Omega \\ \boldsymbol{u}\cdot\boldsymbol{n} &= g & \text{sur } \partial\Omega \\ \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} &= \boldsymbol{h} & \text{sur } \partial\Omega \end{cases}$$

Afin étudier ce problème dans le cadre général de la théorie-$\boldsymbol{L}^p$, les auteurs ont établi une condition **Inf-Sup** permettant ainsi de prouver l'existence de la solution généralisée grâce au théorème de

Babuska-Brezzi. N'ayant pu établir la condition Inf-Sup suivante :

$$\inf_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}^{p'}(\Omega)\\ \boldsymbol{\varphi}\neq\mathbf{0}}} \sup_{\substack{\boldsymbol{\xi}\in \boldsymbol{V}^{p}(\Omega)\\ \boldsymbol{\xi}\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \mathbf{D}(\boldsymbol{\xi}):\mathbf{D}(\boldsymbol{\varphi})\,d\boldsymbol{x}}{\|\,\boldsymbol{\xi}\,\|_{\boldsymbol{W}^{1,p'}(\Omega)}\|\,\boldsymbol{\varphi}\,\|_{\boldsymbol{W}^{1,p}(\Omega)}} \geq \beta, \tag{0.19}$$

qui constitue un des points clefs pour étudier le problème (I.1), nous avons choisi une autre approche pour le résolution : Nous commençons par le cas Hilbertien en utilisant le lemme de Lax-Milgram, la coercivité étant donnée par une l'inégalité de type Korn suivante : pour un ouvert lipschitzien borné

$$\inf_{v\in\mathcal{T}(\Omega)} \|\boldsymbol{u}+\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} \leq C\big(\|\mathbf{D}(\boldsymbol{u})\|^2_{\boldsymbol{L}^2(\Omega)} + \int_\Gamma |\boldsymbol{u}\cdot\boldsymbol{n}|^2\,d\sigma\big), \qquad \forall \boldsymbol{u}\in \boldsymbol{H}^1(\Omega) \tag{0.20}$$

inégalité dite de Poincaré-Morrey. L'étude de la théorie $\boldsymbol{L}^p$ pour $1<p<\infty$, sera faite en deux étapes : Dans un premier temps nous étudions le cas $p>2$ et le cas $1<p<2$se faisant par des arguments de dualité.

Pour le cas $p>2$ nous utilisons la relation (0.13), nous pouvons écrire le problème $(\mathcal{S}_T)$ comme suit :

$$(\mathcal{S}'_T)\begin{cases} -\Delta\,\boldsymbol{u}+\nabla\pi=\boldsymbol{f} \quad \text{et} \quad \operatorname{div}\boldsymbol{u}=\chi & \text{dans} \quad \Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n}=0 \quad \text{et} \quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=-2\boldsymbol{\Lambda}\boldsymbol{u}-\boldsymbol{h} & \text{sur} \quad \Gamma. \end{cases}$$

Pour ce problème, nous avons déjà l'existence d'une solution $(\boldsymbol{u},\pi)\in\boldsymbol{H}^1(\Omega)\times L^2(\Omega)$ qui est unique à un élément de $\mathcal{T}(\Omega)$ additif près pour la vitesse et un réel près pour la pression. Cependant, pour montrer que cette solution est bien dans $\boldsymbol{W}^{1,p}(\Omega)$ nous vérifions tout d'abord que les nouvelles données $\boldsymbol{f}$ et $-2\boldsymbol{\Lambda}\boldsymbol{u}-\boldsymbol{h}$ satisfont la condition de compatibilité suivante : pour tout $p\geqslant 2$ et $\boldsymbol{\varphi}\in\boldsymbol{K}^{p'}_T(\Omega)$

$$\langle \boldsymbol{f},\boldsymbol{\varphi}\rangle_{(\boldsymbol{H}^{p'}_0(\operatorname{div},\Omega))'\times\boldsymbol{H}^{p'}_0(\operatorname{div},\Omega)} + \langle 2\boldsymbol{\Lambda}\boldsymbol{u}+\boldsymbol{h},\boldsymbol{\varphi}\rangle_\Gamma = 0 \tag{0.21}$$

avec

$$\boldsymbol{K}^p_T(\Omega)=\{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega);\ \operatorname{div}\boldsymbol{v}=0,\ \mathbf{curl}\,\boldsymbol{v}=0 \,\text{in}\,\Omega\,\text{and}\,\boldsymbol{v}\cdot\boldsymbol{n}=\mathbf{0}\,\text{on}\,\Gamma\}.$$

L'appartenance de $(\boldsymbol{u},\pi)$ à $\boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$, avec $p\geq 2$, est alors une simple conséquence du Théorème 4.4 dans [55].

La quatrième section de ce chapitre sera consacrée à l'étude de la régularité de la solution. Plus précisément si $\boldsymbol{f}\in\boldsymbol{L}^p(\Omega)$ et $\boldsymbol{h}\in\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$, nous montrons que $\boldsymbol{u}\in\boldsymbol{W}^{2,p}(\Omega)$ et $\pi\in W^{1,p}(\Omega)$.

L'existence de la solution très faible est également étudiée dans ce chapitre dans la section 5. En se servant de nouveau d'un résultat de densité de $\mathcal{D}(\overline{\Omega})$ dans l'espace $\boldsymbol{H}_p(\Delta;\Omega)$ nous montrons que pour $\boldsymbol{v}\in\boldsymbol{H}_p(\Delta;\Omega)$, nous avons $[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_\tau\in\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)$, où

$$\boldsymbol{H}_p(\Delta;\Omega)=\{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega);\ \Delta\boldsymbol{v}\in(\boldsymbol{T}^{p'}(\Omega))'\},$$

avec $(\boldsymbol{T}^{p'}(\Omega))'$ est l'espace dual de

$$\boldsymbol{T}^p(\Omega) = \left\{\boldsymbol{\varphi} \in \boldsymbol{H}_0^p(\mathrm{div}, \Omega);\ \mathrm{div}\,\boldsymbol{\varphi} \in W_0^{1,p}(\Omega)\right\}.$$

De plus nous avons la formule de Green suivante : pour tout $\boldsymbol{v} \in \boldsymbol{H}_p(\Delta; \Omega)$ et tout $\boldsymbol{\varphi} \in \boldsymbol{S}^{p'}(\Omega)$, on a

$$\langle \Delta \boldsymbol{v}, \boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))' \times \boldsymbol{T}^{p'}(\Omega)} = \int_\Omega \boldsymbol{v} \cdot \Delta\boldsymbol{\varphi} d\boldsymbol{x} + \langle 2[\boldsymbol{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma) \times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)} \tag{0.22}$$

avec

$$\boldsymbol{S}^p(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{W}^{2,p}(\Omega);\ \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0,\ \mathrm{div}\,\boldsymbol{\varphi} = 0,\ [\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0}\,\mathrm{on}\,\Gamma\}.$$

Une fois cette trace est proprement définie et la formule de Greeen est bien établie, nous pouvons prouver que pour $\boldsymbol{f} \in (\boldsymbol{T}^{p'}(\Omega))'$ et $\boldsymbol{h} \in \boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)$ qui satisfont les conditions de compatibilité (0.16) et (I.16), le problème $(\mathcal{S})$ admet une unique solution $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)/\boldsymbol{\mathfrak{T}}_T(\Omega)$ et $\pi \in W^{-1,p}(\Omega)/\mathbb{R}$. L'idée de la preuve consiste à passer par le problème dual et à utiliser le résultat obtenu pour la solution forte.

Dans la section 6, nous montrons l'existence d'une base hilbertienne formée par les vecteurs propres de l'opérateur de Stokes. Cette base sera utile dans la suite pour étudier l'existence de la solution faible pour le problème de Navier-Stokes.

Le cas du problème de Stokes avec une condition aux limites de Navier avec friction à la paroi est également traité dans ce chapitre dans le cas hilbertien. En fait pour une friction $\alpha > 0$ bornée, nous montrons l'existence et l'unicité d'une solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$. Contrairement au cas de problème du Stokes avec une condition de Navier dite parfaite, le noyau ne dépend pas de Ω et se réduit à zero.

La dernière partie de ce chapitre sera consacrée à l'étude de problème non linéaire de Navier-Stokes dans le cas hilbertien.

$$(\mathcal{NS}_T)\begin{cases} -\Delta\, \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla\pi = \boldsymbol{f} & \text{dans} \quad \Omega, \\ \mathrm{div}\, \boldsymbol{u} = 0 & \text{dans} \quad \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & \text{sur} \quad \Gamma, \\ 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{sur} \quad \Gamma. \end{cases}$$

Nous commençons par montrer que, pour $\boldsymbol{f} \in (\boldsymbol{H}_0^{6,2}(\mathrm{div}, \Omega))'$ et $\boldsymbol{h} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ qui satisfait la condition (0.16), trouver une solution de problème de Navier-Stokes problem $(\mathcal{NS}_T)$ est équivalent à trouver une solution de la formulation variationnelle suivante :

$$(\mathcal{FNS})\begin{cases} \text{Trouver} \quad \boldsymbol{u} \in \boldsymbol{V}^2(\Omega) \quad \text{telle que,} \\ \\ \forall \boldsymbol{\varphi} \in \boldsymbol{V}^2(\Omega), \quad 2\displaystyle\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\varphi}) = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_\Omega + \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_\Gamma\,, \end{cases}$$

avec $\langle ., .\rangle_\Omega = \langle ., .\rangle_{[\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)]' \times \boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)}$, $\langle ., .\rangle_\Gamma = \langle ., .\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \times \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}$

et

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}\,d\boldsymbol{x}.$$

Constatons que pour cette forme b, nous avons les propriétés d'antisymétries par rapport aux premières et aux deuxièmes variables suivantes : Pour $\boldsymbol{u}\in \boldsymbol{V}^2(\Omega)$ et $\boldsymbol{\eta}\in \mathfrak{T}(\Omega)$, nous avons

$$b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\eta}) = b(\boldsymbol{u},\boldsymbol{\eta},\boldsymbol{\eta}) = b(\boldsymbol{\eta},\boldsymbol{\eta},\boldsymbol{u}) = 0.$$

Par suite, dans le cas d'un ouvert Ω obtenu par rotation autour d'un vecteur fixe, il est nécessaire ici d'imposer une condition de compatibilité de type (I.16) pour résoudre le problème $(\mathcal{NS}_T)$. Ceci est inhabituel dans l'étude des problèmes non linéaires comme par exemple le problème de Navier-Stokes avec des conditions aux limites de Dirichlet.

Remarquons que, sif $\boldsymbol{u} = \boldsymbol{w}+\boldsymbol{\eta}$, avec $\boldsymbol{w}\in \boldsymbol{Z}(\Omega)$ et $\boldsymbol{\eta}\in\mathfrak{T}(\Omega)$, est une solution de $(\mathcal{FNS})$, alors pour tout $\boldsymbol{\psi}\in \boldsymbol{Z}(\Omega)$,

$$2\int_\Omega \mathbf{D}(\boldsymbol{w}):\mathbf{D}(\boldsymbol{\psi})\,d\boldsymbol{x} + b(\boldsymbol{w}+\boldsymbol{\eta},\boldsymbol{w},\boldsymbol{\psi}) + b(\boldsymbol{w},\boldsymbol{\eta},\boldsymbol{\psi}) = \langle \boldsymbol{f},\boldsymbol{\psi}\rangle_\Omega + \langle \boldsymbol{h},\boldsymbol{\psi}\rangle_\Gamma\,, \tag{0.23}$$

avec

$$\boldsymbol{Z}(\Omega) = \{\boldsymbol{v}\in \boldsymbol{H}^1(\Omega);\ \mathrm{div}\,\boldsymbol{v}=0 \quad \mathrm{in}\,\Omega, \quad \boldsymbol{v}\cdot\boldsymbol{n}=0 \quad \mathrm{on}\,\Gamma,\ \int_\Omega \boldsymbol{\beta}\cdot\boldsymbol{v}d\boldsymbol{x}=0\}.$$

Rappelons que pour tout $\boldsymbol{\eta}'\in\mathfrak{T}(\Omega)$, on a

$$b(\boldsymbol{w}+\boldsymbol{\eta},\boldsymbol{w}+\boldsymbol{\eta},\boldsymbol{\psi}+\boldsymbol{\eta}') = b(\boldsymbol{w}+\boldsymbol{\eta},\boldsymbol{w},\boldsymbol{\psi}) + b(\boldsymbol{w},\boldsymbol{\eta},\boldsymbol{\psi}).$$

Ce qui entraine que : Trouver une solutions $\boldsymbol{u}\in \boldsymbol{H}^1(\Omega)$ solution de $(\mathcal{FNS})$ se réduit à trouver $\boldsymbol{w}\in\boldsymbol{Z}(\Omega)$ vérifiant (I.3).

Ainsi, nous montrons l'existence de $(\boldsymbol{u},\pi)\in \boldsymbol{H}^1(\Omega)\times L^2(\Omega)$ solution de problème $(\mathcal{NS}_T)$. Contrairement au cas de problème de Navier-Stokes avec la condition de Dirichlet où on a unicité de la solution pour des données petites, ici on a une infinité de solutions. En effet, $\boldsymbol{u}+c\boldsymbol{\beta}$ sont aussi des solutions pour $c\in\mathbb{R}$. L'idée de la preuve consiste à utiliser la méthode de Galerkin comme pour le cas Dirichlet (voir [61]).

Les résultats de ce chapitre ont fait l'objet d'une publication dans "Journal of Differential Equations" [6] et d'une conférence internationale avec acte [5].

L'objectif du Chapitre II est de généraliser les résultats d'existence de la solution faible pour le problème $(\mathcal{NS}_T)$ obtenus dans le chapitre I dans le cadre la théorie $\boldsymbol{L}^p$. Notons que, à notre connaissance, on ne trouve pas dans la littérature des travaux qui traitent le problème de Navier-Stokes avec des conditions aux limites de Navier dans le cadre de la théorie $\boldsymbol{L}^p$. La plupart des travaux qui existent, étudient le cas hilbertien avec des forces $\boldsymbol{f}$ qui sont uniquement dans $\boldsymbol{L}^2(\Omega)$ (voir [50, 51, 59]). Dans notre travail nous étendrons ces résultats dans un cadre plus général pour obtenir des solutions qui sont dans $(\boldsymbol{u},\pi)\in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$ avec des données dans des espaces fonctionnels plus larges $(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$.

Nous commençons par généraliser les théorèmes démontrés dans le Chapitre I portant sur l'existence de la solution généralisée pour des données $\boldsymbol{f}$ dans des espaces plus large (plus précisément dans $(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$).

Pour étudier l'existence de solution faible pour le problème $(\mathcal{NS}_T)$ pour des données $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$ avec $p \geq 2$, il suffit d'utiliser les résultats de régularité de problème $(\mathcal{S}_T)$. Cependant, le cas $\frac{3}{2} < p < 2$ est plus délicat à étudier. Pour le faire on va utiliser un théorème de point fixe. En premier temps, nous étudions le problème d'Oseen suivant :

$$(\mathcal{O}) \begin{cases} -\Delta\, \boldsymbol{u} + \boldsymbol{v}\cdot\nabla\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{u} = 0 & \text{dans} \quad \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad 2\left[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{sur} \quad \Gamma. \end{cases}$$

On suppose ici que le domaine Ω n'est pas obtenu par rotation autour d'un vecteur et $\boldsymbol{v} \in \boldsymbol{L}^3(\Omega)$ vérifie $\mathrm{div}\,\boldsymbol{v} = 0$ dans Ω et $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ sur Γ.

Pour $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$ et $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ qui satisfait la condition (0.16), nous montrons que le problème $(\mathcal{O})$ admet une unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ et nous avons l'estimation suivante :

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} + \| \pi \|_{L^p(\Omega)/\mathbb{R}} \leq C(1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)})^2 (\| \boldsymbol{f} \|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \| \boldsymbol{h} \|_{W^{-\frac{1}{p},p}(\Gamma)}). \qquad (0.24)$$

Nous montrons également pour $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ et $\boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$ qui satisfait la condition (0.16) que la solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$. De plus, nous avons l'estimation suivante :

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{2,p}(\Omega)} + \| \pi \|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C(1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)})^2 (\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}). \qquad (0.25)$$

Par conséquent, nous montrons que l'application

$$\begin{aligned} T : \boldsymbol{V}^p(\Omega) \quad &\longrightarrow \quad \boldsymbol{V}^p(\Omega) \\ \boldsymbol{v} \quad &\longrightarrow \quad \boldsymbol{u}, \end{aligned}$$

est bien définie. Rappelons que pour $\boldsymbol{v} \in \boldsymbol{V}^p(\Omega)$, $T\boldsymbol{v} = \boldsymbol{u}$ est l'unique solution de problème $(\mathcal{O})$. En outre, en utilisant les estimations (II.46) et (II.39), nous montrons que l'application T est une contraction pour des données $\boldsymbol{f}$ et $\boldsymbol{h}$ qui satisfont :

$$\|\boldsymbol{f}\|_{(\mathbf{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \leq \alpha_1, \qquad (0.26)$$

pour un certain $\alpha_1 > 0$. Appliquant le théorème de point fixe de Banach, on déduit que le problème $(\mathcal{NS}_T)$ admet une solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$. Concernant l'unicité, ici comme dans le cas du problème de Navier-Stokes avec condition Dirichlet, nous avons unicité pour des donnée petites et plus précisément pour :

$$\|\boldsymbol{f}\|_{(\mathbf{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \leq \alpha_2, \qquad (0.27)$$

avec $\alpha_2 \in]0, \alpha_1[$.

Les résultats de ce chapitre à paraitre dans un article accepté au " Mathematical Methods in the

Applied Sciences" [4].

Le troisième chapitre est consacré à la résolution du problème d'évolution de Stokes:

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \qquad & \text{dans } \Omega \times (0,T) \\ \operatorname{div} \boldsymbol{u} = 0 \qquad & \text{dans } \Omega \times (0,T) \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad & \text{dans } \Omega \end{aligned}$$

L'étude de ce problème avec des conditions aux limites de Dirichlet a été faite par un grand nombre d'auteurs. Giga [36, 37] a montré des résultats d'analyticité sur des espaces $\boldsymbol{L}^r$ et il a étudié aussi le domaine de l'opérateur fractionnel de Stokes. Giga et Sohr [38, 39] et Borchers et Sohr [21] ont établi des résultats dans le cas d'un domaine extérieur. Nous pouvons citer le travail de Abe et Giga [1] où ils établissent l'analyticité de semi-groupe dans l'espaces de fonctions bornées. D'autre part, il existe d'autre travaux portant sur les conditions aux limite de type Navier. On peut se référer, par example, au travail de Mitrea et Monniaux [47] où ils montrent l'existence de la solution "mild" pour le problème de Stokes. Récemment Al Baba, Amrouche et Escobedo [10] ont étudié le problème de Stokes avec des conditions de type Navier (0.7), ils ont établit des résultats d'analyticité sur des espaces $\boldsymbol{L}^p$.

Nous nous sommes intéressés ici à la résolution du problème d'évolution homogène de Stokes avec des conditions aux limites de Navier

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0}, & \operatorname{div} \boldsymbol{u} = 0 \quad \text{dans} \quad \Omega \times (0,T), \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} \quad \text{sur} \quad \Gamma \times (0,T), \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{dans} \quad \Omega. \end{cases}$$

et le problème d'évolution non-homogène de Stokes

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 \quad \text{dans} \quad \Omega \times (0,T), \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} \quad \text{sur} \quad \Gamma \times (0,T), \\ & \boldsymbol{u}(0) = \boldsymbol{0} \quad \text{dans} \quad \Omega. \end{cases} \tag{0.28}$$

Concernant l'étude du problème homogène de Stokes, notre approche sera basée sur la théorie des semi-groupes. Nous étudions la résolvante de l'opérateur de Stokes avec des conditions aux limites de Navier. Pour y faire face, nous procédons comme dans [10], nous étudions le problème suivant :

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \ [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma, \end{cases} \tag{0.29}$$

Nous montrons ici, pour $\lambda \in \mathbb{C}^*$, l'existence et l'unicité de la solution faible, de la solution forte et de la solution très faible du problème (III.3). De plus, si Ω est de classe $C^{2,1}$ et $\lambda \in \mathbb{C}^*$ tel que $\operatorname{Re} \lambda \geq 0$, alors nous avons l'estimation suivante :

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{\kappa(\Omega, p)}{|\lambda|} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{0.30}$$

Nous prouvons également des estimations similaires à (0.30) pour les normes de $[\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'_{\sigma,\tau}$ et $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$. Une fois nous avons ces estimations, nous pouvons utiliser la théorie des semi-groupes pour montrer l'existence et l'unicité de la solution faible, de la solution forte et de la solution très faible pour le problème homogène.

Concernant l'étude du problème non-homogène, il est important de noter, sauf dans le cas hilbertien, que l'analyticité ne suffit pas pour obtenir une solution $\boldsymbol{u} \in W^{1,p}(0, T; \boldsymbol{L}^p(\Omega)) \cap L^p(0, T; \boldsymbol{D}(A))$ du problème (0.28) (voir [17]) dans le cadre général de la théorie L^p.

Dans [34, 39], les auteurs ont étudié le problème de Cauchy abstrait suivant :

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{A}\, u(t) = f(t) \quad 0 \leq t \leq T \\ \qquad u(0) = 0, \end{cases} \tag{0.31}$$

avec $-\mathcal{A}$ est un générateur infinitésimal du semi-groupe analytique sur un espace de Banach complexe X. Ils montrent l'existence d'une solution $\boldsymbol{u} \in W^{1,p}(0, T; X) \cap L^p(0, T; \boldsymbol{D}(\mathcal{A}))$ du problème 0.31 ,leur approche est basée sur les estimations des puissances imaginaires pures de l'opérateur $\mathcal{A}$ et sur la notion de ζ-convexité.

Dans notre travail, nous procedons de la même manière. Nous montrons en premier temps l'estimation suivante : Pour un certain t θ_0 on a pour tout $s \in \mathbb{R}$

$$\|(-A)^{i\,s}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \;\leq\; M\, \kappa(\Omega, p)\, e^{|s|\,\theta_0}, \tag{0.32}$$

pour un certain $M > 0$.

Cette estimation va nous permettre d'appliquer le Théorème 2.1 démontré dans [39] et de déduire l'existence de $\boldsymbol{u} \in W^{1,p}(0, T; \boldsymbol{L}^p(\Omega)) \cap L^p(0, T; \boldsymbol{D}(A))$ solution du problème 0.28.

Chapter I

Stokes equations with Navier boundary conditions

1 Introduction

Throughout this work, if we do not say otherwise, we assume that Ω be a bounded domain in $\mathbb{R}^3$ with boundary Γ of class $C^{2,1}$. We consider the stationary Stokes equations:

$$-\Delta\, \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega \tag{I.1}$$

and the stationary Navier-Stokes equations:

$$-\Delta\, \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{I.2}$$

where $\boldsymbol{u}$ denotes a velocity, π a pressure and $\boldsymbol{f}$ are the external forces.

To study these problems, it is necessary to add some boundary conditions. Note that these systems are often studied with Dirichlet boundary condition, also called no-slip boundary condition, which is applicable in the case where the boundary of the flow is solid. Yet, in the physical applications, we are often encounter situations where this conditions does not quite feasible. In this case, it is really important to introduce another boundary conditions to describe the behavior of fluid on the wall. For example, when a part of flow boundary is the air, it is convenient to use a slip boundary condition.

In the literature, in 1827, Navier [53] was the first to propose a Navier with friction boundary condition, in which there is the stagnant layer of fluid close to the wall allowing a fluid to slip, and the tangential component of the strain tensor should be proportional to the tangential component of the fluid velocity on the boundary:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \mathbf{0}, \tag{I.3}$$

where $\mathbf{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\top})$ denotes the deformation tensor associated with the velocity field $\boldsymbol{u}$, α the scalar friction function, $\boldsymbol{n}$ the exterior unit normal and $\boldsymbol{\tau}$ the corresponding tangent vector. Systems (I.1) and (I.2) with (I.3) have been studied by many authors. Note that, the first paper is due to V. A. Solonnikov and V. E. Ščadilov [59] in 1973 without friction function ($\alpha = 0$) in the Hilbert case and only for external force $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$. We also refer to the paper of H. B. da Veiga [30]. We can cite the work of Clopeau, Mikelic and Robert in two dimension [26] or the paper of Verfürth [63], who has studied the mathematical formulation of Newtonian fluid flow with slip boundary condition. In the case of special boundary, such that periodic rough boundary we can mention the works of D. Bucur et al. [23] and the paper of J. Casado-Díaz et al. [33].

The purpose of this work is to study some results of existence, uniqueness and regularity of solution for the stationary Stokes problem (I.1) and also Navier-Stokes problem (I.2) with the boundary condition (I.3) with $\alpha = 0$.

Our method, to study Stokes problem, consists to use Lax-Milgram theorem and Korn's inequality to prove the existence and uniqueness of weak solution in the Hilbert case, to use duality argument for the general case $1 < p < \infty$ and exploit the relationship between slip-Navier boundary condition and the the following boundary condition:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}, \tag{I.4}$$

to prove the regularity. To study Navier-Stokes equation, we use Galerkin method and some compactness results.

The outline of this paper is as follows: In Section 2, we will recall some preliminaries results, introduce an important frameworks and announce a crucial result which gives the relationship between (I.3) and (I.4) in the weak sense on the conventional spaces.

In Section 3, we investigate the existence and uniqueness of weak solution for system (I.1), (I.3). The main idea is to begin by proving the existence and uniqueness in the Hilbert ($p = 2$) case for Ω only of class $C^{1,1}$ then generalizing this result in $\boldsymbol{L}^p$-theory for $1 < p < \infty$ for Ω of class $C^{2,1}$.

In Section 4, we establish the existence and uniqueness of strong solution when we impose more regularity to data.

The purpose of Section 5 is to proof the existence of another class of solutions called very weak solution for less regular data. The idea is based on using the regularity of solution and a duality argument.

In Section 6, we prove the existence of a similar basis of eigenfunctions of Stokes operator to that given by T. Clopeau et al. in [26].

The purpose of Section 7 is to study the Navier-Stokes problem with Navier boundary condition in the Hilbert case. The idea here is to use the Galerkin method and the Brouwer theorem.

2 Preliminaries and functionals spaces

In this section we review such basic notations and functional frameworks. Let us first recall some elementary properties. We note that the vector-valued Laplace operator of a vector field $\boldsymbol{v} = (v_1, v_2, v_3)$ is equivalently defined by

$$\Delta\, \boldsymbol{v} = 2\mathrm{div}\, \mathbf{D}(\boldsymbol{v}) - \mathbf{grad}\, (\mathrm{div}\, \boldsymbol{v}) \tag{I.1}$$

or by

$$\Delta \boldsymbol{v} = \mathbf{grad}\,(\operatorname{div} \boldsymbol{v}) - \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v}. \tag{I.2}$$

Second, we define the following spaces: For all $1 \leq p < \infty$,

$$\boldsymbol{L}^p_0(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \int_\Omega \boldsymbol{v}\,d\boldsymbol{x} = 0\},$$

is equipped with the norm of $\boldsymbol{L}^p(\Omega)$,

$$\boldsymbol{H}^p(\operatorname{div}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \operatorname{div} \boldsymbol{v} \in L^p(\Omega)\},$$

which is equipped with the norm:

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^p(\operatorname{div},\Omega)} = \left(\|\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)}\right)^{\frac{1}{p}}$$

and

$$\boldsymbol{H}^p(\mathbf{curl}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\},$$

which is equipped with the norm:

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^p(\mathbf{curl},\Omega)} = \left(\|\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)}\right)^{\frac{1}{p}}.$$

According to [9], the space $\mathfrak{D}(\overline{\Omega})$ is dense both in $\boldsymbol{H}^p(\operatorname{div}, \Omega)$ and $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$. The closure of $\mathfrak{D}(\Omega)$ in $\boldsymbol{H}^p(\operatorname{div}, \Omega)$ and in $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$ is denoted respectively by $\boldsymbol{H}^p_0(\operatorname{div}, \Omega)$ and $\boldsymbol{H}^p_0(\mathbf{curl}, \Omega)$ and can be characterized respectively by

$$\boldsymbol{H}^p_0(\operatorname{div}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{H}(\operatorname{div}, \Omega); \boldsymbol{v} \cdot \boldsymbol{n} = 0 \,\text{on}\, \Gamma\},$$

$$\boldsymbol{H}^p_0(\mathbf{curl}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega);\ \boldsymbol{v} \times \boldsymbol{n} = 0 \,\text{on}\, \Gamma\}.$$

Let now introduce some notation to describe a boundary. Let us consider any point P on Γ and choose an open neighborhood W of P in Γ small enough to allow the existence of 2 families of $\mathcal{C}^2$ curves on W with these properties: a curve of each family passes through every point of W and the unit tangent vectors to these curves form an orthonormal system (which we assume to have the direct orientation) at every point of W. The lengths s_1, s_2 along each family of curves, respectively, are a possible system of coordinates in W. We denote by $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ the unit tangent vectors to each family of curves, respectively.
With this notation, we have $\boldsymbol{v} = \sum_{k=1}^2 v_k \boldsymbol{\tau}_k + (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}$ where $\boldsymbol{\tau}_k^T = (\tau_{k1}, \tau_{k2}, \tau_{k3})$ and $v_k = \boldsymbol{v} \cdot \boldsymbol{\tau}_k$.

In the sequel, for simplicity of notation, we write

$$\boldsymbol{\Lambda} \boldsymbol{w} = \sum_{k=1}^{2} (\boldsymbol{w}_{\boldsymbol{\tau}} \cdot \frac{\partial \boldsymbol{n}}{\partial s_k})\boldsymbol{\tau}_k. \tag{I.3}$$

In this paper we need a relationship between slip-Navier boundary conditions (I.3) and (I.4), for this reason we state the following result:

Lemma 2.1 *For any $\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$, we have the following equalities:*

$$[2\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = \nabla_{\boldsymbol{\tau}}(\boldsymbol{v}\cdot\boldsymbol{n}) + (\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}})_{\boldsymbol{\tau}} - \boldsymbol{\Lambda}\boldsymbol{v}, \tag{I.4}$$

$$\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n} = \nabla_{\boldsymbol{\tau}}(\boldsymbol{v}\cdot\boldsymbol{n}) - (\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}})_{\boldsymbol{\tau}} - \boldsymbol{\Lambda}\boldsymbol{v}. \tag{I.5}$$

Proof. In this proof we use the local coordinates (similar to Theorem 3.1.1.1 in [40]) and the density of $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ in $\boldsymbol{W}^{2,p}(\Omega)$. Let $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$ and let us start by calculate a gradient of $\boldsymbol{v}$ on the boundary.

$$\begin{aligned}
\nabla \boldsymbol{v} &= \sum_{\ell=1}^{2} \frac{\partial \boldsymbol{v}}{\partial s_\ell}\boldsymbol{\tau}_\ell^T + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}\boldsymbol{n}^T \\
&= \sum_{\ell,k} (v_k \frac{\partial \boldsymbol{\tau}_k}{\partial s_\ell} + \frac{\partial v_k}{\partial s_\ell}\boldsymbol{\tau}_k)\boldsymbol{\tau}_\ell^T + \sum_{\ell=1}^{2} (\frac{\partial (\boldsymbol{v}\cdot\boldsymbol{n})}{\partial s_\ell}\boldsymbol{n} + \boldsymbol{v}\cdot\boldsymbol{n}\frac{\partial \boldsymbol{n}}{\partial s_\ell})\boldsymbol{\tau}_\ell^T \\
&+ \sum_{k=1}^{2} (\frac{\partial v_k}{\partial \boldsymbol{n}}\boldsymbol{\tau}_k + v_k\frac{\partial \boldsymbol{\tau}_k}{\partial \boldsymbol{n}})\boldsymbol{n}^T + (\frac{\partial (\boldsymbol{v}\cdot\boldsymbol{n})}{\partial \boldsymbol{n}}\boldsymbol{n} + \boldsymbol{v}\cdot\boldsymbol{n}\frac{\partial \boldsymbol{n}}{\partial \boldsymbol{n}})\boldsymbol{n}^T.
\end{aligned}$$

As a consequence,

$$(\nabla \boldsymbol{v})\,\boldsymbol{n} = \sum_{k=1}^{2} (\frac{\partial v_k}{\partial \boldsymbol{n}}\boldsymbol{\tau}_k + v_k\frac{\partial \boldsymbol{\tau}_k}{\partial \boldsymbol{n}}) + \frac{\partial (\boldsymbol{v}\cdot\boldsymbol{n})}{\partial \boldsymbol{n}}\boldsymbol{n} + \boldsymbol{v}\cdot\boldsymbol{n}\frac{\partial \boldsymbol{n}}{\partial \boldsymbol{n}}. \tag{I.6}$$

Therefore,

$$[(\nabla \boldsymbol{v})\,\boldsymbol{n}]_{\boldsymbol{\tau}} = \sum_{k=1}^{2} \left(\frac{\partial v_k}{\partial \boldsymbol{n}}\boldsymbol{\tau}_k + v_k(\frac{\partial \boldsymbol{\tau}_k}{\partial \boldsymbol{n}})_{\boldsymbol{\tau}}\right) + \boldsymbol{v}\cdot\boldsymbol{n}\frac{\partial \boldsymbol{n}}{\partial \boldsymbol{n}}. \tag{I.7}$$

Note that $\dfrac{\partial \boldsymbol{n}}{\partial \boldsymbol{n}}\cdot\boldsymbol{n} = 0$, then we have

$$\begin{aligned}
(\nabla \boldsymbol{v})^T &= \sum_{\ell,k=1}^{2} \boldsymbol{\tau}_\ell (v_k (\frac{\partial \boldsymbol{\tau}_k}{\partial s_\ell})^T + \frac{\partial v_k}{\partial s_\ell}\boldsymbol{\tau}_k^T) + \sum_{\ell=1}^{2} \boldsymbol{\tau}_\ell (\frac{\partial (\boldsymbol{v}\cdot\boldsymbol{n})}{\partial s_\ell}\boldsymbol{n}^T + \boldsymbol{v}\cdot\boldsymbol{n}(\frac{\partial \boldsymbol{n}}{\partial s_\ell})^T) \\
&+ \sum_{k=1}^{2} \boldsymbol{n}(\frac{\partial v_k}{\partial \boldsymbol{n}}\boldsymbol{\tau}_k^T + v_k(\frac{\partial \boldsymbol{\tau}_k}{\partial \boldsymbol{n}})^T) + \boldsymbol{n}(\frac{\partial (\boldsymbol{v}\cdot\boldsymbol{n})}{\partial \boldsymbol{n}}\boldsymbol{n}^T + \boldsymbol{v}\cdot\boldsymbol{n}(\frac{\partial \boldsymbol{n}}{\partial \boldsymbol{n}})^T).
\end{aligned}$$

Since $\dfrac{\partial \boldsymbol{n}}{\partial s_k}\cdot\boldsymbol{n} = 0$, we deduce that

$$(\nabla \boldsymbol{v})^T\boldsymbol{n} = \sum_{\ell,k} \boldsymbol{\tau}_\ell v_k (\frac{\partial \boldsymbol{\tau}_k}{\partial s_\ell})^T\boldsymbol{n} + \sum_{\ell=1}^{2} \boldsymbol{\tau}_\ell \frac{\partial (\boldsymbol{v}\cdot\boldsymbol{n})}{\partial s_\ell} + \sum_{k=1}^{2} \boldsymbol{n} v_k (\frac{\partial \boldsymbol{\tau}_k}{\partial \boldsymbol{n}})^T\boldsymbol{n} + \boldsymbol{n}\frac{\partial (\boldsymbol{v}\cdot\boldsymbol{n})}{\partial \boldsymbol{n}}. \tag{I.8}$$

Then,

$$\left[(\nabla \boldsymbol{v})^T\boldsymbol{n}\right]_{\boldsymbol{\tau}} = \sum_{\ell,k=1}^{2} v_k \left((\frac{\partial \boldsymbol{\tau}_k}{\partial s_\ell})^T\boldsymbol{n}\right)\boldsymbol{\tau}_\ell + \sum_{\ell=1}^{2} \frac{\partial (\boldsymbol{v}\cdot\boldsymbol{n})}{\partial s_\ell}\boldsymbol{\tau}_\ell. \tag{I.9}$$

Adding up equality (I.7) to (I.9) we obtain

$$[2\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = \sum_{k=1}^{2} \frac{\partial v_k}{\partial \boldsymbol{n}} \boldsymbol{\tau}_k + v_k (\frac{\partial \boldsymbol{\tau}_k}{\partial \boldsymbol{n}})_{\boldsymbol{\tau}} + \boldsymbol{v} \cdot \boldsymbol{n} \frac{\partial \boldsymbol{n}}{\partial \boldsymbol{n}} + \sum_{\ell,k=1}^{2} \left(v_k (\frac{\partial \boldsymbol{\tau}_k}{\partial s_\ell})^T \boldsymbol{n} \right) \boldsymbol{\tau}_\ell + \sum_{\ell=1}^{2} \frac{\partial (\boldsymbol{v} \cdot \boldsymbol{n})}{\partial s_\ell} \boldsymbol{\tau}_\ell. \quad \text{(I.10)}$$

But we know that

$$(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}})_{\boldsymbol{\tau}} = \sum_{k=1}^{2} \frac{\partial v_k}{\partial \boldsymbol{n}} \boldsymbol{\tau}_k + v_k (\frac{\partial \boldsymbol{\tau}_k}{\partial \boldsymbol{n}})_{\boldsymbol{\tau}} + \boldsymbol{v} \cdot \boldsymbol{n} \frac{\partial \boldsymbol{n}}{\partial \boldsymbol{n}},$$

and

$$\nabla_{\boldsymbol{\tau}} (\boldsymbol{v} \cdot \boldsymbol{n}) = \sum_{\ell=1}^{2} \frac{\partial (\boldsymbol{v} \cdot \boldsymbol{n})}{\partial s_\ell} \boldsymbol{\tau}_\ell.$$

Which implies that

$$[2\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = (\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}})_{\boldsymbol{\tau}} + \nabla_{\boldsymbol{\tau}} (\boldsymbol{v} \cdot \boldsymbol{n}) + \sum_{\ell,k} v_k \frac{\partial \boldsymbol{\tau}_k}{\partial s_\ell} \cdot \boldsymbol{n}\, \boldsymbol{\tau}_\ell. \quad \text{(I.11)}$$

In addition, because $\boldsymbol{\tau}_k^T \cdot \boldsymbol{n} = 0$ on Γ, we have

$$\begin{aligned} \sum_{\ell,k} v_k \frac{\partial \boldsymbol{\tau}_k}{\partial s_\ell} \cdot \boldsymbol{n}\, \boldsymbol{\tau}_\ell &= - \sum_{k=1}^{2} v_k \sum_{\ell=1}^{2} \boldsymbol{\tau}_k \cdot \frac{\partial \boldsymbol{n}}{\partial s_\ell} \boldsymbol{\tau}_\ell, \\ &= - \sum_{\ell=1}^{2} \left(\sum_{k=1}^{2} v_k \boldsymbol{\tau}_k \cdot \frac{\partial \boldsymbol{n}}{\partial s_\ell} \right) \boldsymbol{\tau}_\ell, \\ &= - \sum_{\ell=1}^{2} (\boldsymbol{v}_{\boldsymbol{\tau}} \cdot \frac{\partial \boldsymbol{n}}{\partial s_\ell}) \boldsymbol{\tau}_\ell = -\boldsymbol{\Lambda} \boldsymbol{v}. \end{aligned}$$

Therefore, we have

$$\sum_{\ell,k} v_k \frac{\partial \boldsymbol{\tau}_k}{\partial s_\ell} \cdot \boldsymbol{n}\, \boldsymbol{\tau}_\ell = -\boldsymbol{\Lambda} \boldsymbol{v}.$$

Since Ω is of class $C^{2,1}$, we see that $\dfrac{\partial \boldsymbol{n}}{\partial s_\ell} \in \boldsymbol{W}^{1,\infty}(\Gamma), \ell = 1, 2$. Hence, using (I.11), (I.12) and the density of $\boldsymbol{\mathfrak{D}}(\overline{\Omega})$ in $\boldsymbol{W}^{2,p}(\Omega)$, we obtain (I.4).

In the other hand, we know that

$$\mathbf{curl}\, \boldsymbol{v} = \sum_{k=1}^{2} \frac{\partial \boldsymbol{v}}{\partial s_k} \times \boldsymbol{\tau}_k + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \times \boldsymbol{n}.$$

As consequence, we have

$$\mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n} = \sum_{k=1}^{2} (\frac{\partial \boldsymbol{v}}{\partial s_k} \times \boldsymbol{\tau}_k) \times \boldsymbol{n} + (\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{v}} \times \boldsymbol{n}) \times \boldsymbol{n}.$$

Now, in general

$$(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w} = (\boldsymbol{u} \cdot \boldsymbol{w})\, \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{w})\, \boldsymbol{u}.$$

Using this equality, we have

$$\begin{aligned}\mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n} &= \sum_{k=1}^{2} (\frac{\partial \boldsymbol{v}}{\partial s_k} \cdot \boldsymbol{n})\boldsymbol{\tau}_k - (\boldsymbol{\tau}_k \cdot \boldsymbol{n})\frac{\partial \boldsymbol{v}}{\partial s_k} + (\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n})\boldsymbol{n} - \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}},\\ &= \sum_{k=1}^{2} (\frac{\partial \boldsymbol{v}}{\partial s_k} \cdot \boldsymbol{n})\boldsymbol{\tau}_k + (\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n})\boldsymbol{n} - \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}},\end{aligned}$$

On other hand, we have

$$(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n})\boldsymbol{n} - \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} = -(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}})_{\boldsymbol{\tau}}.$$

Consequently,

$$\begin{aligned}\mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n} &= \sum_{j=1}^{2} (\frac{\partial \boldsymbol{v}}{\partial s_j} \cdot \boldsymbol{n})\boldsymbol{\tau}_j - (\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}})_{\boldsymbol{\tau}},\\ &= \sum_{j=1}^{2} \frac{(\boldsymbol{v} \cdot \boldsymbol{n})}{\partial s_j}\boldsymbol{\tau}_j - \boldsymbol{\Lambda}\boldsymbol{v} - (\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}})_{\boldsymbol{\tau}}.\end{aligned} \tag{I.12}$$

In conclusion, since Ω is of class $C^{2,1}$, we can pass to the limit in (I.12) and we obtain equality (I.5) in $\boldsymbol{W}^{2,p}(\Omega)$. □

Remark 2.2 In the particular case $\boldsymbol{v} \cdot \boldsymbol{n} = 0$, we have the following equality:For all $\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$,

$$[2\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = (\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}})_{\boldsymbol{\tau}} - \boldsymbol{\Lambda}\boldsymbol{v},$$

$$\mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n} = -(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}})_{\boldsymbol{\tau}} - \boldsymbol{\Lambda}\boldsymbol{v}.$$

which implies that

$$[2\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = -\mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n} - 2\boldsymbol{\Lambda}\boldsymbol{v} \quad \text{in } \boldsymbol{W}^{\frac{1}{p},p}(\Gamma). \tag{I.13}$$

Note that we can obtain the equality (I.13) in weak sense. To do this, we must give a sense to $\mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n}$ and to $[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}$ when $\boldsymbol{v}$ belongs to an appropriate space. We need the following space:

$$\boldsymbol{V}^{p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\, \operatorname{div} \boldsymbol{v} = 0 \operatorname{in} \Omega,\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \operatorname{on} \Gamma\},$$

with the norm of $\boldsymbol{W}^{1,p}(\Omega)$and

$$\boldsymbol{E}^{p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\, \Delta \boldsymbol{v} \in [\boldsymbol{H}_0^{p'}(\operatorname{div}, \Omega)]'\},$$

where $[\boldsymbol{H}_0^{p'}(\operatorname{div}, \Omega)]'$ is the dual space of $\boldsymbol{H}_0^{p'}(\operatorname{div}, \Omega)$. $\boldsymbol{E}^{p}(\Omega)$ is equipped with the norm:

$$\| \boldsymbol{v} \|_{\boldsymbol{E}^{p}(\Omega)} = \| \boldsymbol{v} \|_{\boldsymbol{W}^{1,p}(\Omega)} + \| \Delta \boldsymbol{v} \|_{[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'}.$$

We note that $\mathfrak{D}(\overline{\Omega})$ is dense in $\boldsymbol{E}^{p}(\Omega)$ (see [55], Lemma 4.2.1). We need also the following two lemmas. To prove the first one we refer to ([55], Corollary 4.2.2).

Lemma 2.3 *We suppose that* Ω *is of class* $C^{1,1}$*. The linear mapping* $\gamma : \boldsymbol{v} \longrightarrow \mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n}$ *defined on* $\mathfrak{D}(\overline{\Omega})$ *can be extended to a linear and continuous mapping*

$$\gamma : \boldsymbol{E}^p(\Omega) \longrightarrow \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma).$$

Moreover, we have the Green formula: For any $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$ *and* $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega)$,

$$-\langle \Delta \boldsymbol{v}, \boldsymbol{\varphi} \rangle_{\Omega} = \int_{\Omega} \mathbf{curl}\, \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, d\boldsymbol{x} - \langle \mathbf{curl} \boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{\varphi} \rangle_{\Gamma}. \tag{I.14}$$

where $\langle ., . \rangle_{\Gamma}$ *denotes the duality between* $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ *and* $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$ *and* $\langle ., . \rangle_{\Omega}$ *denotes the duality between* $(\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))'$ *and* $\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)$.

Lemma 2.4 *Suppose that* Ω *is of class* $C^{1,1}$*. The linear mapping* $\Theta : \boldsymbol{v} \longrightarrow [\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}|\Gamma}$ *defined on* $\mathfrak{D}(\overline{\Omega})$ *can be extended by continuity to a linear and continuous mapping*

$$\Theta : \boldsymbol{E}^p(\Omega) \longrightarrow \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma).$$

Moreover, we have the Green formula: For any $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$ *and* $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega)$,

$$-\langle \Delta \boldsymbol{v}, \boldsymbol{\varphi} \rangle_{\Omega} = 2 \int_{\Omega} \mathbf{D}(\boldsymbol{v}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} - 2\langle [\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi} \rangle_{\Gamma}. \tag{I.15}$$

Proof. Let $\boldsymbol{v} \in \mathfrak{D}(\overline{\Omega})$ and $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega)$ with $\boldsymbol{\varphi} \cdot \boldsymbol{n} = 0 \,\mathrm{on}\, \Gamma$, then, by using (I.1), we have

$$\begin{aligned} -\langle \Delta \boldsymbol{v}, \boldsymbol{\varphi} \rangle_{\Omega} \quad &= 2 \int_{\Omega} \mathbf{D}(\boldsymbol{v}) : \nabla \boldsymbol{\varphi}\, d\boldsymbol{x} - \langle 2[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi} \rangle_{\Gamma} \\ &\qquad - \int_{\Omega} \mathrm{div}\, \boldsymbol{v}\, \mathrm{div}\, \boldsymbol{\varphi}\, d\boldsymbol{x} \end{aligned}$$

Then, for any $\boldsymbol{v} \in \mathfrak{D}(\overline{\Omega})$ and for any $\boldsymbol{\varphi}$ in $\boldsymbol{V}^{p'}(\Omega)$ we have

$$-\langle \Delta \boldsymbol{v}, \boldsymbol{\varphi} \rangle_{\Omega} = 2 \int_{\Omega} \mathbf{D}(\boldsymbol{v}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} - \langle 2[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi} \rangle_{\Gamma}. \tag{I.16}$$

Now, let $\boldsymbol{\mu}$ be any element of $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$, then there exists an element $\boldsymbol{\varphi}$ in $\boldsymbol{W}^{1,p'}(\Omega)$ such that $\mathrm{div}\, \boldsymbol{\varphi} = 0$ in Ω and $\boldsymbol{\varphi} = \boldsymbol{\mu}_{\boldsymbol{\tau}}$ on Γ with

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'}(\Omega)} \leq C \|\boldsymbol{\mu}_{\boldsymbol{\tau}}\|_{\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)} \leq C \|\boldsymbol{\mu}\|_{\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}. \tag{I.17}$$

Consequently,

$$\begin{aligned}
\left|\langle 2\,[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}\,,\boldsymbol{\mu}\rangle_{\Gamma}\right| &= \left|\langle 2\,[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}\,,\boldsymbol{\mu}_{\boldsymbol{\tau}}\rangle_{\Gamma}\right|\\
&= \left|\langle 2\,[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}\,,\boldsymbol{\varphi}\rangle_{\Gamma}\right|\\
&\leq \|\Delta\boldsymbol{v}\|_{\left(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)\right)'}\|\boldsymbol{\varphi}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}\\
&+ 2\|\mathbf{D}(\boldsymbol{v})\|_{\boldsymbol{L}^p(\Omega)}\|\mathbf{D}(\boldsymbol{\varphi})\|_{\boldsymbol{L}^{p'}(\Omega)}\\
\left|\langle 2\,[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}\,,\boldsymbol{\mu}\rangle_{\Gamma}\right| &\leq \left(\|\Delta\boldsymbol{v}\|^p_{\left[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)\right]'} + 2^{p/2}\|\mathbf{D}(\boldsymbol{v})\|^p_{\boldsymbol{L}^p(\Omega)}\right)^{\frac{1}{p}}\\
&\quad\left(\|\boldsymbol{\varphi}\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)} + 2^{p'/2}\|\mathbf{D}(\boldsymbol{\varphi})\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)}\right)^{\frac{1}{p'}}.
\end{aligned}\tag{I.18}$$

It follows that from Korn's inequality, we have

$$\left|\langle[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}\,,\boldsymbol{\mu}\rangle_{\Gamma}\right| \leq C_p\|\boldsymbol{v}\|_{\boldsymbol{E}^p(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'}(\Omega)}.$$

Thus, by using (I.17), we deduce that

$$\|\,[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}\,\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \leq C\|\boldsymbol{v}\|_{\boldsymbol{E}^p(\Omega)}.$$

Therefore, the linear mapping $\Theta : \boldsymbol{v} \longrightarrow [\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}|\Gamma}$ defined on $\mathfrak{D}(\overline{\Omega})$ is continuous for the norm of $\boldsymbol{E}^p(\Omega)$. Since $\mathfrak{D}(\overline{\Omega})$ is dense in $\boldsymbol{E}^p(\Omega), \Theta$ can be extended by continuity to a mapping still called $\Theta \in \mathfrak{L}(\boldsymbol{E}^p(\Omega),\, \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma))$ and formula (I.16) holds for all $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega)$. □

Owing the previous result, it is possible to extend (I.13) in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and we have the following corollary.

Corollary 2.5 *For any $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$ such that $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on Γ, we have*

$$[2\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = -\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n} - 2\boldsymbol{\Lambda}\boldsymbol{v} \quad \text{in } \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma). \tag{I.19}$$

Remark 2.6 Note that, if Ω is of class $C^{2,1}$, then the slip-Navier boundary condition (I.3) condition differs from (I.4) only by the term $-2\boldsymbol{\Lambda}\boldsymbol{v}$. This term is equal to zero in the case of the flat boundary, consequently we have (I.3) and (I.4) are identical, in this context we cite the paper of H. B. da Veiga et al. [30].

In the sequel, we need the result of existence and uniqueness for the following intermediate problem:

$$(\mathcal{S}_1)\begin{cases}
-\Delta\,\boldsymbol{u} + \nabla\pi = \boldsymbol{f} & \text{in} \quad \Omega,\\
\mathrm{div}\,\boldsymbol{u} = \chi & \text{in} \quad \Omega,\\
\boldsymbol{u}\cdot\boldsymbol{n} = g & \text{on} \quad \Gamma,\\
\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h} & \text{on} \quad \Gamma.
\end{cases}$$

We introduce the following spaces:

$$\boldsymbol{K}_N^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\,\boldsymbol{v} = 0,\ \mathbf{curl}\,\boldsymbol{v} = 0\,\mathrm{in}\,\Omega\,\mathrm{and}\,\boldsymbol{v}\times\boldsymbol{n} = \mathbf{0}\,\mathrm{on}\,\Gamma\},$$

$$\boldsymbol{K}_T^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\,\boldsymbol{v} = 0,\ \mathbf{curl}\,\boldsymbol{v} = 0\,\mathrm{in}\,\Omega\,\mathrm{and}\,\boldsymbol{v}\cdot\boldsymbol{n} = \mathbf{0}\,\mathrm{on}\,\Gamma\}.$$

We now recall a result of existence and uniqueness of weak solution for the problem $(\mathcal{S}_1)$ (see [8] and [55]):

Theorem 2.7 *Suppose that Ω is of class $C^{1,1}$. Let*

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]',\ \chi \in L^p(\Omega),\ g \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma), \boldsymbol{h}\times\boldsymbol{n} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$$

and verifying the following compatibility conditions: For any $\boldsymbol{\varphi} \in \boldsymbol{K}_T^{p'}(\Omega)$,

$$\langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_\Omega + \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_\Gamma = 0, \tag{I.20}$$

$$\int_\Omega \chi\, d\boldsymbol{x} = \int_\Gamma g\, d\boldsymbol{\sigma}. \tag{I.21}$$

Then, Stokes problem $(\mathcal{S}_1)$ has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$ satisfying the estimate:

$$\begin{aligned}\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \\ + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big).\end{aligned}$$

We need also the following result (we refer to [55]):

Theorem 2.8 *Let $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'$ with $\mathrm{div}\,\boldsymbol{f} = 0$ in Ω satisfying the following compatibility condition:*

$$\forall \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega), \qquad \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'\times \boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)} = 0, \tag{I.22}$$

and $\boldsymbol{h}$ be such that $\boldsymbol{h}\times\boldsymbol{n} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$. Then the following problem

$$\begin{cases} -\Delta\,\boldsymbol{u} = \boldsymbol{f} & \text{in} \quad \Omega, \\ \mathrm{div}\,\boldsymbol{u} = 0 & \text{in} \quad \Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on} \quad \Gamma. \end{cases}$$

has a unique solution in $\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{K}_N^p(\Omega)$ and we have

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{K}_N^p(\Omega)} \leq C\big(\|f\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\big).$$

3 Weak solutions of $(\mathcal{S}_T)$

In this paper we are interested in the following problem:

$$(\mathcal{S}_T)\begin{cases} -\Delta\,\boldsymbol{u}+\nabla\pi=\boldsymbol{f} & \text{in} \quad \Omega,\\ \operatorname{div}\boldsymbol{u}=\chi & \text{in} \quad \Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n}=g & \text{on} \quad \Gamma,\\ 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}=\boldsymbol{h} & \text{on} \quad \Gamma. \end{cases}$$

The aim of this section is to study the existence and uniqueness of weak solution for problem $(\mathcal{S}_T)$. First, we consider the Hilbert case. Then, we will study the case of L^p-theory, $1<p<\infty$. We start by the following proposition:

Proposition 3.1 *We suppose that $\chi=0$ and $g=0$. Let $\boldsymbol{f}\in[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'$, $\boldsymbol{h}\in\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ such that*

$$\boldsymbol{h}\cdot\boldsymbol{n}=0 \quad \text{on} \quad \Gamma. \tag{I.1}$$

The problem: Find that $(\boldsymbol{u},\pi)\in\boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$ satisfying $(\mathcal{S}_T)$, in the distribution sense, is equivalent to

$$\begin{cases} \text{Find} \quad \boldsymbol{u}\in\boldsymbol{V}^p(\Omega) \quad \text{such that,}\\ \\ \forall\boldsymbol{\varphi}\in\boldsymbol{V}^{p'}(\Omega), \quad 2\displaystyle\int_\Omega\mathbf{D}(\boldsymbol{u}):\mathbf{D}(\boldsymbol{\varphi})\,d\boldsymbol{x}=\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle_\Omega+\langle\boldsymbol{h},\boldsymbol{\varphi}\rangle_\Gamma\,. \end{cases} \tag{I.2}$$

Proof. First, we note that if $\boldsymbol{h}\in\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$, then we have $\boldsymbol{h}\cdot\boldsymbol{n}\in W^{-\frac{1}{p},p}(\Gamma)$. That means that the relation (I.1) holds in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. In fact, let a be in $W^{\frac{1}{p},p'}(\Gamma)$, or Ω is of class $C^{1,1}$, then $a\boldsymbol{n}\in\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$ since $\boldsymbol{n}\in\boldsymbol{W}^{1,\infty}(\Gamma)$. The regularity $W^{-\frac{1}{p},p}(\Gamma)$ of $\boldsymbol{h}\cdot\boldsymbol{n}$ is then consequence of the relation

$$\langle\boldsymbol{h}\cdot\boldsymbol{n},a\rangle_\Gamma=\langle\boldsymbol{h},a\boldsymbol{n}\rangle_\Gamma.$$

Now, using Green formula (I.15), we deduce that every solution of $(\mathcal{S}_T)$ also solves (I.1). Conversely, let $\boldsymbol{u}$ be a solution of the problem (I.1). Let us take a function $\boldsymbol{\varphi}\in\boldsymbol{\mathcal{D}}(\Omega)$ such that $\operatorname{div}\boldsymbol{\varphi}=0$ as a test function in (I.1). Then we have

$$2\int_\Omega\mathbf{D}(\boldsymbol{u}):\mathbf{D}(\boldsymbol{\varphi})\,d\boldsymbol{x}=\langle-\Delta\boldsymbol{u},\boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)}.$$

As a consequence,

$$\forall\boldsymbol{\varphi}\in\boldsymbol{\mathcal{D}}_\sigma(\Omega), \qquad \langle-\Delta\boldsymbol{u}-\boldsymbol{f},\boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)}=0.$$

So, by the De Rham Theorem, there exists a distribution π in $\mathcal{D}'(\Omega)$ defined uniquely up to an additive constant such that

$$-\Delta\boldsymbol{u}+\nabla\pi=\boldsymbol{f}. \tag{I.3}$$

As $\nabla\pi\in[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'\hookrightarrow\boldsymbol{W}^{-1,p}(\Omega)$, we deduce that $\pi\in L^p(\Omega)$ (we refer to [3]). Moreover, by the fact that $\boldsymbol{u}$ belongs to the space $\boldsymbol{V}^p(\Omega)$, we have $\operatorname{div}\boldsymbol{u}=0$ in Ω and $\boldsymbol{u}.\boldsymbol{n}=0$ on Γ. It remains to prove the Navier boundary condition $2\left[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} = \boldsymbol{h}$ on Γ. We multiply Eq. (I.3) by $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega)$ and we integrate on Ω:

$$2\int_{\Omega} \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} - 2\langle[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\rangle_\Gamma = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_\Omega . \tag{I.4}$$

Using (I.1) and (I.4), we deduce that

$$\forall \boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega), \qquad \langle 2\left[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\rangle_\Gamma = \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_\Gamma$$

Let now $\boldsymbol{\mu}$ be any element of the space $\boldsymbol{W}^{1-\frac{1}{p'},p'}(\Gamma)$. So, there exists $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega)$ such that $\operatorname{div} \boldsymbol{\varphi} = 0$ in Ω and $\boldsymbol{\varphi} = \boldsymbol{\mu}_{\boldsymbol{\tau}}$ on Γ. Its clear that $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega)$ and

$$\begin{aligned}
\langle 2\left[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} - \boldsymbol{h}, \boldsymbol{\mu}\rangle_\Gamma &= \langle 2\left[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} - \boldsymbol{h}, \boldsymbol{\mu}_{\boldsymbol{\tau}}\rangle_\Gamma \\
&= \langle 2\left[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} - \boldsymbol{h}, \boldsymbol{\varphi}\rangle_\Gamma \\
&= 0.
\end{aligned}$$

This implies that

$$2\left[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} = \boldsymbol{h} \quad \text{on} \quad \Gamma,$$

it is that end of the proof. $\square$

First, we denote by $\boldsymbol{a}$ the bilinear form defined on $\boldsymbol{V}^{p}(\Omega) \times \boldsymbol{V}^{p'}(\Omega)$ by

$$\boldsymbol{a}(\boldsymbol{u}, \boldsymbol{\varphi}) = \int_{\Omega} \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x}. \tag{I.5}$$

We introduce the kernel $\mathfrak{T}(\Omega)$ for any $1 < p < \infty$:

$$\mathfrak{T}^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\, \mathbf{D}(\boldsymbol{v}) = \mathbf{0} \,\text{in}\, \Omega,\, \operatorname{div} \boldsymbol{v} = 0 \,\text{in}\, \Omega \,\text{and}\, \boldsymbol{v}\cdot\boldsymbol{n} = 0 \,\text{on}\, \Gamma\}.$$

Observe that, if Ω is obtained by rotation around a constant vector $\boldsymbol{b}$ belongs to $\mathbb{R}^3$, then

$$\mathfrak{T}^p(\Omega) = \operatorname{Span}\{\boldsymbol{\beta}\},$$

where

$$\boldsymbol{\beta}(\boldsymbol{x}) = \boldsymbol{b} \times \boldsymbol{x}, \quad \text{for}\, \boldsymbol{x} \in \Omega.$$

Else, the kernel $\mathfrak{T}^p(\Omega)$ is equal to zero (see [64] for more details). We denote

$$\mathfrak{N}^p(\Omega) = \{(\boldsymbol{u}, c);\, \boldsymbol{u} \in \mathfrak{T}^p(\Omega),\, c \in \mathbb{R}\}.$$

Since $\mathfrak{T}^p(\Omega)$ and $\mathfrak{N}^p(\Omega)$ do not depend on p, we note $\mathfrak{T}(\Omega)$ instead of $\mathfrak{T}^p(\Omega)$ and $\mathfrak{N}(\Omega)$ instead $\mathfrak{N}^p(\Omega)$.

Remark 3.2 By this characterization, we deduce the following compatibility condition:

$$\langle \boldsymbol{f}, \boldsymbol{\beta} \rangle_\Omega + \langle \boldsymbol{h}, \boldsymbol{\beta} \rangle_\Gamma = 0$$

(with $\boldsymbol{h} \cdot \boldsymbol{n} = 0$) that is necessary to solve problem (I.1) and then also to solve $(\mathcal{S}_T)$ with $\chi = 0$ and $g = 0$.

3.1 The case $p = 2$:

In this section we prove the existence and uniqueness of weak solution for problem $(\mathcal{S}_T)$ in the Hilbert case. Our proof is based on the use of Lax-Milgram theorem. Before that, we introduce the following spaces:

$$\boldsymbol{X}(\Omega) = \boldsymbol{H}^1(\Omega)/\boldsymbol{\mathcal{T}}(\Omega). \tag{I.6}$$

To prove the coercivity of the bilinear form $\boldsymbol{a}$, we are now going to give the proof of the following lemma which was cited in [63] without proof.

Lemma 3.3 ***(Poincaré-Morrey Inequality)*** *Let Ω be a Lipschitz bounded domain. Then, we have*

$$\inf_{\boldsymbol{v} \in \boldsymbol{\mathcal{T}}(\Omega)} \|\boldsymbol{u} + \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} \leq C\big(\|\mathbf{D}(\boldsymbol{u})\|^2_{\boldsymbol{L}^2(\Omega)} + \int_\Gamma |\boldsymbol{u} \cdot \boldsymbol{n}|^2 \, d\boldsymbol{\sigma}\big), \qquad \forall \boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \tag{I.7}$$

where the constant C only depends on Ω. In particular, the semi-norm $\|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^2(\Omega)}$ is a norm which is equivalent to the norm $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$ if $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega), \boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ and $\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\beta} d\boldsymbol{x} = 0$.

Proof. Let us P denote the orthogonal projection from $\boldsymbol{L}^2(\Omega)$ onto $\boldsymbol{\mathcal{T}}(\Omega)$ with the product scalar of $\boldsymbol{L}^2(\Omega)$. Then

$$\inf_{\boldsymbol{\rho} \in \boldsymbol{\mathcal{T}}(\Omega)} \|\boldsymbol{u} + \boldsymbol{\rho}\|^2_{\boldsymbol{L}^2(\Omega)} = \|\boldsymbol{u} - P\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}.$$

Therefore, it suffices to show that, there exists $C > 0$ such that

$$\|\boldsymbol{u} - P\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \leq C\big(\|\mathbf{D}(\boldsymbol{u})\|^2_{\boldsymbol{L}^2(\Omega)} + \int_\Gamma |\boldsymbol{u} \cdot \boldsymbol{n}|^2 \, d\boldsymbol{\sigma}\big), \quad \text{for any } \boldsymbol{u} \in \boldsymbol{H}^1(\Omega). \tag{I.8}$$

To establish the existence of such constant, assume the contrary. Then, for any $k \geq 1$, there exists $\boldsymbol{u}_k$, such that

$$\|\boldsymbol{u}_k - P\boldsymbol{u}_k\|^2_{\boldsymbol{L}^2(\Omega)} > k\big(\|\mathbf{D}(\boldsymbol{u}_k)\|^2_{\boldsymbol{L}^2(\Omega)} + \int_\Gamma |\boldsymbol{u}_k \cdot \boldsymbol{n}|^2 \, d\boldsymbol{\sigma}\big).$$

We can suppose that

$$\|\boldsymbol{u}_k - P\boldsymbol{u}_k\|^2_{\boldsymbol{L}^2(\Omega)} = 1.$$

As consequence,

$$\frac{1}{k} > \|\mathbf{D}(\boldsymbol{u}_k)\|^2_{\boldsymbol{L}^2(\Omega)} + \int_\Gamma |\boldsymbol{u}_k \cdot \boldsymbol{n}|^2 \, d\boldsymbol{\sigma}, \qquad \forall k \in \mathbb{N}^*.$$

Setting $\boldsymbol{w}_k = \boldsymbol{u}_k - P\boldsymbol{u}_k$ and using the Korn inequality, we deduce that $\boldsymbol{w}_k$ is bounded in $\boldsymbol{H}^1(\Omega)$. Then, by Rellich theorem, there exists a subsequence still called $\boldsymbol{w}_k$ that converges to $\boldsymbol{w}$ in $\boldsymbol{L}^2(\Omega)$ and weakly in $\boldsymbol{H}^1(\Omega)$. Since $\|\mathbf{D}(\boldsymbol{w})\|_{\boldsymbol{L}^2(\Omega)} = 0$ and $\boldsymbol{w} \cdot \boldsymbol{n} = 0$, we deduce that $\boldsymbol{w}$ belongs to $\boldsymbol{\mathcal{T}}(\Omega)$.

Moreover, we have $\boldsymbol{w} \in \boldsymbol{\mathcal{T}}(\Omega)^{\perp}$, where $\boldsymbol{\mathcal{T}}(\Omega)^{\perp}$ is the orthogonal complement of $\boldsymbol{\mathcal{T}}(\Omega)$ in $\boldsymbol{L}^2(\Omega)$. Hence $\boldsymbol{w} = \boldsymbol{0}$, in contradiction with the relation $\|\boldsymbol{w}_k\|_{\boldsymbol{L}^2(\Omega)} = 1$, $\forall\, k \geq 1$. The proof of lemma is completed. □

Now, we can solve the Stokes problem.

Theorem 3.4 *Suppose that* $\chi = 0$ *and* $g = 0$. *Let* $\boldsymbol{f} \in [\boldsymbol{H}_0^2(\mathrm{div}, \Omega)]'$ *and* $\boldsymbol{h} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$, *satisfying* (I.1) *and*

$$\langle \boldsymbol{f}, \boldsymbol{\beta} \rangle_{[\boldsymbol{H}_0^2(\mathrm{div},\Omega)]' \times \boldsymbol{H}_0^2(\mathrm{div},\Omega)} + \langle \boldsymbol{h}, \boldsymbol{\beta} \rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)} = 0. \tag{I.9}$$

Then, Stokes problem $(\mathcal{S}_T)$ *has a unique solution* $(\boldsymbol{u}, \pi) \in (\boldsymbol{H}^1(\Omega) \times L^2(\Omega))/\boldsymbol{\mathcal{N}}(\Omega)$. *In addition, we have the following estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)/\boldsymbol{\mathcal{T}}(\Omega)} + \|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^2(\mathrm{div},\Omega)]'} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\big). \tag{I.10}$$

Proof. It is clear that the bilinear form $\boldsymbol{a}(.,.)$ given by (I.5) is continuous on $\boldsymbol{H}^1(\Omega)$ and using Poincaré-Morrey inequality (II.5) we deduce that it is also coercive on $\boldsymbol{X}(\Omega)$. Moreover, we have the linear form $\ell : \boldsymbol{X}(\Omega) \to \mathbb{R}$, which is defined by

$$\ell(\boldsymbol{\varphi}) = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{[\boldsymbol{H}_0^2(\mathrm{div},\Omega)]' \times \boldsymbol{H}_0^2(\mathrm{div},\Omega)} + \langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)}.$$

is continuous over $\boldsymbol{H}^1(\Omega)/\boldsymbol{\mathcal{T}}(\Omega)$. Thus, we deduce by Lax-Milgram's theorem that problem (I.1) given in Proposition 3.1 has a unique solution $\boldsymbol{u} \in \boldsymbol{X}(\Omega)$ and $\pi \in L^2(\Omega)/\mathbb{R}$.
Using the variational problem and Poincaré-Morrey inequality, we have

$$\|\boldsymbol{u}\|_{\boldsymbol{X}(\Omega)} \leq C_0\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^2(\mathrm{div},\Omega)]'} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\big). \tag{I.11}$$

On the other hand,

$$\|\nabla \pi\|_{\boldsymbol{H}^{-1}(\Omega)} \leq \|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)} + \|\Delta \boldsymbol{u}\|_{\boldsymbol{H}^{-1}(\Omega)}.$$

We know that

$$\|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)} \leq C_1 \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^2(\mathrm{div},\Omega)]'}$$

and

$$\|\Delta \boldsymbol{u}\|_{\boldsymbol{H}^{-1}(\Omega)} \leq C \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}.$$

Therefore,

$$\inf_{k \in \mathbb{R}} \|\pi + k\|_{L^2(\Omega)} \leq C_2\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^2(\mathrm{div},\Omega)]'} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\big). \tag{I.12}$$

The estimation (I.10) follows easily from (I.11) and (I.12). □

We can also solve the Stokes problem when the divergence does not vanish and we have the following corollary the proof of which is given later in Corollary 2.3.

Corollary 3.5 *Let* $\boldsymbol{f}, \chi, g$ and $\boldsymbol{h}$ *such that*

$$\boldsymbol{f} \in [\boldsymbol{H}_0^2(\mathrm{div}, \Omega)]', \; \chi \in L^2(\Omega), \; g \in H^{\frac{1}{2}}(\Omega) \text{ and } \boldsymbol{h} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma),$$

satisfying the compatibility conditions (I.1), (I.9) *and*

$$\int_\Omega \chi d\boldsymbol{x} = \int_\Gamma g \, d\boldsymbol{\sigma}. \tag{I.13}$$

Then, problem $(\mathcal{S}_T)$ *has a unique solution* $(\boldsymbol{u}, \pi) \in (\boldsymbol{H}^1(\Omega) \times L^2(\Omega))/\mathbf{N}(\Omega)$*. Moreover we have the following estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)/\mathbf{\mathcal{T}}(\Omega)} + \|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^2(\mathrm{div},\Omega)]'} + \|\chi\|_{L^2(\Omega)} + \|g\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\big). \tag{I.14}$$

Remark 3.6 In the case $g = 0$, we can write problem $(\mathcal{S}_T)$ as follows:

$$(\mathcal{S}_T')\begin{cases} -\Delta \, \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \mathrm{div}\, \boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = -2\boldsymbol{\Lambda}\boldsymbol{u} - \boldsymbol{h} & \text{on } \Gamma. \end{cases}$$

3.2 The general case $1 < p < \infty$

We now investigate the case $1 < p < \infty$. We start by showing the existence and uniqueness of weak solution for $(\mathcal{S}_T)$. We start by studying the case $p \geq 2$.

Theorem 3.7 *Assume that* $p \geqslant 2$*. Suppose that* $\chi = 0$ *and* $g = 0$*. Then, for any* $\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))'$ *and* $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ *satisfying* (I.1) *the following compatibility conditions are satisfied*

$$\langle \boldsymbol{f}, \boldsymbol{\beta} \rangle_\Omega + \langle \boldsymbol{h}, \boldsymbol{\beta} \rangle_\Gamma = 0, \tag{I.15}$$

where $\langle .,. \rangle_\Gamma = \langle .,. \rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \times \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}$ *and* $\langle .,. \rangle_\Omega = \langle .,. \rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]' \times \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}$.
Then, problem $(\mathcal{S}_T)$ *has a unique solution* $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega))/\mathbf{N}(\Omega)$*. In addition, we have the following estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)/\mathbf{\mathcal{T}}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big).$$

Proof. We know that for all $p \geqslant 2$ we have

$$\big(\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)\big)' \hookrightarrow \big(\boldsymbol{H}_0^2(\mathrm{div}, \Omega)\big)' \quad \text{and} \quad \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \hookrightarrow \boldsymbol{H}^{-\frac{1}{2}}(\Gamma).$$

Then according to the Hilbert case the problem $(\mathcal{S}_T)$ has a unique solution $(\boldsymbol{u}, \pi) \in (\boldsymbol{H}^1(\Omega) \times L^2(\Omega))/\mathbf{N}(\Omega)$. Applying Corollary 2.5, we have

$$\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} + 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = -2\boldsymbol{\Lambda}\boldsymbol{u} \quad \text{on} \quad \boldsymbol{H}^{-\frac{1}{2}}(\Gamma),$$

because $\boldsymbol{u} \in \boldsymbol{E}^2(\Omega)$ and Ω is of class $C^{2,1}$. As consequence, $(\boldsymbol{u}, \pi)$ is solution of $(\mathcal{S}'_T)$. Therefore, $(\boldsymbol{u}, \pi)$ verifying the following problem: For all $\boldsymbol{\varphi} \in \boldsymbol{V}^2(\Omega)$,

$$\int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\; d\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{(\boldsymbol{H}^2_0(\mathrm{div},\Omega))' \times \boldsymbol{H}^2_0(\mathrm{div},\Omega)} + \langle 2\boldsymbol{\Lambda u} + \boldsymbol{h}, \boldsymbol{\varphi} \rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)}.$$

In particular, we have

$$\langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{(\boldsymbol{H}^2_0(\mathrm{div},\Omega))' \times \boldsymbol{H}^2_0(\mathrm{div},\Omega)} + \langle 2\boldsymbol{\Lambda u} + \boldsymbol{h}, \boldsymbol{\varphi} \rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)} = 0, \quad \text{for all} \quad \boldsymbol{\varphi} \in \boldsymbol{K}^2_T(\Omega).$$

Or more generally, for all $p \geqslant 2$ and $\boldsymbol{\varphi} \in \boldsymbol{K}^{p'}_T(\Omega)$

$$\langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{(\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega))' \times \boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)} + \langle 2\boldsymbol{\Lambda u} + \boldsymbol{h}, \boldsymbol{\varphi} \rangle_\Gamma = 0 \tag{I.16}$$

In the other hand, we have $\boldsymbol{u_\tau} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{6},6}(\Gamma)$. Therefore, by (I.16) and Theorem 2.7, we have $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$, for $2 \leq p \leq 6$. Now, suppose that $p \geq 6$, then repeated application of Theorem 2.7 enables us to assume that $\boldsymbol{u_\tau} \in \boldsymbol{W}^{1-\frac{1}{6},6}(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ for all $p \geq 6$. Clearly, $-2\boldsymbol{\Lambda u} - \boldsymbol{h}$ belongs to $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. Consequently, by the same reasoning, we deduce that the solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$. □

We can also solve the Stokes problem when the divergence does not vanish.

Corollary 3.8 *Let* $p \geqslant 2$. *Let* $\boldsymbol{f}, \chi, g$ and $\boldsymbol{h}$ *such that*

$$\boldsymbol{f} \in [\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]',\ \chi \in L^p(\Omega),\ g \in W^{1-\frac{1}{p},p}(\Gamma) \quad \text{and} \quad \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma),$$

satisfying (I.1), (I.13) *and* (I.15). *Then, Stokes problem* $(\mathcal{S}_T)$ *has a unique solution* $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega))/\mathbf{N}(\Omega)$. *Moreover we have the following estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{\mathcal{T}}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big).$$

Proof. We solve the following Neumann problem:

$$\Delta\theta = \chi \quad \text{in}\Omega \quad \text{and} \quad \frac{\partial \theta}{\partial \boldsymbol{n}} = g \quad \text{on}\Gamma. \tag{I.17}$$

For $\chi \in L^p(\Omega)$ and $g \in W^{1-\frac{1}{p},p}(\Gamma)$, problem (I.17) has a unique solution $\theta \in W^{2,p}(\Omega)/\mathbb{R}$ satisfying the following estimate:

$$\|\theta\|_{W^{2,p}(\Omega)/\mathbb{R}} \leq C\big(\|\chi\|_{L^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Omega)}\big).$$

Setting $\boldsymbol{z} = \boldsymbol{u} - \nabla\theta$, then $(\mathcal{S}_T)$ becomes: Find $(\boldsymbol{z}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ solution of problem

$$\begin{cases} -\Delta\,\boldsymbol{z} + \nabla\pi = \boldsymbol{f} + \nabla\chi & \text{in} \quad \Omega, \\ \operatorname{div}\boldsymbol{z} = 0 & \text{in} \quad \Omega, \\ \boldsymbol{z}\cdot\boldsymbol{n} = 0 & \text{on} \quad \Gamma, \\ 2\,[\mathbf{D}(\boldsymbol{z})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{H} & \text{on} \quad \Gamma. \end{cases} \tag{I.18}$$

with $\boldsymbol{H} = \boldsymbol{h} - 2\,[\mathbf{D}(\nabla\theta)\boldsymbol{n}]_{\boldsymbol{\tau}}$. Observe that $\boldsymbol{f} + \nabla\chi$ belongs to $[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'$ and $\boldsymbol{H}$ belongs to $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and satisfies $\boldsymbol{H}\cdot\boldsymbol{n} = 0$. Using Remark 3.2, to prove that problem (I.18) has a solution, it is necessary to prove the following compatibility condition:

$$\langle \boldsymbol{f} + \nabla\chi, \boldsymbol{\beta}\rangle_\Omega + \langle \boldsymbol{H}, \boldsymbol{\beta}\rangle_\Gamma = 0, \tag{I.19}$$

which is equivalent by (I.15) to the condition

$$\langle \nabla\chi, \boldsymbol{\beta}\rangle_\Omega - \langle 2\,[\mathbf{D}(\nabla\theta)\boldsymbol{n}]_{\boldsymbol{\tau}}\,, \boldsymbol{\beta}\rangle_\Gamma = 0$$

In fact, it is clear that $\nabla\theta$ belongs to $\boldsymbol{E}^p(\Omega)$. Then applying Green formula (I.15) with $\boldsymbol{v} = \nabla\theta$ and $\boldsymbol{\varphi} = \boldsymbol{\beta}$ we obtain

$$\begin{aligned} -\langle 2\,[\mathbf{D}(\nabla\theta)\boldsymbol{n}]_{\boldsymbol{\tau}}\,, \boldsymbol{\beta}\rangle_\Gamma = & \; -\langle \Delta(\nabla\theta), \boldsymbol{\beta}\rangle_\Omega \\ = & \; -\langle \nabla\chi, \boldsymbol{\beta}\rangle_\Omega\,. \end{aligned}$$

Thus, due to Theorem 3.7, problem (I.18) has a unique solution $(\boldsymbol{z}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbf{N}(\Omega)$. $\square$

The following theorem will be proved by duality argument.

Theorem 3.9 *Assume that* $1 < p < 2$. *Let* $\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega))'$, $\chi \in L^p(\Omega)$, $g \in W^{1-\frac{1}{p},p}(\Gamma)$ *and* $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$, *satisfying the compatibility conditions* (I.1), (I.13) *and* (I.15). *Then, problem* $(\mathcal{S}_T)$ *has a unique solution* $(\boldsymbol{u}, \pi) \in \big(\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)\big)/\mathbf{N}(\Omega)$. *In addition, we have the following estimate:*

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{\mathcal{T}}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \le C\big(&\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} \\ &+ \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big). \end{aligned}$$

Proof. The proof will be divided into two steps.
First Step: We suppose that $g = 0$. Using Green formula (I.15), we deduce that problem $(\mathcal{S}_T)$ has the following equivalent variational formulation: Find $(\boldsymbol{u}, \pi)$ in $\big(\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)\big)/\mathbf{N}(\Omega)$ satisfying $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ such that: $\forall\eta \in L^{p'}(\Omega)$, $\forall\boldsymbol{w} \in \boldsymbol{E}^{p'}(\Omega)$ satisfying $\boldsymbol{w}\cdot\boldsymbol{n} = 0$ and $[\boldsymbol{D}(\boldsymbol{w})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0}$ on Γ

$$\begin{aligned} &\langle \boldsymbol{u}, -\Delta\boldsymbol{w} + \nabla\eta\rangle_{\boldsymbol{H}_0^p(\operatorname{div},\Omega)\times(\boldsymbol{H}_0^p(\operatorname{div},\Omega))'} - \int_\Omega \pi\operatorname{div}\boldsymbol{w}\,d\boldsymbol{x} \\ &= \langle \boldsymbol{f}, \boldsymbol{w}\rangle_{(\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega))'\times\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)} + \langle \boldsymbol{h}, \boldsymbol{w}\rangle_\Gamma - \int_\Omega \chi\eta d\boldsymbol{x}. \end{aligned} \tag{I.20}$$

According to Corollary 2.3 for any pair $(\boldsymbol{F}, \varphi)$ in $((\boldsymbol{H}_0^p(\mathrm{div},\Omega))' \perp \boldsymbol{\mathcal{T}}(\Omega)) \times L_0^{p'}(\Omega)$ there exists a unique solution $(\boldsymbol{w}, \eta) \in \big(\boldsymbol{W}^{1,p'}(\Omega) \times L^{p'}(\Omega)\big)/\boldsymbol{\mathcal{N}}(\Omega)$ such that

$$-\Delta \boldsymbol{w} + \nabla \eta = \boldsymbol{F} \quad \text{and} \quad \operatorname{div} \boldsymbol{w} = \varphi \quad \text{in}\, \Omega, \quad \boldsymbol{w}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad [\boldsymbol{D}(\boldsymbol{w})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} \quad \text{on}\, \Gamma$$

and

$$\inf_{(\boldsymbol{\lambda},\mu)\in\boldsymbol{\mathcal{N}}(\Omega)} \big(\|\boldsymbol{w}+\boldsymbol{\lambda}\|_{\boldsymbol{W}^{1,p'}(\Omega)} + \|\eta+\mu\|_{L^{p'}(\Omega)}\big) \leq C\big(\|\boldsymbol{F}\|_{(\boldsymbol{H}_0^p(\mathrm{div},\Omega))'} + \|\varphi\|_{L^{p'}(\Omega)}\big). \tag{I.21}$$

Let T be a linear form defined from $((\boldsymbol{H}_0^p(\mathrm{div},\Omega))' \perp \boldsymbol{\mathcal{T}}(\Omega)) \times L_0^{p'}(\Omega)$ onto $\mathbb{R}$ by

$$T : (\boldsymbol{F},\, \varphi) \mapsto \langle \boldsymbol{f}, \boldsymbol{w}\rangle_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'\times \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} + \langle \boldsymbol{h}, \boldsymbol{w}\rangle_\Gamma - \int_\Omega \chi\eta d\boldsymbol{x}.$$

Note that, using (I.21), for any $(\boldsymbol{\lambda}, k) \in \boldsymbol{\mathcal{N}}(\Omega)$ we have

$$\begin{aligned}|T(\boldsymbol{F},\varphi)| &\leq |\langle \boldsymbol{f}, \boldsymbol{w}+\boldsymbol{\lambda}\rangle_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'\times \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} + \langle \boldsymbol{h}, \boldsymbol{w}+\boldsymbol{\lambda}\rangle_\Gamma - \int_\Omega \chi(\eta+k) d\boldsymbol{x}| \\ &\leq C\big(\|f\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'} + \|\chi\|_{L^{p'}(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big)\big(\|\boldsymbol{F}\|_{(\boldsymbol{H}_0^p(\mathrm{div},\Omega))'} + \|\varphi\|_{L^{p'}(\Omega)}\big)\end{aligned}$$

Thus the linear form T is continuous on $\big((\boldsymbol{H}_0^p(\mathrm{div},\Omega))' \perp \boldsymbol{\mathcal{T}}(\Omega)\big) \times L_0^{p'}(\Omega)$ and we deduce that there exists a unique $(\boldsymbol{u}, \pi)$ in $\boldsymbol{H}_0^p(\mathrm{div},\Omega)_{/\boldsymbol{\mathcal{T}}(\Omega)} \times L^p(\Omega)_{/\mathbb{R}}$ such that

$$T(\boldsymbol{F},\varphi) = \langle \boldsymbol{u}, \boldsymbol{F}\rangle_{\boldsymbol{H}_0^p(\mathrm{div},\Omega)\times(\boldsymbol{H}_0^p(\mathrm{div},\Omega))'} + \int_\Omega \pi\varphi d\boldsymbol{x}$$

To finish, we shall prove that $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$. We recall that $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ and $\Delta\boldsymbol{u} = \nabla\pi - \boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))' \hookrightarrow \boldsymbol{W}^{-1,p}(\Omega)$. Now, we introduce the following spaces:

$$\boldsymbol{X}_p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega);\, \operatorname{div}\boldsymbol{v} \in W_0^{1,p}(\Omega)\}$$

and

$$\boldsymbol{T}_p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \Delta\boldsymbol{v} \in (\boldsymbol{X}_{p'}(\Omega))'\}.$$

It is clear that $\boldsymbol{W}^{-1,p}(\Omega) \hookrightarrow (\boldsymbol{X}_{p'}(\Omega))'$ and thus we have $\boldsymbol{u} \in \boldsymbol{T}_p(\Omega)$. Therefore, according to Lemma 12 of [7], we have $\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and also $-2\boldsymbol{\Lambda}\boldsymbol{u} - \boldsymbol{h}$ belongs to $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. Finally, using Remark 3.6 and Theorem 2.7, we deduce that $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$.

Second Step: We solve Neumann problem (I.17) with $\chi \in L^p(\Omega), g \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$ satisfying (I.13). There exists a unique $\theta \in W^{2,p}(\Omega)/\mathbb{R}$ solution of (I.17). Setting $\boldsymbol{z} = \boldsymbol{u} - \nabla\theta$ the rest of the proof runs as in proof of Corollary 2.3. □

Remark 3.10 We are so far interested in the boundary conditions on the stress tensor. However, it is also interesting to consider the boundary conditions on the tangential components of the normal derivative:

$$\begin{cases} -\Delta\, \boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = \chi & \text{in}\, \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = g \quad \text{and} \quad (\dfrac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}})_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on}\, \Gamma. \end{cases} \tag{I.22}$$

As in Corollary 2.5, using the relation (I.5), we can write

$$(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}})_{\boldsymbol{\tau}} = \nabla_{\boldsymbol{\tau}}(\boldsymbol{u}\cdot\boldsymbol{n}) - \mathbf{curl}\, \boldsymbol{u}\times\boldsymbol{n} - \boldsymbol{\Lambda}\boldsymbol{u} \qquad \text{in } \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma). \tag{I.23}$$

As consequence, in the same way as for problem $(\mathcal{S}_T)$ we can solve the problem (I.22) and we have the following theorem:

Theorem 3.11 *Let $\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega))'$, $\chi \in L^p(\Omega)$, $g \in W^{1-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$, satisfying the compatibility conditions* (I.1) *and* (I.13)*. Then, problem* (I.22) *has a unique solution* $(\boldsymbol{u},\pi)$ *which belongs to* $\boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$*. In addition, we have the following estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq C(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}).$$

4 Strong Solutions of $(\mathcal{S}_T)$

We prove the existence of strong solutions $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{2,p}(\Omega)\times L^p(\Omega)$ for the Stokes problem.

Theorem 4.1 *Let $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \chi \in W^{1,p}(\Omega)$, $g \in W^{2-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$, satisfying the compatibility conditions* (I.1), (I.13) *and* (I.15)*. Then, problem $(\mathcal{S}_T)$ has a unique solution $(\boldsymbol{u},\pi)$ belongs to $(\boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega))/\mathbf{N}(\Omega)$ and satisfies the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)/\mathcal{T}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|g\|_{W^{2-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{\frac{1}{p},p}(\Gamma)}). \tag{I.1}$$

Proof. Before, we note that under the hypothesis of Theorem 2.4, the problem $(\mathcal{S}_T)$ has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbf{N}(\Omega)$.

To prove the regularity of the velocity, we set $\boldsymbol{z} = \mathbf{curl}\, \boldsymbol{u}$. Observe that $-\Delta\boldsymbol{z} = \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{z}$ and $\mathbf{curl}\, \boldsymbol{z} = -\Delta\boldsymbol{u} + \nabla\chi = \boldsymbol{f} + \nabla(\chi - \pi)$. Using Remark 2.2, we deduce that $\boldsymbol{z}$ satisfies the following problem:

$$\begin{cases} -\Delta\, \boldsymbol{z} = \mathbf{curl}\, \boldsymbol{f} & \text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{z} = 0 & \text{in} \quad \Omega, \\ \boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{H} & \text{on} \quad \Gamma. \end{cases}$$

where $\boldsymbol{H} = -2\boldsymbol{\Lambda}\boldsymbol{u} - \boldsymbol{h}$. Since, Ω is of class $\mathcal{C}^{2,1}$ and $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, we have $\boldsymbol{\Lambda}\boldsymbol{u} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$. Consequently, $\boldsymbol{H} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$ and satisfies the compatibility condition (I.1). Since, $\mathbf{curl}\, \boldsymbol{f}$ belongs to $[\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'$ and satisfies the following compatibility condition

$$\langle \mathbf{curl}\, \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{[\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]' \times \boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)} = 0, \quad \text{for all}\, \boldsymbol{\varphi} \in \boldsymbol{K}_N^{p'}(\Omega),$$

we deduce from Theorem 2.8 that $\boldsymbol{z} \in \boldsymbol{W}^{1,p}(\Omega)$. Then $\boldsymbol{u} \in \boldsymbol{X}^{2,p}(\Omega)$, where

$$\boldsymbol{X}^{2,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\, \mathrm{div}\, \boldsymbol{v} \in W^{1,p}(\Omega),\, \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)\, \text{and}\, \boldsymbol{v} \cdot \boldsymbol{n} \in W^{2-\frac{1}{p},\frac{1}{p}}(\Gamma)\}.$$

As a consequence, thanks to the imbedding of $\boldsymbol{X}^{2,p}(\Omega)$ in $\boldsymbol{W}^{2,p}(\Omega)$ (see [9]), the solution $\boldsymbol{u}$ of the problem $(\mathcal{S}_T)$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$. Finally, since

$$\nabla \pi = \Delta \boldsymbol{u} + \boldsymbol{f} \in \boldsymbol{L}^p(\Omega),$$

we deduce that $\pi \in W^{1,p}(\Omega)$. □

5 Very weak solutions of $(\mathcal{S}_T)$

In this section we want to prove the existence of a very weak solution for the Stokes problem $(\mathcal{S}_T)$. To prove this, we shall apply a technique used in [7] for the Dirichlet boundary conditions and in [8] for the Navier boundary conditions. First, we start by introducing the following space:

$$\boldsymbol{T}^p(\Omega) = \left\{ \boldsymbol{\varphi} \in \boldsymbol{H}_0^p(\mathrm{div}, \Omega);\, \mathrm{div}\, \boldsymbol{\varphi} \in W_0^{1,p}(\Omega) \right\}.$$

We recall now some preliminary results which we shall use in the sequel (for instance see [8]).

Lemma 5.1 *The space $\mathcal{D}(\Omega)$ is dense in $\boldsymbol{T}^p(\Omega)$ and for all $\chi \in W^{-1,p}(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{T}^{p'}(\Omega)$, we have*

$$\langle \nabla \chi, \boldsymbol{\varphi} \rangle_{(\boldsymbol{T}^{p'}(\Omega))' \times \boldsymbol{T}^{p'}(\Omega)} = -\langle \chi, \mathrm{div}\, \boldsymbol{\varphi} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p}(\Omega)}. \tag{I.1}$$

Lemma 5.2 *A distribution $\boldsymbol{f}$ belongs to $(\boldsymbol{T}^p(\Omega))'$ if and only if there exist $\boldsymbol{\Psi} \in \boldsymbol{L}^{p'}(\Omega)$ and $f_0 \in W^{-1,p'}(\Omega)$, such that*

$$\boldsymbol{f} = \boldsymbol{\Psi} + \nabla f_0.$$

Moreover, we have

$$\|\boldsymbol{f}\|_{(\boldsymbol{T}^p(\Omega))'} = \mathrm{Max}\{\|\boldsymbol{\Psi}\|_{\boldsymbol{L}^{p'}(\Omega)}, \|f_0\|_{W^{-1,p'}(\Omega)}\}.$$

Giving a meaning to Navier boundary condition of a very weak solution of a Stokes problem is not easy. For this reason, we need to introduce the space

$$\boldsymbol{H}_p(\Delta; \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\, \Delta \boldsymbol{v} \in (\boldsymbol{T}^{p'}(\Omega))'\},$$

which is a Banach space for the norm:

$$\|\boldsymbol{v}\|_{\boldsymbol{H}_p(\Delta;\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\Delta \boldsymbol{v}\|_{(\boldsymbol{T}^{p'}(\Omega))'}.$$

The following lemma (see [8]) is important to prove a trace result.

Lemma 5.3 *The space* $\mathcal{D}(\overline{\Omega})$ *is dense in* $\boldsymbol{H}_p(\Delta;\Omega)$.

Let us introduce the space

$$\boldsymbol{S}^{p}(\Omega)=\{\boldsymbol{\varphi}\in\ \boldsymbol{W}^{2,p}(\Omega);\ \boldsymbol{\varphi}\cdot\boldsymbol{n}=0,\ \operatorname{div}\boldsymbol{\varphi}=0,\ [\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}=\boldsymbol{0}\,\text{on}\,\Gamma\}$$

and recall the following formula (see [3]):

$$\operatorname{div}\boldsymbol{v}=\operatorname{div}_\Gamma\boldsymbol{v}_{\boldsymbol{\tau}}+\boldsymbol{K}\boldsymbol{v}\cdot\boldsymbol{n}+\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}}\cdot\boldsymbol{n}\quad\text{on}\quad\Gamma. \tag{I.2}$$

The following lemma proves that for any $\boldsymbol{v}$ which belongs to $\boldsymbol{H}_p(\Delta;\Omega)$, we have $[\boldsymbol{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}$ is well defined in $\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)$.

Lemma 5.4 *The mapping* $\Upsilon:\boldsymbol{u}\mapsto[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}$ *on the space* $\mathcal{D}(\overline{\Omega})$ *can be extended by continuity to a linear mapping still denoted by* Υ*, from* $\boldsymbol{H}_p(\Delta;\Omega)$ *into* $\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)$ *and we have the following Green formula: For any* $\boldsymbol{u}\in\boldsymbol{H}_p(\Delta;\Omega)$ *and* $\boldsymbol{\varphi}\in\boldsymbol{S}^{p'}(\Omega)$,

$$\langle\Delta\boldsymbol{u},\boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)}=\int_\Omega\boldsymbol{u}\cdot\Delta\boldsymbol{\varphi}d\boldsymbol{x}+\langle 2[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}},\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times\boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}. \tag{I.3}$$

Proof. First, let $\boldsymbol{u}\in\mathcal{D}(\overline{\Omega})$. Then for any $\boldsymbol{\varphi}\in\boldsymbol{S}^{p'}(\Omega)$, we have on one hand

$$\begin{aligned}\langle\Delta\boldsymbol{u},\boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)} &= \int_\Omega\operatorname{div}\boldsymbol{u}\operatorname{div}\boldsymbol{\varphi}d\boldsymbol{x}-2\int_\Omega\boldsymbol{D}(\boldsymbol{u}):\boldsymbol{D}(\boldsymbol{\varphi})d\boldsymbol{x}\\ &+\ \langle 2[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}},\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times\boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}\end{aligned}$$

and on the other hand

$$\int_\Omega\boldsymbol{u}\cdot\Delta\boldsymbol{\varphi}d\boldsymbol{x}\ =\ \int_\Omega\operatorname{div}\boldsymbol{u}\operatorname{div}\boldsymbol{\varphi}d\boldsymbol{x}-2\int_\Omega\boldsymbol{D}(\boldsymbol{u}):\boldsymbol{D}(\boldsymbol{\varphi})d\boldsymbol{x}$$

Thus, for any $\boldsymbol{u}\in\mathcal{D}(\overline{\Omega})$ and $\boldsymbol{\varphi}\in\boldsymbol{S}^{p'}(\Omega)$

$$\langle\Delta\boldsymbol{u},\boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)}=\int_\Omega\boldsymbol{u}\cdot\Delta\boldsymbol{\varphi}d\boldsymbol{x}+\langle 2[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}},\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times\boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}.$$

Now, let $\boldsymbol{\mu}\in\ \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)$. We know that there exits $\boldsymbol{\varphi}\in\ \boldsymbol{W}^{2,p'}(\Omega)$ such that

$$\boldsymbol{\varphi}=\boldsymbol{\mu}_{\boldsymbol{\tau}}\quad\text{and}\quad\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}=\boldsymbol{\Lambda}\boldsymbol{\mu}-\boldsymbol{n}\operatorname{div}_\Gamma\boldsymbol{\mu}_{\boldsymbol{\tau}}\quad\text{on}\quad\Gamma.$$

In addition, we have the following estimate

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{2,p'}(\Omega)}\le C\|\boldsymbol{\mu}_{\boldsymbol{\tau}}\|_{\boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}\le C\|\boldsymbol{\mu}\|_{\boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}. \tag{I.4}$$

As $\boldsymbol{\Lambda\mu}\cdot\boldsymbol{n}=0$ on Γ (see (I.3)), we have

$$\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\cdot\boldsymbol{n}=-\mathrm{div}_\Gamma\,\boldsymbol{\mu_\tau}\quad\text{on}\quad\Gamma.$$

Using identity (I.2), we deduce that

$$\mathrm{div}\boldsymbol{\varphi}=0\quad\text{on}\quad\Gamma.$$

Also, using (I.4), we have

$$\begin{aligned}[\boldsymbol{D}(\boldsymbol{\varphi})\boldsymbol{n}]_{\boldsymbol{\tau}} &= (\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}})_{\boldsymbol{\tau}}-\boldsymbol{\Lambda\varphi}\\ &= \boldsymbol{\Lambda\mu}-\boldsymbol{\Lambda\mu}\\ &= 0.\end{aligned}$$

To recapitulate, $\boldsymbol{\varphi}$ belongs to $\boldsymbol{S}^{p'}(\Omega)$ and satisfies

$$\boldsymbol{\varphi}=\boldsymbol{\mu_\tau}\quad\text{and}\quad\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}=\boldsymbol{\Lambda\mu}-\boldsymbol{n}\mathrm{div}_\Gamma\,\boldsymbol{\mu_\tau}\quad\text{on}\,\Gamma.\tag{I.5}$$

Therefore, we can bound the boundary term as follows: For functions $\boldsymbol{\varphi}$ belonging to $\boldsymbol{S}^{p'}(\Omega)$

$$\begin{aligned}|\langle[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}},\boldsymbol{\mu_\tau}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times\boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}| &= |\langle[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}},\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times\boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}|\\ &\leq |\langle\Delta\boldsymbol{u},\boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)}-\int_\Omega\boldsymbol{u}\cdot\Delta\boldsymbol{\varphi}d\boldsymbol{x}|\\ &\leq \|\boldsymbol{\varphi}\|_{\boldsymbol{T}^{p'}(\Omega)}\|\Delta\boldsymbol{u}\|_{(\boldsymbol{T}^{p'}(\Omega))'}+\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{2,p'}(\Omega)}\\ &\leq C\|\boldsymbol{u}\|_{\boldsymbol{H}_p(\Delta;\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{2,p'}(\Omega)}\\ &\leq C\|\boldsymbol{u}\|_{\boldsymbol{H}_p(\Delta;\Omega)}\|\boldsymbol{\mu}\|_{\boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}.\end{aligned}$$

Consequently, we obtain for any $\boldsymbol{u}\in\boldsymbol{\mathfrak{D}}(\overline{\Omega})$:

$$\|[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}\|_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)}\leq C\|\boldsymbol{u}\|_{\boldsymbol{H}_p(\Delta;\Omega)}.$$

It follows that, the linear mapping $\Upsilon:\boldsymbol{u}\mapsto[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}$ defined in $\boldsymbol{\mathfrak{D}}(\overline{\Omega})$ is continuous for the norm of $\boldsymbol{H}_p(\Delta;\Omega)$. Finally, by density of $\boldsymbol{\mathfrak{D}}(\overline{\Omega})$ in $\boldsymbol{H}_p(\Delta;\Omega)$, we can extend continuously this mapping from $\boldsymbol{H}_p(\Delta;\Omega)$ into $\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)$ and the Green formula (I.3) holds. □

Now, we are in position to prove the existence and uniqueness of a very weak solution for Stokes problem $(\mathcal{S}_T)$. The proof of the following theorem is similar to Theorem 4.15 in [8].

Theorem 5.5 *Given any* $\boldsymbol{f},\chi,g$ *and* $\boldsymbol{h}$ *with*

$$\boldsymbol{f}\in(\boldsymbol{T}^{p'}(\Omega))',\chi\in L^p(\Omega),g\in W^{-\frac{1}{p},p}(\Gamma),\boldsymbol{h}\in W^{-1-\frac{1}{p},p}(\Gamma),$$

and satisfying the compatibility conditions (I.1), (I.15) *and*

$$\int_\Omega \chi d\,\boldsymbol{x} = \langle g, 1\rangle_{W^{-\frac{1}{p},p}(\Gamma)\times W^{\frac{1}{p},p'}(\Gamma)}. \tag{I.6}$$

Then, problem $(\mathcal{S}_T)$ *has a unique solution*

$$\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)/\boldsymbol{\mathcal{T}}(\Omega) \quad \text{and} \quad \pi \in W^{-1,p}(\Omega)/\mathbb{R}.$$

Moreover, we have the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)/\boldsymbol{\mathcal{T}}(\Omega)} + \|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq C(\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\|_{W^{-1-\frac{1}{p},p}(\Gamma)}).$$

Proof. The basic idea of this proof is to use the duality argument and the strong solution of the adjoint problem with Navier boundary condition. The proof falls into three parts: in the first part we will write the variational formulation, after that we prove the existence of a very weak solution when $g = 0$ and, in the third part, we will finish with the case that g is not vanishing.

First Step : Observe that if $\boldsymbol{u} \in \boldsymbol{H}_p(\Delta;\Omega)$ and $\pi \in W^{1,p}(\Omega)$, then according to (I.1) and (I.3) we have for any $\boldsymbol{\varphi} \in \boldsymbol{S}^{p'}(\Omega)$ and for any $q \in W^{1,p'}(\Omega)$,

$$\begin{aligned}
\langle -\Delta\boldsymbol{u} + \nabla\pi, \boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times \boldsymbol{T}^{p'}(\Omega)} &= -\int_\Omega \boldsymbol{u}\cdot\Delta\boldsymbol{\varphi} d\boldsymbol{x} - \langle 2[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)} \\
&- \langle \pi, \operatorname{div}\boldsymbol{\varphi}\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p'}(\Omega)}, \\
\int_\Omega \boldsymbol{u}\cdot\nabla q d\boldsymbol{x} &= -\int_\Omega q \operatorname{div}\boldsymbol{u} d\boldsymbol{x} + \langle \boldsymbol{u}\cdot\boldsymbol{n}, q\rangle_\Gamma.
\end{aligned}$$

Thus, if $(\boldsymbol{u},\pi) \in \boldsymbol{L}^p(\Omega)\times W^{-1,p}(\Omega)$ solution of $(\mathcal{S}_T)$, then $\boldsymbol{u} \in \boldsymbol{H}_p(\Delta;\Omega)$ and for any $\boldsymbol{\varphi} \in \boldsymbol{S}^{p'}(\Omega)$ and for any $q \in W^{1,p'}(\Omega)$ we have

$$\begin{aligned}
-\int_\Omega \boldsymbol{u}\cdot\Delta\boldsymbol{\varphi} d\boldsymbol{x} - \langle \pi, \operatorname{div}\boldsymbol{\varphi}\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p'}(\Omega)} &= \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times \boldsymbol{T}^{p'}(\Omega)} \\
&+ \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}, \\
\int_\Omega \boldsymbol{u}\cdot\nabla q d\boldsymbol{x} &= -\int_\Omega \chi q d\boldsymbol{x} + \langle g, q\rangle_\Gamma.
\end{aligned} \tag{I.7}$$

Conversely, let $(\boldsymbol{u},\pi) \in \boldsymbol{L}^p(\Omega)\times W^{-1,p}(\Omega)$ be a solution of (I.7). Therefore, it follows immediately that,

$$-\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = \chi \quad \text{in} \quad \Omega.$$

Now, writing

$$\Delta\boldsymbol{u} = \nabla\pi - \boldsymbol{f}$$

and using Lemma 5.2 we deduce that $\boldsymbol{u}$ belongs to $\boldsymbol{H}_p(\Delta;\Omega)$. As consequence, applying Lemma 5.4 we have $2[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} \in \boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)$. We repeat application of (I.3) and using Lemma 5.1 enables us to write: for any $\boldsymbol{\varphi} \in \boldsymbol{S}^{p'}(\Omega)$,

$$\int_\Omega \boldsymbol{u} \cdot \Delta \boldsymbol{\varphi} d\boldsymbol{x} + \langle 2[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)} = -\langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times \boldsymbol{T}^{p'}(\Omega)} - \langle \pi, \operatorname{div} \boldsymbol{\varphi}\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p'}(\Omega)}. \tag{I.8}$$

Comparing (I.7) with (I.8), we get

$$\langle 2[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)} = \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}.$$

Let $\boldsymbol{\mu} \in \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)$. Analysis similar to that in the proof of Lemma 5.4, there exists a function $\boldsymbol{\varphi} \in \boldsymbol{S}^{p'}(\Omega)$ such that $\boldsymbol{\varphi} = \boldsymbol{\mu}_{\boldsymbol{\tau}}$.
Consequently,

$$\langle 2[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\mu}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)} = \langle \boldsymbol{h}, \boldsymbol{\mu}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)},$$

Accounting on the above equality, we have

$$2[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h} \quad \text{on} \quad \Gamma.$$

It still remains to prove that $\boldsymbol{u}\cdot\boldsymbol{n} = g \quad \text{on} \quad \Gamma$. For this, we consider the equation $\operatorname{div} \boldsymbol{u} = \chi$ in Ω, we multiply this equation by $q \in W^{1,p'}(\Omega)$, do the integration by parts and compare with (I.7). Then we get

$$\langle \boldsymbol{u}\cdot\boldsymbol{n}, q\rangle_\Gamma = \langle g, q\rangle_\Gamma.$$

As consequence,

$$\boldsymbol{u}\cdot\boldsymbol{n} = g \quad \text{in} \quad W^{-\frac{1}{p},p}(\Gamma).$$

This finishes the first step.
<u>**Second Step :**</u> We suppose that

$$g = 0 \quad \text{on} \quad \Gamma \quad \text{and} \quad \int_\Omega \chi d\boldsymbol{x} = 0.$$

According to Theorem 2.4, for any $(\boldsymbol{F}, \xi) \in (\boldsymbol{L}^{p'}(\Omega)\perp \boldsymbol{\mathcal{T}}(\Omega)) \times (W_0^{1,p'}(\Omega)\cap L_0^{p'}(\Omega))$ there exists a unique solution $(\boldsymbol{\varphi}, q) \in \big(\boldsymbol{W}^{2,p'}(\Omega) \times W^{1,p'}(\Omega)\big)/\boldsymbol{\mathcal{N}}(\Omega)$ such that

$$-\Delta\boldsymbol{\varphi} + \nabla q = \boldsymbol{F} \quad \text{and} \quad \operatorname{div}\boldsymbol{\varphi} = \xi \quad \text{in}\,\Omega, \quad \boldsymbol{\varphi}\cdot\boldsymbol{n} = 0, \quad \text{and} \quad [\boldsymbol{D}(\boldsymbol{\varphi})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} \quad \text{on}\,\Gamma$$

and

$$\inf_{(\boldsymbol{\lambda},k)\in\boldsymbol{\mathcal{N}}(\Omega)} \big(\|\boldsymbol{\varphi}+\boldsymbol{\lambda}\|_{\boldsymbol{W}^{2,p'}(\Omega)} + \|q+k\|_{W^{1,p'}(\Omega)}\big) \leq C\big(\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\varphi\|_{W^{1,p'}(\Omega)}\big). \tag{I.9}$$

Let T be a linear form defined from $\big(\boldsymbol{L}^{p'}(\Omega)\perp\boldsymbol{\mathcal{T}}(\Omega)\big) \times \big(W_0^{1,p'}(\Omega)\cap L_0^{p'}(\Omega)\big)$ onto $\mathbb{R}$ by

$$T : (\boldsymbol{F}, \xi) \mapsto \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times \boldsymbol{T}^{p'}(\Omega)} + \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)} - \int_\Omega \chi q d\boldsymbol{x}.$$

Note that for any $(\boldsymbol{\lambda}, k) \in \mathbf{N}(\Omega)$,

$$\begin{aligned} |T(\boldsymbol{F}, \xi)| &\leq |\langle \boldsymbol{f}, \boldsymbol{\varphi} + \boldsymbol{\lambda} \rangle_{(\boldsymbol{T}^{p'}(\Omega))' \times \boldsymbol{T}^{p'}(\Omega)} + \langle \boldsymbol{h}, \boldsymbol{\varphi} + \boldsymbol{\lambda} \rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma) \times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)} - \int_\Omega \chi(q+k)| \\ &\leq C\big(\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'} + \|\chi\|_{L^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)}\big)\big(\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\xi\|_{W^{1,p'}(\Omega)}\big). \end{aligned}$$

Thus, the linear form T is continuous on $\big(\boldsymbol{L}^{p'}(\Omega) \perp \boldsymbol{\mathcal{T}}(\Omega)\big) \times \big(W_0^{1,p'}(\Omega) \cap L_0^{p'}(\Omega)\big)$ and we deduce that there exists a unique $(\boldsymbol{u}, \pi)$ in $\big(\boldsymbol{L}^p(\Omega)_{/\boldsymbol{\mathcal{T}}(\Omega)} \times W^{-1,p}(\Omega)/_{\mathbb{R}}\big)$ solution of problem (I.7).

It remains to prove the existence and uniqueness of very weak solution when g is not vanish.

Third Step : Now, we suppose that $g \neq 0$ satisfying the compatibility condition (I.6). Then, there exists a unique $\theta \in W^{1,p}(\Omega)/\mathbb{R}$ solution of Neumann problem (I.17). Now, setting $\boldsymbol{z} = \boldsymbol{u} - \nabla\theta$, then $(\mathcal{S}_T)$ becomes: Find $(\boldsymbol{z}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ satisfying (I.18), with $\boldsymbol{H} = \boldsymbol{h} - 2\,[\mathbf{D}(\nabla\theta)\boldsymbol{n}]_{\boldsymbol{\tau}}$. Since $\nabla\theta \in \boldsymbol{H}_p(\Delta; \Omega)$, it is clear that $\boldsymbol{H}$ belongs to $\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)$ and satisfies $\boldsymbol{H} \cdot \boldsymbol{n} = 0$. On the other hand, we have $\boldsymbol{f} + \nabla\chi$ belongs to $(\boldsymbol{T}^{p'}\Omega)'$. Thus, due to the second step, there exists a $(\boldsymbol{z}, \pi) \in (\boldsymbol{L}^p(\Omega) \times W^{1,p}(\Omega))/\mathbf{N}(\Omega)$ solution of (I.18), and the proof is complete. $\square$

6 Eigenfunctions of the Stokes problem

In [26] T. Clopeau et al. showed, in two dimensions, the existence of an orthonormal basis formed by the eigenfunctions of Stokes operator with Navier boundary condition. It was the idea to invest in a relationship between $\mathbf{curl}\,\boldsymbol{u}$ and $[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau$ on the boundary and reduce to the resolution of bi-Laplacian problem. By this method they were able to show the existence of the basis without solving the stationary Stokes problem. Yet, this technique is not valid in the case of three dimension.

Our objective in this section is to show the existence of the Hilbertian basis of $\boldsymbol{V}^2(\Omega)$ formed by a sequence of eigenfunctions $(\boldsymbol{v}_k)_k \in \boldsymbol{H}^1(\Omega)$ of problem $(\mathcal{S}_T)$. Let us introduce the following problem:

$$\begin{cases} -\Delta\,\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = 0 & \text{in} \quad \Omega, \\ \qquad \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0} & \text{on} \quad \Gamma, \\ \qquad \displaystyle\int_\Omega \boldsymbol{u}\cdot\boldsymbol{\beta}\,d\boldsymbol{x} = 0. \end{cases} \tag{I.1}$$

We start by the following lemma:

Lemma 6.1 *Let* $\boldsymbol{f} \in \big(\boldsymbol{H}_0^2(\operatorname{div}, \Omega)\big)'$ *satisfying the following compatibility condition:*

$$\langle \boldsymbol{f}, \boldsymbol{\beta} \rangle_{\left(\boldsymbol{H}_0^2(\operatorname{div},\Omega)\right)' \times \boldsymbol{H}_0^2(\operatorname{div},\Omega)} = 0. \tag{I.2}$$

Then, problem (I.1) *has a unique solution* $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ *and* $\pi \in L^2(\Omega)/\mathbb{R}$ *satisfying the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{f}\|_{\left(\boldsymbol{H}_0^2(\operatorname{div},\Omega)\right)'}$$

Proof. Let $(\boldsymbol{u}, \pi)$ be one of solutions given by Theorem 3.4. We know that $\boldsymbol{u}$ can be written as

$\boldsymbol{u} = P\boldsymbol{u}+c\boldsymbol{\beta}$, where $P\boldsymbol{u}$ is the orthogonal projection of $\boldsymbol{u}$ onto $\boldsymbol{L}^2(\Omega)\perp\boldsymbol{\mathcal{T}}(\Omega)$ and $c \in \mathbb{R}$. It is immediate that $P\boldsymbol{u}$ is the unique solution of (I.1). This finishes the proof. $\square$

We introduce the following spaces:

$$\boldsymbol{Z}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega);\ \mathrm{div}\,\boldsymbol{v} = 0 \quad \text{in}\,\Omega, \quad \boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{on}\,\Gamma,\ \int_\Omega \boldsymbol{\beta}\cdot\boldsymbol{v}d\boldsymbol{x} = 0\}$$

and

$$\boldsymbol{H}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ \mathrm{div}\,\boldsymbol{v} = 0 \quad \text{in}\,\Omega, \quad \boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{on}\,\Gamma,\ \int_\Omega \boldsymbol{\beta}\cdot\boldsymbol{v}d\boldsymbol{x} = 0\}.$$

We have the following theorem:

Theorem 6.2 *There exists a sequence $(\boldsymbol{v}_j)_j$ of $\boldsymbol{V}^2(\Omega)$ and $(\lambda_j)_j \subset \mathbb{R}$ such that: The sequence $(\boldsymbol{v}_j)_j$ is a Hilbert basis of $\boldsymbol{H}(\Omega)$. Moreover we have*

i) $\lambda_j > C(\Omega) > 0$ and $\lim_{j\to+\infty}\lambda_j = +\infty$

ii) $2\int_\Omega \mathbf{D}(\boldsymbol{v}_j) : \mathbf{D}(\boldsymbol{\varphi})\ d\boldsymbol{x} = \lambda_j \int_\Omega \boldsymbol{v}_j\cdot\boldsymbol{\varphi}\,d\boldsymbol{x}, \quad \forall\,\boldsymbol{\varphi} \in \boldsymbol{V}^2(\Omega)$

iii) $\int_\Omega \mathbf{D}(\boldsymbol{v}_j) : \mathbf{D}(\boldsymbol{v}_k)\,d\boldsymbol{x} = \lambda_j\delta_{jk}.$

Proof. We introduce the operator

$$\begin{aligned}\Lambda : \boldsymbol{H}(\Omega) &\longrightarrow \boldsymbol{Z}(\Omega) \longrightarrow \boldsymbol{H}(\Omega)\\ \boldsymbol{f} &\longmapsto \ \ \boldsymbol{u} \quad \longmapsto \boldsymbol{u},\end{aligned}$$

where $\boldsymbol{u}$ is the solution given by Lemma 6.1. Note that, $\boldsymbol{H}(\Omega)$ is a Hilbert separable space and Λ is compact. Moreover, Λ is self-adjoint operator, indeed

$$\int_\Omega \Lambda\boldsymbol{f}_1\cdot\boldsymbol{f}_2\,d\boldsymbol{x} = 2\int_\Omega \mathbf{D}(\boldsymbol{u}_1) : \mathbf{D}(\boldsymbol{u}_2)\,d\boldsymbol{x} = \int_\Omega \boldsymbol{f}_1\cdot\Lambda\boldsymbol{f}_2\,d\boldsymbol{x},$$

where $\Lambda\boldsymbol{f}_i = \boldsymbol{u}_i, i = 1, 2$. To summarize, Λ is a compact and self-adjoint operator, consequently $\boldsymbol{H}(\Omega)$ possesses a Hilbert basis formed by a sequence of eigenfunctions $\boldsymbol{u}_k$:

$$\Lambda\boldsymbol{u}_j = \mu_j\boldsymbol{u}_j, \quad \mu_j > 0,\ j \geq 1 \quad \text{and} \quad \lim_{j\to+\infty}\mu_j = 0.$$

Particularly,

$$\int_\Omega \boldsymbol{u}_j\cdot\boldsymbol{u}_k\,d\boldsymbol{x} = \delta_{jk} \quad \text{and} \quad \int_\Omega \Lambda\boldsymbol{u}_j\cdot\boldsymbol{v}\,d\boldsymbol{x} = \int_\Omega \mu_j\boldsymbol{u}_j\cdot\boldsymbol{v}\,d\boldsymbol{x} \quad \forall\boldsymbol{v} \in \boldsymbol{V}^2(\Omega).$$

So,

$$\forall\boldsymbol{v} \in \boldsymbol{V}^2(\Omega), \quad \int_\Omega \mu_j\mathbf{D}(\boldsymbol{u}_j) : \mathbf{D}(\boldsymbol{v})\,d\boldsymbol{x} = \int_\Omega \boldsymbol{u}_j\cdot\boldsymbol{v}\,d\boldsymbol{x}$$

We set $\lambda_j = \frac{1}{\mu_j}, \quad \forall j \geq 1$. Then we have $\lim_{j\to+\infty}\lambda_j = +\infty$
and

$$\forall\boldsymbol{v} \in \boldsymbol{V}^2(\Omega), \quad \int_\Omega \mathbf{D}(\boldsymbol{u}_j) : \mathbf{D}(\boldsymbol{v})\,d\boldsymbol{x} = \lambda_j\int_\Omega \boldsymbol{u}_j\cdot\boldsymbol{v}\,d\boldsymbol{x}.$$

As consequence, by applying Lemma 3.2 we have

$$\|\mathbf{D}(\boldsymbol{u}_j)\|^2_{\boldsymbol{L}^2(\Omega)} = \lambda_j \geq \frac{1}{C^2}\|\boldsymbol{u}_j\|^2_{\boldsymbol{L}^2(\Omega)} = \frac{1}{C^2}.$$

$\square$

This basis will be very useful in the study of problem of existence of weak solution for Navier-Stokes equation in the Hilbert case.

7 Stokes equation with friction slip boundary condition

In this section we consider Stokes problem with friction slip boundary condition and we give some results as the case of Stokes problem with Navier boundary condition.

$$(\mathcal{S}_F)\begin{cases} -\Delta\, \boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{u} = \chi \quad \text{in} \quad \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = g \quad \text{on} \quad \Gamma, \\ 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{h} \quad \text{on} \quad \Gamma, \end{cases}$$

where $\alpha > 0$ is friction coefficient. We proceed in the same way as in the Proposition 3.1, we obtain the following proposition:

Proposition 7.1 *We suppose that $\chi = 0$, $g = 0$. Let $\boldsymbol{f}\in [\boldsymbol{H}^2_0(\operatorname{div},\Omega)]'$, $\boldsymbol{h}\in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ such that*

$$\boldsymbol{h}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma.$$

The problem $(\mathcal{S}_F)$ is equivalent to the following variational problem:

$$\begin{cases} \text{Find} \quad \boldsymbol{u}\in \boldsymbol{V}^2(\Omega) \quad \text{such that,} \\ \forall\boldsymbol{\varphi}\in \boldsymbol{V}^2(\Omega),\; 2\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} + \int_\Gamma \alpha(x)\boldsymbol{\varphi}_{\boldsymbol{\tau}}\boldsymbol{u}_{\boldsymbol{\tau}}\, d\sigma = \langle \boldsymbol{f},\boldsymbol{\varphi}\rangle_{[\boldsymbol{H}^2_0(\operatorname{div},\Omega)]'\times\boldsymbol{H}^2_0(\operatorname{div},\Omega)} \\ \qquad\qquad - \langle \boldsymbol{h},\boldsymbol{\varphi}\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}. \end{cases} \tag{I.1}$$

Now, we show the following inequality which will be useful in the proof of coercivity.

Lemma 7.2 *Assume that Ω is of class $C^{1,1}$. Suppose that $\alpha\in L^\infty(\Gamma)$ and $\alpha > 0$. Then, there exists a constant $C = C(\Omega) > 0$ such that*

$$\int_\Omega |\mathbf{D}(\boldsymbol{u})|^2 d\boldsymbol{x} + \int_\Gamma \alpha|\boldsymbol{u}_{\boldsymbol{\tau}}|^2 d\boldsymbol{\sigma} \geq C\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}, \quad \forall\, \boldsymbol{u}\in \boldsymbol{V}^2(\Omega). \tag{I.2}$$

Proof. Assume that inequality (I.2) is not true. Then there is a sequence $(\boldsymbol{u}_n)_n$ of $\boldsymbol{V}^2(\Omega)$ such that

$$\|\boldsymbol{u}_n\|_{\boldsymbol{H}^1(\Omega)} = 1$$

and

$$\int_\Omega |\mathbf{D}(\boldsymbol{u}_n)|^2 d\boldsymbol{x} + \int_\Gamma \alpha |(\boldsymbol{u}_n)_{\boldsymbol{\tau}}|^2 d\boldsymbol{\sigma} < \frac{1}{n}, \quad \forall n \geq 1. \tag{I.3}$$

Hence, there is a subsequence, again denoted by $(\boldsymbol{u}_n)_n$ and $\boldsymbol{u} \in \boldsymbol{V}^2(\Omega)$ such that

$$(\boldsymbol{u}_n)_n \text{converge to } \boldsymbol{u} \text{ weakly in } \boldsymbol{H}^1(\Omega), \text{ strongly in } \boldsymbol{L}^2(\Omega),$$

and

$$\mathbf{D}(\boldsymbol{u}_n) \longrightarrow 0 \quad \text{in} \quad \boldsymbol{L}^2(\Omega)$$

It follows from Korn inequality that for all $\varepsilon > 0$ we have

$$\begin{aligned} \|\boldsymbol{u}_p - \boldsymbol{u}_n\|^2_{\boldsymbol{H}^1(\Omega)} &\leqslant C\{\|\boldsymbol{u}_p - \boldsymbol{u}_n\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{D}(\boldsymbol{u}_p) - \mathbf{D}(\boldsymbol{u}_n)\|^2_{\boldsymbol{L}^2(\Omega)}\} \\ &\leqslant \varepsilon. \end{aligned}$$

Thus, the subsequence $(\boldsymbol{u}_n)_n$ is a Cauchy sequence in $\boldsymbol{H}^1(\Omega)$ and then

$$\boldsymbol{u}_n \longrightarrow \boldsymbol{u} \quad \text{strongly in } \boldsymbol{H}^1(\Omega).$$

Therefore, $\mathbf{D}(\boldsymbol{u}) = 0$, which means that $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}$, $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$. But by Inequality (I.3) we have $\boldsymbol{u}_{\boldsymbol{\tau}} = 0$ on Γ, then $\boldsymbol{u} = 0$ on Γ. Using the second left-hand side of (I.3) , we deduce that $\boldsymbol{u} = 0$. Which is a contradiction with $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} = 1$. □

Now, we can solve the Stokes problem with Navier boundary condition.

Theorem 7.3 *(**Weak solution for** $(\mathcal{S}_F)$)*
Suppose that $\chi = 0$ and $g = 0$. Let $\boldsymbol{f} \in [\boldsymbol{H}_0^2(\mathrm{div}, \Omega)]'$ and $\boldsymbol{h} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$, satisfying

$$\boldsymbol{h} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma.$$

Then, the Stokes problem $(\mathcal{S}_F)$ has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ and we have the following estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^2(\mathrm{div},\Omega)]'} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}. \tag{I.4}$$

Proof. Thanks to the Proposition 7.1, the problem $(\mathcal{S}_F)$ is equivalent to the following variational formulation:

$$\begin{cases} \text{Find} \quad \boldsymbol{u} \in \boldsymbol{V} \quad \text{such that,} \\ \\ \forall \boldsymbol{\varphi} \in \boldsymbol{V}, \quad 2\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} + \int_\Gamma \alpha(x) \boldsymbol{\varphi}_{\boldsymbol{\tau}} \boldsymbol{u}_{\boldsymbol{\tau}}\, d\sigma = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{[\boldsymbol{H}_0^2(\mathrm{div},\Omega)]' \times \boldsymbol{H}_0^2(\mathrm{div},\Omega)} \\ \qquad\qquad\qquad\qquad - \langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)}. \end{cases} \tag{I.5}$$

We consider the Hilbert space $\boldsymbol{V}^2(\Omega)$ endowed with the norm $\|\boldsymbol{v}\|_{\boldsymbol{V}^2(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|D(\boldsymbol{v})\|_{\boldsymbol{L}^2(\Omega)}$,

$$E(\boldsymbol{u},\boldsymbol{\varphi}) = 2\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})\,d\boldsymbol{x} + \int_\Gamma \alpha(x)\boldsymbol{\varphi_\tau}\boldsymbol{u_\tau}\,d\sigma \quad \text{for } \boldsymbol{u},,\ \boldsymbol{\varphi} \in \boldsymbol{V}^2(\Omega),$$

$$l(\boldsymbol{\varphi}) = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{[\boldsymbol{H}^2_0(\mathrm{div},\Omega)]' \times \boldsymbol{H}^2_0(\mathrm{div},\Omega)} - \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)}.$$

Its easy to see that the bilinear form $E(.,.)$ (resp. the linear form $l(.)$)is continuous on $\boldsymbol{V}^2(\Omega) \times \boldsymbol{V}^2(\Omega)$(resp. on $\boldsymbol{V}^2(\Omega)$) since we have

$$E(\boldsymbol{u},\boldsymbol{\varphi}) \leq C\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{V}^2(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{V}^2(\Omega)},$$

$$l(\boldsymbol{\varphi}) \leq (\|\boldsymbol{f}\|_{[\boldsymbol{H}^2_0(\mathrm{div},\Omega)]'}\| + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)})\|\boldsymbol{\varphi}\|_{\boldsymbol{V}^2(\Omega)}.$$

Furthermore, the bilinear form $E(.,.)$ is $\boldsymbol{V}^2(\Omega)-$elliptic, as we have

$$\forall \boldsymbol{u} \in \boldsymbol{V}^2(\Omega), \quad E(\boldsymbol{u},\boldsymbol{u}) \geq C\|\boldsymbol{u}\|_{\boldsymbol{V}}.$$

thanks to the Lemma 7. According to Lax-Milgram theorem, there exists a unique function $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and $\pi \in L^2(\Omega)$ solution of the Stokes problem $(\mathcal{S}_F)$. □

Remark 7.4 *We can also solve the Stokes problem $(\mathcal{S}_F)$ when the divergence operator is not vanish and $g \neq 0$. To do this, simply we consider the following Neumann problem:*

$$\Delta\theta = \chi \quad \text{in}\Omega \quad \text{and} \quad \frac{\partial\theta}{\partial\boldsymbol{n}} = g \quad \text{on} \quad \Gamma. \tag{I.6}$$

and setting $\boldsymbol{z} = \boldsymbol{u} - \nabla\theta$.

We prove now the existence of strong solution $(\boldsymbol{u},\pi) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$ for the Stokes problem $(\mathcal{S}_T)$. The proof of the following theorem is based on results given in Lemma2 and Lemma3.

Theorem 7.5 *Assume that Ω is of class $C^{2,1}$. Suppose that $\chi = 0$. Let $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, $g \in H^{\frac{3}{2}}(\Omega)$ and $\boldsymbol{h} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$, satisfying*

$$\int_\Gamma g\,d\sigma = 0 \quad \text{and} \quad \boldsymbol{h}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma.$$

Then, the solution $(\boldsymbol{u},\pi)$ given by Theorem 7.3 belongs to $\boldsymbol{H}^2(\Omega)\times H^1(\Omega)/\mathbb{R}$ and satisfies the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leq C(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|g\|_{H^{\frac{3}{2}}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}). \tag{I.7}$$

Proof. Before, we note that under the hypothesis of Theorem 7.3, the data $\boldsymbol{f}$ satisfies also the hypothesis of Theorem 8.

So, this implies that problem $(\mathcal{S}_F)$ has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$. Now, to prove the regularity of the velocity, we set $\boldsymbol{z} = \mathbf{curl}\, \boldsymbol{u}$. We deduce that $\boldsymbol{z}$ satisfies the following problem:

$$\begin{cases} -\Delta\, \boldsymbol{z} = \mathbf{curl}\, \boldsymbol{f} & \text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{z} = 0 & \text{in} \quad \Omega, \\ \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{H} & \text{on} \quad \Gamma. \end{cases}$$

where $\boldsymbol{H} = \alpha \boldsymbol{u}_{\boldsymbol{\tau}} - 2 \sum_{k=1}^{2} (\boldsymbol{u}_{\boldsymbol{\tau}} \cdot \dfrac{\partial \boldsymbol{n}}{\partial s_k}) \boldsymbol{\tau}_k - \boldsymbol{h}$.

Because Ω is of class $\mathcal{C}^{2,1}$ and $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, we have $\sum_{k=1}^{2} (\boldsymbol{u}_{\boldsymbol{\tau}} \cdot \dfrac{\partial \boldsymbol{n}}{\partial s_k}) \boldsymbol{\tau}_k \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$, it follows that $\boldsymbol{H} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ and $\boldsymbol{H} \cdot \boldsymbol{n} = 0$. Since $\mathbf{curl}\, \boldsymbol{f}$ belongs to $[\boldsymbol{H}_0^2(\mathbf{curl}, \Omega)]'$ and satisfies the compatibility condition (I.22) for $p = 2$, as result from Theorem 2.8 that $\boldsymbol{z} \in \boldsymbol{H}^1(\Omega)$. Then $\boldsymbol{u} \in \boldsymbol{X}^{2,2}(\Omega)$. As a consequence, thanks the imbedding of $\boldsymbol{X}^{2,2}(\Omega)$ in $\boldsymbol{H}^2(\Omega)$ (see[9]), the solution $\boldsymbol{u}$ of the problem $(\mathcal{S}_T)$ belongs to $\boldsymbol{H}^2(\Omega)$.
Where

$$\boldsymbol{X}^{2,2}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega); \operatorname{div} \boldsymbol{v} \in H^1(\Omega), \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \,\text{and}\, \boldsymbol{v} \cdot \boldsymbol{n} \in H^{\frac{3}{2}}(\Gamma)\}.$$

Moreover, since

$$\nabla \pi = \Delta \boldsymbol{u} + \boldsymbol{f} \in \boldsymbol{L}^2(\Omega),$$

we deduce that $\pi \in H^1(\Omega)$. $\square$

We have the following result concerning strong solution of the Stokes problem $(\mathcal{S}_T)$, where the divergence operator does not vanish.

Corollary 7.6 *Assume that Ω is of class $C^{2,1}$. For every $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, $\chi \in H^1(\Omega)$, $g \in H^{\frac{3}{2}}(\Omega)$ and $\boldsymbol{h} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$, satisfying*

$$\int_\Gamma g\, d\sigma = \int_\Omega \chi dx \quad \text{and} \quad \boldsymbol{h} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma.$$

The solution $(\boldsymbol{u}, \pi)$ given by Theorem8 belongs to $\boldsymbol{H}^2(\Omega) \times H^1(\Omega)/\mathbb{R}$ and satisfies the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leq C(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\chi\|_{H^1(\Omega)} + \|g\|_{H^{\frac{3}{2}}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}). \tag{I.8}$$

Proof. We solve the following Neumann problem:

$$\Delta \theta = \chi \quad \text{in}\Omega \quad \text{and} \quad \frac{\partial \theta}{\partial \boldsymbol{n}} = g \quad \text{on} \quad \Gamma. \tag{I.9}$$

This problem has a unique solution this problem has a unique solution $\theta \in H^3(\Omega)/\mathbb{R}$ satisfying the estimate:

$$\|\theta\|_{H^3(\Omega)} \leq C\{\|\chi\|_{H^1(\Omega)} + \|g\|_{H^{\frac{3}{2}}(\Omega)}\}. \tag{I.10}$$

Setting $\boldsymbol{z} = \boldsymbol{u} - \nabla\theta$, then $(\mathcal{S}_T)$ becomes: Find $(\boldsymbol{z}, \pi) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)/\mathbb{R}$ such that

$$\begin{cases} -\Delta\, \boldsymbol{z} + \nabla\pi = \boldsymbol{f} + \nabla\chi & \text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{z} = 0 & \text{in} \quad \Omega, \\ \boldsymbol{z} \cdot \boldsymbol{n} = 0 & \text{on} \quad \Gamma, \\ 2\,[\mathbf{D}(\boldsymbol{z})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{z}_{\boldsymbol{\tau}} = \boldsymbol{H} & \text{on} \quad \Gamma. \end{cases} \tag{I.11}$$

where $\boldsymbol{H} = \boldsymbol{h} - 2\,[\mathbf{D}(\nabla\theta)\boldsymbol{n}]_{\boldsymbol{\tau}} - \alpha(\nabla\theta)_{\boldsymbol{\tau}}$ and belongs to $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$. Observe that $\boldsymbol{f} + \nabla\chi$ belongs to $\boldsymbol{L}^2(\Omega)$. Thus Due to Theorem 7.5, this problem has a unique solution $(\boldsymbol{z}, \pi) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)/\mathbb{R}$. Therefore $\boldsymbol{u} = \boldsymbol{z} + \nabla\theta$ belongs to $\boldsymbol{H}^2(\Omega)$. $\square$

Using the same reasoning of the Section 3, Section 4 and Section 5, we have the following results:

Theorem 7.7 *Let* $\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\operatorname{div}, \Omega))'$, $\chi \in L^p(\Omega)$, $g \in W^{1-\frac{1}{p},p}(\Gamma)$ *and* $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$, *satisfying the compatibility conditions satisfying the compatibility conditions* (I.1) *and* (I.13). *Then, problem* $(\mathcal{S}_F)$ *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. *In addition, we have the following estimate:*

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} \\ + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big). \end{aligned}$$

8 Navier-Stokes Equation

We consider the following Navier-Stokes problem:

$$(\mathcal{NS}_T) \begin{cases} -\Delta\, \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla\pi = \boldsymbol{f} & \text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & \text{on} \quad \Gamma, \\ 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on} \quad \Gamma. \end{cases}$$

The aim of this section is to prove the existence of weak solution and strong solution of problem $(\mathcal{NS}_T)$. To prove the existence of weak solution, we need to introduce some spaces. First, we introduce the following space:

$$\boldsymbol{H}_0^{\frac{6}{5},2}(\operatorname{div}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega);\ \operatorname{div} \boldsymbol{v} \in L^2(\Omega),\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \,\text{on}\, \Gamma\},$$

which is a Banach space for the norm

$$||\boldsymbol{v}||_{\boldsymbol{H}_0^{\frac{6}{5},2}(\operatorname{div},\Omega)} = ||\boldsymbol{v}||_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + ||\operatorname{div} \boldsymbol{v}||_{L^2(\Omega)}.$$

We define also the space

$$\boldsymbol{E}^{\frac{6}{5},2}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega);\ \Delta\boldsymbol{v} \in [\boldsymbol{H}_0^{6,2}(\operatorname{div}, \Omega)]'\}.$$

We note that $\mathfrak{D}(\overline{\Omega})$ is dense in $\boldsymbol{H}_0^{\frac{6}{5},2}(\mathrm{div},\Omega)$. As consequence, we have the following lemma, which has a similar proof as Lemma 2.3.

Lemma 8.1 *Suppose that Ω is of class $C^{1,1}$. The linear mapping $\Upsilon : \boldsymbol{v} \longrightarrow [\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}|\Gamma}$ defined on $\mathfrak{D}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping*

$$\Upsilon : \boldsymbol{E}^{\frac{6}{5},2}(\Omega) \longrightarrow \boldsymbol{H}^{-\frac{1}{2}}(\Gamma).$$

Moreover, we have the Green formula: for any $\boldsymbol{v} \in \boldsymbol{E}^{\frac{6}{5},2}(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{V}^2(\Omega)$,

$$-\left\langle \Delta \boldsymbol{v}, \boldsymbol{\varphi}\right\rangle_{[\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)]' \times \boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)} = 2\int_\Omega \mathbf{D}(\boldsymbol{v}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} - 2\langle [\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\rangle_\Gamma. \tag{I.1}$$

Thanks to this lemma, we can show that the Navier-Stokes problem $(\mathcal{NS}_T)$ is equivalent to the following formulation:

$$(\mathcal{FNS}) \begin{cases} \text{Find} \quad \boldsymbol{u} \in \boldsymbol{V}^2(\Omega) \quad \text{such that,} \\ \\ \forall \boldsymbol{\varphi} \in \boldsymbol{V}^2(\Omega), \quad 2\displaystyle\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} + \int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\varphi} d\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_\Omega + \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_\Gamma\,, \end{cases}$$

where $\langle .,.\rangle_\Omega = \langle .,.\rangle_{[\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)]' \times \boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)}$ and $\langle .,.\rangle_\Gamma = \langle .,.\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}$.
In the sequel we write $b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \displaystyle\int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w} d\boldsymbol{x}$. To facilitate the work, we give some properties of the operator b.

Lemma 8.2 *For any $\boldsymbol{u} \in \boldsymbol{V}^2(\Omega)$ and $\boldsymbol{\eta} \in \mathfrak{T}(\Omega)$, the following identities hold*

$$b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\eta}) = b(\boldsymbol{u},\boldsymbol{\eta},\boldsymbol{\eta}) = b(\boldsymbol{\eta},\boldsymbol{\eta},\boldsymbol{u}) = 0.$$

Proof. We know that $b(\boldsymbol{u},\boldsymbol{\eta},\boldsymbol{\eta}) = 0$. Since, $\boldsymbol{D}(\boldsymbol{\eta}) = 0$, we have $\frac{\partial \eta_i}{\partial x_j} = -\frac{\partial \eta_j}{\partial x_i}$, then, for any $\boldsymbol{u} \in \boldsymbol{V}^2(\Omega)$

$$\begin{aligned} b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\eta}) &= \sum_{i,j=1}^3 \int_\Omega \boldsymbol{u}_i \frac{\partial \boldsymbol{u}_j}{\partial \boldsymbol{x}_i}\boldsymbol{\eta}_j d\boldsymbol{x} = -\sum_{i,j=1}^3 \int_\Omega \boldsymbol{u}_i \frac{\partial \boldsymbol{\eta}_j}{\partial \boldsymbol{x}_i}\boldsymbol{u}_j d\boldsymbol{x} \\ &= \sum_{i,j=1}^3 \int_\Omega \boldsymbol{u}_i \frac{\partial \boldsymbol{\eta}_i}{\partial \boldsymbol{x}_j}\boldsymbol{u}_j d\boldsymbol{x} = -\sum_{i,j=1}^3 \int_\Omega \boldsymbol{u}_j \frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_j}\boldsymbol{\eta}_i d\boldsymbol{x} \\ &= -b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\eta}). \end{aligned}$$

Consequently, $b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\eta}) = 0$. Finally, because $\boldsymbol{D}(\boldsymbol{\eta}) = 0$, we have

$$b(\boldsymbol{\eta},\boldsymbol{\eta},\boldsymbol{u}) = \frac{1}{2}\int_\Omega \boldsymbol{u}\cdot\nabla|\boldsymbol{\eta}|^2\, d\boldsymbol{x} = 0.$$

□

Using Lemma 8.2, we deduce that the following compatibility condition is necessary to solve problem $(\mathcal{FNS})$:

$$\langle \boldsymbol{f}, \boldsymbol{\beta} \rangle_{\Omega} + \langle \boldsymbol{h}, \boldsymbol{\beta} \rangle_{\Gamma} = 0. \tag{I.2}$$

Due to all these results we can now solve the problem $(\mathcal{FNS})$.

Theorem 8.3 *Let* $\boldsymbol{f} \in \big(\boldsymbol{H}_0^{6,2}(\mathrm{div}, \Omega)\big)'$ *and* $\boldsymbol{h} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$, *satisfying* $\boldsymbol{h}\cdot \boldsymbol{n} = 0$ *on* Γ *and the compatibility condition* (I.2). *Then problem* $(\mathcal{FNS})$ *has solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)$. *Moreover, we have the following estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{\left(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)\right)'} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\big).$$

Proof. To show the existence of $\boldsymbol{u}$, we start by constructing the approximate solutions of the problem $(\mathcal{FNS})$ by Galerkin method and then thanks to compactness arguments, we prove, by passing to the limit, some convergence properties. We note that, if $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{\eta}$, with $\boldsymbol{w} \in \boldsymbol{Z}(\Omega)$ and $\boldsymbol{\eta} \in \boldsymbol{\mathcal{T}}(\Omega)$, is a solution of $(\mathcal{FNS})$, then for any $\boldsymbol{\psi} \in \boldsymbol{Z}(\Omega)$,

$$2\int_{\Omega} \mathbf{D}(\boldsymbol{w}) : \mathbf{D}(\boldsymbol{\psi})\, d\boldsymbol{x} + b(\boldsymbol{w} + \boldsymbol{\eta}, \boldsymbol{w}, \boldsymbol{\psi}) + b(\boldsymbol{w}, \boldsymbol{\eta}, \boldsymbol{\psi}) = \langle \boldsymbol{f}, \boldsymbol{\psi} \rangle_{\Omega} + \langle \boldsymbol{h}, \boldsymbol{\psi} \rangle_{\Gamma}\,. \tag{I.3}$$

Note that for any $\boldsymbol{\eta}' \in \boldsymbol{\mathcal{T}}(\Omega)$, we have

$$b(\boldsymbol{w} + \boldsymbol{\eta}, \boldsymbol{w} + \boldsymbol{\eta}, \boldsymbol{\psi} + \boldsymbol{\eta}') = b(\boldsymbol{w} + \boldsymbol{\eta}, \boldsymbol{w}, \boldsymbol{\psi}) + b(\boldsymbol{w}, \boldsymbol{\eta}, \boldsymbol{\psi}).$$

It follows that problem $(\mathcal{FNS})$ is reduced to finding $\boldsymbol{w} \in \boldsymbol{Z}(\Omega)$ verifying (I.3).

Now, for each fixed integer $m \geq 1$, we define an approximate solution $\boldsymbol{w}_m$ of (I.3) by

$$\begin{cases} \boldsymbol{w}_m \in \boldsymbol{V}_m, \\ \\ 2\displaystyle\int_{\Omega} \mathbf{D}(\boldsymbol{w}_m) : \mathbf{D}(\boldsymbol{v}_k)\, d\boldsymbol{x} + b(\boldsymbol{w}_m + \boldsymbol{\eta}, \boldsymbol{w}_m, \boldsymbol{v}_k) + b(\boldsymbol{w}_m, \boldsymbol{\eta}, \boldsymbol{v}_k) \\ \qquad\qquad\qquad\qquad = \langle \boldsymbol{f}, \boldsymbol{v}_k \rangle_{\Omega} + \langle \boldsymbol{h}, \boldsymbol{v}_k \rangle_{\Gamma}\,,\ k = 1, ..., m \end{cases} \tag{I.4}$$

where $\boldsymbol{V}_m = \langle \boldsymbol{v}_1, ..., \boldsymbol{v}_m \rangle$ is the space spanned by the vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_m$ and $\{\boldsymbol{v}_i\}_i$ is the Hilbertian basis of $\boldsymbol{H}(\Omega)$ given by eigenfunctions of Stokes problem. We define a scalar product on $\boldsymbol{Z}(\Omega)$ by

$$\forall \boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{Z}(\Omega), \quad ((\boldsymbol{u}_1, \boldsymbol{u}_2)) = \int_{\Omega} \mathbf{D}(\boldsymbol{u}_1) : \mathbf{D}(\boldsymbol{u}_2) d\boldsymbol{x}.$$

With an aim to establish the existence of the solutions of the problem (I.4), we consider the following operator

$$\begin{aligned} \boldsymbol{P}_m : \boldsymbol{V}_m &\longrightarrow \boldsymbol{V}_m \\ \boldsymbol{w} &\longrightarrow \boldsymbol{P}_m(\boldsymbol{w}) \end{aligned}$$

defined, for each $\boldsymbol{\eta} \in \mathfrak{T}(\Omega)$ fixed, by

$$((\boldsymbol{P}_m(\boldsymbol{w}), \boldsymbol{z})) = 2((\boldsymbol{w}, \boldsymbol{z})) + b(\boldsymbol{w} + \boldsymbol{\eta}, \boldsymbol{w}, \boldsymbol{z}) + b(\boldsymbol{w}, \boldsymbol{\eta}, \boldsymbol{z}) - \langle \boldsymbol{f}, \boldsymbol{z} \rangle_{\Omega} - \langle \boldsymbol{h}, \boldsymbol{z} \rangle_{\Gamma}\,.$$

Let us note that

$$\forall \boldsymbol{w} \in \boldsymbol{Z}(\Omega), \qquad b(\boldsymbol{w} + \boldsymbol{\eta}, \boldsymbol{w}, \boldsymbol{w}) = b(\boldsymbol{w}, \boldsymbol{\eta}, \boldsymbol{w}) = 0.$$

Thus, using inequality (II.5), we show that

$$((\boldsymbol{P}_m(\boldsymbol{w}), \boldsymbol{w})) \geq \|\boldsymbol{w}\|_{\boldsymbol{V}_m}(2\|\boldsymbol{w}\|_{\boldsymbol{V}_m} - C(\|\boldsymbol{f}\|_{\left(\mathbf{H}_0^{6,2}(\mathrm{div},\Omega)\right)'} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)})),$$

where the norm on $\boldsymbol{V}_m$ is induced by the norm on $\boldsymbol{Z}(\Omega)$.

As consequence, $((\boldsymbol{P}_m(\boldsymbol{w}), \boldsymbol{w})) > 0$ for $\|\boldsymbol{w}\|_{\boldsymbol{V}_m} > \frac{C}{2}(\|\boldsymbol{f}\|_{\left(\mathbf{H}_0^{6,2}(\mathrm{div},\Omega)\right)'} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)})$. We know that $\boldsymbol{P}_m : \boldsymbol{V}_m \longrightarrow \boldsymbol{V}_m$ is continuous. Therefore, the hypothesis of Brouwer theorem is satisfied and there exists a solution $\boldsymbol{w}_m$ of (I.4).

Passage to the limit: Since $\boldsymbol{w}_m$ is a solution of problem (I.4), we have

$$2\|\mathbf{D}(\boldsymbol{w}_m)\|^2_{\boldsymbol{L}^2(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{w}_m \rangle_{\Omega} + \langle \boldsymbol{h}, \boldsymbol{w}_m \rangle_{\Gamma}.$$

Using compatibility condition (I.2) and Lemma 3.2, we obtain the *a priori* estimate:

$$\|\boldsymbol{w}_m\|_{\boldsymbol{V}_m} \leq C(\|\boldsymbol{f}\|_{\left(\mathbf{H}_0^{6,2}(\mathrm{div},\Omega)\right)'} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}).$$

Since the sequence $\boldsymbol{w}_m$ remains bounded in $\boldsymbol{Z}(\Omega)$, we can extract a subsequence $\boldsymbol{w}_k$ such that

$$\boldsymbol{w}_k \rightharpoonup \boldsymbol{w} \quad \text{weakly in } \boldsymbol{Z}(\Omega),$$

The injection of $\boldsymbol{Z}(\Omega)$ into $\boldsymbol{H}(\Omega)$ is compact, so we have

$$\boldsymbol{w}_k \longrightarrow \boldsymbol{w} \quad \text{in } \boldsymbol{H}(\Omega).$$

Therefore, we can pass to the limit in (I.4) and we obtain

$$2\int_{\Omega} \mathbf{D}(\boldsymbol{w}) : \mathbf{D}(\boldsymbol{\psi})\, d\boldsymbol{x} + b(\boldsymbol{w} + \boldsymbol{\eta}, \boldsymbol{w}, \boldsymbol{\psi}) + b(\boldsymbol{w}, \boldsymbol{\eta}, \boldsymbol{\psi}) = \langle \boldsymbol{f}, \boldsymbol{\psi} \rangle_{\Omega} + \langle \boldsymbol{h}, \boldsymbol{\psi} \rangle_{\Gamma}$$

for any $\boldsymbol{\psi} \in \boldsymbol{Z}(\Omega)$.

Finally, we conclude that $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{\eta}$ is a solution of $(\mathcal{NS}_T)$. $\square$

Remark 8.4 We proved that, in the case of symmetric domain, the Navier-Stokes problem $(\mathcal{NS}_T)$ has a infinity of solutions for all data satisfying the compatibility condition (I.2). Then, we have a situation where we don't have a uniqueness of solution even if the data is sufficiently small. Unlike the Navier-Stokes problem with Dirichlet boundary condition where we have a uniqueness of solution for data sufficiently small.

Theorem 8.5 *Let* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ *and* $\boldsymbol{h} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$, *satisfying* $\boldsymbol{h}\cdot\boldsymbol{n} = 0$ *on* Γ *and the compatibility condition* (I.2). *Then problem* $(\mathcal{NS}_T)$ *has solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$. *Moreover, we have the following estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}\big).$$

Proof. According to Theorem 4.3, there exists $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)$ solution of $(\mathcal{NS}_T)$. So, $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ belongs to $\boldsymbol{L}^{\frac{3}{2}}(\Omega)$. Thus, $\boldsymbol{f} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$ and satisfies the following condition:

$$\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\beta}\, d\boldsymbol{x} - b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\beta}) - \int_\Gamma \boldsymbol{h}\cdot\boldsymbol{\beta}\, d\boldsymbol{\sigma} = 0. \tag{I.5}$$

Furthermore, $\boldsymbol{u}$ satisfies the following problem:

$$\begin{cases} -\Delta\, \boldsymbol{u} + \nabla\pi = \boldsymbol{f} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad [2\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on} \quad \Gamma. \end{cases} \tag{I.6}$$

Thanks to Theorem 2.4 we deduce that $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,\frac{3}{2}}(\Omega)$. Thus, $\nabla\boldsymbol{u} \in \boldsymbol{W}^{1,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$. Therefore $\boldsymbol{f} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ satisfying again the condition (I.5). As consequence, we apply again Theorem 2.4, we deduce that $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$ solves (I.6). $\square$

Chapter II

Navier-Stokes equations with Navier boundary conditions

1 Introduction and functional framework

Let Ω be a bounded domain of class $C^{1,1}$ of $\mathbb{R}^3$ with boundary Γ. We are interested to the study of solutions to the Navier-Stokes equations:

$$(\mathcal{NS}) \begin{cases} -\Delta\, \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = \chi & \text{in} \quad \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = g \quad \text{and} \quad 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on} \quad \Gamma, \end{cases}$$

where $\boldsymbol{u}$ denote the velocity, π a pressure, $\boldsymbol{f}$ are the external forces, $\boldsymbol{n}$ the exterior unit normal, $\boldsymbol{\tau}$ the corresponding tangent vector and $\mathbf{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\top})$ denotes the deformation tensor associated with the velocity field $\boldsymbol{u}$. Note that this type of boundary condition is introduced by Navier in 1827 (see [53]). They are used in various physical situations, as for instance in case when the stagnant layer of fluid close to the wall allowing a fluid to slip.

Note that the mathematical literature on the Navier-Stokes equations with Navier boundary conditions is less wide than that on the Navier-Stokes equations with classical Dirichlet boundary conditions. However, there exist some works that are interested to study these types of problem. Indeed, one can cite the work of V. A. Solonnikov and V. E. Ščadilov [59] where the authors studied the Hilbert case for data $\boldsymbol{f}$ belongs to $\boldsymbol{L}^2(\Omega)$. Another example is the work of G. Mulone and F. Salemi [50, 51] where authors studied the existence of weak solution in $\boldsymbol{H}^1(\Omega)$ for stationary Navier-Stokes system (the evolution system is also studied). In our works, we extend these results and the result obtained in [6] for the case of $\boldsymbol{L}^p$ theory not only for large values of p, but also sor small values ($\frac{3}{2} < p < 2$). It is important to note that, to our knowledge, there are no works that are studying the problem $\boldsymbol{L}^p$ theory.

The purpose of this paper is to prove the existence of weak solutions and some regularity results in L^p-theory with $1 < p < \infty$. Note that the existence of solution in $\boldsymbol{H}^1(\Omega)\times L^2(\Omega)$ has been proved in [6] with a suitable choice of data. For $p \geq 2$ we can easily prove the existence by using regularity of Stokes problem. We can next, obtain a weak solution in $\boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$ for $\frac{3}{2} < p < 2$ by using the fixed point theorem after studying the Oseen problem. The difficult part in this works is to prove the existence of weak solution in $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ for $\frac{3}{2} < p < 2$. To do this we study before the Oseen problem with Navier boundary condition and we can conclude the existence result for Navier-Stokes problem with fixed point theorem.

Let us first introduce some functional spaces and recall some elementary proprieties. We note that the vector-valued Laplace operator of a vector field $\boldsymbol{v} = (v_1, v_2, v_3)$ is equivalently defined by

$$\Delta\, \boldsymbol{v} = 2\mathrm{div}\, \mathbf{D}(\boldsymbol{v}) - \mathbf{grad}\,(\mathrm{div}\, \boldsymbol{v}). \tag{II.1}$$

Second, we define the following spaces: For all $1 < r, p < \infty$,

$$\boldsymbol{H}_0^{r,p}(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^r(\Omega);\, \mathrm{div}\, \boldsymbol{v} \in L^p(\Omega),\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \mathrm{on}\, \Gamma\},$$

which is the Banach space for the norm:

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^{r,p}(\mathrm{div},\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^r(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|_{L^p(\Omega)}$$

and we suppose that

$$r \le p \quad \text{and} \qquad \frac{1}{r} \le \frac{1}{p} + \frac{1}{3} \quad \text{if}\, p > \frac{3}{2}. \tag{II.2}$$

With this choice of p and r, we have

$$\boldsymbol{W}^{1,r}(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega) \quad \text{if} \quad p > \frac{3}{2} \quad \text{and} \quad \frac{1}{r} \le \frac{1}{p} + \frac{1}{3} \tag{II.3}$$

and

$$\boldsymbol{W}^{1,r}(\Omega) \hookrightarrow \boldsymbol{W}^{1,1}(\Omega) \hookrightarrow \boldsymbol{L}^{\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega) \quad \text{if} \quad p \le \frac{3}{2} \quad \text{and} \quad 1 < r \le p. \tag{II.4}$$

If $r = p$ we denote $\boldsymbol{H}_0^p(\mathrm{div}, \Omega)$ instead of $\boldsymbol{H}_0^{p,p}(\mathrm{div}, \Omega)$.

The proof of the following lemma is similar to that of Lemma 8 in [7].

Lemma 1.1 *The space $\mathfrak{D}(\Omega)$ is dense in $\boldsymbol{H}_0^{r,p}(\mathrm{div}, \Omega)$.*

In the next lemma we characterize the dual space of $\boldsymbol{H}_0^{r',p'}(\mathrm{div}, \Omega)$.

Lemma 1.2 *A distribution $\boldsymbol{f}$ belongs to $\big(\boldsymbol{H}_0^{r',p'}(\mathrm{div}, \Omega)\big)'$ if and only if there exist $\boldsymbol{F} \in \boldsymbol{L}^r(\Omega)$ and $\chi \in L^p(\Omega)$, such that $\boldsymbol{f} = \boldsymbol{F} + \nabla\chi$ with the following norm*

$$\| \boldsymbol{f} \|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} = \max\{\| \boldsymbol{F} \|_{\boldsymbol{L}^r(\Omega)}, \| \chi \|_{L^p(\Omega)}\}.$$

Let introduce the space:

$$\boldsymbol{V}^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\, \mathrm{div}\, \boldsymbol{v} = 0 \,\mathrm{in}\, \Omega,\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \,\mathrm{on}\, \Gamma\} \tag{II.5}$$

and

$$\boldsymbol{E}^{r,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\ \Delta\, \boldsymbol{v} \in \big(\boldsymbol{H}_0^{r',p'}(\mathrm{div}, \Omega)\big)'\}.$$

The proof of the following lemma is similar to the case $r = p$ (see ([55], Corollary 4.2.2)).

Lemma 1.3 *The space* $\mathfrak{D}(\overline{\Omega})$ *is dense in* $\boldsymbol{E}^{r,p}(\Omega)$.

As consequence, we have the following lemma.

Lemma 1.4 *The linear mapping* $\Theta : \boldsymbol{v} \longrightarrow [\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}|\Gamma}$ *defined on* $\mathfrak{D}(\overline{\Omega})$ *can be extended by continuity to a linear and continuous mapping*

$$\Theta : \boldsymbol{E}^{r,p}(\Omega) \longrightarrow \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma).$$

Moreover, we have the Green formula: for any $\boldsymbol{v} \in \boldsymbol{E}^{r,p}(\Omega)$ *and* $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega)$,

$$-\langle \Delta \boldsymbol{v}, \boldsymbol{\varphi}\rangle_{\Omega} = 2\int_{\Omega} \mathbf{D}(\boldsymbol{v}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} - 2\langle [\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\rangle_{\Gamma}. \tag{II.6}$$

where $\langle .,.\rangle_{\Gamma}$ *denotes the duality between* $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ *and* $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$ *and* $\langle .,.\rangle_{\Omega}$ *denotes the duality besttween* $(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$ *and* $\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)$.

The plan of the paper is the following. In section 2 we recall some results and we will try to extend a result of existence for Stokes problem with data $\boldsymbol{f}$ only in $\big(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)\big)'$ (proved in [6]) to the case $\big(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)\big)'$. In section 3, we give some results of existence, uniqueness and regularity for Oseen problem with Navier boundary condition and we establish some estimates of solution that will be useful to show the existence for solution for problem $(\mathcal{NS})$ for small values of p ($\frac{3}{2} < p < 2$). The last section is devoted to study of the Navier-Stokes problem in $\boldsymbol{L}^p$-theory for $\frac{3}{2} < p < \infty$, we give some results of existence and regularity.

2 Stokes problem

We first study results of existence for weak solution for the following Stokes problem:

$$(\mathcal{S})\begin{cases} -\Delta\, \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \mathrm{div}\, \boldsymbol{u} = \chi & \text{in} \quad \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = g \quad \text{and} \quad 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on} \quad \Gamma. \end{cases}$$

Note that in [6] we have shown the results of existence and uniqueness of the weak solution for data $\boldsymbol{f} \in \big(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)\big)'$. In this section we show the same results for data $\boldsymbol{f}$ in a larger space such as the space $\big(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)\big)'$. This will help us in the study of Oseen equations and Navier-Stokes equations.

Remark 2.1 We can always return to solve this problem with $\mathrm{div}\,\boldsymbol{u} = 0$ and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$. Indeed, for $\chi \in L^p(\Omega)$ and $g \in W^{1-\frac{1}{p},p}(\Gamma)$, we consider $\theta \in W^{2,p}(\Omega)$ a solution of the following problem:

$$\Delta\theta = \chi \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial\theta}{\partial\boldsymbol{n}} = g \quad \text{on } \Gamma. \tag{II.1}$$

Now, setting $\boldsymbol{z} = \boldsymbol{u} - \nabla\,\theta$, then $(\mathcal{S})$ becomes: Find $(\boldsymbol{z}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ solution of problem

$$\begin{cases} -\Delta\,\boldsymbol{z} + \nabla\pi = \boldsymbol{f} + \nabla\chi & \text{in} \quad \Omega, \\ \operatorname{div}\boldsymbol{z} = 0 & \text{in} \quad \Omega, \\ \boldsymbol{z}\cdot\boldsymbol{n} = 0 & \text{on} \quad \Gamma, \\ 2\,[\mathbf{D}(\boldsymbol{z})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{H} & \text{on} \quad \Gamma, \end{cases} \tag{II.2}$$

with $\boldsymbol{H} = \boldsymbol{h} - 2\,[\mathbf{D}(\nabla\theta)\boldsymbol{n}]_{\boldsymbol{\tau}}$.

Before study the problem $(\mathcal{S})$, we give the variational problem that will be useful to characterize the kernel of Stokes operator. We have the following proposition whose proof is similar to proof of Proposition 3.1 in [6].

Proposition 2.2 *We suppose that $\chi = 0$ and $g = 0$. Let $\boldsymbol{f} \in \big(\boldsymbol{H}_0^{r',p'}(\operatorname{div},\Omega)\big)'$, $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ such that*

$$\boldsymbol{h}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma. \tag{II.3}$$

The Stokes problem $(\mathcal{S})$ is equivalent to the variational formulation:

$$\begin{cases} \text{Find} \quad \boldsymbol{u} \in \boldsymbol{V}^p(\Omega) \quad \text{such that,} \\ \forall \boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega), \quad 2\displaystyle\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})\,d\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_\Omega + \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_\Gamma\,. \end{cases} \tag{II.4}$$

We introduce the kernel $\boldsymbol{\mathcal{T}}^p(\Omega)$ for any $1 < p < \infty$:

$$\boldsymbol{\mathcal{T}}^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\, \mathbf{D}(\boldsymbol{v}) = \boldsymbol{0} \quad \text{in}\ \Omega, \quad \operatorname{div}\boldsymbol{v} = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{on}\ \Gamma\}.$$

Observe that, if Ω is obtained by rotation around a constant vector $\boldsymbol{b}$ belonging to $\mathbb{R}^3$, then

$$\boldsymbol{\mathcal{T}}^p(\Omega) = \operatorname{Span}\{\boldsymbol{\beta}\}, \text{ where } \boldsymbol{\beta}(\boldsymbol{x}) = \boldsymbol{b}\times\boldsymbol{x}, \quad \text{for}\ \boldsymbol{x} \in \Omega.$$

Else, the kernel $\boldsymbol{\mathcal{T}}^p(\Omega)$ is equal to zero (see [64] for more details). We denote

$$\boldsymbol{\mathcal{N}}^p(\Omega) = \{(\boldsymbol{u}, c);\ \boldsymbol{u} \in \boldsymbol{\mathcal{T}}^p(\Omega),\ c \in \mathbb{R}\}.$$

Since $\boldsymbol{\mathcal{T}}^p(\Omega)$ and $\boldsymbol{\mathcal{N}}^p(\Omega)$ do not depend on p, we note $\boldsymbol{\mathcal{T}}(\Omega)$ instead of $\boldsymbol{\mathcal{T}}^p(\Omega)$ and $\boldsymbol{\mathcal{N}}(\Omega)$ instead $\boldsymbol{\mathcal{N}}^p(\Omega)$. The following two theorems are proved in [6].

Theorem 2.3 *Let $\boldsymbol{f}, \chi, g$ and $\boldsymbol{h}$ such that*

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]',\ \chi \in L^p(\Omega),\ g \in W^{1-\frac{1}{p},p}(\Gamma) \quad \text{and} \quad \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma),$$

satisfying (II.3),

$$\int_\Omega \chi d\boldsymbol{x} = \int_\Gamma g\,d\sigma. \tag{II.5}$$

and

$$\langle \boldsymbol{f}, \boldsymbol{\beta}\rangle_{\Omega} + \langle \boldsymbol{h}, \boldsymbol{\beta}\rangle_{\Gamma} = 0, \tag{II.6}$$

where $\langle .,.\rangle_{\Gamma} = \langle .,.\rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}$ *and* $\langle .,.\rangle_{\Omega} = \langle .,.\rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\times \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}$. *Then, the Stokes problem* $(\mathcal{S})$ *has a unique solution* $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega))/\mathbf{N}(\Omega)$. *Moreover we have the following estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)/\mathcal{T}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big).$$

We recall also the following regularity result:

Theorem 2.4 *Let*

$$\boldsymbol{f}\in \boldsymbol{L}^p(\Omega), \chi \in W^{1,p}(\Omega),\ g \in W^{2-\frac{1}{p},p}(\Gamma) \quad \text{and} \quad \boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma),$$

satisfying the compatibility conditions (II.3), (II.5) *and* (II.6). *Then, problem* $(\mathcal{S})$ *has a unique solution* $(\boldsymbol{u}, \pi)$ *belongs to* $(\boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega))/\mathbf{N}(\Omega)$ *and satisfies the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)/\mathcal{T}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\chi\|_{W^{1,p}(\Omega)} + \|g\|_{W^{2-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{\frac{1}{p},p}(\Gamma)}\big).$$

Now it is easy to extend Theorem 2.3 to the case $\boldsymbol{f} \in \big(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)\big)'$ and we have the following result.

Theorem 2.5 *Assume that* r *and* p *satisfying* (II.2). *Let*

$$\boldsymbol{f}\in \big(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)\big)', \chi \in L^p(\Omega), g \in W^{1-\frac{1}{p},p}(\Gamma) \quad \text{and} \quad \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$$

satisfying (II.3), (II.5) *and*

$$\langle \boldsymbol{f}, \boldsymbol{\beta}\rangle_{\Omega} + \langle \boldsymbol{h}, \boldsymbol{\beta}\rangle_{\Gamma} = 0, \tag{II.7}$$

where $\langle .,.\rangle_{\Omega} = \langle .,.\rangle_{[\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)]'\times \boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)}$. *Then, the Stokes problem* $(\mathcal{S})$ *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbf{N}(\Omega)$ *satisfying the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)/\mathcal{T}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq C\big(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big) \tag{II.8}$$

In particular, if $\chi = 0$, $g = 0$ and $\boldsymbol{h} = \boldsymbol{0}$ *then* $\boldsymbol{u}\in \boldsymbol{W}^{2,r}(\Omega)$ *with the corresponding estimate.*

Proof. Observe that, due to Lemma 1.2, we can write $\boldsymbol{f}$ as $\boldsymbol{f} = \boldsymbol{F} + \nabla\psi$, where $\boldsymbol{F} \in \boldsymbol{L}^r(\Omega)$ and $\psi \in L^p(\Omega)$. Since, $\langle \nabla\psi, \boldsymbol{\beta}\rangle_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'\times \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} = 0$, by Theorem 2.3, we deduce that the following problem:

$$\begin{cases} -\Delta\, \boldsymbol{v} + \nabla\pi_0 = \nabla\psi \quad \text{and} \quad \mathrm{div}\, \boldsymbol{v} = \chi & \text{in} \quad \Omega,\\ \boldsymbol{v}\cdot\boldsymbol{n} = g \quad \text{and} \quad 2\,[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on} \quad \Gamma, \end{cases}$$

has a unique solution $(\boldsymbol{v}, \pi_0) \in (\boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega))/\mathbf{N}(\Omega)$.

Now, since $\int_\Omega \boldsymbol{F}\cdot\boldsymbol{\beta}\,dx = 0$, by Theorem 2.4, the following problem:

$$\begin{cases} -\Delta\,\boldsymbol{w} + \nabla\pi_1 = \boldsymbol{F} \quad \text{and} \quad \operatorname{div}\boldsymbol{w} = 0 \qquad \text{in} \quad \Omega, \\ \qquad \boldsymbol{w}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad 2\left[\mathbf{D}(\boldsymbol{w})\boldsymbol{n}\right]_{\boldsymbol{\tau}} = \boldsymbol{0} \quad \text{on} \quad \Gamma, \end{cases}$$

has a unique solution $(\boldsymbol{w},\pi_1) \in \left(\boldsymbol{W}^{2,r}(\Omega)\times W^{1,r}(\Omega)\right)/\mathbf{N}(\Omega)$. Observe that, according to (II.2), we have $\boldsymbol{w} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\pi_1 \in L^p(\Omega)$. Setting $\boldsymbol{u} = \boldsymbol{v}+\boldsymbol{w}$ and $\pi = \pi_0+\pi_1$, then $(\boldsymbol{u},\pi)$ is solution of the problem $(\mathcal{S})$. The estimate (II.8) is immediate. Notice that, if $\chi = 0$ in Ω and $g = 0$ and $\boldsymbol{h} = \boldsymbol{0}$ on Γ, then $\pi_0 = \psi$ and $\boldsymbol{v} = c\boldsymbol{\beta}$ with $c \in \mathbb{R}$. $\square$

Remark 2.6 If Ω is not obtained by rotation around a vector, we have the following result: Let

$$\boldsymbol{f} \in \left(\boldsymbol{H}_0^{r',p'}(\operatorname{div},\Omega)\right)', \chi \in L^p(\Omega), g \in W^{1-\frac{1}{p},p}(\Gamma) \,\text{and}\, \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$$

satisfying (II.3) and (II.5). Then, the Stokes problem $(\mathcal{S})$ has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \le C\big(\|\boldsymbol{f}\|_{\left(\boldsymbol{H}_0^{r',p'}(\operatorname{div},\Omega)\right)'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big).$$

3 Oseen problem

In this section we take

$$\boldsymbol{v} \in \boldsymbol{L}^3(\Omega), \quad \operatorname{div}\boldsymbol{v} = 0 \quad \text{in}\,\Omega \quad \text{and} \quad \boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{on}\,\Gamma \tag{II.1}$$

and we are interested into the following type-Oseen problem:

$$(\mathcal{O})\begin{cases} -\Delta\,\boldsymbol{u} + \boldsymbol{v}\cdot\nabla\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = 0 \qquad \text{in} \quad \Omega, \\ \qquad \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad 2\left[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}\right]_{\boldsymbol{\tau}} = \boldsymbol{h} \quad \text{on} \quad \Gamma. \end{cases}$$

The aim of this section is to study the existence and uniqueness of weak and strong solutions for Oseen problem $(\mathcal{O})$. These results will be used in the following to show the existence of weak solution for Navier-Stokes problem for $\frac{3}{2} < p < 2$ by fixed point theorem.

Proposition 3.1 *Let* $\boldsymbol{f} \in \left(\boldsymbol{H}_0^{r',p'}(\operatorname{div},\Omega)\right)'$, $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ *satisfying* (II.3). *The variational formulation:*

$$\begin{cases} \text{Find} \quad \boldsymbol{u} \in \boldsymbol{V}^p(\Omega) \quad \text{such that,} \\ \\ \forall\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega), \quad 2\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})\,d\boldsymbol{x} + \int_\Omega (\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\varphi}\,dx = \langle\boldsymbol{f},\boldsymbol{\varphi}\rangle_\Omega + \langle\boldsymbol{h},\boldsymbol{\varphi}\rangle_\Gamma\,. \end{cases} \tag{II.2}$$

is equivalent to find $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$ *solution of problem* $(\mathcal{O})$.

Proof. Using Green formula (II.6), we deduce that every solution of $(\mathcal{O})$ also solves (II.2). Conversely, let $\boldsymbol{u}$ be a solution of the problem (II.2). Let us take a function $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega)$ such that $\operatorname{div} \boldsymbol{\varphi} = 0$ as a test function in (II.2). We have

$$2\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} + \int_\Omega (\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\varphi}\, dx = \langle -\Delta \boldsymbol{u} + \boldsymbol{v}\cdot\nabla\boldsymbol{u}, \boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)}.$$

As a consequence,

$$\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega) \qquad \langle -\Delta \boldsymbol{u} + \boldsymbol{v}\cdot\nabla\boldsymbol{u} - \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)} = 0.$$

So, by the De Rham's Theorem, there exists a distribution π in $\mathcal{D}^{'}(\Omega)$ defined uniquely up to an additive constant such that

$$-\Delta\boldsymbol{u} + \boldsymbol{v}\cdot\nabla\boldsymbol{u} + \nabla\pi = \boldsymbol{f}. \tag{II.3}$$

As $\nabla\pi \in [\boldsymbol{H}_0^{r',p'}(\operatorname{div},\Omega)]' \hookrightarrow \boldsymbol{W}^{-1,p}(\Omega)$, we deduce that $\pi \in L^p(\Omega)$ (we refer to [3]). Moreover, by the fact that $\boldsymbol{u}$ belongs to the space $\boldsymbol{V}^p(\Omega)$, we have $\operatorname{div}\boldsymbol{u} = 0$ in Ω and $\boldsymbol{u}.\boldsymbol{n} = 0$ on Γ. It remains to prove $2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h}$ on Γ. We multiply the equation (II.3) by $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega)$ and we integrate on Ω:

$$2\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} + \int_\Omega (\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\varphi}\, dx - 2\langle [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\rangle_\Gamma = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_\Omega\,. \tag{II.4}$$

Comparing (II.2) and (II.4), we deduce that

$$\forall \boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega), \qquad \langle 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi}\rangle_\Gamma = \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_\Gamma$$

Let now $\boldsymbol{\mu}$ any element of the space $\boldsymbol{W}^{1-\frac{1}{p'},p'}(\Gamma)$. So, there exists $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega)$ such that $\operatorname{div}\boldsymbol{\varphi} = 0$ in Ω and $\boldsymbol{\varphi} = \boldsymbol{\mu}_{\boldsymbol{\tau}}$ on Γ. It is clear that $\boldsymbol{\varphi} \in \boldsymbol{V}^2(\Omega)$ and

$$\langle 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} - \boldsymbol{h}, \boldsymbol{\mu}\rangle_\Gamma = \langle 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} - \boldsymbol{h}, \boldsymbol{\mu}_{\boldsymbol{\tau}}\rangle_\Gamma = \langle 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} - \boldsymbol{h}, \boldsymbol{\varphi}\rangle_\Gamma = 0.$$

This implies that

$$2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h} \quad \text{on} \quad \Gamma,$$

which is the end of the proof. □

We recall the following inequality whose the proof is given in [6].

Lemma 3.2 ***(Poincaré-Morrey Inequality)*** *Let Ω be a Lipschitz bounded domain. Then, we have*

$$\forall \boldsymbol{u} \in \boldsymbol{H}^1(\Omega), \qquad \inf_{\boldsymbol{k}\in\mathcal{T}(\Omega)} \|\boldsymbol{u}+\boldsymbol{k}\|^2_{\boldsymbol{L}^2(\Omega)} \leq C(\|\mathbf{D}(\boldsymbol{u})\|^2_{\boldsymbol{L}^2(\Omega)} + \int_\Gamma |\boldsymbol{u}\cdot\boldsymbol{n}|^2\, d\sigma), \tag{II.5}$$

where the constant C only depends on Ω. In particular, the semi-norm $\|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^2(\Omega)}$ is a norm which is equivalent to the norm $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$ if $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ and $\int_\Omega \boldsymbol{u}\cdot\boldsymbol{\beta} d\boldsymbol{x} = 0$.

Now, we state a result of existence of weak solution in $\boldsymbol{H}^1(\Omega)$ for Oseen problem with $\boldsymbol{f} \in (\boldsymbol{H}_0^{6,2}(\operatorname{div},\Omega))'$.

Theorem 3.3 ***(Weak solution in*** $\boldsymbol{H}^1(\Omega)$***)*** *Let* $\boldsymbol{f} \in (\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'$ *and* $\boldsymbol{h} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ *satisfying* (II.3). *Then the problem* $(\mathcal{O})$ *has a unique solution* $(\boldsymbol{u},\pi) \in (\boldsymbol{H}^1(\Omega)/\boldsymbol{\mathcal{T}}(\Omega)) \times (L^2(\Omega)/\mathbb{R})$ *and we have the following estimates:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)/\boldsymbol{\mathcal{T}}(\Omega)} \leq C(\|\boldsymbol{f}\|_{\left(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)\right)'} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}), \tag{II.6}$$

$$\| \pi \|_{L^2(\Omega)/\mathbb{R}} \leq C(1+ \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)})(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}). \tag{II.7}$$

Moreover, if $\boldsymbol{f} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega)$ *and* $\boldsymbol{h} \in \boldsymbol{W}^{\frac{1}{6},\frac{6}{5}}(\Gamma)$, *then* $\boldsymbol{u} \in \boldsymbol{W}^{2,\frac{6}{5}}(\Omega)$ *and* $\pi \in W^{1,\frac{6}{5}}(\Omega)$ *and we have the following estimates:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,\frac{6}{5}}(\Omega)/\boldsymbol{\mathcal{T}}(\Omega)} \leq C(1+ \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)})(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{\frac{1}{6},\frac{6}{5}}(\Gamma)}), \tag{II.8}$$

$$\| \pi \|_{W^{1,\frac{6}{5}}(\Omega)/\mathbb{R}} \leq C(1+ \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)})(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{\frac{1}{6},\frac{6}{5}}(\Gamma)}). \tag{II.9}$$

Proof. We know according to Proposition 3.1 that the Oseen problem $(\mathcal{O})$ is equivalent to (II.2). Let us introduce the bilinear continuous form: $a(.,.) : \boldsymbol{V}^2(\Omega)/\boldsymbol{\mathcal{T}}(\Omega) \times \boldsymbol{V}^2(\Omega)/\boldsymbol{\mathcal{T}}(\Omega) \longrightarrow \mathbb{R}$ defined as follows:

$$\forall \boldsymbol{u}, \boldsymbol{\varphi} \in \boldsymbol{V}^2(\Omega)/\boldsymbol{\mathcal{T}}(\Omega), \quad a(\boldsymbol{u},\boldsymbol{\varphi}) = 2\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} + \int_\Omega (\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\varphi}\, dx. \tag{II.10}$$

Since $\mathrm{div}\, \boldsymbol{v} = 0$ in Ω and $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on Γ, we have

$$\int_\Omega (\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{u}\, dx = 0.$$

As consequence, using inequality (II.5), the form $a(.,.)$ is coercive. Therefore, by Lax-Milgram theorem, problem $(\mathcal{O})$ has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{V}^2(\Omega)/\boldsymbol{\mathcal{T}}(\Omega) \times L^2(\Omega)/\mathbb{R}$.

Using the variational problem and Poincaré-Morrey Inequality, we have

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)/\boldsymbol{\mathcal{T}}(\Omega)} \leq C_0\big(\|\boldsymbol{f}\,\|_{[\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)]'} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\big).$$

On the other hand,

$$\|\nabla\pi\,\|_{\boldsymbol{H}^{-1}(\Omega)} \leq \|\boldsymbol{f}\,\|_{\boldsymbol{H}^{-1}(\Omega)} + \|\Delta\boldsymbol{u}\|_{\boldsymbol{H}^{-1}(\Omega)} + \| \boldsymbol{v}\cdot\nabla\boldsymbol{u} \,\|_{\boldsymbol{H}^{-1}(\Omega)}\ .$$

We know that,

$$\|\boldsymbol{f}\,\|_{\boldsymbol{H}^{-1}(\Omega)} \leq C_1\|\boldsymbol{f}\,\|_{[\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)]'}$$

,

$$\| \boldsymbol{v}\cdot\nabla\boldsymbol{u} \,\|_{\boldsymbol{H}^{-1}(\Omega)} \leq C \,\| \boldsymbol{v}\cdot\nabla\boldsymbol{u} \,\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} \leq \| \boldsymbol{v} \,\|_{\boldsymbol{L}^3(\Omega)} \| \boldsymbol{u} \,\|_{\boldsymbol{H}^1(\Omega)}$$

and

$$\|\Delta\boldsymbol{u}\|_{\boldsymbol{H}^{-1}(\Omega)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}.$$

Hence,

$$\inf_{k\in\mathbb{R}} \| \pi + k \|_{L^2(\Omega)} \leq C(1+ \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)})(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}).$$

If $\boldsymbol{f} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega)$ and $\boldsymbol{h} \in \boldsymbol{W}^{\frac{1}{6},\frac{6}{5}}(\Gamma)$, then $\boldsymbol{u}$ belongs to $\boldsymbol{H}^1(\Omega)$ and we deduce that $\boldsymbol{v}\cdot\nabla\boldsymbol{u}$ belongs to $\boldsymbol{L}^{\frac{6}{5}}(\Omega)$. Using the regularity of Stokes problem, we conclude that $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{2,\frac{6}{5}}(\Omega) \times W^{1,\frac{6}{5}}(\Omega)$. $\square$

In the remainder of this section, we suppose that Ω is not obtained by rotation around a vector. This implies that $\mathcal{T}(\Omega)$ is reduced to zero.

Theorem 3.4 ***(Strong solution in $\boldsymbol{W}^{2,p}(\Omega)$, with $p \geq \frac{6}{5}$)*** *Let $p \geq \frac{6}{5}$,*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega),\ \boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \quad \text{satisfying} \quad \boldsymbol{h}\cdot\boldsymbol{n} = 0 \quad \text{on}\Gamma.$$

Then, the solution $(\boldsymbol{u},\pi)$ given by Theorem 3.3 belongs to $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and satisfies the estimates:

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{2,p}(\Omega)} + \| \pi \|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C(1+ \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)})(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}). \tag{II.11}$$

Further, if $p \geq 3$, we have $\boldsymbol{v}\cdot\nabla\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$.

Proof. We know by Theorem 3.3 that the solution $(\boldsymbol{u},\pi)$ belongs to $\boldsymbol{W}^{2,\frac{6}{5}}(\Omega) \times W^{1,\frac{6}{5}}(\Omega)$. As $\boldsymbol{v}\cdot\nabla\boldsymbol{u} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega)$ only, we will search a priori estimates by considering $\boldsymbol{v}$ more regular in first time.

1) **First step:** We consider $\boldsymbol{v} \in \boldsymbol{L}^s(\Omega)$ with

$$s = 3 \quad \text{if} \quad p < 3, \quad s = 3+\varepsilon \quad \text{if} \quad p = 3 \quad \text{and} \quad s = p \quad \text{if} \quad p > 3,$$

for some arbitrary $\varepsilon > 0$. We know that there exists a sequence $\boldsymbol{v}_\lambda \in \mathcal{D}_\sigma(\Omega)$ such that $\boldsymbol{v}_\lambda$ converges to $\boldsymbol{v}$ in $\boldsymbol{L}^s(\Omega)$ as $\lambda \to 0$ (see [3]). Therefore, we search for $(\boldsymbol{u}_\lambda,\pi_\lambda) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ solution of the problem:

$$(\mathcal{O}_\lambda)\begin{cases} -\Delta\, \boldsymbol{u}_\lambda + \boldsymbol{v}_\lambda\cdot\nabla\boldsymbol{u}_\lambda + \nabla\pi_\lambda = \boldsymbol{f} \quad \text{and} \quad \mathrm{div}\, \boldsymbol{u}_\lambda = 0 & \text{in} \quad \Omega, \\ \boldsymbol{u}_\lambda\cdot\boldsymbol{n} = 0 \quad \text{and} \quad 2\,[\mathbf{D}(\boldsymbol{u}_\lambda)\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on} \quad \Gamma. \end{cases}$$

Recall that, from above, we can obtain a unique solution $(\boldsymbol{u}_\lambda,\pi_\lambda)$ belonging to $\boldsymbol{W}^{2,\frac{6}{5}}(\Omega) \times W^{1,\frac{6}{5}}(\Omega)$.

If $\frac{6}{5} \leq p \leq 6$, then $\boldsymbol{f} - \boldsymbol{v}_\lambda\cdot\nabla\boldsymbol{u}_\lambda$ belongs to $\boldsymbol{L}^2(\Omega)$, using the regularity of Stokes problem, we deduce that $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ and also $\boldsymbol{v}_\lambda\cdot\nabla\boldsymbol{u}_\lambda$ belongs to $\boldsymbol{L}^6(\Omega)$. As consequence, applying the regularity of Stokes problem, we conclude that $(\boldsymbol{u}_\lambda,\pi_\lambda) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.

If $p \geq 6$, then $\boldsymbol{u}_\lambda \in \boldsymbol{W}^{2,6}(\Omega)$ and $\boldsymbol{v}_\lambda\cdot\nabla\boldsymbol{u}_\lambda$ belongs to $\boldsymbol{L}^\infty(\Omega)$. Apply again the regularity of Stokes problem we deduce that $(\boldsymbol{u}_\lambda,\pi_\lambda) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.

Thus, we focus on obtaining a strong estimate for $(\boldsymbol{u}_\lambda,\pi_\lambda)$ independently from λ. Let $\varepsilon > 0$ with $0 < \lambda < \frac{\varepsilon}{2}$. We consider:

$$\boldsymbol{v}_\lambda = \boldsymbol{v}_1^\varepsilon + \boldsymbol{v}_{\lambda,2}^\varepsilon \quad \text{where} \quad \boldsymbol{v}_1^\varepsilon = \tilde{\boldsymbol{v}} * \rho_{\frac{\varepsilon}{2}} \quad \text{and} \quad \boldsymbol{v}_{\lambda,2}^\varepsilon = \boldsymbol{v}_\lambda - \boldsymbol{v}_1^\varepsilon \tag{II.12}$$

$\tilde{\boldsymbol{v}}$ being the extension by zero of $\boldsymbol{v}$ to $\mathbb{R}^3$ and $\rho_{\frac{\varepsilon}{2}}$ the classical mollifier. By regularity estimates for the Stokes problem, we have

$$\| \boldsymbol{u}_\lambda \|_{\boldsymbol{W}^{2,p}(\Omega)} + \| \pi_\lambda \|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\big(\| \boldsymbol{f} \|_{\boldsymbol{L}^p(\Omega)} + \| \boldsymbol{h} \|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)} + \| \boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^p(\Omega)} \big). \tag{II.13}$$

Now, we use the decomposition (II.12) in order to bound the term $\| \boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^p(\Omega)}$. We observe first that

$$\| \boldsymbol{v}^\varepsilon_{\lambda,2} \|_{\boldsymbol{L}^s(\Omega)} \leq \| \boldsymbol{v}_\lambda - \boldsymbol{v} \|_{\boldsymbol{L}^s(\Omega)} + \| \boldsymbol{v} - \tilde{\boldsymbol{v}} * \rho_{\frac{\varepsilon}{2}} \|_{\boldsymbol{L}^s(\Omega)} \leq \lambda + \frac{\varepsilon}{2} < \varepsilon. \tag{II.14}$$

Recall that

$$\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{W}^{1,k}(\Omega) \tag{II.15}$$

for any $k \in [1, p^*]$ if $p < 3$, for any $k \in [1, \infty[$ if $p = 3$ and for any $k \in [1, \infty]$ if $p > 3$. Then, using Hölder inequality and Sobolev embedding, we obtain

$$\| \boldsymbol{v}^\varepsilon_{\lambda,2} \cdot \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^p(\Omega)} \leq \| \boldsymbol{v}^\varepsilon_{\lambda,2} \|_{\boldsymbol{L}^s(\Omega)} \| \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^k(\Omega)} \leq C\varepsilon \| \boldsymbol{u}_\lambda \|_{\boldsymbol{W}^{2,p}(\Omega)} \tag{II.16}$$

where $\frac{1}{k} = \frac{1}{p} - \frac{1}{s}$, which is well defined because the definition of the real number s.

a) Case $p \leq 2$. There exists $r \in]3, +\infty]$ such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$ and we have

$$\| \boldsymbol{v}^\varepsilon_1 \cdot \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^p(\Omega)} \leq \| \boldsymbol{v}^\varepsilon_1 \|_{\boldsymbol{L}^r(\Omega)} \| \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^2(\Omega)} .$$

Now, using Young inequality, with $1 + \frac{1}{r} = \frac{1}{3} + \frac{1}{t}$ such that $t \geq 1$, we deduce that

$$\| \boldsymbol{v}^\varepsilon_1 \|_{\boldsymbol{L}^r(\Omega)} \leq \| \tilde{\boldsymbol{v}} \|_{\boldsymbol{L}^3(\mathbb{R}^3)} \| \rho_{\frac{\varepsilon}{2}} \|_{\boldsymbol{L}^t(\mathbb{R}^3)} .$$

As consequence,

$$\| \boldsymbol{v}^\varepsilon_1 \cdot \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^p(\Omega)} \leq C_\varepsilon \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)} \| \boldsymbol{u}_\lambda \|_{\boldsymbol{H}^1(\Omega)} .$$

Using estimate (II.6), we have

$$\| \boldsymbol{v}^\varepsilon_1 \cdot \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^p(\Omega)} \leq C_\varepsilon \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)} \big(\|\boldsymbol{f}\|_{\left(\boldsymbol{H}^{6,2}_0(\mathrm{div},\Omega)\right)'} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\big).$$

Because $p \geq \frac{6}{5}$, thanks the following embedding

$$\boldsymbol{L}^p(\Omega) \hookrightarrow \big(\boldsymbol{H}^{6,2}_0(\mathrm{div}, \Omega)\big)', \quad \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \hookrightarrow \boldsymbol{H}^{-\frac{1}{2}}(\Gamma),$$

we obtain that

$$\| \boldsymbol{v}^\varepsilon_1 \cdot \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^p(\Omega)} \leq C_\varepsilon \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)} \big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\big). \tag{II.17}$$

Using (II.16) and (II.17), choosing $\varepsilon > 0$ small enough, we deduce from (II.13) that

$$\| \boldsymbol{u}_\lambda \|_{\boldsymbol{W}^{2,p}(\Omega)} + \| \pi_\lambda \|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C(1+ \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)})(\| \boldsymbol{f} \|_{\boldsymbol{L}^p(\Omega)} + \| \boldsymbol{h} \|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}). \tag{II.18}$$

b) Case $p \geq 2$. We know that the embedding

$$\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{W}^{1,q}(\Omega) \tag{II.19}$$

is compact for any $q \in [1, p^*[$ if $p < 3$, for any $q \in [1, \infty[$ if $p = 3$ and for any $q \in [1, \infty]$ if $p > 3$. We choose the exponent q such that $q > 2$. Therefore,

$$\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{W}^{1,q}(\Omega) \hookrightarrow \boldsymbol{H}^1(\Omega).$$

As consequence, for any $\varepsilon' > 0$ there exists $C'_\varepsilon > 0$ such that

$$\| \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^q(\Omega)} \leq \varepsilon' \| \boldsymbol{u}_\lambda \|_{\boldsymbol{W}^{2,p}(\Omega)} + C_{\varepsilon'} \| \boldsymbol{u}_\lambda \|_{\boldsymbol{H}^1(\Omega)} .$$

Now, we want to estimate the term $\| \boldsymbol{v}_1^\varepsilon \cdot \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^p(\Omega)}$. Choosing $2 < q < p^*$, then there exist $r > p$ such that $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$ and $t > 1$ such that $1 + \frac{1}{r} = \frac{1}{3} + \frac{1}{t}$. Using Young's inequality, we have

$$\begin{aligned} \| \boldsymbol{v}_1^\varepsilon \cdot \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^p(\Omega)} &\leq \| \boldsymbol{v}_1^\varepsilon \|_{\boldsymbol{L}^r(\Omega)} \| \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^q(\Omega)} \\ &\leq \| \tilde{\boldsymbol{v}} \|_{\boldsymbol{L}^3(\mathbb{R}^3)} \| \rho_{\frac{\varepsilon}{2}} \|_{L^t(\mathbb{R}^3)} \| \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^q(\Omega)} . \end{aligned}$$

Thus,

$$\| \boldsymbol{v}_1^\varepsilon \cdot \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^p(\Omega)} \leq C_\varepsilon \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)} (\varepsilon' \| \boldsymbol{u}_\lambda \|_{\boldsymbol{W}^{2,p}(\Omega)} + C_{\varepsilon'} \| \boldsymbol{u}_\lambda \|_{\boldsymbol{H}^1(\Omega)}). \tag{II.20}$$

Using (II.16) and (II.20), choosing ε and ε' such that $\varepsilon' C_\varepsilon \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)} < \frac{1}{2}$, we deduce from (II.13) that

$$\| \boldsymbol{u}_\lambda \|_{\boldsymbol{W}^{2,p}(\Omega)} + \| \pi_\lambda \|_{W^{1,p}(\Omega)} \leq C_\varepsilon C_{\varepsilon'} (1+ \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)})(\| \boldsymbol{f} \|_{\boldsymbol{L}^p(\Omega)} + \| \boldsymbol{h} \|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}).$$

In all cases, we have the estimate uniform on λ. Therefore, we can extract subsequences, that we still call $(\boldsymbol{u}_\lambda)_\lambda$ and $(\pi_\lambda)_\lambda$, such that if $\lambda \longrightarrow 0$,

$$\boldsymbol{u}_\lambda \rightharpoonup \boldsymbol{u} \quad \text{weakly in} \quad \boldsymbol{W}^{2,p}(\Omega)$$

and

$$\pi_\lambda + k_\lambda \rightharpoonup \pi \quad \text{weakly in} \quad W^{1,p}(\Omega), \quad \text{with} \quad k_\lambda \in \mathbb{R}.$$

It is easy to verify that $(\boldsymbol{u}, \pi)$ is a solution of $(\mathcal{O})$ satisfying estimate (II.11).

2) Second step: We consider now $\boldsymbol{v} \in \boldsymbol{L}^3(\Omega)$ satisfying (II.1). Let $\boldsymbol{v}_\lambda \in \boldsymbol{\mathcal{D}}_\sigma(\Omega)$ such that

$\boldsymbol{v}_\lambda \longrightarrow \boldsymbol{v}$ in $\boldsymbol{L}^3(\Omega)$. Consequently, the problem $(\mathcal{O}_\lambda)$ in under conditions of $a)$ and its solution $(\boldsymbol{u}_\lambda, \pi_\lambda) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ verifies:

$$\| \boldsymbol{u}_\lambda \|_{\boldsymbol{W}^{2,p}(\Omega)} + \| \pi_\lambda \|_{W^{1,p}(\Omega)} \leq C(1 + \| \boldsymbol{v}_\lambda \|_{\boldsymbol{L}^3(\Omega)})(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}).$$

As consequence, when $\lambda \longrightarrow 0$ the limit function $(\boldsymbol{u}, \pi)$ is solution of problem $(\mathcal{O})$ and satisfies (II.11).

$\square$

Now we solve problem $(\mathcal{O})$ with $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div}, \Omega))'$ in order to find the generalized solutions for $p \geq 2$.

Theorem 3.5 ***(Generalized solution in $\boldsymbol{W}^{1,p}(\Omega)$ with $p \geq 2$)*** *Suppose that $p \geq 2$ and verifies* (II.2). *Let $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div}, \Omega))'$, $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ satisfying $\boldsymbol{h} \cdot \boldsymbol{n} = 0$ on Γ. Then, problem $(\mathcal{O})$ has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. Moreover, if $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$, there exists some constant $C > 0$ such that:*

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} + \| \pi \|_{L^p(\Omega)/\mathbb{R}} \leq C(1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)})(\| \boldsymbol{f} \|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \| \boldsymbol{h} \|_{W^{-\frac{1}{p},p}(\Gamma)}). \tag{II.21}$$

In particular, if $\boldsymbol{h} = \boldsymbol{0}$, then $\boldsymbol{u} \in \boldsymbol{W}^{2,t}(\Omega)$ with the corresponding estimate, where $t = \min(r, 3)$ if $r \neq 3$ and $t < 3$ if $r = 3$.

Proof.

1) If $\boldsymbol{h} \neq \boldsymbol{0}$.

A) Existence: We note that, if $p \geq 2$ and $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$, then $r \geq \frac{6}{5}$. Applying Theorem 2.5, there exists a unique $(\boldsymbol{u}_0, q_0) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ solution of the following problem:

$$\begin{cases} -\Delta\, \boldsymbol{u}_0 + \nabla q_0 = \boldsymbol{f} \quad \text{and} \quad \mathrm{div}\, \boldsymbol{u}_0 = 0 & \text{in} \quad \Omega, \\ \boldsymbol{u}_0 \cdot \boldsymbol{n} = 0 \quad \text{and} \quad 2\, [\mathbf{D}(\boldsymbol{u}_0)\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on} \quad \Gamma \end{cases} \tag{II.22}$$

verifying the estimate:

$$\| \boldsymbol{u}_0 \|_{\boldsymbol{W}^{1,p}(\Omega)} + \| q_0 \|_{L^p(\Omega)/\mathbb{R}} \leq C(\| \boldsymbol{f} \|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \| \boldsymbol{h} \|_{W^{-\frac{1}{p},p}(\Gamma)}). \tag{II.23}$$

We have $\boldsymbol{v} \cdot \nabla \boldsymbol{u}_0$ belongs to $\boldsymbol{L}^k(\Omega)$ with $\frac{1}{k} = \frac{1}{p} + \frac{1}{3}$. Since $k \geq \frac{6}{5}$, by Theorem 3.4 , we deduce the existence of $(\boldsymbol{z}, \theta) \in \boldsymbol{W}^{2,k}(\Omega) \times W^{1,k}(\Omega)/\mathbb{R}$ solution of problem:

$$\begin{cases} -\Delta\, \boldsymbol{z} + \boldsymbol{v} \cdot \nabla \boldsymbol{z} + \nabla \theta = -\boldsymbol{v} \cdot \nabla \boldsymbol{u}_0 \quad \text{and} \quad \mathrm{div}\, \boldsymbol{z} = 0 & \text{in} \quad \Omega, \\ \boldsymbol{z} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad 2\, [\mathbf{D}(\boldsymbol{z})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on} \quad \Gamma \end{cases} \tag{II.24}$$

and verifying the following estimate:

$$\| \boldsymbol{z} \|_{\boldsymbol{W}^{2,k}(\Omega)} + \| \theta \|_{W^{1,k}(\Omega)/\mathbb{R}} \leq C(1+ \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)}) \| \boldsymbol{v} \cdot \nabla \boldsymbol{u}_0 \|_{\boldsymbol{L}^k(\Omega)}$$
$$\leq C \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)} (1+ \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)})(\| \boldsymbol{f} \|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \| \boldsymbol{h} \|_{W^{-\frac{1}{p},p}(\Gamma)}). \tag{II.25}$$

Now, because $\frac{1}{k} = \frac{1}{p} + \frac{1}{3}$, we have $\boldsymbol{W}^{2,k}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$ and $W^{1,k}(\Omega) \hookrightarrow L^p(\Omega)$ and $(\mathbf{u}_0 + \boldsymbol{z}, q_0 + \theta) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ verifies the problem $(\mathcal{O})$. Estimate (II.21) follows from (II.23) and (II.25).

B) Estimates: In order to prove the estimate of solution obtained in the existence part, let us write for $\varepsilon > 0$

$$\boldsymbol{v} = \boldsymbol{v}_1^\varepsilon + \boldsymbol{v}_2^\varepsilon \quad \text{where} \quad \boldsymbol{v}_1^\varepsilon = \tilde{\boldsymbol{v}} * \rho_{\varepsilon/2}{}_{/\Omega} \quad \text{and} \quad \boldsymbol{v}_2^\varepsilon = \boldsymbol{v} - \boldsymbol{v}_1^\epsilon$$

where $\tilde{\boldsymbol{v}}$ being the extension by zero of $\boldsymbol{v}$ to $\mathbb{R}^3$ and $\rho_{\varepsilon/2}$ the classical mollifier. Let $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ be the solution of $(\mathcal{O})$. Using the Stokes regularity we have

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} + \| \pi \|_{L^p(\Omega)/\mathbb{R}} \leq C(\| \boldsymbol{f} \|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \| \boldsymbol{h} \|_{W^{-\frac{1}{p},p}(\Gamma)} + \| \boldsymbol{v} \cdot \nabla \boldsymbol{u} \|_{\boldsymbol{L}^r(\Omega)}). \tag{II.26}$$

For the term depending on $\boldsymbol{v}_2^\varepsilon$, we have as in the proof of Theorem 3.4

$$\| \boldsymbol{v}_2^\varepsilon \cdot \nabla \boldsymbol{u} \|_{\boldsymbol{L}^r(\Omega)} \leq C\varepsilon \| \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} . \tag{II.27}$$

In order to estimate the term depending on $\boldsymbol{v}_1^\varepsilon$, there are two cases:

i) If $p \in [2, 6)$: For the term depending on $\boldsymbol{v}_1^\varepsilon$, we use:

$$\| \boldsymbol{v}_1^\varepsilon \cdot \nabla \boldsymbol{u} \|_{\boldsymbol{L}^r(\Omega)} \leq \| \boldsymbol{v}_1^\varepsilon \|_{\boldsymbol{L}^k(\Omega)} \| \nabla \boldsymbol{u} \|_{\boldsymbol{L}^2(\Omega)}$$
$$\leq \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)} \| \rho_{\varepsilon/2} \|_{\boldsymbol{L}^t(\mathbb{R}^3)} \| \nabla \boldsymbol{u} \|_{\boldsymbol{L}^2(\Omega)},$$

where $k = \frac{6p}{6-p}$ and $t = \frac{2p}{p+2}$. Now, using estimate (II.6), we deduce that:

$$\| \boldsymbol{v}_1^\varepsilon \cdot \nabla \boldsymbol{u} \|_{\boldsymbol{L}^r(\Omega)} \leq C_\varepsilon \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)} (\|\boldsymbol{f}\|_{\left(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)\right)'} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}) \tag{II.28}$$

As consequence, estimate (II.21) is deduced from (II.26), (II.27) and (II.28).

ii) If $p \geq 6$: We note that estimate (II.27) is still valid. For the term depending on $\boldsymbol{v}_1^\varepsilon$, we have:

$$\| \boldsymbol{v}_1^\varepsilon \cdot \nabla \boldsymbol{u} \|_{\boldsymbol{L}^r(\Omega)} \leq \| \boldsymbol{v}_1^\varepsilon \|_{\boldsymbol{L}^k(\Omega)} \| \nabla \boldsymbol{u} \|_{\boldsymbol{L}^q(\Omega)}$$
$$\leq \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)} \| \rho_{\varepsilon/2} \|_{\boldsymbol{L}^t(\mathbb{R}^3)} \| \nabla \boldsymbol{u} \|_{\boldsymbol{L}^q(\Omega)},$$

where $\frac{1}{r} = \frac{1}{k} + \frac{1}{q}$ and $\frac{1}{t} = 1 + \frac{1}{p} - \frac{1}{q}$. Choosing $3 < q < p$, we have $3 < k < p$ and $t > 1$.

Use the interpolation estimate and young inequality, we deduce that:

$$\| \nabla \boldsymbol{u} \|_{\boldsymbol{L}^q(\Omega)} \leq \| \boldsymbol{u} \|^{\alpha}_{\boldsymbol{H}^1(\Omega)} \| \boldsymbol{u} \|^{1-\alpha}_{\boldsymbol{W}^{1,p}(\Omega)} \leq C_{\varepsilon'} \| \boldsymbol{u} \|_{\boldsymbol{H}^1(\Omega)} + \varepsilon' \| \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)}$$

where $\alpha = \frac{2(p-q)}{q(p-2)}$. With an adequate choice of small ε and ε' we obtain estimate (II.21).

2) If $\boldsymbol{h} = \boldsymbol{0}$ on Γ, then by Theorem 2.5, we have $\boldsymbol{u}_0 \in \boldsymbol{W}^{2,r}(\Omega) \hookrightarrow \boldsymbol{W}^{1,r^*}(\Omega)$ with $\frac{1}{r^*} = \frac{1}{r} - \frac{1}{3}$ if $r < 3$, $1 < r^* < \infty$ if $r = 3$ and $r^* = \infty$ if $r > 3$. Therefore, $\boldsymbol{v} \cdot \nabla \boldsymbol{u}_0 \in \boldsymbol{L}^t(\Omega)$ with $t = \min(r, 3)$ if $r \neq 3$ and $t < 3$ if $r = 3$. Again by Theorem 3.4 and Theorem 3.9, we have $\boldsymbol{z} \in \boldsymbol{W}^{2,t}(\Omega)$, which finish the proof.

$\square$

Notice that we can also solve the following non homogeneous problem for $p \geq 2$:

$$(\mathcal{O}') \begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{v} \cdot \nabla \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = \chi \quad \text{in} \quad \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = g \quad \text{and} \quad 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h} \quad \text{on} \quad \Gamma. \end{cases}$$

Corollary 3.6 *Suppose that $p \geq 2$ and verifies* (II.2). *Let*

$$\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\operatorname{div}, \Omega))', \quad \chi \in L^p(\Omega), \quad g \in W^{1-\frac{1}{p},p}(\Gamma) \quad \text{and} \quad \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$$

satisfying $\boldsymbol{h} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma$ *with*

$$\int_\Omega \chi \, dx = \int_\Gamma g \, d\sigma. \tag{II.29}$$

Then, the problem $(\mathcal{O}')$ has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. Moreover, if $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$, there exists some constant $C > 0$ such that:

$$\begin{aligned} \| \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} + \| \pi \|_{L^p(\Omega)/\mathbb{R}} \leq C(1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)}) \big(\| \boldsymbol{f} \|_{(\boldsymbol{H}_0^{r',p'}(\operatorname{div},\Omega))'} + \| \boldsymbol{h} \|_{W^{-\frac{1}{p},p}(\Gamma)} \\ + (1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)}) (\| \chi \|_{L^p(\Omega)} + \| g \|_{W^{1-\frac{1}{p},p}(\Gamma)}) \big). \end{aligned} \tag{II.30}$$

Proof. We solve the following Neumann problem:

$$\Delta \theta = \chi \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial \theta}{\partial \boldsymbol{n}} = g \quad \text{on } \Gamma. \tag{II.31}$$

For $\chi \in L^p(\Omega)$ and $g \in W^{1-\frac{1}{p},p}(\Gamma)$, problem (II.31) has a unique solution $\theta \in W^{2,p}(\Omega)/\mathbb{R}$ satisfying the following estimate:

$$\|\theta\|_{W^{2,p}(\Omega)/\mathbb{R}} \leq C(\|\chi\|_{L^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)}).$$

Setting $\boldsymbol{z} = \boldsymbol{u} - \nabla \theta$, then $(\mathcal{O}')$ becomes: Find $(\boldsymbol{z}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ solution of problem:

$$\begin{cases} -\Delta \boldsymbol{z} + \boldsymbol{v} \cdot \nabla \boldsymbol{z} + \nabla \pi = \boldsymbol{f} + \nabla \chi - \boldsymbol{v} \cdot \nabla(\nabla \theta) \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{z} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad 2\,[\mathbf{D}(\boldsymbol{z})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{H} \quad \text{on} \quad \Gamma \end{cases}$$

with $\boldsymbol{H} = \boldsymbol{h} - 2\,[\mathbf{D}(\nabla\theta)\boldsymbol{n}]_{\boldsymbol{\tau}}$. Observe that $\boldsymbol{H}$ belongs to $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and satisfies $\boldsymbol{H}\cdot\boldsymbol{n} = 0$. Applying Theorem 3.5 we have $(\boldsymbol{z},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$. Moreover, we have the following estimate:

$$\begin{aligned}\|\,\boldsymbol{z}\,\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\,\pi\,\|_{L^p(\Omega)/\mathbb{R}} \leq C(1+\|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)})\big(\,\|\,\boldsymbol{H}\,\|_{W^{-\frac{1}{p},p}(\Gamma)} + \|\,\boldsymbol{f}\,\|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} \\ + \|\,\chi\,\|_{L^p(\Omega)} + \|\,\boldsymbol{v}\cdot\nabla\nabla\theta\,\|_{\boldsymbol{L}^r(\Omega)}\,\big)\end{aligned} \tag{II.32}$$

Using Lemma 1.4, we have

$$\begin{aligned}\|\,\boldsymbol{H}\,\|_{W^{-\frac{1}{p},p}(\Gamma)} &\leq \|\,\boldsymbol{h}\,\|_{W^{-\frac{1}{p},p}(\Gamma)} + C\,\|\,\nabla\theta\,\|_{\boldsymbol{E}^{r,p}(\Omega)} \\ &\leq \|\,\boldsymbol{h}\,\|_{W^{-\frac{1}{p},p}(\Gamma)} + C(\|\,\nabla\theta\,\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\,\nabla\chi\,\|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'}) \\ &\leq C(\|\,\chi\,\|_{L^p(\Omega)} + \|\,g\,\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \|\,\boldsymbol{h}\,\|_{W^{-\frac{1}{p},p}(\Gamma)}).\end{aligned} \tag{II.33}$$

In addition we have

$$\|\,\boldsymbol{v}\cdot\nabla\nabla\theta\,\|_{\boldsymbol{L}^r(\Omega)} \leq C\,\|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)}\,(\|\,\chi\,\|_{L^p(\Omega)} + \|\,g\,\|_{W^{1-\frac{1}{p}}(\Omega)}). \tag{II.34}$$

Estimate (II.30) follows from estimate (II.32), (II.33) and (II.34). □

Now, let us introduce the following space:

$$\boldsymbol{Z}^p(\Omega) = \{(\boldsymbol{v},\theta) \in \boldsymbol{W}^{1,p'}(\Omega)\times L^{p'}(\Omega);\quad \Delta\boldsymbol{v} \in (\boldsymbol{H}_0^{p^*,p}(\mathrm{div},\Omega))'\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \,\text{and}\, [\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\tau} = \boldsymbol{0}\,\text{on}\,\Gamma\}.$$

Lemma 3.7 *We suppose that* $\frac{3}{2} < p < 2$ *and* r *verifying* (II.2)*. Let* $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$*,* $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ *satisfies* $\boldsymbol{h}\cdot\boldsymbol{n} = 0$ *on* Γ*. Then, there exists a unique* $(\boldsymbol{u},\pi) \in \boldsymbol{L}^{p^*}(\Omega)\times L^p(\Omega)$ *solution of* $(\mathcal{O})$*. Moreover, if* $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$*, we have the following estimate:*

$$\|\,\boldsymbol{u}\,\|_{\boldsymbol{L}^{p^*}(\Omega)} + \|\,\pi\,\|_{L^p(\Omega)/\mathbb{R}} \leq C(1+\|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)})(\|\,\boldsymbol{f}\,\|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \|\,\boldsymbol{h}\,\|_{W^{-\frac{1}{p},p}(\Gamma)}). \tag{II.35}$$

Proof.

i) In the first step we prove that: Find $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$ satisfying $(\mathcal{O})$ is equivalent to find $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$ with $\mathrm{div}\,\boldsymbol{v} = 0$ and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ such that:

$$(\mathcal{OD})\begin{cases}\forall(\boldsymbol{w},\theta) \in \boldsymbol{Z}^p(\Omega),\\ \langle\boldsymbol{u}, -\Delta\boldsymbol{w} - \boldsymbol{v}\cdot\nabla\boldsymbol{w} + \nabla\theta\rangle_{\Omega p^*,p} - \displaystyle\int_\Omega \pi\mathrm{div}\,\boldsymbol{w}\,d\boldsymbol{x} = \langle\boldsymbol{f},\boldsymbol{w}\rangle_{\Omega r',p'} + \langle\boldsymbol{h},\boldsymbol{w}\rangle_\Gamma\end{cases}$$

where the brackets $\langle .,.\rangle_{\Omega p^*,p}$ denotes the duality $\boldsymbol{H}_0^{p^*,p}(\mathrm{div},\Omega)\times(\boldsymbol{H}_0^{p^*,p}(\mathrm{div},\Omega))'$ and $\langle .,.\rangle_{\Omega r',p'}$ denotes the duality $(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'\times\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)$. Indeed, let $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$

solution of $(\mathcal{O})$ and let $(\boldsymbol{w},\theta)\in \boldsymbol{Z}^{p}(\Omega)$. Using Green formula (II.6), we obtain:

$$\begin{aligned}
-\langle \Delta \boldsymbol{u}, \boldsymbol{w}\rangle_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'\times \boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)} &= 2\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{w})\,d\boldsymbol{x} - 2\langle [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{w}\rangle_\Gamma \\
&= \langle \boldsymbol{u}, -\Delta \boldsymbol{w}\rangle_{\Omega p^*,p} - \langle \boldsymbol{h}, \boldsymbol{w}\rangle_\Gamma .
\end{aligned}$$

As $\boldsymbol{v}\cdot\nabla\boldsymbol{u}\in(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$, we have

$$\langle \boldsymbol{v}\cdot\nabla\boldsymbol{u}, \boldsymbol{w}\rangle_{\Omega r',p'} = \int_\Omega (\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{w}\,dx,$$

where the integral is well defined because $\frac{1}{3}+\frac{1}{p}+\frac{1}{p'^*}=1$. It is clear that

$$\int_\Omega (\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{w}\,dx = -\int_\Omega (\boldsymbol{v}\cdot\nabla)\boldsymbol{w}\cdot\boldsymbol{u}\,dx.$$

Next, since $\boldsymbol{w}\cdot\boldsymbol{n}=0$ on Γ, we have

$$\langle \nabla\pi, \boldsymbol{w}\rangle_{\Omega r',p'} = -\int_\Omega \pi\,\mathrm{div}\,\boldsymbol{w}\,dx.$$

Now, since $\mathrm{div}\,\boldsymbol{w}=0$ in Ω and $\boldsymbol{u}\cdot\boldsymbol{n}=0$ on Γ, we have for any θ in $L^{p'}(\Omega)$:

$$0 = -\int_\Omega \theta\,\mathrm{div}\,\boldsymbol{u}\,dx = \langle \boldsymbol{u}, \nabla\theta\rangle_{\Omega p^*,p}.$$

Because $\boldsymbol{W}^{1,p}(\Omega)\hookrightarrow \boldsymbol{L}^{p^*}(\Omega)$, we have $\boldsymbol{u}\in \boldsymbol{H}_0^{p^*,p}(\mathrm{div},\Omega)$. To recap, we have obtained that:

$$\langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\Omega r',p'} = \langle \boldsymbol{u}, -\Delta\boldsymbol{w}\rangle_{\Omega p^*,p} + \langle \boldsymbol{u}, \nabla\theta\rangle_{\Omega p^*,p} - \int_\Omega (\boldsymbol{v}\cdot\nabla)\boldsymbol{w}\cdot\boldsymbol{u}\,dx - \int_\Omega \pi\,\mathrm{div}\,\boldsymbol{w}\,dx + \langle \boldsymbol{h}, \boldsymbol{w}\rangle_\Gamma .$$

It means that $(\boldsymbol{u},\pi)$ satisfies $(\mathcal{OD})$. The second implication is similar to that given in the proof of Proposition 3.1.

ii) Let $\boldsymbol{f}_k\in\boldsymbol{\mathfrak{D}}(\Omega)$ and $\boldsymbol{h}_k\in C^\infty(\Gamma)$. Using the density of $\boldsymbol{\mathfrak{D}}(\Omega)$ in $(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$ and the density of $C^\infty(\Gamma)$ in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ we have $\boldsymbol{f}_k\longrightarrow\boldsymbol{f}$ in $(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$ and $\boldsymbol{h}_k\longrightarrow\boldsymbol{h}$ in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. Set $\boldsymbol{H}_k=\boldsymbol{h}_k-(\boldsymbol{h}_k\cdot\boldsymbol{n})\boldsymbol{n}$. Because Ω is of class $C^{1,1}$, then $\boldsymbol{n}\in\boldsymbol{W}^{1,\infty}(\Gamma)$. This means that $\boldsymbol{H}_k$ is regular and verifying: $\boldsymbol{H}_k\cdot\boldsymbol{n}=0$ on Γ and $\boldsymbol{H}_k\longrightarrow\boldsymbol{h}$ in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ because $\boldsymbol{h}\cdot\boldsymbol{n}=0$ on Γ.

Now, we know that there exists $(\boldsymbol{u}_k,\pi_k)\in\boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$ satisfying:

$$\begin{cases} -\Delta\,\boldsymbol{u}_k+\boldsymbol{v}\cdot\nabla\boldsymbol{u}_k+\nabla\pi_k=\boldsymbol{f}_k \quad \text{and} \quad \mathrm{div}\,\boldsymbol{u}_k=0 & \text{in} \quad \Omega,\\ \boldsymbol{u}_k\cdot\boldsymbol{n}=0 \quad \text{and} \quad 2\,[\mathbf{D}(\boldsymbol{u}_k)\boldsymbol{n}]_{\boldsymbol{\tau}}=\boldsymbol{H}_k & \text{on} \quad \Gamma. \end{cases} \tag{II.36}$$

Now, let us define the space $\boldsymbol{E}=\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega)\times L^p(\Omega)/\mathbb{R}$ and for $\boldsymbol{w}=(\boldsymbol{u},\pi)\in\boldsymbol{E}$, we define the following norm:

$$\|\,\boldsymbol{w}\,\|_{\boldsymbol{E}}=\|\,\boldsymbol{u}\,\|_{\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega)}+\|\,\pi\,\|_{L^p(\Omega)/\mathbb{R}}\,.$$

We define also the dual space $\boldsymbol{E}'=(\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega))'\times L_0^{p'}(\Omega)$ which is equipped with the following norm: for $\boldsymbol{G} = (\boldsymbol{F}, \varphi) \in \boldsymbol{E}'$,

$$\| \boldsymbol{G} \|_{\boldsymbol{E}'} = \| \boldsymbol{F} \|_{(\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega))'} + (1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)}) \| \varphi \|_{L^{p'}(\Omega)}$$

Using Hahn-Banach Theorem we can write

$$\begin{aligned} \| \boldsymbol{w} \|_{\boldsymbol{E}} &= \sup_{\boldsymbol{G} \in \boldsymbol{E}',\, \boldsymbol{G} \neq \boldsymbol{0}} \frac{|\langle \boldsymbol{G}, \boldsymbol{w} \rangle_{\boldsymbol{E}' \times \boldsymbol{E}}}{\| \boldsymbol{G} \|_{\boldsymbol{E}'}} \\ &= \sup_{\substack{\boldsymbol{F} \in (\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega))',\, \boldsymbol{F} \neq \boldsymbol{0} \\ \varphi \in L^{p'}(\Omega), \varphi \neq 0}} \frac{|\langle \boldsymbol{F}, \boldsymbol{w} \rangle_{\Omega p*,p} + \int_\Omega \varphi \pi \, dx|}{\| \boldsymbol{F} \|_{(\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega))'} + (1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)}) \| \varphi \|_{L^{p'}(\Omega)}}. \end{aligned}$$

Consequently,

$$\| \boldsymbol{u}_k \|_{\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega)} + \| \pi_k \|_{L^p(\Omega)/\mathbb{R}} = \sup_{\substack{\boldsymbol{F} \in (\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega))', \boldsymbol{F} \neq \boldsymbol{0} \\ \varphi \in L^{p'}(\Omega), \varphi \neq 0}} \frac{|\langle \boldsymbol{u}_k, \boldsymbol{F} \rangle_{\Omega p*,p} + \int_\Omega \pi_k \varphi \, dx|}{\| \boldsymbol{F} \|_{(\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega))'} + (1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)}) \| \varphi \|_{L^{p'}(\Omega)}}.$$

Since $p' > 2$, we know due the Corollary 3.6 that for any $\boldsymbol{F} \in (\boldsymbol{H}_0^{p^*,p}(\mathrm{div},\Omega))'$ and $\varphi \in L^{p'}(\Omega)$, there exits a unique $(\boldsymbol{w}, \theta) \in \boldsymbol{W}^{1,p'}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$ such that:

$$\begin{cases} -\Delta\, \boldsymbol{w} - \boldsymbol{v} \cdot \nabla \boldsymbol{w} + \nabla \theta = \boldsymbol{F} \quad \text{and} \quad \mathrm{div}\, \boldsymbol{w} = -\varphi & \text{in} \quad \Omega, \\ \boldsymbol{w} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad 2\,[\mathbf{D}(\boldsymbol{w})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on} \quad \Gamma, \end{cases}$$

satisfying the estimate:

$$\| \boldsymbol{w} \|_{\boldsymbol{W}^{1,p'}(\Omega)} + \| \theta \|_{L^{p'}(\Omega)/\mathbb{R}} \leq C(1 + \| \boldsymbol{v} \|_{L^3(\Omega)}) \big(\| \boldsymbol{F} \|_{(\boldsymbol{H}_0^{p^*,p}(\mathrm{div},\Omega))'} + (1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)}) \| \varphi \|_{L^{p'}(\Omega)} \big). \tag{II.37}$$

Thus,

$$\| \boldsymbol{u}_k \|_{\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega)} + \| \pi_k \|_{L^p(\Omega)/\mathbb{R}} = \sup_{\substack{\boldsymbol{F} \in (\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega))', \boldsymbol{F} \neq \boldsymbol{0} \\ \varphi \in L^{p'}(\Omega), \varphi \neq 0}} \frac{|\langle \boldsymbol{u}_k, -\Delta \boldsymbol{w} - \boldsymbol{v} \cdot \nabla \boldsymbol{w} + \nabla \theta \rangle_{\Omega p*,p} - \int_\Omega \pi_k \mathrm{div}\, \boldsymbol{w} \, dx|}{\| \boldsymbol{F} \|_{(\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega))'} + (1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)}) \| \varphi \|_{L^{p'}(\Omega)}}$$

$$= \sup_{\substack{\boldsymbol{F} \in (\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega))', \boldsymbol{F} \neq \boldsymbol{0} \\ \varphi \in L^{p'}(\Omega), \varphi \neq 0}} \frac{|\langle -\Delta \boldsymbol{u}_k + \boldsymbol{v} \cdot \nabla \boldsymbol{u}_k + \nabla \pi_k, \boldsymbol{w} \rangle_{\Omega p*,p} - \langle \boldsymbol{h}_k, \boldsymbol{w} \rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \times \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}|}{\| \boldsymbol{F} \|_{(\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega))'} + (1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)}) \| \varphi \|_{L^{p'}(\Omega)}}$$

As consequence,

$$\begin{aligned} | \langle \boldsymbol{f}_k, \boldsymbol{w} \rangle_{\Omega r',p'} - \langle \boldsymbol{h}_k, \boldsymbol{w} \rangle_\Gamma | &\leq \| \boldsymbol{f}_k \|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} \| \boldsymbol{w} \|_{\boldsymbol{W}^{1,p'}(\Omega)} + \| \boldsymbol{h}_k \|_{\boldsymbol{W}^{-1,\frac{1}{p}}(\Gamma)} \| \boldsymbol{w} \|_{\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)} \\ &\leq C(1 + \| \boldsymbol{v} \|_{L^3(\Omega)}) \big(\| \boldsymbol{f}_k \|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \| \boldsymbol{h}_k \|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \big) \\ &\quad \times \big(\| \boldsymbol{F} \|_{(\boldsymbol{H}_0^{p^*,p}(\mathrm{div},\Omega))'} + (1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)}) \| \varphi \|_{L^{p'}(\Omega)} \big). \end{aligned}$$

Hence,

$$\| \boldsymbol{u}_k \|_{\boldsymbol{H}_0^{p^*,p}(\mathrm{div},\Omega)} + \| \pi_k \|_{L^p(\Omega)/\mathbb{R}} \leq C(1+ \| \boldsymbol{v} \|_{L^3(\Omega)})(\| \boldsymbol{f}_k \|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \| \boldsymbol{h}_k \|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}).$$

Then, we deduce that $\boldsymbol{u}_k$ is bounded in $\boldsymbol{L}^{p*}(\Omega)$ and π_k+c_k is bounded in $L^p(\Omega)$ where $(c_k)_k \subset \mathbb{R}$. Now, we introduce the following space:

$$\boldsymbol{S}^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n} = 0,\ \mathrm{div}\,\boldsymbol{v} = 0\,\text{and}\quad [\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\tau} = \boldsymbol{0} \quad \text{on}\,\Gamma\}.$$

Further, we have for any $\boldsymbol{\psi} \in \boldsymbol{S}^{p'}(\Omega)$,

$$\int_\Omega \Delta \boldsymbol{u}_k \cdot \boldsymbol{\psi}\, dx = \int_\Omega \Delta\boldsymbol{\psi}\cdot\boldsymbol{u}_k\, dx + \langle 2\,[\mathbf{D}(\boldsymbol{u}_k)\boldsymbol{n}]_\tau\,, \boldsymbol{\psi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}.$$

Then, we deduce that

$$-\langle \boldsymbol{f}_k, \boldsymbol{\psi}\rangle_{\Omega r',p'} - \int_\Omega \boldsymbol{v}\otimes\boldsymbol{u}_k : \nabla\boldsymbol{\psi}\, dx = \int_\Omega \Delta\boldsymbol{\psi}\cdot\boldsymbol{u}_k\, dx + \langle 2\,[\mathbf{D}(\boldsymbol{u}_k)\boldsymbol{n}]_\tau\,, \boldsymbol{\psi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}. \tag{II.38}$$

Therefore, we can pass to the limit in (II.38) as in the proof of Lemma 5.4 in [6], we deduce that $2\,[\mathbf{D}(\boldsymbol{u}_k)\boldsymbol{n}]_\tau\,, \boldsymbol{\psi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)} \longrightarrow 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau\,, \boldsymbol{\psi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}$. Finally, we can pass to the limit in (II.36) and we deduce that $(\boldsymbol{u},\pi) \in \boldsymbol{L}^{p*}(\Omega)\times L^p(\Omega)$ is the unique solution of $(\mathcal{O})$.

□

Remark 3.8 For $\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$ with $1 < p < 2$, we managed to prove the existence of solution $(\boldsymbol{u},\pi) \in \boldsymbol{L}^{p^*}(\Omega)\times L^p(\Omega)$ for problem $(\mathcal{O})$, it remains to prove that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. We show this later in Theorem 3.10 for $\frac{3}{2} < p < 2$.

Before, we give the following regularity result.

Theorem 3.9 ***(Strong solution with*** $1 < p < \frac{6}{5}$***)*** *Let* $1 < p < \frac{6}{5}$,

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega) \quad \text{and} \quad \boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \quad \text{satisfying} \quad \boldsymbol{h}\cdot\boldsymbol{n} = 0 \quad \text{on}\,\Gamma.$$

Then, problem $(\mathcal{O})$ *has a unique solution* $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)/\mathbb{R}$*. Moreover, there exists* $C > 0$ *such that:*

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{2,p}(\Omega)} + \| \pi \|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C(1+ \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)})^2(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}). \tag{II.39}$$

Proof. Let $\lambda > 0$, $\boldsymbol{f}_\lambda \in \mathcal{D}(\Omega)$ such that $\boldsymbol{f}_\lambda \longrightarrow \boldsymbol{f}$ in $\boldsymbol{L}^p(\Omega)$ and $\boldsymbol{h}_\lambda \in C^\infty(\Gamma)$ such that $\boldsymbol{h}_\lambda \longrightarrow \boldsymbol{h}$ in $\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$. Let $(\boldsymbol{u}_\lambda,\pi_\lambda) \in \boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)/\mathbb{R}$ the unique solution of problem:

$$\begin{cases} -\Delta\,\boldsymbol{u}_\lambda + \boldsymbol{v}\cdot\nabla\boldsymbol{u}_\lambda + \nabla\pi_\lambda = \boldsymbol{f}_\lambda \quad \text{and} \quad \mathrm{div}\,\boldsymbol{u}_\lambda = 0 \quad \text{in} \quad \Omega,\\ \qquad \boldsymbol{u}_\lambda\cdot\boldsymbol{n} = 0 \quad \text{and} \quad 2\,[\mathbf{D}(\boldsymbol{u}_\lambda)\boldsymbol{n}]_\tau = \boldsymbol{h}_\lambda \quad \text{on} \quad \Gamma, \end{cases}$$

satisfying

$$\| \boldsymbol{u}_\lambda \|_{\boldsymbol{W}^{2,p}(\Omega)} + \| \pi_\lambda \|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C_1 \big(\| \boldsymbol{f}_\lambda \|_{\boldsymbol{L}^p(\Omega)} + \| \boldsymbol{h}_\lambda \|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)} + \| \boldsymbol{v} \cdot \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^p(\Omega)} \big). \tag{II.40}$$

Now, we decompose $\boldsymbol{v}$ as in (II.12). Then for a fixed $\varepsilon > 0$ we set $\boldsymbol{v} = \boldsymbol{v}_1^\varepsilon + \boldsymbol{v}_2^\varepsilon$ with $\boldsymbol{v}_1^\varepsilon = \tilde{\boldsymbol{v}} * \rho_{\frac{\varepsilon}{2}/\Omega}$ and $\| \boldsymbol{v}_2^\varepsilon \|_{\boldsymbol{L}^3(\Omega)} \leq \varepsilon$. Therefore,

$$\| \boldsymbol{v}_2^\varepsilon \cdot \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^p(\Omega)} \leq \| \boldsymbol{v}_2^\varepsilon \|_{\boldsymbol{L}^3(\Omega)} \| \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^{p*}(\Omega)} \leq C_2 \varepsilon \| \boldsymbol{u}_\lambda \|_{\boldsymbol{W}^{2,p}(\Omega)}$$

and

$$\| \boldsymbol{v}_1^\varepsilon \cdot \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^p(\Omega)} \leq \| \boldsymbol{v}_1^\varepsilon \|_{\boldsymbol{L}^r(\Omega)} \| \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^q(\Omega)} \leq \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)} \| \rho_{\frac{\varepsilon}{2}} \|_{\boldsymbol{L}^t(\Omega)} \| \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^q(\Omega)}$$

with $\frac{1}{r} + \frac{1}{q} = \frac{1}{p}$ and $1 + \frac{1}{r} = \frac{1}{3} + \frac{1}{t}$. We choose $r > 3$ and $t > 1$ and this implies that $q < p^*$.

Taking $\varepsilon > 0$ such that $C_1 C_2 \varepsilon < \frac{1}{2}$, we obtain

$$\| \boldsymbol{u}_\lambda \|_{\boldsymbol{W}^{2,p}(\Omega)} + \| \pi_\lambda \|_{W^{1,p}(\Omega)/\mathbb{R}} \leq 2C_1 \big(\| \boldsymbol{f}_\lambda \|_{\boldsymbol{L}^p(\Omega)} + \| \boldsymbol{h}_\lambda \|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)} + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)} \| \rho_{\frac{\varepsilon}{2}} \|_{\boldsymbol{L}^t(\Omega)} \| \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^q(\Omega)} \big). \tag{II.41}$$

From (II.41), we prove that there exists $C_3 > 0$ not depending on λ such that for any $\lambda > 0$ we have

$$\| \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^q(\Omega)} \leq C_3 \| \boldsymbol{f}_\lambda \|_{\boldsymbol{L}^p(\Omega)} . \tag{II.42}$$

We know that the embedding $\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{W}^{1,q}(\Omega)$ is compact for any $q < p^*$. As consequence, for any $\eta > 0$ there exists $C_\eta > 0$ such that

$$\begin{aligned} \| \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^q(\Omega)} &\leq \eta \| \boldsymbol{u}_\lambda \|_{\boldsymbol{W}^{2,p}(\Omega)} + C_\eta \| \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^q(\Omega)} \\ &\leq \eta \| \boldsymbol{u}_\lambda \|_{\boldsymbol{W}^{2,p}(\Omega)} + C C_\eta \| \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^{p^*}(\Omega)} . \end{aligned}$$

Now, using estimate (II.35), we deduce that

$$\| \nabla \boldsymbol{u}_\lambda \|_{\boldsymbol{L}^q(\Omega)} \leq \eta \| \boldsymbol{u}_\lambda \|_{\boldsymbol{W}^{2,p}(\Omega)} + C C_\eta (1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)}) (\| \boldsymbol{f}_\lambda \|_{\boldsymbol{L}^p(\Omega)} + \| \boldsymbol{h}_\lambda \|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}). \tag{II.43}$$

From (II.41) and (II.43) and choosing η such that $\eta C_1 C_\varepsilon \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)} < \frac{1}{4}$, we deduce we obtain for any $\lambda > 0$ small enough

$$\| \boldsymbol{u}_\lambda \|_{\boldsymbol{W}^{2,p}(\Omega)} + \| \pi_\lambda \|_{W^{1,p}(\Omega)/\mathbb{R}} \leq 4C_1 \max(1, C C_\varepsilon C_\eta) \big(1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)} \big)^2 (\| \boldsymbol{f}_\lambda \|_{\boldsymbol{L}^p(\Omega)} + \| \boldsymbol{h}_\lambda \|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}). \tag{II.44}$$

Thus we can extract subsequences of $\boldsymbol{u}_\lambda$ and π_λ, still denoted by $\boldsymbol{u}_\lambda$ and π_λ, such that

$$\boldsymbol{u}_\lambda \rightharpoonup \boldsymbol{u} \quad \text{in} \quad \boldsymbol{W}^{2,p}(\Omega) \quad \text{and} \quad \pi_\lambda + c_\lambda \rightharpoonup \pi \quad \text{in} \quad W^{1,p}(\Omega),$$

where $(c_\lambda)_\lambda \subset \mathbb{R}$ and $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$ verifies $(\mathcal{O})$ for $\boldsymbol{h} = \boldsymbol{0}$ and the following estimate:

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{2,p}(\Omega)} + \| \pi \|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C \big(1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)} \big)^2 (\| \boldsymbol{f} \|_{\boldsymbol{L}^p(\Omega)} + \| \boldsymbol{h} \|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}). \tag{II.45}$$

$\square$

Theorem 3.10 ***(Generalized solution with $\frac{3}{2} < p < 2$)*** *Suppose that $\frac{3}{2} < p < 2$ and verifies* (II.2). *Let $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$, $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ satisfying $\boldsymbol{h}\cdot\boldsymbol{n} = 0$ on Γ. Then, problem $(\mathcal{O})$ has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. Moreover, if $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$, there exists some constant $C > 0$ such that:*

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} + \| \pi \|_{L^p(\Omega)/\mathbb{R}} \leq C(1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)})^2 (\| \boldsymbol{f} \|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \| \boldsymbol{h} \|_{W^{-\frac{1}{p},p}(\Gamma)}). \tag{II.46}$$

In particular, if $\boldsymbol{h} = \boldsymbol{0}$, then $\boldsymbol{u} \in \boldsymbol{W}^{2,r}(\Omega)$. Moreover, if $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$, then we have the following estimate:

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{2,r}(\Omega)} + \| \pi \|_{L^p(\Omega)/\mathbb{R}} \leq C(1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)}) \| \boldsymbol{f} \|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'}. \tag{II.47}$$

Proof.

A) Existence:

A1) We note that, if $p > \frac{3}{2}$ and $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$, then $r > 1$. Applying Theorem 2.5, there exists a unique $(\boldsymbol{u}_0, q_0) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ solution of problem (II.22). We have $\boldsymbol{v}\cdot\nabla\boldsymbol{u}_0$ belongs to $\boldsymbol{L}^k(\Omega)$ with $\frac{1}{k} = \frac{1}{p} + \frac{1}{3}$. Since $k > 1$, by Theorem 3.9, we deduce the existence of $(\boldsymbol{z},\theta) \in \boldsymbol{W}^{2,k}(\Omega) \times W^{1,k}(\Omega)/\mathbb{R}$ solution of problem (II.24). Now, because $\frac{1}{k} = \frac{1}{p} + \frac{1}{3}$, we have $\boldsymbol{W}^{2,k}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$ and $W^{1,k}(\Omega) \hookrightarrow L^p(\Omega)$, then $(\mathbf{u}_0 + \boldsymbol{z}, q_0 + \theta) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ verifies the problem $(\mathcal{O})$.

A2) Now, if $\boldsymbol{h} = \boldsymbol{0}$ on Γ, then by Theorem 2.5, we have $\boldsymbol{u}_0 \in \boldsymbol{W}^{2,r}(\Omega)$. Therefore, $\boldsymbol{v}\cdot\nabla\boldsymbol{u}_0 \in \boldsymbol{L}^r(\Omega)$. By Stokes regularity we have $z \in \boldsymbol{W}^{2,r}(\Omega)$.

B) Estimate:

B1) The case of $\boldsymbol{h} = \boldsymbol{0}$ on Γ: Let $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$, then we can write $\boldsymbol{f}$ as follows:

$$\boldsymbol{f} = \boldsymbol{F} + \nabla\psi_0 \quad \text{with} \quad \boldsymbol{F} \in \boldsymbol{L}^r(\Omega) \quad \text{and} \quad \psi_0 \in L^p(\Omega)$$

with

$$\| \boldsymbol{F} \|_{\boldsymbol{L}^r(\Omega)} \leq \| \boldsymbol{f} \|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} \quad \text{and} \quad \| \psi_0 \|_{L^p(\Omega)} \leq \| \boldsymbol{f} \|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'}.$$

We write the first equation of $(\mathcal{O})$ as follows

$$-\Delta\boldsymbol{u} + \boldsymbol{v}\cdot\nabla\boldsymbol{u} + \nabla(\pi - \psi_0) = \boldsymbol{F},$$

and so we can suppose that $\boldsymbol{f} \in \boldsymbol{L}^r(\Omega)$.

Let $(\boldsymbol{f}_k)_k \subset \boldsymbol{\mathcal{D}}(\Omega)$ such that $\boldsymbol{f}_k \longrightarrow \boldsymbol{f}$ in $\boldsymbol{L}^r(\Omega)$. For $t \in (0,1)$. We have

$$\rho_t * \tilde{\boldsymbol{f}}_k \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3), \quad \lim_{t\to 0}\lim_{k\to\infty} \rho_t * \tilde{\boldsymbol{f}}_k = \tilde{\boldsymbol{f}}, \quad \text{and} \quad \rho_t * \tilde{\boldsymbol{f}}_{k/\Omega} \to \boldsymbol{f} \quad \text{when } t \to 0,\ k \to \infty$$

where $\tilde{\boldsymbol{f}}$ is the extension by zero of $\boldsymbol{f}$ to $\mathbb{R}^3$.

Moreover, for $q = \frac{2p}{3p-2}$,

$$\| \rho_t * \tilde{\boldsymbol{f}}_k \|_{\boldsymbol{L}^{\frac{6}{5}}(\mathbb{R}^3)} \leq \| \rho_t \|_{L^q(\mathbb{R}^3)} \| \tilde{\boldsymbol{f}}_k \|_{\boldsymbol{L}^r(\mathbb{R}^3)} \leq \frac{4}{3}\pi t^{\frac{-3}{q'}} \| \tilde{\boldsymbol{f}}_k \|_{\boldsymbol{L}^r(\mathbb{R}^3)} . \tag{II.48}$$

We choose $t = k^{-\alpha}$ with $\alpha > 0$ which will be specified later. Set $\boldsymbol{\mathfrak{F}}_k = \rho_t * \tilde{\boldsymbol{f}}_{k/\Omega}$. Then, $\boldsymbol{\mathfrak{F}}_k \longrightarrow \boldsymbol{f}$ in $\boldsymbol{L}^r(\Omega)$ as $k \to \infty$. According step **A2)**, there exists $(\boldsymbol{u}_k, \pi_k) \in \boldsymbol{W}^{2,r}(\Omega) \times L^p(\Omega)$ solution for the problem:

$$\begin{cases} -\Delta\, \boldsymbol{u}_k + \boldsymbol{v} \cdot \nabla \boldsymbol{u}_k + \nabla \pi_k = \boldsymbol{\mathfrak{F}}_k \quad \text{and} \quad \operatorname{div} \boldsymbol{u}_k = 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{u}_k \cdot \boldsymbol{n} = 0 \quad \text{and} \quad 2\,[\mathbf{D}(\boldsymbol{u}_k)\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} \quad \text{on} \quad \Gamma, \end{cases}$$

Observe also that $(\boldsymbol{u}_k, \pi_k)$ belongs to $\boldsymbol{H}^1(\Omega) \times L^2(\Omega)$. Since $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$, we have $\boldsymbol{W}^{2,r}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$ and using Stokes regularity we obtain:

$$\| \boldsymbol{u}_k \|_{\boldsymbol{W}^{1,p}(\Omega)} + \| \pi_k \|_{L^p(\Omega)/\mathbb{R}} \leq \| \boldsymbol{u}_k \|_{\boldsymbol{W}^{2,r}(\Omega)} + \| \pi_k \|_{L^p(\Omega)/\mathbb{R}} \leq C_1 \big(\| \boldsymbol{\mathfrak{F}}_k \|_{\boldsymbol{L}^r(\Omega)} + \| \boldsymbol{v} \cdot \nabla \boldsymbol{u}_k \|_{\boldsymbol{L}^r(\Omega)} \big). \tag{II.49}$$

From (II.49), we prove that there exists a constant C_2 not depending on k and $\boldsymbol{v}$ such that for any $k \in \mathbb{N}^*$ we have

$$\| \nabla \boldsymbol{u}_k \|_{\boldsymbol{L}^p(\Omega)} \leq C_2 \| \boldsymbol{\mathfrak{F}}_k \|_{\boldsymbol{L}^r(\Omega)} . \tag{II.50}$$

Indeed, assume, per absurdum, the invalidity of (II.50). Then for any $m \in \mathbb{N}^*$ there exists $\ell_m \in \mathbb{N}, \boldsymbol{\mathfrak{F}}_{\ell_m} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega)$ and $\boldsymbol{v}_m \in \boldsymbol{L}^3(\Omega)$ with $\operatorname{div} \boldsymbol{v}_m = 0$ in Ω and $\boldsymbol{v}_m \cdot \boldsymbol{n} = 0$ on Γ such that, if $(\boldsymbol{u}_{\ell_m}, \pi_{\ell_m})$ denotes the solution of the problem:

$$\begin{cases} -\Delta\, \boldsymbol{u}_{\ell_m} + \boldsymbol{v} \cdot \nabla \boldsymbol{u}_{\ell_m} + \nabla \pi_{\ell_m} = \boldsymbol{\mathfrak{F}}_{\ell_m} \quad \text{and} \quad \operatorname{div} \boldsymbol{u}_{\ell_m} = 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{u}_{\ell_m} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad 2\,[\mathbf{D}(\boldsymbol{u}_{\ell_m})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} \quad \text{on} \quad \Gamma \end{cases}$$

the inequality

$$\| \nabla \boldsymbol{u}_{\ell_m} \|_{\boldsymbol{L}^p(\Omega)} > m \| \boldsymbol{\mathfrak{F}}_{\ell_m} \|_{\boldsymbol{L}^r(\Omega)}, \tag{II.51}$$

would hold. We set:

$$\boldsymbol{w}_m = \frac{\boldsymbol{u}_{\ell_m}}{\| \nabla \boldsymbol{u}_{\ell_m} \|_{\boldsymbol{L}^p(\Omega)}}, \quad \theta_m = \frac{\pi_{\ell_m}}{\| \nabla \boldsymbol{u}_{\ell_m} \|_{\boldsymbol{L}^p(\Omega)}} \quad \text{and} \quad \boldsymbol{R}_m = \frac{\boldsymbol{\mathfrak{F}}_{\ell_m}}{\| \nabla \boldsymbol{u}_{\ell_m} \|_{\boldsymbol{L}^p(\Omega)}}.$$

Then for any $m \in \mathbb{N}^*$ we have

$$\begin{cases} -\Delta\, \boldsymbol{w}_m + \boldsymbol{v}_m \cdot \nabla \boldsymbol{w}_m + \nabla \theta_m = \boldsymbol{R}_m \quad \text{and} \quad \operatorname{div} \boldsymbol{w}_m = 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{w}_m \cdot \boldsymbol{n} = 0 \quad \text{and} \quad 2\,[\mathbf{D}(\boldsymbol{w}_m)\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} \quad \text{on} \quad \Gamma. \end{cases}$$

Now, since $\boldsymbol{R}_m \in \boldsymbol{L}^{\frac{6}{5}}(\Omega) \hookrightarrow (\boldsymbol{H}_0^{6,2}(\operatorname{div}, \Omega))'$ for any $m \in \mathbb{N}^*$ and $t > 0$, and applying estimate

(II.6),

$$\| \boldsymbol{w}_m \|_{\boldsymbol{H}^1(\Omega)} \leq C \| \boldsymbol{R}_m \|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} = \frac{C}{\| \nabla \boldsymbol{u}_{\ell_m} \|_{\boldsymbol{L}^p(\Omega)}} \| \boldsymbol{\mathfrak{F}}_{\ell_m} \|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} = \frac{C}{\| \nabla \boldsymbol{u}_{\ell_m} \|_{\boldsymbol{L}^p(\Omega)}} \| \rho_t * \tilde{\boldsymbol{f}}_{\ell_m/\Omega} \|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} .$$

From (II.51), we have

$$\| \boldsymbol{w}_m \|_{\boldsymbol{H}^1(\Omega)} \leq \frac{C}{m \| \boldsymbol{\mathfrak{F}}_{\ell_m} \|_{\boldsymbol{L}^r(\Omega)}} \| \rho_t * \tilde{\boldsymbol{f}}_{\ell_m} \|_{\boldsymbol{L}^{\frac{6}{5}}(\mathbb{R}^3)} .$$

Using (II.48) and choosing $t = \frac{1}{m^\alpha}$ with $0 < \alpha < \frac{q'}{3}$, we deduce that

$$\| \boldsymbol{w}_m \|_{\boldsymbol{H}^1(\Omega)} \leq \frac{4\pi C}{3m^{1-\frac{3\alpha}{q'}} \| \boldsymbol{\mathfrak{F}}_{\ell_m} \|_{\boldsymbol{L}^r(\Omega)}} \| \tilde{\boldsymbol{f}}_{\ell_m} \|_{\boldsymbol{L}^r(\mathbb{R}^3)} .$$

It is clear that the right hand side of the last inequality tends to zero when m goes to ∞, we deduce that

$$\boldsymbol{w}_m \longrightarrow \boldsymbol{0} \quad \text{in} \quad \boldsymbol{H}^1(\Omega).$$

Then, $\nabla \boldsymbol{w}_m \longrightarrow \boldsymbol{0}$ in $\boldsymbol{L}^2(\Omega)$ and in particular in $\boldsymbol{L}^p(\Omega)$. On the other hand, we have $\| \nabla \boldsymbol{w}_m \|_{\boldsymbol{L}^p(\Omega)} = 1$, which leads to a contradiction. Inequality (II.50) is therefore established. As consequence, from (II.49) and (II.50) we obtain for any $k \in \mathbb{N}^*$

$$\| \boldsymbol{u}_k \|_{\boldsymbol{W}^{1,p}(\Omega)} + \| \pi_k \|_{L^p(\Omega)/\mathbb{R}} \leq C(1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)}) \| \boldsymbol{\mathfrak{F}}_k \|_{\boldsymbol{L}^r(\Omega)} .$$

Thus we can extract a subsequence of $\boldsymbol{u}_k$ and π_k, still denoted by $\boldsymbol{u}_k$ and π_k, such that

$$\boldsymbol{u}_k \rightharpoonup \boldsymbol{u} \quad \text{in} \quad \boldsymbol{W}^{1,p}(\Omega) \quad \pi_k + c_k \rightharpoonup \pi \quad \text{in} \quad L^p(\Omega),$$

where $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ verifies $(\mathcal{O})$ for $\boldsymbol{h} = \boldsymbol{0}$ and the following estimate:

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} + \| \pi \|_{L^p(\Omega)/\mathbb{R}} \leq C(1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)}) \| \boldsymbol{f} \|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} .$$

B2) The case of $\boldsymbol{h} \neq \boldsymbol{0}$ on Γ: Applying Theorem 2.5, there exists a unique $(\boldsymbol{u}_0, q_0) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ solution of problem (II.22) verifying the estimate:

$$\| \boldsymbol{u}_0 \|_{\boldsymbol{W}^{1,p}(\Omega)} + \| q_0 \|_{L^p(\Omega)/\mathbb{R}} \leq C(\| \boldsymbol{f} \|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \| \boldsymbol{h} \|_{W^{-\frac{1}{p},p}(\Gamma)}). \tag{II.52}$$

We have $\boldsymbol{v} \cdot \nabla \boldsymbol{u}_0$ belongs to $\boldsymbol{L}^r(\Omega)$. According to step **A1)** and step **B1)**, we deduce the existence of $(\boldsymbol{z}, \theta) \in \boldsymbol{W}^{2,r}(\Omega) \times W^{1,r}(\Omega)/\mathbb{R}$ solution of problem (II.24) and verifying the following estimate:

$$\begin{aligned} \| \boldsymbol{z} \|_{\boldsymbol{W}^{2,r}(\Omega)} + \| \theta \|_{W^{1,r}(\Omega)/\mathbb{R}} &\leq C(1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)}) \| \boldsymbol{v} \cdot \nabla \boldsymbol{u}_0 \|_{\boldsymbol{L}^r(\Omega)} \\ &\leq C \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)} (1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)})(\| \boldsymbol{f} \|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \| \boldsymbol{h} \|_{W^{-\frac{1}{p},p}(\Gamma)}). \end{aligned} \tag{II.53}$$

Now, because $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$, we have $\boldsymbol{W}^{2,r}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$ and $W^{1,r}(\Omega) \hookrightarrow L^p(\Omega)$ and $(\mathbf{u}_0 + \boldsymbol{z}, q_0 + \theta) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ verifies the problem $(\mathcal{O})$. Estimate (II.46) follows from (II.52) and (II.53).

$\square$

Remark 3.11 Notice that if $p \leq \frac{3}{2}$, we have $\boldsymbol{v} \cdot \nabla \boldsymbol{u}_0$ is not integrable. Thus, we can not apply the same reasoning as in the previous proof. We have $\boldsymbol{v} \cdot \nabla \boldsymbol{u}_0 = \mathrm{div}(\boldsymbol{v} \otimes \boldsymbol{u}) \in \boldsymbol{W}^{-1,p}(\Omega)$ but it is not an element of $(\boldsymbol{H}_0^{r',p'}(\mathrm{div}, \Omega))'$. Consequently, it is difficult to prove the existence of generalized solutions of problem $(\mathcal{O})$ when $1 < p < \frac{3}{2}$ unlike the case of Oseen problem with Dirichlet boundary condition where we can use the fact that $\mathrm{div}(\boldsymbol{v} \otimes \boldsymbol{u}) \in \boldsymbol{W}^{-1,p}(\Omega)$.

4 Navier Stokes equation

The aim of this section is to prove the existence of weak solution and strong solution of problem $(\mathcal{NS})$. Take $\chi = 0$ and $g = 0$. Thanks to Lemma 1.4, we can show that the Navier-Stokes problem $(\mathcal{NS})$, is equivalent to the following formulation:

$$(\mathcal{FNS}) \begin{cases} \text{Find} \quad \boldsymbol{u} \in \boldsymbol{V}^p(\Omega) \quad \text{such that,} \\ \\ \forall \boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega), \quad 2\displaystyle\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\varphi}) = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega + \langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_\Gamma, \end{cases}$$

where $\langle ., . \rangle_\Omega = \langle ., . \rangle_{[\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)]' \times \boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)}$ and $b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \displaystyle\int_\Omega (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} \cdot \boldsymbol{w} d\boldsymbol{x}$. To facilitate the work, we give some properties of the operator b. The proof of the following lemma is similar to the case $p = 2$.

Lemma 4.1 *For any $\boldsymbol{u} \in \boldsymbol{V}^p(\Omega)$ and $\boldsymbol{\eta} \in \boldsymbol{\mathcal{T}}(\Omega)$, the following identities hold*

$$b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\eta}) = b(\boldsymbol{u}, \boldsymbol{\eta}, \boldsymbol{\eta}) = b(\boldsymbol{\eta}, \boldsymbol{\eta}, \boldsymbol{u}) = 0.$$

Remark 4.2 Using Lemma 4.1, in the case where Ω is obtained by rotation around a constant vector $\boldsymbol{a}$, we deduce that the following condition is necessary to solve the problem $(\mathcal{NS})$,

$$\langle \boldsymbol{f}, \boldsymbol{\beta} \rangle_\Omega + \langle \boldsymbol{h}, \boldsymbol{\beta} \rangle_\Gamma = 0. \tag{II.1}$$

An existence result is given by the following theorem, whose proof is given in [6].

Theorem 4.3 *Let*

$$\boldsymbol{f} \in \left(\boldsymbol{H}_0^{6,2}(\mathrm{div}, \Omega)\right)' \quad \text{and} \quad \boldsymbol{h} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma),$$

satisfying $\boldsymbol{h} \cdot \boldsymbol{n} = 0$ on Γ and the condition (II.1). *Then Problem $(\mathcal{FNS})$ has solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$. Moreover, we have the following estimate:*

$$\| \boldsymbol{u} \|_{\boldsymbol{H}^1(\Omega)} + \| \pi \|_{L^2(\Omega)/\mathbb{R}} \leq C(\| \boldsymbol{f} \|_{\left(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)\right)'} + \| \boldsymbol{h} \|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}).$$

Now, we treat the general case $1 < p < \infty$. We start with the case $p \geq 2$.

Theorem 4.4 ***(Generalized solution for*** $p \geq 2$***)*** *We suppose that $p \geq 2$ and $r \geq \frac{6}{5}$. Let $\boldsymbol{f} \in \big(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)\big)'$ and $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$, satisfying $\boldsymbol{h}\cdot\boldsymbol{n} = 0$ on Γ and the compatibility condition* (II.1)*. Then, Problem* $(\mathcal{NS})$ *has solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$. Moreover, we have the following estimate:*

$$\|\,\boldsymbol{u}\,\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\,\pi\,\|_{L^p(\Omega)/\mathbb{R}} \leq C\big(\,\|\,\boldsymbol{f}\,\|_{\left(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)\right)'} + \|\,\boldsymbol{h}\,\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\,\big).$$

Proof. Since $p \geq 2$ and $r \geq \frac{6}{5}$, we have $\big(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)\big)' \hookrightarrow \big(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)\big)'$ and $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \hookrightarrow \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$. Thus, problem $(\mathcal{NS})$ with $\chi = 0$ and $g = 0$ has a solution $(\boldsymbol{u},\pi) \in \boldsymbol{H}^1(\Omega)\times L^2(\Omega)$. As consequence, $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ belongs to $\boldsymbol{L}^{\frac{3}{2}}(\Omega)$. Now, we write

$$-\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f} - \boldsymbol{u}\cdot\nabla\boldsymbol{u}.$$

At first, according to Lemma 4.1, we note that $(\boldsymbol{f} - \boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{h})$ satisfies the compatibility condition (II.7). Using Lemma 1.2, we can decompose $\boldsymbol{f}$ as follows:

$$\mathbf{f} = \boldsymbol{F} + \nabla\psi, \quad \text{with} \quad \boldsymbol{F} \in \boldsymbol{L}^r(\Omega)\,\text{and}\,\psi \in L^p(\Omega).$$

As consequence, we distinguish according to the values of r and p the different following cases:

i) If $r \leq \frac{3}{2}$, then $p \leq 3$ and $\boldsymbol{f} - \boldsymbol{u}\cdot\nabla\boldsymbol{u}$ belongs to $\big(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)\big)'$. As consequence, using Theorem 2.5, we deduce that $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$.

ii) If $r > \frac{3}{2}$ and $p \leq 3$, then $\frac{1}{r} \leq \frac{2}{3} \leq \frac{1}{p} + \frac{1}{3}$ and we have $\boldsymbol{f} - \boldsymbol{u}\cdot\nabla\boldsymbol{u}$ belongs to $\big(\boldsymbol{H}_0^{3,p'}(\mathrm{div},\Omega)\big)'$. We apply for the second time the Theorem 2.5, we deduce that $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$.

iii) If $p > 3$, since (II.2), then $r \geq \frac{3}{2}$. Using the same reasoning, we deduce that $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$.

□

Now, we give some regularity results.

4.1 Strong solution

Theorem 4.5 ***(Strong solution with*** $p \geq \frac{6}{5}$***)*** *We suppose that $p \geq \frac{6}{5}$. Let $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$, satisfying* (II.3) *and the condition* (II.1)*. Then, Problem* $(\mathcal{NS})$ *has solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$. Moreover, we have the following estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\big).$$

Proof. The idea of the proof is to use the result of existence for Stokes problem. According to Theorem 4.3, there exists $(\boldsymbol{u},\pi) \in \boldsymbol{H}^1(\Omega)\times L^2(\Omega)$ solution of $(\mathcal{NS})$. We consider two cases:

a) $\frac{6}{5} \leq p \leq \frac{3}{2}$. We have $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ belonging to $\boldsymbol{L}^{\frac{3}{2}}(\Omega)$, the result follows directly from the regularity of Stokes problem.

b) $\frac{3}{2} < p < 3$. We have $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \times W^{1,\frac{3}{2}}(\Omega)$. Recall that

$$\boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{W}^{1,3}(\Omega) \hookrightarrow \boldsymbol{L}^{t}(\Omega)$$

with $t \geq 1$. Then, $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ belongs to $\boldsymbol{L}^{q}(\Omega)$ for any $q < 3$. To conclude we apply the regularity of Stokes problem.

c) $p \geq 3$. We know that $\boldsymbol{u} \in \boldsymbol{W}^{2,3}(\Omega)$ and $\pi \in W^{1,3}(\Omega)$. Using the fact $\boldsymbol{W}^{2,3}(\Omega) \hookrightarrow \boldsymbol{L}^{\infty}(\Omega)$, we deduce that $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ belongs to $\boldsymbol{L}^{q}(\Omega)$ for any $q \in [1, \infty[$. Again, according the regularity of Stokes, we have $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.

□

4.2 Study of Existence result of problem $(\mathcal{NS})$ for $\frac{3}{2} < p < 2$

In this subsection we study the existence of weak solution of problem $(\mathcal{NS})$ with domain Ω not obtained by rotation around a vector, with data $\boldsymbol{f} \in \big(\boldsymbol{H}_0^{r',p'}(\mathrm{div}, \Omega)\big)'$ and $\frac{3}{2} < p < 2$. We give the following Theorem whose proof is similar to Theorem 4.1 in [9].

Theorem 4.6 *Assume that $\frac{3}{2} < p \leq 2$ and $r = \frac{3p}{3+p}$. We suppose that $\chi = 0$ and $g = 0$ on Γ. Let*

$$\boldsymbol{f} \in \big(\boldsymbol{H}_0^{r',p'}(\mathrm{div}, \Omega)\big)' \quad \text{and} \quad \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$$

satisfying $\boldsymbol{h} \cdot \boldsymbol{n} = 0$ on Γ.

i) *There exists a constant α_1 such that, if*

$$\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \leq \alpha_1, \tag{II.2}$$

then, problem $(\mathcal{NS})$ has solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^{p}(\Omega)/\mathbb{R}$. Moreover, we have the following estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^{p}(\Omega)/\mathbb{R}} \leq C\big(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big).$$

ii) *Moreover, there exists a constant $\alpha_2 \in]0, \alpha_1]$ such that this solution is unique if*

$$\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \leq \alpha_2, \tag{II.3}$$

Proof.

i) Existence: We start to prove the existence of generalized solutions. We want to apply Banach's fixed point theorem. The idea is to apply this fixed point over theorem the Oseen problem $(\mathcal{O})$. We are searching for a fixed point for the mapping T,

$$\begin{aligned} T : \boldsymbol{V}^{p}(\Omega) &\longrightarrow \boldsymbol{V}^{p}(\Omega) \\ \boldsymbol{v} &\longrightarrow \boldsymbol{u}, \end{aligned}$$

where $\boldsymbol{V}^p(\Omega)$ is given by (II.5) and for $\boldsymbol{v} \in \boldsymbol{V}^p(\Omega)$, $T\boldsymbol{v} = \boldsymbol{u}$ is the unique solution of Oseen problem $(\mathcal{O})$. In order to apply the fixed point theorem, we have to define a neighborhood $\boldsymbol{B}_\lambda$, in the form:

$$\boldsymbol{B}_\lambda = \{\boldsymbol{v} \in \boldsymbol{V}^p(\Omega), \quad \boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma \quad \text{and} \quad \| \boldsymbol{v} \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq \lambda\}.$$

Let $\boldsymbol{v}_1$ and $\boldsymbol{v}_2 \in \boldsymbol{B}_\lambda$. We choose a contraction method, we must prove that there exists $\theta \in]0,1[$ such that

$$\| T\boldsymbol{v}_1 - T\boldsymbol{v}_2 \|_{\boldsymbol{W}^{1,p}(\Omega)} = \| \boldsymbol{u}_1 - \boldsymbol{u}_2 \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq \theta \| \boldsymbol{v}_1 - \boldsymbol{v}_2 \|_{\boldsymbol{W}^{1,p}(\Omega)} . \tag{II.4}$$

In order to estimate $\| \boldsymbol{u}_1 - \boldsymbol{u}_2 \|_{\boldsymbol{W}^{1,p}(\Omega)}$, we observe that for each $k = 1, 2$, we have $(\boldsymbol{u}_k, \pi_k) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ verifies:

$$\begin{cases} -\Delta\, \boldsymbol{u}_k + \boldsymbol{v}_k \cdot \nabla \boldsymbol{u}_k + \nabla \pi_k = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u}_k = 0 \quad \text{in} \quad \Omega, \\ \qquad \boldsymbol{u}_k \cdot \boldsymbol{n} = 0 \quad \text{and} \quad 2\,[\mathbf{D}(\boldsymbol{u}_k)\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h} \quad \text{on} \quad \Gamma. \end{cases}$$

with the estimate

$$\| \boldsymbol{u}_k \|_{\boldsymbol{W}^{1,p}(\Omega)} + \| \pi_k \|_{L^p(\Omega)/\mathbb{R}} \leq C(1 + \| \boldsymbol{v}_k \|_{\boldsymbol{L}^3(\Omega)})^2 \big(\| \boldsymbol{f} \|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \| \boldsymbol{h} \|_{W^{-\frac{1}{p},p}(\Gamma)} \big).$$

Now, in order to estimate the difference $\boldsymbol{u}_1 - \boldsymbol{u}_2$, we observe the problem verified by $(\boldsymbol{u}, \pi) = (\boldsymbol{u}_1 - \boldsymbol{u}_2, \pi_1 - \pi_2)$, which is the following one:

$$\begin{cases} -\Delta\, \boldsymbol{u} + \boldsymbol{v}_1 \cdot \nabla \boldsymbol{u} + \nabla \pi = -\boldsymbol{v} \cdot \nabla \boldsymbol{u}_2 \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \Omega, \\ \qquad \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} \quad \text{on} \quad \Gamma, \end{cases}$$

where $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$. Using the estimate made for Oseen problem successively for $\boldsymbol{u}$ and $\boldsymbol{u}_2$, we obtain that:

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(1 + \| \boldsymbol{v}_1 \|_{\boldsymbol{L}^3(\Omega)}) \| \boldsymbol{v} \cdot \nabla \boldsymbol{u}_2 \|_{\boldsymbol{L}^r(\Omega)} .$$

Then

$$\begin{aligned} \| \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} &\leq C(1 + \| \boldsymbol{v}_1 \|_{\boldsymbol{L}^3(\Omega)}) \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)} \| \nabla \boldsymbol{u}_2 \|_{\boldsymbol{L}^p(\Omega)} \\ &\leq C^2(1 + \| \boldsymbol{v}_1 \|_{\boldsymbol{L}^3(\Omega)})(1 + \| \boldsymbol{v}_2 \|_{\boldsymbol{L}^3(\Omega)})^2 \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)} \\ &\times \big(\| \boldsymbol{f} \|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \| \boldsymbol{h} \|_{W^{-\frac{1}{p},p}(\Gamma)} \big). \end{aligned} \tag{II.5}$$

Now, we set $\alpha = \| \boldsymbol{f} \|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \| \boldsymbol{h} \|_{W^{-\frac{1}{p},p}(\Gamma)}$. Then, (II.5) becomes:

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C^2 \alpha (1 + \| \boldsymbol{v}_1 \|_{\boldsymbol{L}^3(\Omega)})(1 + \| \boldsymbol{v}_2 \|_{\boldsymbol{L}^3(\Omega)})^2 \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)}$$

Now, let $C_1 > 0$ such that

$$\forall \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega), \quad \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)} \leq C_1 \| \boldsymbol{v} \|_{\boldsymbol{W}^{1,p}(\Omega)} .$$

As consequence, we have

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C^2 C_1 \alpha (1 + C_1 \| \boldsymbol{v}_1 \|_{\boldsymbol{W}^{1,p}(\Omega)})(1 + C_1 \| \boldsymbol{v}_2 \|_{\boldsymbol{W}^{1,p}(\Omega)})^2 \| \boldsymbol{v} \|_{\boldsymbol{W}^{1,p}(\Omega)} . \tag{II.6}$$

Therefore, we can obtain estimate (II.4) if we consider λ such that

$$C_1 C^2 \alpha (1 + C_1 \lambda)^3 < 1,$$

that is verified, for example, taking:

$$\lambda = \frac{1}{C_1}((2C_1 C^2 \alpha)^{-\frac{1}{3}} - 1) \quad \text{and} \quad \alpha < \frac{1}{2C_1 C^2}. \tag{II.7}$$

Thus, if (II.7) is verified, the fixed point $\boldsymbol{u}* \in \boldsymbol{W}^{1,p}(\Omega)$ satisfies the estimate:

$$\| \boldsymbol{u}* \|_{\boldsymbol{W}^{1,p}(\Omega)} \quad \leq \quad C\alpha (1 + C_1 \| \boldsymbol{u}* \|_{\boldsymbol{W}^{1,p}(\Omega)})^2$$

i.e

$$\| \boldsymbol{u}* \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq x*,$$

where $x*$ is the smallest solution of the equation: $aC_1 x^2 + (2a-1)x + \frac{a}{C_1} = 0$ with $a = CC_1\alpha$, *i.e* $x* = \frac{1-2a-\sqrt{1-4a}}{2aC_1}$. We take $a < \frac{1}{4}$ and we have

$$\| \boldsymbol{u}* \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq \frac{2a}{C_1(1 - 2a + \sqrt{1-4a})} \leq \frac{4a}{C_1},$$

which finishes the proof of existence.

ii) Uniqueness: We shall next prove uniqueness. Let us denote by $(\boldsymbol{u}_1, \pi_1)$ the solution obtained in step i) and by $(\boldsymbol{u}_2, \pi_2)$ any other solution of $(\mathcal{NS})$ corresponding to the same data. Set $\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{u}_2$ and $\pi = \pi_1 - \pi_2$. We find that:

$$\begin{cases} -\Delta\, \boldsymbol{u} + \boldsymbol{u}_1 \cdot \nabla \boldsymbol{u} + \nabla \pi = -\boldsymbol{u} \cdot \nabla \boldsymbol{u}_2 & \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \qquad \text{in} \quad \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & \text{and} \quad 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} \qquad \text{on} \quad \Gamma, \end{cases}$$

As consequence,

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(1 + \| \boldsymbol{u}_1 \|_{\boldsymbol{L}^3(\Omega)}) \| \boldsymbol{u} \cdot \nabla \boldsymbol{u}_2 \|_{\boldsymbol{L}^r(\Omega)},$$

as in step i), we have

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq CC_1(1 + C_1 \| \boldsymbol{u}_1 \|_{\boldsymbol{W}^{1,p}(\Omega)}) \| \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} \| \boldsymbol{u}_2 \|_{\boldsymbol{W}^{1,p}(\Omega)},$$

We know that, for $i = 1, 2$

$$\| \boldsymbol{u}_i \|_{\boldsymbol{W}^{1,p}(\Omega)} \quad \leq \quad C\alpha(1 + C_1 \| \boldsymbol{u}_i \|_{\boldsymbol{W}^{1,p}(\Omega)})^2,$$

so

$$\| \boldsymbol{u}_i \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq 4C\alpha.$$

Consequently,

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq 4C_1 C^2 \alpha(1 + 4CC_1\alpha) \| \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} .$$

Then, if $4C_1C^2\alpha(1+4CC_1\alpha)^2 < 1$, we deduce that $\boldsymbol{u} = \boldsymbol{0}$, which finish the proof of uniqueness.

$\square$

Chapter III

Semi-group theory for the Stokes operator with Navier boundary condition.

1 Introduction

Let us consider in a bounded cylindrical domain $\Omega \times (0,T)$ the linearised evolution Navier-Stokes problem

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega \times (0,T), \\ \qquad\qquad\qquad\qquad \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in} \qquad \Omega, \end{cases} \tag{III.1}$$

where the unknowns $\boldsymbol{u}$ and π stand respectively for the velocity field and the pressure of a fluid occupying a domain Ω. Given data are the external force $\boldsymbol{f}$ and the initial velocity $\boldsymbol{u}_0$.

To study these problems, it is necessary to add some boundary conditions. Note that these systems are often studied with Dirichlet boundary condition, also called no-slip boundary condition, which is applicable in the case where the boundary of the flow is solid. Yet, in the physical applications, we are often encounter situations where this conditions does not quite feasible. In this case, it is really important to introduce another boundary conditions to describe the behaviour of fluid on the wall. For example, when a part of flow boundary is the air, it is convenient to use a slip boundary condition.

In the literature, in 1827, Navier [53] was the first to propose a Navier-with friction boundary condition, in which there is the stagnant layer of fluid close to the wall allowing a fluid to slip, and the tangential component of the strain tensor should be proportional to the tangential component of the fluid velocity on the boundary:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \mathbf{0}, \tag{III.2}$$

where $\mathbf{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\top})$ denotes the deformation tensor associated with the velocity field $\boldsymbol{u}$, α the scalar friction function, $\boldsymbol{n}$ the exterior unit normal and $\boldsymbol{\tau}$ the corresponding tangent vector.

In this paper we deal with the Stokes operator with the Navier-type boundary conditions (III.2) for $\alpha = 0$. Our goal is to obtain a good semi-group theory for the Stokes operator with Navier-type boundary conditions (III.2) for $\alpha = 0$ on $\boldsymbol{L}^p$-spaces as it is well known for Dirichlet boundary condition (for instance see [36–39]).

One of our main result is the following:

Theorem 1.1 *The Stokes operator with Navier-type boundary conditions generates a bounded analytic semi-group on* $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$*for all* $1 < p < \infty$*.*

This analyticity allows us to solve the Problem (III.1)-(III.2) and to get strong, weak and very weak solutions. To prove Theorem 1.1 we use a classical approach. We study the resolvent of the Stokes operator:

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \ [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0} & \text{on } \Gamma, \end{cases} \tag{III.3}$$

where $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re} \lambda \geq 0$. We prove the existence of weak, strong and very weak solutions to Problem (III.3) satisfying the resolvent estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C(\Omega, p)}{|\lambda|} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{III.4}$$

We also prove estimate (III.4) for the norms of $[\boldsymbol{H}^{p'}_0(\operatorname{div}, \Omega)]'_{\sigma,\tau}$ and $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$. Notice that for $p = 2$ one has estimate (III.4) in a sector $\lambda \in \Sigma_\varepsilon$ for a fixed $\varepsilon \in]0, \pi[$, where

$$\Sigma_\varepsilon = \{\lambda \in \mathbb{C}^*; \ |\arg \lambda| \leq \pi - \varepsilon\}.$$

2 Preliminary results

In this section we review such basic notations and functional frameworks. Let us first recall some elementary properties.

Let us consider the following spaces:

$$\boldsymbol{X}^{m,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \ \operatorname{div} \boldsymbol{v} \in W^{m-1,p}(\Omega), \ \mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{W}^{m-1,p}(\Omega) \text{ and } \boldsymbol{v} \cdot \boldsymbol{n} \in W^{m-1/p,p}(\Gamma)\} \tag{III.1}$$

and

$$\boldsymbol{Y}^{m,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \operatorname{div} \boldsymbol{v} \in W^{m-1,p}(\Omega), \ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{W}^{m-1,p}(\Omega) \text{ and } \boldsymbol{v} \times \boldsymbol{n} \in \boldsymbol{W}^{m-1/p,p}(\Gamma)\}.$$

We have the following lemma (see [9]).

Lemma 2.1 *Let* $m \in \mathbb{N}^*$*. Assume that* Ω *is of class* $C^{m,1}$*, then the spaces* $\boldsymbol{X}^{m,p}(\Omega)$ *and* $\boldsymbol{Y}^{m,p}(\Omega)$ *are continuously embedded in* $\boldsymbol{W}^{m,p}(\Omega)$*.*

Second, we define the following spaces: For all $1 \leq p < \infty$,

$$\boldsymbol{L}^p_{\sigma,\tau}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega \text{ and } \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\},$$

is equipped with the norm of $\boldsymbol{L}^p(\Omega)$. We use the some notations as in Chapter I and Chapter II and n the sequel, for simplicity of notations, we write

$$\boldsymbol{\Lambda} \boldsymbol{w} = \sum_{k=1}^{2} (\boldsymbol{w}_{\boldsymbol{\tau}} \cdot \frac{\partial \boldsymbol{n}}{\partial s_k}) \boldsymbol{\tau}_k. \tag{III.2}$$

We recall that for any $\boldsymbol{v} \in \mathfrak{D}(\overline{\Omega})$ the following formulas holds:

$$[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = \nabla_{\boldsymbol{\tau}}(\boldsymbol{v}\cdot\boldsymbol{n}) + (\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}})_{\boldsymbol{\tau}} - \boldsymbol{\Lambda}\boldsymbol{v} \tag{III.3}$$

and

$$\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n} = \nabla_{\boldsymbol{\tau}}(\boldsymbol{v}\cdot\boldsymbol{n}) - (\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}})_{\boldsymbol{\tau}} - \boldsymbol{\Lambda}\boldsymbol{v}. \tag{III.4}$$

Remark 2.2 In particular if $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ and $[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0}$ we have $\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n} = -2\boldsymbol{\Lambda}\boldsymbol{v}$. As consequence

$$\Delta\boldsymbol{v}\cdot\boldsymbol{n} = \mathrm{div}_\Gamma\boldsymbol{\Lambda}\boldsymbol{v} \quad \text{on}\,\Gamma,$$

where div_Γ denote the surface divergence on Γ.

Consider now the space

$$\boldsymbol{E}^p(\Omega) = \{\boldsymbol{v}\in \boldsymbol{W}^{1,p}(\Omega);\, \Delta\boldsymbol{v}\in[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\},$$

which is a Banach space for the norm:

$$\|\,\boldsymbol{v}\,\|_{\boldsymbol{E}^p(\Omega)} = \|\,\boldsymbol{v}\,\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\,\Delta\boldsymbol{v}\,\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}\,.$$

We know that $\mathfrak{D}(\overline{\Omega})$ is dense in $\boldsymbol{E}^p(\Omega)$ (see [8, Lemma 4.1]). In addition we have the following lemma (see [6, Lemma 2.4]).

Lemma 2.3 *Suppose that Ω is of class $C^{1,1}$. The linear mapping $\Theta : \boldsymbol{v} \longrightarrow [\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}|\Gamma}$ defined on $\mathfrak{D}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping*

$$\Theta : \boldsymbol{E}^p(\Omega) \longrightarrow \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma).$$

Moreover, we have the Green formula: for any $\boldsymbol{v}\in\boldsymbol{E}^p(\Omega)$ and $\boldsymbol{\varphi}\in\boldsymbol{V}^{p'}(\Omega)$,

$$-\langle\Delta\boldsymbol{v},\boldsymbol{\varphi}\rangle_\Omega = 2\int_\Omega \mathbf{D}(\boldsymbol{v}) : \mathbf{D}(\boldsymbol{\varphi})\,d\boldsymbol{x} - 2\langle[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}},\boldsymbol{\varphi}\rangle_\Gamma, \tag{III.5}$$

where $\langle.,.\rangle_\Gamma$ denotes the anti-duality between $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$ and $\langle.,.\rangle_\Omega$ denotes the anti-duality between $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$ and $\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)$.

We have also the following result:

Lemma 2.4 *Assume that Ω is of class $C^{1,1}$. Let $\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega)$ such that $\Delta\boldsymbol{v}\in\boldsymbol{L}^p(\Omega)$. Then $[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}$ and $\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n}$ belongs to $\boldsymbol{W}^{-1/p,p}(\Gamma)$ and satisfy respectively formulas* (III.3) *and* (III.4)*.*

Next we consider the problem:

$$\mathrm{div}\,(\mathbf{grad}\,\pi\,-\,\boldsymbol{f}) = 0 \quad \text{in}\ \Omega, \qquad (\mathbf{grad}\,\pi\,-\,\boldsymbol{f})\cdot\boldsymbol{n} = 0 \quad \text{on}\ \Gamma. \tag{III.6}$$

We recall the following lemma concerning the weak Neumann problem without giving the proof.

Lemma 2.5 ***(i)*** *Let $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ (see [57] for instance), the Problem (III.6) has a unique solution $\pi \in W^{1,p}(\Omega)/\mathbb{R}$ satisfying the estimate*

$$\|\mathbf{grad}\,\pi\|_{\boldsymbol{L}^p(\Omega)} \;\leq\; C_1(\Omega)\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)},$$

for some constant $C_1(\Omega) > 0$.

(ii) *Let $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$, the problem (III.6) has a unique solution $\pi \in L^p(\Omega)/\mathbb{R}$ satisfying the estimate*

$$\|\pi\|_{L^p(\Omega)/\mathbb{R}} \;\leq\; C_2(\Omega,p)\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}.$$

(iii) *Let $\boldsymbol{f} \in (\boldsymbol{T}^{p'}(\Omega))'$, where $\boldsymbol{T}^p(\Omega)$ is given by (III.47). The Problem (III.6) has a unique solution $\pi \in W^{-1,p}(\Omega)/\mathbb{R}$ satisfying the estimate*

$$\|\pi\|_{W^{-1,p}(\Omega)} \;\leq\; C(\Omega,p)\,\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'}.$$

We give now the definition of a sectorial operator. Let $0 \leq \theta < \pi/2$ and let Σ_θ be the sector

$$\Sigma_\theta \;=\; \Big\{\lambda \in \mathbb{C}^*;\;\; |\arg\lambda| < \pi - \theta\Big\}.$$

Definition 2.6 *Let X be a Banach space. We say that a linear densely defined operator $\mathcal{A} : D(\mathcal{A}) \subseteq X \longmapsto X$ is sectorial if there exists a constant $M > 0$ such that*

$$\forall\,\lambda \in \Sigma_\theta, \qquad \|R(\lambda,\,\mathcal{A})\|_{\mathcal{L}(X)} \;\leq\; \frac{M}{|\lambda|}, \tag{III.7}$$

where $R(\lambda,\,\mathcal{A}) \;=\; (\lambda\,I \;-\; \mathcal{A})^{-1}$.

This means that the resolvent of a sectorial operator contains a sector Σ_θ for some $0 \leq \theta < \pi/2$ and for every $\lambda \in \Sigma_\theta$ one has estimate (III.7). Moreover thanks to [35, Chapter 2, Theorem 4.6, page 101] we have the following theorem:

Theorem 2.7 *An operator $\mathcal{A}$ generates a bounded analytic semi-group if and only if $\mathcal{A}$ is sectorial.*

However, it is not always easy to prove that an operator is sectorial in the sense of Definition 2.6. Although, Yosida [65] has proved that it is suffices to prove (III.7) in the half plane $\{\lambda \in \mathbb{C}^*;\, \mathrm{Re}\,\lambda \geq w\}$, for some $w \geq 0$. This result is stated in [12, Chapter 1, Theorem 3.2, page 30] and proved by K. Yosida.

Proposition 2.8 *Let $\mathcal{A}$: $\mathrm{D}(\mathcal{A}) \subseteq X \longmapsto X$ be a linear densely defined operator, let $w \geq 0$ and $M > 0$ such that*

$$\forall\,\lambda \in \mathbb{C}^*,\;\; \mathrm{Re}\,\lambda \geq w, \qquad \|R(\lambda,\,\mathcal{A})\|_{\mathcal{L}(X)} \;\leq\; \frac{M}{|\lambda|}. \tag{III.8}$$

Then $\mathcal{A}$ is sectorial.

Remark 2.9 Proposition 2.8 means that there exists an angle $0 < \theta_0 < \pi/2$ such that the resolvent set of the operator $\mathcal{A}$ contains the sector

$$\Sigma_{\theta_0} = \big\{\lambda \in \mathbb{C};\ \ |\arg\lambda| \leq \pi - \theta_0\big\}$$

where estimate (III.8) is satisfied.

To show the resolvent estimates of type (III.4), we need the following lemma (see [11] for the proof). Before we recall that the vectors $\overline{\boldsymbol{u}}$ and $\mathrm{Re}\,\boldsymbol{u}$ given by

$$\overline{\boldsymbol{u}} = (\ \overline{u_1},\ \ \overline{u_2},\ \ \overline{u_3}\ ), \qquad \mathrm{Re}\,\boldsymbol{u} = (\ \mathrm{Re}\,u_1,\ \ \mathrm{Re}\,u_2,\ \ \mathrm{Re}\,u_3\ )$$

are the conjugate and the real part of the vector $\boldsymbol{u}$

Lemma 2.10 *Let $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ such that $\Delta\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. Then*

$$\begin{aligned} -\ \langle\Delta\boldsymbol{u},\, |\boldsymbol{u}|^{p-2}\boldsymbol{u}\rangle_{\boldsymbol{L}^p(\Omega)\times\boldsymbol{L}^{p'}(\Omega)} \ &=\ \int_\Omega |\boldsymbol{u}|^{p-2}|\nabla\boldsymbol{u}|^2\,\mathrm{d}\,x \ +\ 4\,\frac{p-2}{p^2}\int_\Omega \Big|\nabla|\boldsymbol{u}|^{p/2}\Big|^2\,\mathrm{d}\,x \\ &+\ (p-2)\,i\sum_{k=1}^{3}\int_\Omega |\boldsymbol{u}|^{p-4}\,\mathrm{Re}\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\mathrm{Im}\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\,\mathrm{d}\,x \ -\ \Big\langle\frac{\partial\,\boldsymbol{u}}{\partial\boldsymbol{n}},\,|\boldsymbol{u}|^{p-2}\boldsymbol{u}\Big\rangle_\Gamma, \end{aligned} \qquad \text{(III.9)}$$

where $\langle .\,,\,.\rangle_\Gamma$ is the antiduality between $\boldsymbol{W}^{-1/p,p}(\Gamma)$ and $\boldsymbol{W}^{1/p,p'}(\Gamma)$.

Remark 2.11 Notice that if Ω is of class $C^{1,1}$ then $\boldsymbol{n} \in \boldsymbol{W}^{1,\infty}(\Gamma)$. This means that, if $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ such that $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ and $[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0}$ on Γ, then using formula (III.3), we deduce that $(\dfrac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}})_{\boldsymbol{\tau}} = \boldsymbol{\Lambda}\boldsymbol{u}$ belongs to $\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \hookrightarrow \boldsymbol{L}^p(\Gamma)$. In addition, we have $|\boldsymbol{u}|^{p-2}\overline{\boldsymbol{u}} \in \boldsymbol{W}^{1,p'}(\Omega)$ and then its trace belongs to $\boldsymbol{W}^{1-1/p',p'}(\Gamma) \hookrightarrow \boldsymbol{L}^{p'}(\Gamma)$. As consequence, the integral $\displaystyle\int_\Gamma |\boldsymbol{u}|^{p-2}(\frac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}})_{\boldsymbol{\tau}}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,\sigma$ is well defined and we can replace in (III.9) the term $\Big\langle\frac{\partial\,\boldsymbol{u}}{\partial\boldsymbol{n}},\,|\boldsymbol{u}|^{p-2}\boldsymbol{u}\Big\rangle_\Gamma$ by $\displaystyle\int_\Gamma |\boldsymbol{u}|^{p-2}(\frac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}})_{\boldsymbol{\tau}}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,\sigma$.

3 The Stokes operator

In this section we define the Stokes operator. This will be useful in the following to study the analyticity. Will proceed as in the case of the Stokes operator with Dirichlet boundary conditions.

Before we recall the following space:

$$\boldsymbol{V}^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\ \ \mathrm{div}\,\boldsymbol{u} = 0 \text{ in } \Omega \ \text{ and } \ \boldsymbol{u}\cdot\boldsymbol{n} \,=\, 0, \quad \text{on } \Gamma\}.$$

The Stokes operator with Navier-type boundary conditions is defined by

$$\forall\,\boldsymbol{u} \in \boldsymbol{V}^p(\Omega),\ \ \forall\,\boldsymbol{v} \in \boldsymbol{V}^{p'}(\Omega), \qquad \langle A\,\boldsymbol{u}\,,\,\boldsymbol{v}\rangle_{(\boldsymbol{V}^{p'}(\Omega))'\times\boldsymbol{V}^{p'}(\Omega)} \ =\ \int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\overline{\boldsymbol{v}})\,\mathrm{d}\,x.$$

Notice that, we can also define the Stokes problem with Navier boundary condition by $A\,:\,\mathbf{D}_p(A) \subset$

$\boldsymbol{L}^p_{\sigma,\tau}(\Omega) \longmapsto \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, where

$$\mathbf{D}_p(A) = \{\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega);\ \Delta\boldsymbol{u} \in \boldsymbol{L}^p(\Omega),\ \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega, \boldsymbol{u}\cdot\boldsymbol{n} = 0,\ [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0} \text{ on } \Gamma\} \qquad \text{(III.1)}$$

and $A\boldsymbol{u} = -P\Delta\boldsymbol{u}$, for all $\boldsymbol{u} \in \mathbf{D}_p(A)$. We recall that $P : \boldsymbol{L}^p(\Omega) \longmapsto \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ is the Helmholtz projection defined by, for all $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $P\boldsymbol{f} = \boldsymbol{f} - \mathbf{grad}\,\pi$, where π is the unique solution of Problem (III.6). This means that, the Stokes operator is defined by :

$$\boldsymbol{u} \in \mathbf{D}_p(A), \qquad A\boldsymbol{u} = -P\Delta\boldsymbol{u} = -\Delta\boldsymbol{u} + \mathbf{grad}\,\pi,$$

where π is the unique solution up to an additive constant of the problem

$$\operatorname{div}(\mathbf{grad}\,\pi - \Delta\boldsymbol{u}) = 0 \quad \text{in } \Omega, \qquad (\mathbf{grad}\,\pi - \Delta\boldsymbol{u})\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma. \qquad \text{(III.2)}$$

Now, we remember the following density result:

Proposition 3.1 *The space $\mathbf{D}_p(A)$ is dense in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$.*

Proof. The proof is obvious. Indeed, we have $\mathfrak{D}_\sigma(\Omega) \subset \mathbf{D}_p(A) \subset \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. Because $\mathfrak{D}_\sigma(\Omega)$ is dense in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, we have $\mathbf{D}_p(A)$ is dense in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. □

The following proposition gives a simple definition of the domain of A.

Proposition 3.2 *Suppose that Ω is of class $C^{2,1}$, then*

$$\mathbf{D}_p(A) = \Big\{\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega);\ \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u}\cdot\boldsymbol{n} = 0,\ [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0} \text{ on } \Gamma\Big\}. \qquad \text{(III.3)}$$

Proof. For $\boldsymbol{u} \in \mathbf{D}_p(A)$, we set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{u}$. It is easy to see that $\boldsymbol{z} \in \boldsymbol{L}^p(\Omega)$, $\operatorname{div}\boldsymbol{z} = 0$ in Ω and $\mathbf{curl}\,\boldsymbol{z} = -\Delta\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. In addition since Ω is of class $C^{2,1}$, then $\boldsymbol{z}\times\boldsymbol{n} = -2\boldsymbol{\Lambda}\boldsymbol{u} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$. Therefore, applying Lemma 2.1 we deduce that $\boldsymbol{z} \in \boldsymbol{Y}^{1,p}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$. Finally observe that $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$, $\mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, $\operatorname{div}\boldsymbol{u} = 0$ in Ω and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ. Again thanks to Lemma 2.1, we conclude that $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$, which ends the proof. □

Remark 3.3 In [11] authors studied the Stokes operator with the following boundary condition:

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0}. \qquad \text{(III.4)}$$

In this case the pressure is constant and the Stokes operator coincide with le Laplacian operator. In our case the pressure cannot be constant since it is solution of problem (III.54). So, we are in the same situation of the Stokes operator with Dirichlet boundary condition.

4 Analyticity results

This section devoted to prove the estimate of type (III.4), for data in $\boldsymbol{L}^p(\Omega)$, $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$ or $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$.

4.1 Analyticity in $\mathbf{L}^p(\Omega)$

The main goal of this subsection is to prove that the Stokes operator with Navier boundary conditions generates a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ for all $1 < p < \infty$. Our study will be done in two parts. In the first part we treat the Hilbertian case and in the second step we treat the general $\boldsymbol{L}^p$-theory.

4.1.1 The Hilbertian case

Let us recall the following lemma (see [11] for the proof).

Lemma 4.1 *Let $\varepsilon \in \,]0,\pi[$ be fixed. There exists a constant $C_\varepsilon > 0$ such that for every positive real numbers a and b one has:*

$$\forall\, \lambda \in \Sigma_\varepsilon, \qquad |\lambda\, a\, +\, b| \geq C_\varepsilon(|\lambda| a\, +\, b). \tag{III.1}$$

We want to study the resolvent of the Stokes problem. For that we consider the problem (III.3) and we have the following theorem:

Theorem 4.2 *Let $\varepsilon \in \,]0,\pi[$ be fixed, $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\lambda \in \Sigma_\varepsilon$.*
(i) *The problem (III.3) has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$.*
(ii) *There exist a constant $C'_\varepsilon > 0$ independent of $\boldsymbol{f}$ and λ such that the solution $\boldsymbol{u}$ satisfies the estimates*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C'_\varepsilon}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \tag{III.2}$$

and

$$\|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C'_\varepsilon}{\sqrt{2|\lambda|}}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}. \tag{III.3}$$

($C'_\varepsilon = 1/C_\varepsilon$, where C_ε is the constant in (III.1)).
(iii) *If Ω is of class $C^{2,1}$ then $(\boldsymbol{u},\pi) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$ and satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \leq \frac{C(\Omega,\lambda,\varepsilon)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}, \tag{III.4}$$

where $C(\Omega,\lambda,\varepsilon) = C(\Omega)(C'_\varepsilon + 1)(|\lambda| + 1)$.

Proof. The proof is in two parts. We start by the proof of existence and uniqueness.

(i) Existence and uniqueness: We consider the variational problem: find $\boldsymbol{u} \in \boldsymbol{V}^2(\Omega)$ such that for any $\boldsymbol{v} \in \boldsymbol{V}^2(\Omega)$

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \overline{\boldsymbol{v}}\,\mathrm{d}\,x, \tag{III.5}$$

where

$$a(\boldsymbol{u},\boldsymbol{v}) = \lambda\int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x + 2\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\overline{\boldsymbol{v}})\,\mathrm{d}x.$$

We can easily verify that the sesqui-linear form a is a continuous form on $\boldsymbol{V}^2(\Omega)$. For the coercivity, observe that since $\lambda \in \Sigma_\varepsilon$, thanks to Lemma 4.1 there exists a constant C_ε such that

$$\begin{aligned} |\, a(\boldsymbol{v}, \boldsymbol{v})| &= \left|\lambda \, \|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + 2\|\mathbf{D}(\boldsymbol{v})\|^2_{\boldsymbol{L}^2(\Omega)}\right| \\ &\geq C_\varepsilon \, (|\lambda| \, \|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + 2\|\mathbf{D}(\boldsymbol{v})\|^2_{\boldsymbol{L}^2(\Omega)}). \end{aligned}$$

Now, using Korn's inequality we deduce that

$$|\, a(\boldsymbol{v}, \boldsymbol{v})| \geq C_\varepsilon \, \min(|\lambda| \, , 2)\|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\Omega)}.$$

This means that, for all $\lambda \in \Sigma_\varepsilon$ a is a sesqui-linear continuous coercive form on $\boldsymbol{V}^2(\Omega)$. As consequence, due to Lax-Milgram Lemma, problem (III.5) has a unique solution $\boldsymbol{u} \in \boldsymbol{V}^2(\Omega)$ since the right-hand side belongs to the anti-dual $(\boldsymbol{V}^2(\Omega))'$.

Now, we prove that the two problems (III.3) and (III.5) are equivalent. Using Green formula (III.5), we deduce that every solution of (III.3) also solves (III.5). Conversely, let $\boldsymbol{u}$ a solution of the problem (III.5). Let us take a function $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega)$ such that $\operatorname{div} \boldsymbol{v} = 0$ as a test function in (III.5) we have

$$\lambda \int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{v}} \, \mathrm{d}\, x + 2 \int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\overline{\boldsymbol{v}}) \, \mathrm{d}\, x = \langle \lambda \boldsymbol{u} - \Delta \boldsymbol{u}, \boldsymbol{v} \rangle_{\boldsymbol{\mathcal{D}}'(\Omega) \times \boldsymbol{\mathcal{D}}(\Omega)}.$$

As a consequence,

$$\forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega), \qquad \langle \lambda \boldsymbol{u} - \Delta \boldsymbol{u} - \boldsymbol{f}, \boldsymbol{v} \rangle_{\boldsymbol{\mathcal{D}}'(\Omega) \times \boldsymbol{\mathcal{D}}(\Omega)} = 0.$$

So, by De Rham's Theorem, there exists a distribution π in $\boldsymbol{\mathcal{D}}'(\Omega)$ defined uniquely up to an additive constant such that

$$\lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}. \tag{III.6}$$

As $\nabla \pi \in \boldsymbol{H}^{-1}(\Omega)$, we deduce that $\pi \in L^2(\Omega)$ (we refer to [3]). Moreover, by the fact that $\boldsymbol{u}$ belongs to the space $\boldsymbol{V}^2(\Omega)$, we have $\operatorname{div} \boldsymbol{u} = 0$ in Ω and $\boldsymbol{u}.\boldsymbol{n} = 0$ on Γ. It remains to prove the Navier boundary condition $[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0}$ on Γ. We multiply the equation (III.6) by $\boldsymbol{v} \in \boldsymbol{V}^2(\Omega)$ and we integrate on Ω:

$$\lambda \int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{v}} \, \mathrm{d}\, x + 2 \int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\overline{\boldsymbol{v}}) \, \mathrm{d}\, x - 2\langle [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} \, , \boldsymbol{v} \rangle_\Gamma = \int_\Omega \boldsymbol{f} \cdot \overline{\boldsymbol{v}} \, \mathrm{d}\, x. \tag{III.7}$$

Using (III.5) and (III.7), we deduce that

$$\forall \boldsymbol{v} \in \boldsymbol{V}^2(\Omega), \qquad \langle [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} \, , \boldsymbol{v} \rangle_\Gamma = 0.$$

Let now $\boldsymbol{\mu}$ any element of the space $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$. So, there exists $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ such that $\operatorname{div} \boldsymbol{v} = 0$ in Ω and $\boldsymbol{v} = \boldsymbol{\mu}_{\boldsymbol{\tau}}$ on Γ. Its clear that $\boldsymbol{v} \in \boldsymbol{V}^2(\Omega)$ and

$$\langle [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} \, , \boldsymbol{\mu} \rangle_\Gamma = \langle [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} \, , \boldsymbol{\mu}_{\boldsymbol{\tau}} \rangle_\Gamma = \langle [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} \, , \boldsymbol{v} \rangle_\Gamma = 0.$$

This implies that

$$[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} \quad \text{on} \quad \Gamma.$$

This prove that the two problems (III.3) and (III.5) are equivalent. Thus we obtain the existence and the uniqueness of solution to problem (III.3).

(ii) Estimates: Now we prove the estimates. For this, multiplying the first equation of System (III.3) by $\overline{\boldsymbol{u}}$ and integrating both sides one gets

$$\lambda \int_{\Omega} |\boldsymbol{u}|^2 \,\mathrm{d}\, x \,+\, 2\int_{\Omega} |\mathbf{D}(\boldsymbol{u})|^2 \,\mathrm{d}\, x \,=\, \int_{\Omega} \boldsymbol{f}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\, x.$$

Now as described above, since $\lambda\in\Sigma_\varepsilon$, there exists a constant $C'_\varepsilon = 1/C_\varepsilon$ such that

$$\begin{aligned}
|\lambda|\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + 2\|\mathbf{D}(\boldsymbol{u})\|^2_{\boldsymbol{L}^2(\Omega)} &\leq C'_\varepsilon\big|\lambda\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + 2\|\mathbf{D}(\boldsymbol{u})\|^2_{\boldsymbol{L}^2(\Omega)}\big|\\
&= C'_\varepsilon\Big|\int_{\Omega}\boldsymbol{f}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\, x\Big|\\
&\leq C'_\varepsilon\,\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}$$

As a result

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C'_\varepsilon}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)},$$

which is estimate (III.2). In addition, it is clear that

$$\begin{aligned}
\|\mathbf{D}(\boldsymbol{u})\|^2_{\boldsymbol{L}^2(\Omega)} &\leq \frac{C'_\varepsilon}{2}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\\
&\leq \frac{C'^2_\varepsilon}{2|\lambda|}\,\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)},
\end{aligned}$$

which is estimate (III.3). We recall that C_ε is the constant in (III.1). □

Remark 4.3 The previous theorem is true in particular for $\boldsymbol{f}\in\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$.

4.1.2 L^p-theory

In this section we extend Theorem 4.2 to every $1 < p < \infty$. In the first theorem we show the existence, uniqueness and regularity of the solution to the problem (III.3) in the case of $\boldsymbol{L}^p$-theory. The proof of resolvent estimates will be divided en two step: In the first step, we prove the estimate for $\lambda\in\mathbb{C}^*$ such that $\operatorname{Re}\lambda\geq 0$ and $|\lambda|\geq\lambda_0$, where $\lambda_0 = \lambda_0(\Omega, p)$ is defined in (III.23). In the second step we prove the resolvent estimate for $0 < |\lambda| \leq \lambda_0$.

Theorem 4.4 *Let $\lambda\in\mathbb{C}\in\Sigma_\varepsilon$ and let $\boldsymbol{f}\in\boldsymbol{L}^p(\Omega)$. The problem* (III.3) *has a unique solution $(\boldsymbol{u},\pi)\in\boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$. Moreover, if Ω is of class $C^{2,1}$ then $(\boldsymbol{u},\pi)\in\boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$.*

Proof. Existence: As in proof of Theorem 4.2, we prove that problem (III.3) is equivalent to the following variational problem: Find $\boldsymbol{u}\in\boldsymbol{V}^p(\Omega)$ such that for all $\boldsymbol{v}\in\boldsymbol{V}^{p'}(\Omega)$

$$\lambda\int_{\Omega}\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x + 2\int_{\Omega}\mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\overline{\boldsymbol{v}})\,\mathrm{d}x = \int_{\Omega}\boldsymbol{f}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x.$$

To prove existence, we distinguish three cases:

(i) Case $2 \leq p \leq 6$. Let $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ be the unique solution of problem (III.3). We write problem (III.3) in the form:

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{F}, \;\; \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \;\; [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0} & \text{on } \Gamma, \end{cases} \tag{III.8}$$

where $\boldsymbol{F} = \boldsymbol{f} - \lambda \boldsymbol{u}$. Thanks to the embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$ one has $\boldsymbol{F} \in \boldsymbol{L}^p(\Omega)$. It remains to verify (see [6, Theorem 3.7]) that $\boldsymbol{F}$ satisfies the following compatibility condition

$$\int_\Omega \boldsymbol{F} \cdot \overline{\boldsymbol{\beta}} \,\mathrm{d}\, x = 0. \tag{III.9}$$

Thanks to Lemma 2.3, we have

$$\int_\Omega \boldsymbol{F} \cdot \overline{\boldsymbol{\beta}} \,\mathrm{d}\, x = \int_\Omega (-\Delta \boldsymbol{u} + \nabla \pi) \cdot \overline{\boldsymbol{\beta}} \,\mathrm{d}\, x = 2 \int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\overline{\boldsymbol{\beta}}) \,\mathrm{d}\, x - 2\langle [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}\,, \boldsymbol{\beta}\rangle_\Gamma = 0.$$

As consequence, applying [6, Theorem 3.7], we deduce that $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$.

(ii) Case $p \geq 6$. Since $\boldsymbol{f} \in \boldsymbol{L}^6(\Omega)$, according to the point **(i)**, problem (III.3) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,6}(\Omega \hookrightarrow \boldsymbol{L}^\infty(\Omega)$. Again, applying [6, Theorem 3.7]), we have $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$.

(iii) Case $p \leq 2$. Thanks to step **(i)** and **(ii)**, the operator $\lambda I + A$ is an isomorphism from $\boldsymbol{V}^p(\Omega)$ to $(\boldsymbol{V}^{p'}(\Omega))'$ for $p \geq 2$. Then the adjoint operator which is equal to $\lambda I + A$ is also an isomorphism from $\boldsymbol{V}^{p'}(\Omega)$ to $(\boldsymbol{V}^p(\Omega))'$ for $p' \leq 2$. As consequence, the operator $\lambda I + A$ is an isomorphism for $p \leq 2$. Notice that the operator $\lambda I + A \in \mathcal{L}(\boldsymbol{V}^p(\Omega), (\boldsymbol{V}^{p'}(\Omega))')$ is defined by: for all $\boldsymbol{\varphi} \in \boldsymbol{V}^p(\Omega)$, for all $\boldsymbol{\xi} \in \boldsymbol{V}^{p'}(\Omega)$

$$\langle (\lambda I + A)\boldsymbol{\varphi}\,, \boldsymbol{\xi}\rangle_{(\boldsymbol{V}^p(\Omega))' \times \boldsymbol{V}^p(\Omega)} = \lambda \int_\Omega \boldsymbol{\varphi} \cdot \overline{\boldsymbol{\xi}} \,\mathrm{d}x + 2 \int_\Omega \mathbf{D}(\boldsymbol{\varphi}) : \mathbf{D}(\overline{\boldsymbol{\xi}}) \,\mathrm{d}x.$$

Regularity: If Ω is of class $C^{2,1}$, we consider the problem (III.8) and we apply the Stokes regularity ([6, Theorem 4.1]), which ends the proof. $\square$

Remark 4.5 **1.** The pressure π given by Theorem 4.4, satisfies also the following problem:

$$\Delta \pi = \operatorname{div} \boldsymbol{f} \quad \text{in } \Omega, \qquad \frac{\partial \pi}{\partial \boldsymbol{n}} = \Delta\, \boldsymbol{u} \cdot \boldsymbol{n} = 2 \operatorname{div}_\Gamma(\boldsymbol{\Lambda u}) + \boldsymbol{f} \cdot \boldsymbol{n} \quad \text{on } \Gamma.$$

Moreover, we have the estimate:

$$\|\nabla \pi\|_{\boldsymbol{L}^p(\Omega)} \leq C(\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}). \tag{III.10}$$

2. The result of the previous theorem holds in particular for $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. In this case, notice that the pressure π is a solution of the problem

$$\Delta\pi = 0 \qquad \text{in } \Omega, \qquad \frac{\partial \pi}{\partial \boldsymbol{n}} = \Delta\,\boldsymbol{u}\cdot\boldsymbol{n} = 2\,\mathrm{div}_\Gamma(\boldsymbol{\Lambda u}) \qquad \text{on } \Gamma.$$

Moreover, we have the estimate:

$$\|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}. \tag{III.11}$$

The latter estimates will be useful in the sequel to show the resolvent estimates.

In the following, we want to prove the a resolvent estimate similarly to the estimate (III.2) for all $1 < p < \infty$. But this case will not be easy to show as in the case p = 2 and the proof will be done in several steps.

Resolvent estimates for $|\lambda| \geq \lambda_0$

Proposition 4.6 *Suppose that $2 \leq p \leq 4$. Assume that Ω is of class $C^{2,1}$. Let $\lambda \in \mathbb{C}^*$ such that $\mathrm{Re}\,\lambda \geq 0$ and $|\lambda| \geq \lambda_0$, where $\lambda_0 = \lambda_0(\Omega, p)$ is defined in (III.23). Moreover, let $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, where $1 < p < \infty$ and let $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ be the unique solution of problem (III.3). Then $\boldsymbol{u}$ satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \;\leq\; \frac{\kappa_1(\Omega,\, p)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{III.12}$$

where the constant $\kappa_1(\Omega,\, p)$ is independent of λ and $\boldsymbol{f}$.

Proof. The proof will be divided en two step. Note that the calculation in the first step is true for all $p \geq 2$.

First step: We proceed as in [11, Proposition 4.11]). Multiplying the first equation of problem (III.3) by $|\boldsymbol{u}|^{p-2}\,\overline{\boldsymbol{u}}$ and integrating both sides one gets thanks to Lemma 2.10

$$\begin{aligned}
\lambda \int_\Omega |\boldsymbol{u}|^p \,\mathrm{d}\,x \,+\, &\int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2 \,\mathrm{d}\,x \,+\, 4\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x \\
&+\,(p-2)\,i\sum_{k=1}^{3}\int_\Omega |\boldsymbol{u}|^{p-4}\,\mathrm{Re}\,\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\mathrm{Im}\,\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\,\mathrm{d}\,x \\
&= \int_\Gamma |\boldsymbol{u}|^{p-2}\big(\frac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}}\big)_{\boldsymbol{\tau}}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,\sigma \,+\, \int_\Omega |\boldsymbol{u}|^{p-2}\,\boldsymbol{f}\cdot\bar{\boldsymbol{u}}\,\mathrm{d}\,x - \int_\Omega |\boldsymbol{u}|^{p-2}\;\nabla\pi\cdot\bar{\boldsymbol{u}}.
\end{aligned} \tag{III.13}$$

Notice that, according to Remark 2.11, we have the integral on Γ is well defined.

Since $[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0}$ on Γ, we have

$$\begin{aligned}
\Big(\frac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}}\Big)_{\boldsymbol{\tau}}\cdot\overline{\boldsymbol{u}}_{\boldsymbol{\tau}} \;&=\; \sum_{j=1}^{2}\Big(\frac{\partial\,\boldsymbol{n}}{\partial s_j}\cdot\boldsymbol{u}_{\boldsymbol{\tau}}\Big)\boldsymbol{\tau}_j\cdot\sum_{k=1}^{2}\overline{u}_k\,\boldsymbol{\tau}_k \\
&=\; \sum_{j=1}^{2}\sum_{k=1}^{2}\overline{u}_j\,u_k\,\boldsymbol{\tau}_k\cdot\frac{\partial\,\boldsymbol{n}}{\partial s_j} \;=\; -\boldsymbol{n}\cdot\sum_{j,k=1}^{2}\overline{u}_j\,u_k\,\frac{\partial\,\boldsymbol{\tau}_k}{\partial s_j}.
\end{aligned} \tag{III.14}$$

Now putting together (III.13) and (III.14) one gets

$$\begin{aligned}
\lambda \int_{\Omega} |\boldsymbol{u}|^{p} \,\mathrm{d}\, x \,+\, \int_{\Omega} |\boldsymbol{u}|^{p-2}\, |\nabla\, \boldsymbol{u}|^{2} \,\mathrm{d}\, x \,+\, 4\, \frac{p-2}{p^{2}} \int_{\Omega} |\nabla\, |\boldsymbol{u}|^{p/2}|^{2} \,\mathrm{d}\, x \\
+\, (p-2)\, i \sum_{k=1}^{3} \int_{\Omega} |\boldsymbol{u}|^{p-4}\, \mathrm{Re}\, \Big(\frac{\partial\, \boldsymbol{u}}{\partial x_k} \cdot \overline{\boldsymbol{u}}\Big) \mathrm{Im}\, \Big(\frac{\partial\, \boldsymbol{u}}{\partial x_k} \cdot \overline{\boldsymbol{u}}\Big) \,\mathrm{d}\, x \\
= \int_{\Gamma} |\boldsymbol{u}|^{p-2} \sum_{j=1}^{2} \sum_{k=1}^{2} \overline{u}_j\, u_k\, \boldsymbol{\tau}_k \cdot \frac{\partial\, \boldsymbol{n}}{\partial s_j} \,\mathrm{d}\, \sigma + \int_{\Omega} |\boldsymbol{u}|^{p-2}\, \boldsymbol{f} \cdot \overline{\boldsymbol{u}} \,\mathrm{d}\, x - \int_{\Omega} |\boldsymbol{u}|^{p-2}\, \nabla\pi \cdot \bar{\boldsymbol{u}}.
\end{aligned} \tag{III.15}$$

As consequence

$$\begin{aligned}
\mathrm{Re}\, \lambda \int_{\Omega} |\boldsymbol{u}|^{p} \,\mathrm{d}\, x \,+\, \int_{\Omega} |\boldsymbol{u}|^{p-2}\, |\nabla\, \boldsymbol{u}|^{2} \,\mathrm{d}\, x \,+\, 4\, \frac{p-2}{p^{2}} \int_{\Omega} |\nabla\, |\boldsymbol{u}|^{p/2}|^{2} \,\mathrm{d}\, x \\
=\, \mathrm{Re} \Big[\int_{\Gamma} |\boldsymbol{u}|^{p-2} \Big(\frac{\partial\, \boldsymbol{u}}{\partial\, \boldsymbol{n}}\Big)_{\boldsymbol{\tau}} \cdot \overline{\boldsymbol{u}}_{\boldsymbol{\tau}} \,\mathrm{d}\, \sigma \Big] \,+\, \mathrm{Re} \Big[\int_{\Omega} |\boldsymbol{u}|^{p-2} \boldsymbol{f} \cdot \overline{\boldsymbol{u}} \,\mathrm{d}\, x \Big] \,-\, \mathrm{Re} \Big[\int_{\Omega} |\boldsymbol{u}|^{p-2}\, \nabla\pi \cdot \bar{\boldsymbol{u}} \,\mathrm{d}\, x \Big]
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Im}\, \lambda \int_{\Omega} |\boldsymbol{u}|^{p} \,\mathrm{d}\, x \,+\, (p-2) \sum_{k=1}^{3} \int_{\Omega} |\boldsymbol{u}|^{p-4}\, \mathrm{Re}\, \Big(\frac{\partial\, \boldsymbol{u}}{\partial x_k} \cdot \overline{\boldsymbol{u}}\Big) \mathrm{Im}\, \Big(\frac{\partial\, \boldsymbol{u}}{\partial x_k} \cdot \overline{\boldsymbol{u}}\Big) \,\mathrm{d}\, x \\
=\, \mathrm{Im} \Big[\int_{\Gamma} |\boldsymbol{u}|^{p-2} \Big(\frac{\partial\, \boldsymbol{u}}{\partial\, \boldsymbol{n}}\Big)_{\boldsymbol{\tau}} \cdot \overline{\boldsymbol{u}}_{\boldsymbol{\tau}} \,\mathrm{d}\, \sigma \Big] \,+\, \mathrm{Im} \Big[\int_{\Omega} |\boldsymbol{u}|^{p-2} \boldsymbol{f} \cdot \overline{\boldsymbol{u}} \,\mathrm{d}\, x \Big] \,-\, \mathrm{Im} \Big[\int_{\Omega} |\boldsymbol{u}|^{p-2}\, \nabla\pi \cdot \bar{\boldsymbol{u}} \,\mathrm{d}\, x \Big].
\end{aligned}$$

Using (III.14), we have

$$\begin{aligned}
\mathrm{Re}\, \lambda\, \|\boldsymbol{u}\|^{p}_{\boldsymbol{L}^p(\Omega)} \,+\, \int_{\Omega} |\boldsymbol{u}|^{p-2}\, |\nabla\, \boldsymbol{u}|^{2} \,\mathrm{d}\, x \,+\, 4\, \frac{p-2}{p^{2}} \int_{\Omega} |\nabla\, |\boldsymbol{u}|^{p/2}|^{2} \,\mathrm{d}\, x \\
\leq\, C_1(\Omega) \int_{\Gamma} |\boldsymbol{u}|^{p} \,\mathrm{d}\, \sigma \,+\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)} \,+\, \|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}
\end{aligned} \tag{III.16}$$

and

$$\begin{aligned}
|\mathrm{Im}\, \lambda|\, \|\boldsymbol{u}\|^{p}_{\boldsymbol{L}^p(\Omega)} \quad \leq& \quad \frac{p-2}{2} \int_{\Omega} |\boldsymbol{u}|^{p-2}\, |\nabla\, \boldsymbol{u}|^{2} \,\mathrm{d}\, x \,+\, C_1(\Omega) \int_{\Gamma} |\boldsymbol{u}|^{p} \,\mathrm{d}\, \sigma \\
+& \quad \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)} \,+\, \|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)},
\end{aligned} \tag{III.17}$$

for some constant $C_1(\Omega) > 0$. Now putting together (III.16) and (III.17) one has

$$\begin{aligned}
(\mathrm{Re}\, \lambda \,+\, |\mathrm{Im}\, \lambda|) \|\boldsymbol{u}\|^{p}_{\boldsymbol{L}^p(\Omega)} \quad +& \quad \int_{\Omega} |\boldsymbol{u}|^{p-2}\, |\nabla\, \boldsymbol{u}|^{2} \,\mathrm{d}\, x \,+\, 4\, \frac{p-2}{p^{2}} \int_{\Omega} |\nabla\, |\boldsymbol{u}|^{p/2}|^{2} \,\mathrm{d}\, x \\
\leq& \quad \frac{p-2}{2} \int_{\Omega} |\boldsymbol{u}|^{p-2}\, |\nabla\, \boldsymbol{u}|^{2} \,\mathrm{d}\, x \,+\, 2\, C_1(\Omega) \int_{\Gamma} |\boldsymbol{u}|^{p} \,\mathrm{d}\, \sigma \\
+& \quad 2\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)} \,+\, 2 \|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}.
\end{aligned} \tag{III.18}$$

Now putting together (III.16) and (III.17) one has

$$\begin{aligned}(\operatorname{Re}\lambda + |\operatorname{Im}\lambda|)\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} &+ \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla \boldsymbol{u}|^2 \,\mathrm{d}\,x + 4\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x \\ &\leq \frac{p-2}{2}\int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla \boldsymbol{u}|^2 \,\mathrm{d}\,x + 2\,C_1(\Omega)\int_\Gamma |\boldsymbol{u}|^p \,\mathrm{d}\,\sigma \\ &+ 2\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)} + 2\|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \end{aligned} \tag{III.19}$$

To estimate the integral on Γ, we need the following inequality (see [40, Chapter 1, Theorem 1.5.1.10, page 41]):

$$\int_\Gamma |w|^2 \,\mathrm{d}\,\sigma \leq \varepsilon \int_\Omega |\nabla w|^2 \,\mathrm{d}\,x + C_\varepsilon \int_\Omega |w|^2 \,\mathrm{d}\,x, \tag{III.20}$$

for all $w \in H^1(\Omega)$ and for all $\varepsilon \in]0,1[$. Applying formula (III.20) to $w = |\boldsymbol{u}|^{p/2}$ and substituting in (III.19) one gets

$$\begin{aligned}|\lambda|\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} &+ \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla \boldsymbol{u}|^2 \,\mathrm{d}\,x + 4\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x \\ &\leq \frac{p-2}{2}\int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla \boldsymbol{u}|^2 \,\mathrm{d}x + 2\,C_1(\Omega)\Big[\varepsilon\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x + C_\varepsilon\int_\Omega |\boldsymbol{u}|^p \,\mathrm{d}\,x\Big] \\ &+ 2\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)} + 2\|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \end{aligned} \tag{III.21}$$

We chose $\varepsilon > 0$ such that $\varepsilon\, C_1(\Omega) = \frac{p-2}{p^2}$. As a result the constant C_ε in (III.21) depends on p and Ω. Then by setting $C_\varepsilon = C_2(\Omega, p)$ one has

$$\begin{aligned}|\lambda|\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} &+ \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla \boldsymbol{u}|^2 \,\mathrm{d}\,x + 2\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x \\ &\leq C_3(\Omega,p)\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \frac{p-2}{2}\int_\Omega |\boldsymbol{u}|^{p-2}|\nabla \boldsymbol{u}|^2 \,\mathrm{d}\,x + 2\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)}\right)\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}, \end{aligned}$$

where

$$C_3(\Omega,p) = 2\,C_1(\Omega)C_2(\Omega,p). \tag{III.22}$$

We define

$$\lambda_0 = 2\,C_3(\Omega,p). \tag{III.23}$$

Now, for $|\lambda| \geq \lambda_0$ one has

$$\begin{aligned}\frac{|\lambda|}{2}\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} &+ \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla \boldsymbol{u}|^2 \,\mathrm{d}\,x + 2\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x \\ &\leq \frac{p-2}{2}\int_\Omega |\boldsymbol{u}|^{p-2}|\nabla \boldsymbol{u}|^2 \,\mathrm{d}\,x + 2\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)}\right)\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \end{aligned}$$

Second step: Since $2 \leq p \leq 4$, according to the first step, we can write

$$\begin{aligned}\frac{|\lambda|}{2}\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} \;&+\; \frac{4-p}{2}\int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2\,\mathrm{d}\,x \,+\, 2\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2\,\mathrm{d}\,x\\ &\le\; 2\,(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \,+\, \|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)})\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}.\end{aligned}$$

Now, to estimate the pressure term, we proceed as follows: Since Ω is of class $C^{2,1}$, we have $\boldsymbol{u}\in \boldsymbol{H}^{\,2}(\Omega)\hookrightarrow \boldsymbol{W}^{\,1,p}(\Omega)$ for all $p\le 6$. Using (III.11), we deduce that

$$\|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)}\le C\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)}.$$

In addition, using the regularity estimates of Stokes problem [6], we deduce that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)}\le C\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{f}-\lambda\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}.$$

Therefore

$$\begin{aligned}\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} &\le C(1+\frac{1}{|\lambda|}+C'_\varepsilon)\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\\ &\le C(1+\frac{1}{\lambda_0}+C'_\varepsilon)\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}.\end{aligned}$$

As consequence

$$\|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)}\le C(1+\frac{1}{\lambda_0}+C'_\varepsilon)\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}$$

Thus

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\le\frac{C_4(\Omega,p)}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)},\qquad\text{(III.24)}$$

where $C_4(\Omega)=4(1+C(1+\frac{1}{\lambda_0}+C'_\varepsilon))$. □

Proposition 4.7 *Suppose that* $p>4$. *With the same assumptions of Proposition 4.6, let* $\boldsymbol{u}\in\boldsymbol{W}^{2,p}(\Omega)$ *be the unique solution of problem (III.3). Then* $\boldsymbol{u}$ *satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\;\le\;\frac{\kappa_1(\Omega,\,p)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)},\qquad\text{(III.25)}$$

where the constant $\kappa_1(\Omega,\,p)$ *is independent of* λ *and* $\boldsymbol{f}$.

Proof. We have

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,4}(\Omega)}\le C\|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)}+\|\boldsymbol{f}-\lambda\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)}.$$

Therefore

$$\begin{aligned}\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,4}(\Omega)} &\le (1+\frac{1}{|\lambda|}+C_4(\Omega))\|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)}\\ &\le C_5(\Omega)\|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)},\end{aligned}$$

where $C_5(\Omega)=1+\frac{1}{\lambda_0}+C_4(\Omega)$.

Now, since $\boldsymbol{W}^{2,4}(\Omega) \hookrightarrow \boldsymbol{W}^{1,\infty}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$, then

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C_6(\Omega)\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,4}(\Omega)} \leq C_5(\Omega)C_6(\Omega)\|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)} \leq C_7(\Omega)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}$$

where $C_7(\Omega) = C_5(\Omega)C_6(\Omega)(\operatorname{mes}\Omega)^{(p-4)/4p}$.

As consequence

$$\|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)} \leq CC_7(\Omega)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \tag{III.26}$$

Notice that

$$\begin{aligned} \|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} &= \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)} \\ &\leq C_7(\Omega)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \end{aligned} \tag{III.27}$$

Now, putting together (III.16), (III.20) and (III.27) one has

$$\begin{aligned} \operatorname{Re}\lambda\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2\,\mathrm{d}\,x + 2\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2\,\mathrm{d}\,x \\ \leq (\frac{C_3(\Omega,p)}{2}C_7(\Omega) + CC_7(\Omega) + 1)\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \end{aligned}$$

As a result one has

$$\operatorname{Re}\lambda\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \;\leq\; C_8(\Omega,p)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{III.28}$$

$$\int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2\,\mathrm{d}\,x \;\leq\; C_8(\Omega,p)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}, \tag{III.29}$$

and

$$2\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2\,\mathrm{d}\,x \;\leq\; C_8(\Omega,p)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}, \tag{III.30}$$

where

$$C_8(\Omega,p) = \frac{C_3(\Omega,p)}{2}C_7(\Omega) + CC_7(\Omega) + 1. \tag{III.31}$$

In addition, using (III.17), (III.29) and (III.30) one has

$$|\operatorname{Im}\lambda|\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \;\leq\; C_9(\Omega,p)\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{III.32}$$

Thus putting together (III.28) and (III.32) one gets for $p > 4$

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \;\leq\; \frac{C_{10}(\Omega,p)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{III.33}$$

which ends the case $p > 4$. □

To recapitulate, from (III.24) and (III.33), we deduce that for all $p \geq 2$ we have

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{\kappa_1(\Omega, p)}{|\lambda|} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{III.34}$$

with

$$\kappa_1(\Omega, p) = \max(C_4(\Omega, p), C_{10}(\Omega, p)). \tag{III.35}$$

By duality we obtain estimate (III.34) for all $1 < p < 2$. And we have the similar results as in Proposition 4.6 and Proposition 4.7. To summarize, we state the following theorem:

Theorem 4.8 *For $1 \leq p \leq \infty$. Assume that Ω is of class $C^{2,1}$. Let $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re}\lambda \geq 0$ and $|\lambda| \geq \lambda_0$, where $\lambda_0 = \lambda_0(\Omega, p)$ is defined in (III.23). Moreover, let $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, where $1 < p < \infty$ and let $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ be the unique solution of problem (III.3). Then $\boldsymbol{u}$ satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{\kappa_1(\Omega,\, p)}{|\lambda|} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{III.36}$$

where the constant $\kappa_1(\Omega,\, p)$ is independent of λ and $\boldsymbol{f}$.

Remark 4.9 The estimates (III.36) remain valid for $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$. To obtain this estimate we proceed in the same way as in the proofs of Proposition 4.6 and Proposition 4.7, just using the estimate (III.10) instead of (III.11) to estimate the pressure.

Resolvent estimates for $0 < |\lambda| \leq \lambda_0$

Proposition 4.10 *Let $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re}\lambda \geq 0$ and $0 < |\lambda| \leq \lambda_0$, with λ_0 as in proposition 4.6. Moreover, let $1 < p < \infty$, $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and let $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ be the unique solution of problem* (III.3). *Then $\boldsymbol{u}$ satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{\kappa_2(\Omega, p)}{|\lambda|} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{III.37}$$

For some constant $\kappa_2(\Omega, p)$ independent of λ and $\boldsymbol{f}$.

Proof. Thanks to (III.2) with $\varepsilon = \frac{\pi}{2}$ we have

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C_{11}}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}$$

and

$$\|\mathbf{D}(\boldsymbol{u})\|^2_{\boldsymbol{L}^2(\Omega)} \leq \frac{C^2_{11}}{|\lambda|}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)}.$$

Moreover, using Korn's inequality, we deduce that

$$\begin{aligned}
\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} &\leq C_{12}(\Omega)(\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{D}(\boldsymbol{u})\|^2_{\boldsymbol{L}^2(\Omega)}) \\
&\leq C_{12}(\Omega)C^2_{11}\frac{1+|\lambda|}{|\lambda|^2}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}$$

Since $|\lambda| \leq \lambda_0$, we have

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq \frac{C_{13}(\Omega)}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)},$$

where

$$C_{13}(\Omega) = C_{11}\sqrt{C_{12}(\Omega)(1+\lambda_0)}.$$

For the rest of the proof we distinguish two cases:

(i) Case $2 \leq p \leq 6$. Because $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$ we have

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_{14}(\Omega,p)\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq \frac{C_{13}(\Omega,p)C_{14}(\Omega)}{|\lambda|}\|\boldsymbol{f}\|_{L^2(\Omega)} \leq \frac{C_{15}(\Omega,p)}{|\lambda|}\|\boldsymbol{f}\|_{L^p(\Omega)},$$

where

$$C_{15}(\Omega,p) = (\operatorname{mes}\Omega)^{(p-2)/2p}\, C_{13}(\Omega,p)C_{14}(\Omega).$$

(ii) Case $p \geq 6$. We have

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,6}(\Omega)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} + \|\boldsymbol{f}-\lambda\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)}$$

Consequently

$$\begin{aligned}\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,6}(\Omega)} &\leq C(1+\frac{1}{|\lambda|}+C_{15}(\Omega))\|\boldsymbol{f}\|_{\boldsymbol{L}^6(\Omega)}\\ &\leq \frac{C_{16}(\Omega)}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^6(\Omega)}\end{aligned}$$

where $C_{16}(\Omega) = C(1+\lambda_0+\lambda_0 C_{15}(\Omega))$.

Now, since $\boldsymbol{W}^{2,6}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$, we have

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_{17}(\Omega,p)\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,6}(\Omega)} \leq C_{17}(\Omega,p)\frac{C_{16}(\Omega)}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^6(\Omega)} \leq \frac{C_{18}(\Omega,p)}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.$$

Finally, we set $\kappa_2(\Omega,p) = \max(C_{15}(\Omega,p), C_{18}(\Omega,p)$, and estimate (III.37) is established. □

Theorem 4.11 *Let $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re}\lambda \geq 0$, let $1 < p < \infty$, $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and let $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ be the unique solution of problem* (III.3)*. Then $\boldsymbol{u}$ satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{\kappa_3(\Omega,p)}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{III.38}$$

where $\kappa_3(\Omega,p) = \max(\kappa_1(\Omega,p), \kappa_2(\Omega,p))$.

We have also the following estimate

$$\|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{\kappa_4(\Omega,p)}{\sqrt{|\lambda|}}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \tag{III.39}$$

and

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq \kappa_5(\Omega,p)\,\frac{1+|\lambda|}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{III.40}$$

Proof. The proof of estimate (III.38) is a conclusion of Propositions 4.6 and Proposition 4.10. Let us prove estimate (III.39). The proof is done in two steps.

(i) Case $\displaystyle\int_\Omega \boldsymbol{u}\cdot\boldsymbol{\beta}\,dx = 0$. Thanks to regularity estimate of the Stokes problem [6] we know that $\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \simeq \|-\Delta\boldsymbol{u}+\nabla\pi\|_{\boldsymbol{L}^p(\Omega)}$ and using the Gagliardo-Nirenberg inequality (see [22, Chapter 9, page 195] for instance) we have

$$\begin{aligned}
\|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^p(\Omega)} &\leq C(\Omega,p)\,\|\boldsymbol{u}\|^{1/2}_{\boldsymbol{W}^{2,p}(\Omega)}\|\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\\
&\leq C'(\Omega,p)\,\|\boldsymbol{f}-\lambda\,\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\\
&\leq \frac{C(\Omega,p)}{\sqrt{|\lambda|}}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.
\end{aligned}$$

(ii) General case. Let $\boldsymbol{u}\in\mathbf{D}_p(A)$ be the unique solution of problem (III.3) and set

$$\widetilde{\boldsymbol{u}} = \boldsymbol{u} - \frac{\displaystyle\int_\Omega \boldsymbol{u}\cdot\boldsymbol{\beta}\,dx}{\|\boldsymbol{\beta}\|_{\boldsymbol{L}^2(\Omega)}}\boldsymbol{\beta}.$$

As a result, thanks to the previous case we have

$$\|\mathbf{D}(\widetilde{\boldsymbol{u}})\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,\|A\widetilde{\boldsymbol{u}}i\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\|\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}.$$

Thus

$$\|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^p(\Omega)} = \|\mathbf{D}(\widetilde{\boldsymbol{u}})\|_{\boldsymbol{L}^p(\Omega)} \leq \|A\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\|\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)} = \|-A\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\|\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}.$$

Moreover, it is clear that

$$\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

As a consequence we deduce estimate (III.39).

□

Theorem 4.12 *The operator $-A$ generated a bounded analytic semigroup on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ for all $1<p<\infty$.*

Proof. Recall that by Theorem 2.7 to prove that $-A$ generated a bounded analytic semigroup, it suffices to prove that $-A$ is sectoriel. Now applying Theorem 4.4 and Theorem 4.11, we deduce that the operator $-A$ satisfies the assumptions of Proposition 2.8. Thus, we deduce that the anlayticity of the semigroup generated by the operator $-A$ on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ for all $1<p<\infty$. □

4.2 Analyticity in $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$

In this subsection we establish the resolvant estimate of type (III.4) for data $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$. For this, let us introduce the space:

$$\boldsymbol{E} = \{\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]';\ \mathrm{div}\,\boldsymbol{f} \in L^p(\Omega)\},$$

which is a Banach space with the norm

$$\|\boldsymbol{f}\|_{\boldsymbol{E}} = \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} + \|\mathrm{div}\,\boldsymbol{f}\|_{L^p(\Omega)}. \tag{III.41}$$

Also let us recall the following results whose proofs are given in [11].

Lemma 4.13 *The space $\mathcal{D}(\overline{\Omega})$ is dense in $\boldsymbol{E}$.*

As consequence we have the following corollary:

Corollary 4.14 *The linear mapping $\gamma\,:\,\boldsymbol{f} \longmapsto \boldsymbol{f}\cdot\boldsymbol{n}$ defined on $\mathcal{D}(\overline{\Omega})$ can be extended to a linear continuous mapping still denoted by $\gamma\,:\,\boldsymbol{E} \longmapsto W^{-1-1/p,p}(\Gamma)$. Moreover we have the following Green formula: for all $\boldsymbol{f} \in \boldsymbol{E}$ and for all $\chi \in W^{2,p'}(\Omega)$ such that $\frac{\partial\chi}{\partial\boldsymbol{n}} = 0$ on Γ,*

$$\int_\Omega (\mathrm{div}\,\boldsymbol{f})\,\overline{\chi}\,dx \;=\; -\langle \boldsymbol{f}\,,\,\nabla\chi\rangle_\Omega \;+\; \langle \boldsymbol{f}\cdot\boldsymbol{n}\,,\,\chi\rangle_\Gamma, \tag{III.42}$$

where $\langle\cdot\,,\,\cdot\rangle_\Omega = \langle\cdot\,,\,\cdot\rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\times\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}$ and $\langle\cdot\,,\,\cdot\rangle_\Gamma \;=\; \langle\cdot\,,\,\cdot\rangle_{W^{-1-1/p,p}(\Gamma)\times W^{1+1/p,p'}(\Gamma)}$.

We proceeding as in [10], we prove the following theorem.

Theorem 4.15 *Let $\lambda \in \mathbb{C}^*$ such that $\mathrm{Re}\,\lambda \geq 0$ and let $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$. The problem (III.3) has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$. Moreover, if Ω is of class $C^{2,1}$, then we have the following estimate:*

$$\|\boldsymbol{u}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \;\leq\; \frac{C(\Omega,p)}{|\lambda|}\,\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \tag{III.43}$$

for some constant $C(\Omega,p) > 0$ independent of λ and $\boldsymbol{f}$.

Proof. (i) Existence: For the existence of solutions for problem (III.3) we proceed in the same way as in the proof of Theorem 4.4.
(ii) Estimate: We know that,according to Theorem 4.4, for $\boldsymbol{F} \in \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)$ and $\lambda \in \mathbb{C}^*$ such that $\mathrm{Re}\,\lambda \geq 0$ the following problem

$$\begin{cases} \lambda\boldsymbol{v} - \Delta\boldsymbol{v} \,+\, \nabla\theta = \boldsymbol{F}, & \mathrm{div}\,\boldsymbol{v} = 0 \qquad \text{in} \quad \Omega, \\ \qquad \boldsymbol{v}\cdot\boldsymbol{n} = 0, & [\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} \qquad \text{on} \quad \Gamma, \end{cases} \tag{III.44}$$

has a unique solution $(\boldsymbol{v},\theta) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$. Moreover, if Ω is of class $C^{2,1}$, applying Theorem 4.11, we have the following estimate:

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)} \;\leq\; \frac{C(\Omega,p')}{|\lambda|}\,\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Thus

$$\|\boldsymbol{v}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} \leq \frac{C(\Omega,p')}{|\lambda|}\,\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}.$$

Now let $(\boldsymbol{u},\pi)\in \boldsymbol{W}^{1,p}(\Omega)\times \boldsymbol{L}^p(\Omega)/\mathbb{R}$ be the solution of problem (III.3), then by using (III.42) we have:

$$\begin{aligned}
\|\boldsymbol{u}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} &= \sup_{\boldsymbol{F}\in\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega),\boldsymbol{F}\neq 0} \frac{|\langle \boldsymbol{u}\,,\,\boldsymbol{F}\rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}}\\
&= \sup_{\boldsymbol{F}\in\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega),\boldsymbol{F}\neq 0} \frac{|\langle \boldsymbol{u}\,,\,\lambda\,\boldsymbol{v}-\Delta\boldsymbol{v}-\nabla\theta\rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}}\\
&= \sup_{\boldsymbol{F}\in\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega),\boldsymbol{F}\neq 0} \frac{|\langle \lambda\,\boldsymbol{u}-\Delta\boldsymbol{u}-\nabla\pi\,,\,\boldsymbol{v}\rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}}\\
&= \sup_{\boldsymbol{F}\in\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega),\boldsymbol{F}\neq 0} \frac{|\langle \boldsymbol{f}\,,\,\boldsymbol{v}\rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}}\\
&\leq \frac{C(\Omega,p')}{|\lambda|}\,\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'},
\end{aligned}$$

which is estimate (III.43) and this finish the proof of theorem. □

Stokes operator on $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$

Let us introduce the following space:

$$[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau} = \{\boldsymbol{f}\in[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]';\ \mathrm{div}\,\boldsymbol{f}=0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{f}\cdot\boldsymbol{n}=0 \quad \text{on } \Gamma\}.$$

Notice that, we can also define the Stokes problem with Navier boundary condition by $B:\mathbf{D}_p(B)\subset [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}\longmapsto [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$, where

$$\begin{aligned}\mathbf{D}_p(B)=\big\{\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega);\ \Delta\boldsymbol{u}\in[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]',\ \mathrm{div}\,\boldsymbol{u}=0 \text{ in } \Omega,&\\
\boldsymbol{u}\cdot\boldsymbol{n}=0,\ [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}=\mathbf{0} \text{ on } \Gamma\big\}&\end{aligned} \tag{III.45}$$

and $B\,\boldsymbol{u}=-P\Delta\boldsymbol{u}$, for all $\boldsymbol{u}\in\mathbf{D}_p(B)$. We recall that $P:[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\longmapsto[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$ is the Helmholtz projection defined by, for all $\boldsymbol{f}\in[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$, $P\boldsymbol{f}=\boldsymbol{f}-\mathbf{grad}\,\pi$, where π is the unique solution of Problem (III.6). This means that, the Stokes operator is defined by:

$$\boldsymbol{u}\in\mathbf{D}_p(B),\qquad B\boldsymbol{u}=-P\Delta\boldsymbol{u}=-\Delta\boldsymbol{u}+\mathbf{grad}\,\pi,$$

where π is the unique solution up to an additive constant of the problem

$$\mathrm{div}(\mathbf{grad}\,\pi-\Delta\boldsymbol{u})=0 \qquad \text{in } \Omega, \qquad (\mathbf{grad}\,\pi-\Delta\boldsymbol{u})\cdot\boldsymbol{n}=0 \qquad \text{on } \Gamma. \tag{III.46}$$

Proposition 4.16 *The operator B is a densely defined operator.*

Proof. Just use the density of $\mathcal{D}_\sigma(\Omega)$ in $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$(see [11]). $\square$

We proceed as in the proof of Theorem4.12, we have the following result.

Theorem 4.17 *The operator $-B$ generates a bounded analytic semi-group on $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$.*

4.3 Analyticity in $(\boldsymbol{T}^{p'}(\Omega))'$

In this section we prove the first time the existence of very weak solution for problem (III.3). To prove this we shall apply the same argument using in the proof of ([6, Theorem 5.5]). In the second time, we establish also the estimate of type (III.4) for the force $\boldsymbol{f} \in \boldsymbol{T}^{p'}(\Omega))'$.

First le us introduce the following space

$$\boldsymbol{T}^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}_0^p(\mathrm{div},\Omega);\ \ \mathrm{div}\,\boldsymbol{v} \in W_0^{1,p}(\Omega)\}. \tag{III.47}$$

Thanks to [8, Lemma 4.11, Lemma 4.12] we know that $\mathcal{D}(\Omega)$ is dense in $\boldsymbol{T}^p(\Omega)$ and a distribution $\boldsymbol{f} \in (\boldsymbol{T}^p(\Omega))'$ if and only if there exists a function $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$ and a function $f_0 \in W^{-1,p'}(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\psi} + \nabla f_0$. Moreover we have the estimate

$$\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|f_0\|_{W^{-1,p'}(\Omega)} \leq \|\boldsymbol{f}\|_{(\boldsymbol{T}^p(\Omega))'}.$$

We introduce the following space

$$\boldsymbol{G} = \{\boldsymbol{f} \in (\boldsymbol{T}^{p'}(\Omega))';\ \mathrm{div}\,\boldsymbol{f} \in L^p(\Omega)\},$$

equipped with the graph norm.

In the following, it is important to recall the two following results given in [11].

Lemma 4.18 *The space $\mathcal{D}(\overline{\Omega})$ is dense in $\boldsymbol{G}$.*

In the following corollary, we prove that for $\boldsymbol{f} \in \boldsymbol{G}$, the normal component of $\boldsymbol{f}$ is well defined and belongs to $W^{-2-1/p,p}(\Gamma)$ and we have the appropriate Green formula:

Corollary 4.19 *The linear mapping $\gamma\ :\ \boldsymbol{f} \longmapsto \boldsymbol{f}\cdot\boldsymbol{n}$ defined on $\mathcal{D}(\overline{\Omega})$ can be extended to a linear continuous mapping still denoted by $\gamma\ :\ \boldsymbol{G} \longmapsto W^{-2-1/p,p}(\Gamma)$. Moreover we have the following Green formula: for all $\boldsymbol{f} \in \boldsymbol{G}$ and for all $\chi \in W^{3,p'}(\Omega)$ such that $\frac{\partial\chi}{\partial\boldsymbol{n}} = 0$ on Γ and $\Delta\chi = 0$ on Γ,*

$$\int_\Omega (\mathrm{div}\,\boldsymbol{f})\,\overline{\chi}\,dx = -\langle \boldsymbol{f}\,,\,\nabla\chi\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)} + \langle \boldsymbol{f}\cdot\boldsymbol{n}\,,\,\chi\rangle_\Gamma. \tag{III.48}$$

We recall that $\langle .\,,\,.\rangle_\Gamma = \langle .\,,\,.\rangle_{W^{-2-1/p,p}(\Gamma)\times W^{2+1/p,p'}(\Gamma)}$.

Let also introduce the following space:

$$\boldsymbol{H}^p(\Delta,\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \ \Delta\boldsymbol{v} \in (\boldsymbol{T}^{p'}(\Omega))'\},$$

which is a Banach space for the graph norm. According to [8, Lemma 4.13, Lemma 4.14] we know that $\mathfrak{D}(\overline{\Omega})$ is dense in $\boldsymbol{H}^p(\Delta, \Omega)$. As consequence, thanks to [6, Lemma 5.4] for every $\boldsymbol{v}$ in $\boldsymbol{H}^p(\Delta, \Omega)$ the trace $[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}$ belongs to $\boldsymbol{W}^{-1-1/p,p}(\Gamma)$. Moreover we have the Green formula: for any $\boldsymbol{u} \in \boldsymbol{H}_p(\Delta; \Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{S}^{p'}(\Omega)$,

$$\langle \Delta \boldsymbol{u}, \boldsymbol{\varphi} \rangle_{(\boldsymbol{T}^{p'}(\Omega))' \times \boldsymbol{T}^{p'}(\Omega)} = \int_\Omega \boldsymbol{u} \cdot \Delta \overline{\boldsymbol{\varphi}} d\boldsymbol{x} + \langle 2[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi} \rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma) \times \boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}, \tag{III.49}$$

where

$$\boldsymbol{S}^{p'}(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{W}^{2,p'}(\Omega);\ \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0,\ \operatorname{div} \boldsymbol{\varphi} = 0,\ [\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} \operatorname{on} \Gamma\}.$$

We recall also the following De Rham result:

Lemma 4.20 *Let $\boldsymbol{F} \in \boldsymbol{W}^{-2,p}(\Omega)$ verifying $\langle \boldsymbol{F}\,,\, \boldsymbol{v} \rangle_{\mathfrak{D}'(\Omega) \times \mathfrak{D}(\Omega)} = 0$ for all $\boldsymbol{v} \in \mathfrak{D}_\sigma(\Omega)$. Then there exists $\chi \in W^{-1,p}(\Omega)$ such that $\boldsymbol{F} = \nabla \chi$.*

Now, we can prove the existence of very weak solution and we established the resolvent estimate.

Theorem 4.21 *Let $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re} \lambda \geq 0$ and let $\boldsymbol{f} \in (\boldsymbol{T}^{p'}(\Omega))'$ then the problem* (III.3) *has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$. Moreover we have the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C(\Omega, p)}{|\lambda|} \|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'}, \tag{III.50}$$

for some constant $C(\Omega, p) > 0$ independent of λ and $\boldsymbol{f}$.

Proof. By Green formula (III.49), as in ([6, Theorem 5.5]), it is easy to prove that find $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ solution of problem (III.3) is equivalent to the problem: Find $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ such that for all $\boldsymbol{\varphi} \in \boldsymbol{S}^{p'}(\Omega)$ and for all $q \in W^{1,p'}(\Omega)$

$$\begin{aligned} \lambda \int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{\varphi}} \, \mathrm{d}\, x \, - \int_\Omega \boldsymbol{u} \cdot \Delta \overline{\boldsymbol{\varphi}} \, \mathrm{d}\, x &= \langle \boldsymbol{f},\, \boldsymbol{\varphi} \rangle_{(\boldsymbol{T}^{p'}(\Omega))' \times \boldsymbol{T}^{p'}(\Omega)} \\ \int_\Omega \boldsymbol{u} \cdot \nabla \overline{q} \, \mathrm{d}\, x &= 0. \end{aligned} \tag{III.51}$$

Note that the pressure can be recovered by De Rham argument given by Lemma 4.20.

Applying Theorem 4.4, for any $\boldsymbol{F} \in \boldsymbol{L}^{p'}(\Omega)$ there exists a unique solution $(\boldsymbol{\varphi}, q) \in \boldsymbol{W}^{2,p'}(\Omega) \times W^{1,p'}(\Omega)/\mathbb{R}$ such that

$$\lambda \boldsymbol{\varphi} - \Delta \boldsymbol{\varphi} + \nabla q = \boldsymbol{F} \quad \text{and} \quad \operatorname{div} \boldsymbol{\varphi} = 0 \quad \text{in}\, \Omega, \quad \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0, \quad \text{and} \quad [\boldsymbol{D}(\boldsymbol{\varphi})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} \quad \text{on}\, \Gamma. \tag{III.52}$$

In addition, thanks to Theorem 4.11 we have

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)} \leq \frac{C(\Omega, p')}{|\lambda|} \|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Let T be a linear form defined from $\boldsymbol{L}^{p'}(\Omega)$ onto $\mathbb{C}$ by:

$$T : (\boldsymbol{F},\, \xi) \longmapsto \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{(\boldsymbol{T}^{p'}(\Omega))' \times \boldsymbol{T}^{p'}(\Omega)}.$$

where $\boldsymbol{\varphi}$ is the unique solution of (III.52). Note that

$$|L(\boldsymbol{F})| \leq \|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'}\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)} \leq \frac{C(\Omega,p')}{|\lambda|}\,\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'}\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Thus, the linear form T is continuous on $\boldsymbol{L}^{p'}(\Omega)$ and we deduce that there exists a unique $(\boldsymbol{u},\pi)$ in $\boldsymbol{L}^p(\Omega)$ satisfies

$$L(\boldsymbol{F}) = \int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{F}}\,\mathrm{d}\,x = \langle \boldsymbol{f}\,,\,\boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)}$$

and satisfying the estimate (III.50). On other worlds $\boldsymbol{u}$ is the unique solution of problem (III.51). Which completes the proof of the theorem. □

Stokes operator on $(\boldsymbol{T}^{p'}(\Omega))'$

Let us introduce the following space:

$$[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau} = \{\boldsymbol{f}\in(\boldsymbol{T}^{p'}(\Omega))';\ \operatorname{div}\boldsymbol{f}=0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{f}\cdot\boldsymbol{n}=0 \quad \text{on } \Gamma\}.$$

Notice that, we can also define the Stokes operator with Navier boundary condition by $B : \mathbf{D}_p(C)\subset [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}\longmapsto[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$, where

$$\mathbf{D}_p(C) = \{\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega);\ \Delta\boldsymbol{u}\in[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau},\ \operatorname{div}\boldsymbol{u}=0 \text{ in } \Omega,$$
$$\boldsymbol{u}\cdot\boldsymbol{n} = 0,\ [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0} \text{ on } \Gamma\} \qquad \text{(III.53)}$$

and $C\,\boldsymbol{u} = -P\Delta\boldsymbol{u}$, for all $\boldsymbol{u}\in\mathbf{D}_p(C)$. We recall that $P : [\boldsymbol{T}^{p'}(\Omega)]'\longmapsto[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ is the Helmholtz projection defined by, for all $\boldsymbol{f}\in[\boldsymbol{T}^{p'}(\Omega)]'$, $P\boldsymbol{f} = \boldsymbol{f} - \mathbf{grad}\,\pi$, where π is the unique solution of Problem (III.6). This means that, the Stokes operator is defined by:

$$\boldsymbol{u}\in\mathbf{D}_p(C), \qquad C\boldsymbol{u} = -P\Delta\boldsymbol{u} = -\Delta\boldsymbol{u} + \mathbf{grad}\,\pi,$$

where π is the unique solution up to an additive constant of the problem

$$\operatorname{div}(\mathbf{grad}\,\pi - \Delta\boldsymbol{u}) = 0 \quad \text{in } \Omega, \qquad (\mathbf{grad}\,\pi - \Delta\boldsymbol{u})\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma. \qquad \text{(III.54)}$$

Proposition 4.22 *The operator C is a densely defined operator.*

Proof. Just use the density of $\mathfrak{D}_\sigma(\Omega)$ in $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$(see [11]). □

We proceed as in the proof of Theorem4.12, we have the following result.

Theorem 4.23 *The operator $-C$ generates a bounded analytic semi-group on $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$.*

5 Pure imaginary powers of the Stokes operator

The main goal of this section is to study the pure imaginary powers for the Stokes operator A defined in section 3. Before that, we recall some definitions. For a complex Banach space X, we consider a linear operator $\mathcal{A}$. We give the following definition:

Definition 5.1 *The operator $\mathcal{A}$ is said to be a nonnegative operator if its resolvent set contains all negative real numbers and*

$$\sup_{t>0} t\, \|(t\, I + \mathcal{A})^{-1}\|_{\mathcal{L}(X)} < \infty.$$

We give also the following definition.

Definition 5.2 *Let $\mathcal{A}$ a bounded operator such that the inverse $\mathcal{A}^{-1}$ exists and it is bounded (i.e. $0 \in \rho(\mathcal{A})$), the complex power $\mathcal{A}^z$ can be defined for all $z \in \mathbb{C}$ by the means of the Dunford integral:*

$$\mathcal{A}^z f = \frac{1}{2\,\pi\, i} \int_{\Gamma_\theta} (-\lambda)^z\, (\lambda\, I + \mathcal{A})^{-1} f \mathrm{d}\,\lambda, \tag{III.1}$$

where Γ_θ runs in the resolvent set of $-\mathcal{A}$ from $\infty\, e^{i(\theta-\pi)}$ to zero and from zero to $\infty\, e^{i(\pi-\theta)}$, $0<\theta<\pi/2$ in $\mathbb{C}$ avoiding the non negative real axis. The branch of $(-\lambda)^z$ is taken so that $\mathrm{Re}((-\lambda)^z) > 0$ for $\lambda < 0$.

Remark 5.3 *For more details about a complex power of a nonnegative operator, you can consult [43, 62].*

We recall the following property that will be used to study the abstract Cauchy-problem (see [39] for more details).

Definition 5.4 *Let $\theta \geq 0$ and $K \geq 1$. A nonnegative operator $\mathcal{A}$ belongs to $E^\theta_K(X)$ if $\mathcal{A}^{is} \in \mathcal{L}(X_{\mathcal{A}})$ for every $s \in \mathbb{R}$ and its norm in $\mathcal{L}(X_{\mathcal{A}})$ satisfies the estimate*

$$\|\mathcal{A}^{is}\|_{\mathcal{L}(X_{\mathcal{A}})} \leq\; K\, e^{\theta\, |s|}. \tag{III.2}$$

If in addition $D(\mathcal{A})$ and $R(\mathcal{A})$ are dense in X, we say that $\mathcal{A} \in \mathcal{E}^\theta_K(X)$.

We proceed in the same way as in [11]. We start by prove the same result the same result given by the [11, Proposition 6.3] and we obtain the two following estimates:

(i) There exists an angle $0 < \theta_0 < \pi/2$ such that the resolvent set of the operator $-(I+A)$ contains the sector

$$\Sigma_{\theta_0} = \{\lambda \in \mathbb{C};\ \ |\arg \lambda| \leq \pi - \theta_0\}. \tag{III.3}$$

Moreover, we have the estimate

$$\forall\, \lambda \in \Sigma_{\theta_0},\ \ \lambda \neq 0, \qquad \|(\lambda\, I + I + A)^{-1}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \leq \frac{\kappa_3(\Omega, p)}{|\lambda|}, \tag{III.4}$$

where $\kappa_3(\Omega, p)$ is the constant in (III.38).

(ii) Now, let $0 < \alpha < 1$ be fixed, then for $\lambda \in \Sigma_{\theta_0}$ such that $\lambda \neq 0$ and $|\lambda| \leq \frac{1}{2\,\kappa_3(\Omega,p)}$ one has

$$\|(\lambda\, I + I + A)^{-1}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \leq 2^{\alpha}\,\kappa_3^{\alpha}(\Omega,p)\,|\lambda|^{\alpha-1}. \tag{III.5}$$

Now, using the estimate (III.4), estimate (III.5) and the Dunford integral for the operator $(I - A)$ we obtain the following estimate as in [11, Proposition 6.4]: For all $s \in \mathbb{R}$ we have

$$\|(I + A)^{i\,s}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \leq M\,\kappa_3(\Omega,p)\,e^{|s|\,\theta_0}, \tag{III.6}$$

for some constant $M > 0$.

As consequence we proceed as in the proof of [39, Theorem A1] and [11, Theorem 6.5], we can proof the main result of this section:

Theorem 5.5 *Let θ_0 as given in previous point **(i)**. For all $s \in \mathbb{R}$ we have*

$$\|(-A)^{i\,s}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \leq M\,\kappa_3(\Omega,p)\,e^{|s|\,\theta_0}, \tag{III.7}$$

for some constant $M > 0$. On other words we have

$$\forall\, 1 < p < \infty,\;\; s \in \mathbb{R}, \qquad (-A)^{i\,s} \in \mathcal{E}^{\theta_0}_K(\boldsymbol{L}^p_{\sigma,\tau}(\Omega)),$$

where $K = M\,\kappa_3(\Omega,p)$.

6 The evolutionary Stokes problem

6.1 The homogeneous Problem

In this section we solve the homogeneous evolutionary Stokes problem

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0}, & \operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \Omega \times (0,T), \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} \quad \text{on} \quad \Gamma \times (0,T), \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega. \end{cases} \tag{III.1}$$

In fact, for the homogeneous Stokes Problem, the analyticity of the semi-group give us a unique solutions satisfying all the regularity desired.

Before we state our theorem, we recall the followings theorem whose proofs is given [35, Chapter 2, Theorem 4.6].

Theorem 6.1 *For an operator $(A, \mathbf{D}(A))$ on a Banach space $\mathbf{X}$, the following statements are equivalent:*

a) A generates a bounded analytic semigroup $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$ on $\mathbf{X}$.

b) There exists $\vartheta \in (0, \frac{\pi}{2})$ such that the operators $e^{\pm i\vartheta}A$ generate bounded strongly continuous semigroups on $\mathbf{X}$.

c) A generates a bounded strongly semigroup $(T(t))_{t\geq 0}$ on $\mathbf{X}$ such that $rg(T(t)) \subset \mathbf{D}(A)$ for all $t > 0$ and

$$M := \sup_{t>0} \|tAT(t)\| < \infty.$$

c) A generates a bounded strongly continuous semigroup $(T(t))_{t\geq 0}$ on $\mathbf{X}$, and there exits a constant $C > 0$ such that

$$\|R(r+is, A)\| \leq \frac{C}{|s|}$$

for all $r > 0$ and $0 \neq s \in \mathbb{R}$.

d) A is sectorial.

We also recall the following result (see [35, Chapter 2, Proposition 4.3]).

Proposition 6.2 *Let $(A, \mathbf{D}(A))$ be a sectorial operator of angle δ. Then, for all $z \in \Sigma_\delta$, the maps $T(z)$ are bounded linear operators on $\mathbf{X}$ satisfying the following proprieties.*

1) $\|T(z)\|$ is uniformly bounded for $z \in \Sigma_{\delta'}$, if $0 < \delta' < \delta$.

2) The map $z \mapsto T(z)$ is analytic in Σ_δ.

3) $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \Sigma_\delta$.

4) The map $z \mapsto T(z)$ is strongly continuous in $\Sigma_{\delta'} \cup \{0\}$ if $0 < \delta' < \delta$.

Applying the semi-group theory to the Stokes operator on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $[\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'_{\sigma,\tau}$ and $(\boldsymbol{T}^{p'}(\Omega))'_{\sigma,\tau}$ respectively, one gets the strong, weak and very weak solutions to the homogeneous Stokes Problem (III.1). As in [11], the proofs of the following theorems are based on Theorem 6.1 and Proposition 6.2.

Theorem 6.3 *Let $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, then problem (III.1) has a unique solution $\boldsymbol{u}(t)$ satisfying*

$$\boldsymbol{u} \in C([0, +\infty[, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)) \cap C(]0, +\infty[, \mathbf{D}_p(A)) \cap C^1(]0, +\infty[, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)), \tag{III.2}$$

$$\boldsymbol{u} \in C^k(]0, +\infty[, \mathbf{D}_p(A^\ell)), \qquad \forall\, k \in \mathbb{N},\ \forall\, \ell \in \mathbb{N}^*. \tag{III.3}$$

Moreover we have the estimates

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq C_{19}(\Omega, p)\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{III.4}$$

and

$$\Big\|\frac{\partial \boldsymbol{u}(t)}{\partial t}\Big\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C_{20}(\Omega, p)}{t}\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}. \tag{III.5}$$

In addition, if Ω is of class $C^{2,1}$ the following estimates hold

$$\|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C_{21}(\Omega, p)}{\sqrt{t}}\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{III.6}$$

and

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C_{22}(\Omega,p)\,(1+\frac{1}{t})\,\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}. \tag{III.7}$$

Theorem 6.4 *Suppose that the given data* $\boldsymbol{u}_0 \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$ *then the Problem (III.1) has a unique solution* $\boldsymbol{u}$ *satisfying*

$$\boldsymbol{u} \in C([0,\,+\infty[,\,[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}) \cap C(]0,\,+\infty[,\,\mathbf{D}_p(B)) \cap C^1(]0,\,+\infty[,\,[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}), \tag{III.8}$$

$$\boldsymbol{u} \in C^k(]0,\,+\infty[,\,\mathbf{D}_p(B^\ell)), \qquad \forall\, k\in\mathbb{N},\ \forall\,\ell\in\mathbb{N}^*. \tag{III.9}$$

Moreover we have the estimates

$$\|\boldsymbol{u}(t)\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \leq C(\Omega,p)\,\|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}, \tag{III.10}$$

$$\Big\|\frac{\partial \boldsymbol{u}(t)}{\partial t}\Big\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \leq \frac{C(\Omega,p)}{t}\,\|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \tag{III.11}$$

and

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\Omega,p)\,(1+\frac{1}{t})\|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}. \tag{III.12}$$

Theorem 6.5 *Suppose that the given data* $\boldsymbol{u}_0 \in [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ *the Problem (III.1) has a unique solution* $\boldsymbol{u}$ *satisfying*

$$\boldsymbol{u} \in C([0,\,+\infty[,\,[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}) \cap C(]0,\,+\infty[,\,\mathbf{D}_p(C)) \cap C^1(]0,\,+\infty[,\,[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}), \tag{III.13}$$

$$\boldsymbol{u} \in C^k(]0,\,+\infty[,\,\mathbf{D}_p(C^\ell)), \qquad \forall\, k\in\mathbb{N},\ \forall\,\ell\in\mathbb{N}^*. \tag{III.14}$$

Moreover we have the estimates

$$\|\boldsymbol{u}(t)\|_{[\boldsymbol{T}^{p'}(\Omega)]'} \leq C(\Omega,p)\,\|\boldsymbol{u}_0\|_{[\boldsymbol{T}^{p'}(\Omega)]'}, \tag{III.15}$$

$$\Big\|\frac{\partial \boldsymbol{u}(t)}{\partial t}\Big\|_{[\boldsymbol{T}^{p'}(\Omega)]'} \leq \frac{C(\Omega,p)}{t}\,\|\boldsymbol{u}_0\|_{[\boldsymbol{T}^{p'}(\Omega)]'}. \tag{III.16}$$

and

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,(1\,+\,\frac{1}{t})\,\|\boldsymbol{u}_0\|_{[\boldsymbol{T}^{p'}(\Omega)]'}. \tag{III.17}$$

6.2 The inhomogeneous Stokes Problem

In this section we solve the inhomogeneous Stokes problem

$$\left\{\begin{array}{rlll} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f}, & \mathrm{div}\,\boldsymbol{u} = 0 & \text{in} & \Omega\times(0,T),\\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, & [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0} & \text{on} & \Gamma\times(0,T),\\ & \boldsymbol{u}(0) = \mathbf{0} & \text{in} & \Omega. \end{array}\right. \tag{III.18}$$

For the inhomogeneous Stokes Problem, the boundedness of the pure imaginary powers of the Stokes operator give us the maximal $L^p - L^q$- regularity for the solutions for the Problem.

Now, we recall some definition and important result. We start by the following definition (for instance see [32, Theorem 2]).

Definition 6.6 *A banach space X is $\zeta-$ convex if there is a symmetric biconvex function ζ on $X \times X$ such that $\zeta(0,0) > 0$ and*

$$\forall x, y \in X, \quad ||x||_X \geq 1, \qquad \zeta(x,y) \leq ||x+y||_X. \tag{III.19}$$

As consequence, we have the following proposition:

Theorem 6.7 *Every Hilbert space is ζ-convex, ζ-convexity is preserved by Banach space's isomorphism and closed subspaces. Cartesian products and quotients of ζ-convex spaces are also ζ-convex.*

As mentioned in [34, 39], we can use the following essential property of a ζ-convex Banach space:

Proposition 6.8 *A Banach space X is ζ-convex if and only if the truncated Hilbert transform*

$$(H_\varepsilon f)(t) = \frac{1}{\pi}\int_{|\tau|>\varepsilon} \frac{f(t-\tau)}{\tau}\, d\tau, \qquad f \in L^s(\mathbb{R}; X)$$

converges as $\varepsilon \to 0$ for almost $t \in \mathbb{R}$. Moreover, there is a constant $C = C(s, X)$ independents of f such that

$$||Hf||_{L^s(\mathbb{R};X)}, \qquad 1 < s < \infty,$$

where $(Hf)(t) = \lim_{\varepsilon\to 0}(H_\varepsilon f)(t)$.

Remark 6.9 Applying proposition 6.8, we deduce that $L^p(\Omega)$ is ζ-convex if and only if $1 < p < \infty$. (see [32, Proposition 3] for the proof). And we have also the dual spaces $(\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))'$ and $(\boldsymbol{T}^{p'}(\Omega))'$ are ζ-convex Banach space for $1 < p < \infty$.

Now, we consider the abstract Cauchy-Problem on a complex Banach space X:

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{A}\, u(t) = f(t) \quad 0 \leq t \leq T \\ \qquad u(0) = 0, \end{cases} \tag{III.20}$$

where $-\mathcal{A}$ is the infinitesimal generator of an analytic semi-group in X. The following theorem will be used. See [39, Theorem 3.3] for the proof.

Theorem 6.10 *Let X be a ζ-convex Banach space. Assume that $0 < T \leq \infty$, $1 < p < \infty$ and that $\mathcal{A} \in \mathcal{E}_K^\theta(X)$ for some $K \geq 1$, $0 \leq \theta < \pi/2$ and $\mathcal{E}_K^\theta(X)$ as in Definition 5.4. Then for every $f \in L^p(0,T;\, X)$ there exists a unique solution $\boldsymbol{u}$ of the abstract Cauchy-Problem (III.20) satisfying the properties:*

$$u \in L^p(0, T_0;\, D(\mathcal{A})), \;\; T_0 \leq T \;\; \textit{if}\; T < \infty \;\; \text{and} \;\; T_0 < T \;\; \textit{if}\; T = \infty,$$

$$\frac{\partial u}{\partial t} \in L^p(0,T;\, X)$$

and
$$\int_0^T \Big\|\frac{\partial u}{\partial t}\Big\|_X^p \,\mathrm{d}\,t + \int_0^T \|\mathcal{A}u(t)\|_X^p \,\mathrm{d}\,t \leq C \int_0^T \|f(t)\|_X^p \,\mathrm{d}\,t$$
with $C = C(p, \theta, K, X)$ independent of f and T.

As consequence, as in [11], using Remark 6.9, Theorem 5.5 and Theorem 6.10, we deduce the following the three following results:

Theorem 6.11 (Strong Solutions for the inhomogeneous Stokes Problem) *Suppose that Ω is of class $C^{2,1}$ and let $0 < T \leq \infty$, $1 < p, q < \infty$, $\boldsymbol{f} \in \boldsymbol{L}^q(0, T; \boldsymbol{L}^p(\Omega))$ and $\boldsymbol{u}_0 = \mathbf{0}$. The Problem (III.18) has a unique solution $(\boldsymbol{u}, \pi)$ such that*

$$\boldsymbol{u} \in L^q(0, T_0;\, \boldsymbol{W}^{2,p}(\Omega)), \quad T_0 \leq T \ \textit{if}\ T < \infty \ \text{ and } \ T_0 < T \ \textit{if}\ T = \infty, \tag{III.21}$$

$$\pi \in L^q(0, T;\, W^{1,p}(\Omega)/\mathbb{R}), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T;\, \boldsymbol{L}^p(\Omega)) \tag{III.22}$$

and

$$\int_0^T \Big\|\frac{\partial \boldsymbol{u}}{\partial t}\Big\|_{\boldsymbol{L}^p(\Omega)}^q \mathrm{d}\,t + \int_0^T \|A\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)}^q \mathrm{d}\,t + \int_0^T \|\pi(t)\|_{W^{1,p}(\Omega)/\mathbb{R}}^q \mathrm{d}\,t \leq C(p, q, \Omega) \int_0^T \|\boldsymbol{f}(t)\|_{\boldsymbol{L}^p(\Omega)}^q \mathrm{d}\,t. \tag{III.23}$$

Theorem 6.12 (Weak Solutions for the inhomogeneous Stokes Problem) *Let $1 < p, q < \infty$, $\boldsymbol{u}_0 = 0$ and let $\boldsymbol{f} \in L^q(0, T;\, [\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]')$, $0 < T \leq \infty$. The Problem (III.18) has a unique solution $(\boldsymbol{u}, \pi)$ satisfying*

$$\boldsymbol{u} \in L^q(0, T_0;\, \boldsymbol{W}^{1,p}(\Omega)), \quad T_0 \leq T \ \textit{if}\ T < \infty \ \text{ and } \ T_0 < T \ \textit{if}\ T = \infty,$$

$$\pi \in L^q(0, T;\, L^p(\Omega)/\mathbb{R}), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T; \in [\boldsymbol{H}_0^{p'}(\mathrm{div}\Omega)]'_{\sigma,T})$$

and

$$\int_0^T \Big\|\frac{\partial \boldsymbol{u}}{\partial t}\Big\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div}\Omega)]'}^q \mathrm{d}\,t + \int_0^T \|A\boldsymbol{u}(t)\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div}\Omega)]'}^q \mathrm{d}\,t + \int_0^T \|\pi(t)\|_{L^p(\Omega)/\mathbb{R}}^q \mathrm{d}\,t \leq C(p, q, \Omega) \int_0^T \|\boldsymbol{f}(t)\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div}\Omega)]'}^q \mathrm{d}\,t.$$

Theorem 6.13 (Very weak solutions for the inhomogeneous Stokes Problem) *Suppose that Ω is of class $C^{2,1}$, let $0 < T \leq \infty$, $1 < p, q < \infty$, $\boldsymbol{u}_0 = 0$ and $\boldsymbol{f} \in L^q(0, T;\, [\boldsymbol{T}^{p'}(\Omega)]')$. Then the evolutionary Stokes Problem (III.18) has a unique solution $(\boldsymbol{u}, \pi)$ satisfying*

$$\boldsymbol{u} \in L^q(0, T_0;\, \boldsymbol{L}^p(\Omega)), \quad T_0 \leq T \ \textit{if}\ T < \infty \ \text{ and } \ T_0 < T \ \textit{if}\ T = \infty,$$

$$\pi \in L^q(0, T;\, W^{-1,p}(\Omega)/\mathbb{R}), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T; \in [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau})$$

and

$$\int_0^T \Big\| \frac{\partial \boldsymbol{u}}{\partial t} \Big\|^q_{[\boldsymbol{T}^{p'}(\Omega)]'} \,\mathrm{d}\,t + \int_0^T \|A\boldsymbol{u}(t)\|^q_{[\boldsymbol{T}^{p'}(\Omega)]'} \,\mathrm{d}\,t + \int_0^T \|\pi(t)\|^q_{W^{-1,p}(\Omega)/\mathbb{R}} \,\mathrm{d}\,t \\ \leq C(p,q,\Omega) \int_0^T \|\boldsymbol{f}(t)\|^q_{[\boldsymbol{T}^{p'}(\Omega)]'} \,\mathrm{d}\,t.$$

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Résumé: Cette thèse est consacrée à l'étude des équations de Stokes et de Navier-Stokes avec des conditions aux limites de Navier dans un ouvert borné de $\mathbb{R}^3$. Le manuscrit ici est composé de trois chapitres. Dans le premier, nous considérons les équations de Stokes stationnaires avec des conditions aux limites de Navier. Nous démontrons l'existence, l'unicité et la régularité de la solution d'abord dans un cadre hilbertien puis dans le cadre de la théorie L^p. Nous traitons aussi le cas de solutions très faibles. Dans le deuxième chapitre, nous nous intéressons aux équations de Navier-Stokes avec la condition de Navier. Sous certaines hypothèses sur les données, nous démontrons l'existence de solution faible dans $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$, avec $\frac{3}{2} < p < \infty$ en utilisant un théorème du point fixe appliqué à un problème d'Oseen. Nous démontrons examinons ensuite les questions de régularité des solutions en particulier dans $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$. Dans le dernier chapitre, nous étudions le problème d'évolution de Stokes avec la condition de Navier. La résolution de ce problème se fait au moyen de la théorie des semi-groupes analytiques qui jouent un rÃ´le important pour établir l'existence et l'unicité de la solution dans le cas homogène. Nous traitons le cas du problème non homogène par le biais des puissances imaginaires de l'opérateur de Stokes.
Mots clés : Équations de Stokes - Équations de Navier-Stokes - Équations d'Oseen - Conditions de Navier - Inégalité de Korn - ThéorieL^p - Théorie des semi-groupes.

Abstract: This thesis is devoted to the study of the Stokes equations and Navier-Stokes equations with Navier boundary conditions in a bounded domain of $\mathbb{R}^3$. The work contains three chapters: In the first chapter, we consider the stationary Stokes equations with Navier boundary condition. We show the existence, uniqueness and regularity of the solution in the Hilbert case and in the L^p-theory. We prove also the case of very weak solutions. In the second chapter, we focus on the Navier-Stokes equations with the Navier boundary condition. We show the existence of the weak solution in $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$, with $\frac{3}{2} < p < \infty$ by a fixed point theorem over the Oseen equation. We show also the existence of the strong solution in $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$, for $\frac{6}{5} \leq p < \infty$. In chapter three, we study the evolution Stokes problem with Navier boundary condition. For this, we apply the analytic semi-groups theory, which plays a crucial role in the study of existence and uniqueness of solution in the case of the homogeneous evolution problem. We treat the case of non-homogeneous problem through imaginary powers of the Stokes operator.
Keywords : Stokes equations - Navier-Stokes equations - Oseen equations - Navier boundary conditions - Korn inequality - L^p-theory - Semi-groups theory.