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par

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Etude qualitative des équations de Hamilton-Jacobi avec diffusion non linéaire

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"Si vous touchez aux maths, vous ne devez être ni pressés, ni cupides, fussiez-vous roi ou reine."

Euclide

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Résumé

Cette thèse est consacrée à l'étude des propriétés qualitatives de solutions d'une équation d'évolution de type Hamilton-Jacobi avec une diffusion donnée par l'opérateur p-Laplacien. On s'attache principalement à l'étude de l'effet de la diffusion non-linéaire sur le phénomène d'explosion du gradient. Les principales questions qu'on étudie portent sur l'existence locale, régularité, profil spatial d'explosion et la localisation des points d'explosion. En particulier on montre un résultat d'explosion en seul point du bord. Dans le chapitre 4, on utilise une approche de solutions de viscosité pour prolonger la solution explosive au delà des singularités et on étudie son comportement en temps grands. Dans l'avant dernier chapitre on s'intéresse au caractère borné des solutions globales du problème unidimensionnel. Dans le dernier chapitre on démontre une estimation de gradient locale en espace et on l'utilise pour obtenir un résultat de type Liouville. On s'inspire et on compare nos résultats avec les résultats connus pour le cas de la diffusion linéaire.

Mots-clés: Problémes paraboliques-dégénérés, estimations de gradient, explosion en un seul point, solutions de viscosité, comportement asymptotique, Théorème de type Liouville.

Title : A qualitative study of a Hamilton-Jacobi equation with a nonlinear diffusion

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Abstract

This thesis is devoted to the study of qualitative properties of solutions of an evolution equation of Hamilton-Jacobi type with a p-Laplacian diffusion. It is mainly concerned with the study of the effect of the non-linear diffusion on the gradient blow-up phenomenon. The main issues we are studying are: local existence and uniqueness, regularity, spatial profile of gradient blow-up and localization of the singularities. We provide examples where the gradient blow-up set is reduced to a single point. In Chapter 4, a viscosity solution approach is used to extend the blowing-up solutions beyond the singularities and an ergodic problem is also analyzed in order to study their long time behavior. In the penultimate chapter, we address the question of boundedness of global solutions to the one-dimensional problem. In the last chapter we prove a local in space, gradient estimate and we use it to obtain a Liouville-type theorem.

Keywords: Degenerate parabolic equations, gradient estimates, single point gradient blow-up, viscosity solutions, asymptotic behavior, Liouville-type theorem.

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Chapitre 1

Introduction générale

1.1 Motivation et présentation générale du problème

L'objectif de ce travail de thèse est d'étudier l'effet d'une diffusion non linéaire sur le comportement qualitatif des solutions des équations de type Hamilton-Jacobi. Plus précisément nous nous intéressons à l'équation suivante

$$u_t - \Delta_p u = |\nabla u|^q \qquad \text{dans} \quad \Omega \times (0,T), \tag{1.1}$$

où $\Delta_p u = div\,(|\nabla u|^{p-2}\nabla u)$.

Les équations (1.1), notamment dans le cas de la diffusion linéaire $p = 2$, sont importantes de plusieurs points de vue :

a. Elles fournissent une méthode de construction des solutions de l'équation de Hamilton-Jacobi $u_t = |\nabla u|^q$, par régularisation parabolique évanescente (i.e. on met un paramètre ϵ devant le terme de diffusion et on regarde la limite $\epsilon \to 0^+$). Cette dernière équation est fondamentale en théorie du contrôle optimal [54].

b. Elles interviennent dans le modèle physique de Kardar-Parisi-Zhang (KPZ) qui décrit des processus de croissance d'interfaces rugueuses par déposition de particules sur une surface [74]. L'équation KPZ est à l'origine une EDP stochastique qui décrit l'évolution au cours du temps de la hauteur h d'une interface

$$h_t = \lambda\Delta h + \frac{\lambda}{2}|\nabla h|^2 + \eta(x,t).$$

La diffusion tient compte de la relaxation, le terme en gradient vient de l'effet de croissance par déposition de nouvelles particules et $\eta(x,t)$ est un bruit blanc de moyenne nulle produit par des forces stochastiques avec faible corrélation. L'équation de KPZ déterministe a ensuite été généralisée par Krug et Spohn [78] afin d'étudier l'effet d'une non-linéarité plus forte ($|\nabla u|^q$ avec $q > 2$) sur le comportement de la solution.

L'équation (1.1) présente aussi un très grand intérêt mathématique en elle-même. En effet la compétition entre le terme source qui dépend seulement du gradient et le terme de diffusion est à l'origine d'une grande richesse de phénomènes : structure des états stationnaires, si $\Omega \neq \mathbb{R}^N$ existence de solutions globales non bornées en norme C^1, pour $\Omega = \mathbb{R}^N$ comportement en temps grand de type diffusif pour certaines valeur de q et p ou bien de type hyperbolique ou le terme hamiltonien l'emporte... Le phénomène qui nous intéresse le plus ici est celui de l'apparition d'une singularité en gradient et non en amplitude : si $T_{max}(u_0) < \infty$ (ici $T_{max}(u_0)$ est le temps maximal d'existence de la solution classique), alors u reste bornée en norme L^∞ (ceci découle d'une application simple du principe du maximum faible) mais pour $q > p$ et certaines données initiales, avec des conditions de Dirichlet homogènes au bord, on a $\lim\limits_{t \to T_{max}(u_0)} |\nabla u(t)| = \infty$. Ce phénomène d'explosion dépend fortement de la donnée initiale, de la taille de la non-linéarité, du domaine et des conditions aux limites. Rappelons que quand on s'intéresse à des équations quasi-linéaires paraboliques de la forme :

$$u_t - \sum_{i,j=1}^{N} a_{i,j}(x, u, \nabla u) \frac{\partial^2 u}{\partial x_i \partial x_j} - F(x, u, \nabla u) = 0 \tag{1.2}$$

avec des coefficients $a_{i,j}$ et F assez réguliers, les questions d'existence locale et d'unicité pour le problème de Cauchy-Dirichlet sont assez bien comprises (du moins pour le cas uniformément parabolique) [90, 80, 55]. Sous certaines conditions sur les termes non linéaires $a_{i,j}$ et F, des estimations a priori des solutions ont été obtenues. Il a été observé que si ces conditions ne sont pas satisfaites alors on a apparition de solutions "explosives". Il est connu que la condition de croissance de Bernstein-Nagumo [80, 116]

$$\frac{F(x, u, p)}{A(x, u, p)} \leq K(u) h(|p|)$$

avec h satisfaisant la restriction

$$\int_1^\infty \frac{s\, ds}{h(s)} = \infty,$$

garantit une borne sur le gradient de la solution une fois que l'on dispose d'une estimation de la norme L^∞ de la solution. Si cette condition n'est pas satisfaite, alors on peut avoir des solutions bornées dont le gradient explose soit au bord du domaine soit à l'intérieur. Cette condition a été affaiblie en écrivant F comme la somme de deux fonctions qui vérifient une certaine monotonie et des conditions de croissance relaxées (voir [115, 23]).

Dans cette thèse on se placera donc principalement dans le cas $q > p > 2$ et on s'attachera à répondre à certaines questions relatives au phénomène d'explosion du gradient (conditions suffisantes d'explosion, profil spatial, localisation des points d'explosion...) en étudiant l'influence de l'opérateur p-Laplacien sur ce phénomène. Le cas de la diffusion linéaire $p = 2$ pour l'équation (1.1) ayant fait l'objet d'un certain nombre d'études, on essaiera de s'en inspirer et on les comparera au cas de la diffusion non-linéaire.

Signalons que l'étude de la formation de singularités en temps fini pour les solutions de certaines EDP paraboliques semi-linéaires a fait l'objet de nombreux travaux ces dernières années. La grande majorité d'entre eux s'est concentrée sur l'explosion de la solution en amplitude, c'est à dire en norme L^∞. Ces travaux regroupent des résultats sur les critères d'explosion, localisation des points d'explosion, vitesse et profil spatial de d'explosion, etc. En particulier pour l'équation de la chaleur semi-linéaire bien connue

$$u_t - \Delta u = u^q \tag{1.3}$$

nous renvoyons le lecteur à [100, 102] et les références qui s'y trouvent.

Étant donné la dégénérescence de l'équation (1.1) aux points où $\nabla u = 0$, on ne peut espérer obtenir en général des solutions régulières sans imposer certaines conditions sur les données initiale et au bord. Ainsi selon les problèmes qu'on tentera de résoudre, différentes notions de solutions, telles que les solutions faibles, classiques ou de viscosité doivent être considérées. Dans un premier temps on s'intéressera aux solutions faibles Lipschitziennes en espace pour avoir un bon cadre de travail sur les singularités en gradient. Dans un second temps on se placera dans un cas particulier ou la théorie classique des EDP paraboliques quasi-linéaires nous donne l'existence de solutions classiques, ce qui facilitera l'étude de la localisation des points d'explosion. Dans le chapitre 4 nous verrons une autre approche qui est basée sur la notion de solution de viscosité et qui permettra l'étude de la continuation des solutions explosives.

1.2 Etat de l'art et résultats du chapitre 2 : Théorie locale et profil spatial de l'explosion

Dans ce chapitre on s'intéresse au problème de Cauchy-Dirichlet associé à l'équation (1.1) :

$$\begin{cases} \partial_t u - \mathrm{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^q, & x \in \Omega, t > 0, \\ u(x,t) = g(x), & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

où

- $\Omega \subset \mathbb{R}^N$ est un domaine borné régulier,
- $p > 2$ et $q > p-1$,
- $u_0 \in W^{1,\infty}(\Omega)$, g est la trace d'une fonction C^2. On suppose que u_0 satisfait la condition de compatibilité

$$u_0(x) = g(x) \qquad \text{pour} \quad x \in \partial\Omega. \tag{1.5}$$

D'après les résultats de [50, 59, 106, 110] nous savons que pour $p = 2$ les solutions positives du problème (1.4) existent globalement lorsque $1 < q < 2$ tandis qu'elles peuvent exploser en temps fini pour $q > 2$ (Ceci est clairement détaillé dans le livre de Quittner-Souplet et les références qui s'y trouvent [100]).

Pour le cas $p > 2$ où l'équation peut être dégénérée (pour $|\nabla u| = 0$), J. Zhao [126] a étudié les problèmes d'existence et de non existence de solutions faibles de (1.4) pour $q \le p-1$. Dans ce cas il prouve l'existence de solutions (globales) pour toute donnée initiale u_0. Pour $q < p$ une borne L^∞ locale du gradient des solutions est prouvée dans [33]. Pour le cas $q > p-1$, Chen, Nakao et Ohara [36] ont montré l'existence de solutions faibles globales mais seulement sous une condition de petitesse de la donnée initiale et avec l'hypothèse d'une courbure moyenne positive du bord $\partial\Omega$.

Notre première contribution vient compléter ces résultats en montrant l'existence et l'unicité d'une solution maximale en temps dans $W^{1,\infty}(\Omega)$ sans restriction de taille sur la donnée initiale et en mettant en évidence l'alternative d'explosion dans $W^{1,\infty}$.

Tout d'abord la notion de solution faible pour le problème (1.4) est définie de manière standard comme suit :

Définition 1.2.1. *Soit $r = \max(p, q)$. Une fonction $u(x,t)$ est appelée solution faible du problème (1.4) dans $Q_T := \Omega \times (0,T)$ si*

$$u \in C(\overline{\Omega} \times [0,T)) \cap L^r((0,T); W^{1,r}(\Omega)),$$

$$u_t \in L^2((0,T); L^2(\Omega)),$$

$u(x,0) = u_0(x)$, $u = g$ sur $\partial\Omega$ et l'égalité

$$\int\int_{Q_T} u_t\psi + |\nabla u|^{p-2}\nabla u \cdot \nabla\psi \, dx\, dt = \int\int_{Q_T} |\nabla u|^q \psi \, dx\, dt \tag{1.6}$$

est satisfaite pour tout $\psi \in C^0(\overline{Q_T}) \cap L^p((0,T); W^{1,p}(\Omega))$ telle que $\psi = 0$ sur $\partial\Omega \times (0,T)$.

Rappelons que la notion de solution faible (ou au sens des distributions) n'est pas la seule notion dont on dispose pour résoudre le problème (1.1). En effet, la théorie des solutions de viscosité fournit un cadre plus général pour les problèmes d'existence et d'unicité, mais celui-ci est moins adapté pour l'étude des problèmes de singularités qui nous intéressent. En effet ces solutions ne sont a priori pas assez régulières pour "voir" les singularités du gradient. La notion de solution faible qui fait intervenir les espaces de Sobolev et qui demande un peu plus de régularité est certainement une meilleure alternative.

Théorème 1.2.1. *On suppose que $q > p-1 > 1$. Soient $M > 0$ et u_0, g satisfaisant la condition de compatibilité (1.5) et $\|\nabla u_0\|_\infty \leq M$. Alors*

(i) Il existe un temps $T = T\left(M, p, q, N, \|g\|_{C^2}\right) > 0$ et une solution faible u de (1.4) dans $[0,T)$, qui de plus satisfait $u \in L^\infty_{loc}([0,T); W^{1,\infty}(\Omega))$.

(ii) Pour tout $\mathcal{T} > 0$ le problème (1.4) admet au plus une solution u telle que $u \in L^\infty_{loc}([0,\mathcal{T}); W^{1,\infty}(\Omega))$.

(iii) Il existe une (unique) solution faible, maximale de (1.4), notée u. Soit $T_{max}(u_0)$ le temps maximal d'existence, alors

$$\min_\Omega u_0 \leq u \leq \max_\Omega u_0 \qquad \text{dans} \quad \Omega \times (0, T_{max}(u_0)) \tag{1.7}$$

et

$$\text{si} \quad T_{max}(u_0) < \infty, \quad \text{alors} \quad \lim_{t\to T_{max}(u_0)} \|\nabla u\|_{L^\infty(\Omega)} = \infty.$$

La méthode classique pour montrer l'existence locale de solutions faibles est d'introduire un problème approché uniformément parabolique permettant de construire des solutions classiques u_ε. Le but est de démontrer que les suites u_ε et ∇u_ε sont uniformément bornées dans un espace L^m. Avec ces estimations et quitte à extraire une sous-suite, on a la convergence forte des u_ε par le théorème d'Ascoli mais seulement une convergence faible des ∇u_ε qui est insuffisante pour passer à la limite dans le terme non-linéaire. La principale difficulté consiste alors à avoir de meilleures estimations sur ∇u_ε. Pour venir à bout de cette difficulté, notre principale nouveauté par rapport à [36] est un contrôle du gradient des solutions approchées près du bord pour un temps petit via des fonctions barrière bien choisies. L'invariance par translation en espace de l'équation (1.4) nous permet ensuite d'avoir un contrôle du gradient sur tout le domaine via un principe de comparaison. On utilise alors un résultat fort de Friedman-DiBenedetto sur la régularité Hölderienne du gradient de certaines EDP paraboliques dégénérées [47, 48].

Notre deuxième contribution porte sur un effet régularisant sur u_t. En utilisant l'homogénéité du p-Laplacien et un résultat de comparaison on a pu établir le résultat suivant. On note $osc(u_0) = \max_{\overline{\Omega}} u_0 - \min_{\overline{\Omega}} u_0$.

Théorème 1.2.2. *On suppose que $q > p-1 > 1$ et $u_0 \in W^{1,\infty}(\Omega)$. Soit u l'unique solution faible de (1.4) dans $L^\infty_{loc}([0, T_{max}(u_0)); W^{1,\infty}(\Omega))$. Alors*

$$u_t \leq \frac{1}{p-2}\frac{osc(u_0)}{t} \qquad \text{dans} \;\; \mathcal{D}'(\Omega \times (0,T)). \tag{1.8}$$

Remarquons que grâce à l'estimation unilatérale de u_t on a une estimation de $\Delta_p u$ dans l'esprit des estimation de Bénilan et Crandall. En effet on a

$$\Delta_p u = u_t - |\nabla u|^q \leq \frac{1}{p-2}\frac{\|u_0\|_\infty}{t}. \tag{1.9}$$

L'estimation de semi-concavité (1.9) a été montré dans [51] seulement pour $q = p$ et dans le cas du problème de Cauchy $\Omega = \mathbb{R}^N$. Elle est aussi valide pour le problème de Cauchy pour $1 < q \leq p$ (voir [84]). Rappelons que les estimations de ce type peuvent servir pour démontrer des inégalités de Harnack ou bien obtenir des résultats de régularité. Signalons qu'en utilisant des arguments similaires à ceux de [28], il est possible de montrer que pour $q < p-1$

$$u_t \geq \frac{-1}{p-2}\frac{\|u_0\|_\infty}{t}.$$

Dans les travaux de Aronson-Bénilan on rencontre aussi des estimations unilatérales similaires. Dans le chapitre 5 cette estimation est complétée par une estimation inférieure sur u_t.

Théorème 1.2.3. *On suppose que $q > p-1 > 1$. Soit u l'unique solution faible de (1.4) dans $L^\infty_{loc}([0, T_{max}(u_0)); W^{1,\infty}(\Omega))$. Fixons $t_0 \in (0, T_{max})$, alors*

$$u_t \geq -\left(\frac{q-p+1}{p-2}\right)\sup_{[0,t_0]\times\Omega}|\nabla u|^q - \left(\frac{1}{p-2}\right)\frac{osc(u_0)}{t} \qquad \textit{dans} \;\; \mathcal{D}'\left((0,t_0)\times\Omega\right). \tag{1.10}$$

L'estimation (1.8) et la méthode de Bernstein (qu'on détaillera un peu plus loin dans le dernière section de l'introduction) nous ont permis d'établir une estimation locale sur le gradient de la solution donnant le profil spatial de l'explosion. On note $\delta(x) = dist(x, \Omega)$.

Théorème 1.2.4. *On suppose que* $q > p - 1 > 1$. *Soient* $M > 0$ *et* u_0 *satisfaisant les conditions de compatibilité et* $\|\nabla u_0\|_\infty \leq M$. *Soit* u *l'unique solution faible de* (1.4) *dans* $L^\infty_{loc}([0, T_{max}(u_0)); W^{1,\infty}(\Omega))$. *Alors*

$$|\nabla u| \leq C_1 \delta^{-1/(q-p+1)}(x) + C_2. \qquad \textit{dans} \quad \Omega \times (0, T_{max}(u_0)) \,. \tag{1.11}$$

où $C_1 = C_1(q, p, N) > 0$ *et* $C_2 = C_2(q, p, \Omega, M, \|g\|_{C^2}) > 0$.

Comme on le verra dans les chapitres 3 et 5, cette estimation est optimale. Pour terminer nous donnons une condition suffisante qui induit l'explosion du gradient en temps fini.

Proposition 1.2.1 (Méthode de la fonction propre). *On suppose que* $q > p > 2$. *Soit* u *l'unique solution faible de* (1.4) *dans* $L^\infty_{loc}([0, T_{max}(u_0)); W^{1,\infty}(\Omega))$. *Soit* $\alpha \geq 1$ *telle que* $\frac{p-1}{q-p+1} < \alpha < q - 1$, *alors il existe une constante* $C = C(q, p, \alpha, \Omega, \|g\|_\infty) > 0$ *telle que si* $\int_\Omega u_0 \, \varphi_1^\alpha dx \geq C$, *alors* $T_{max}(u_0) < \infty$.

Dans [85] un autre critère suffisant qui repose sur une autre méthode est fourni : pour tout $r \in [1, \infty)$, il existe une constante $C_1 = C_1(p, q, r, \Omega) > 0$ telle que, si $u_0 \in W^{1,\infty}$ et $\|u_0\|_r \geq C_1$, alors $T_{max}(u_0) < \infty$.

Questions ouvertes ou avec des réponses partielles

Avec quelle vitesse ?

Après avoir obtenu le profil spatial de l'explosion, il est naturel de s'intéresser à la vitesse avec laquelle ce phénomène se produit. Un première réponse a été donnée dans les travaux de [65, 42] pour le cas de la diffusion linéaire.

Théorème 1.2.5 (J.S. Guo et B. Hu). *Soit le problème* (1.4) *avec* $q > 2 = p$. *Si le gradient explose en temps fini* T^*, *alors il existe une constante* $C_0 >$ *telle que :*

$$\sup_{x \in \overline{\Omega}, 0 \leq \tau \leq t} |\nabla u(x, \tau)| \geq C_0 (T^* - t)^{\frac{-1}{q-2}}.$$

Il est intéressant de constater que pour cette équation la vitesse d'explosion ne coïncide pas avec celle suggérée par l'invariance de l'équation. En effet, soit u une solution de (1.4), la fonction $u_\lambda = \lambda^{-k} u(\lambda x, \lambda^2 t)$; $k = \dfrac{q-2}{q-1}$ résout aussi (1.4). On peut alors envisager l'existence de solutions auto-similaires de la forme

$$w(t, x) = (T - t)^{\frac{k}{2}} V \left(\frac{x}{\sqrt{T - t}} \right), \tag{1.12}$$

dans un demi-espace, avec la condition $w = 0$ sur le bord (ceci ne peut se produire dans l'espace entier, car on sait que toutes les solution existent globalement cf. [100]). S'il existe des solutions w de la forme (1.12) avec $\nabla V \in L^\infty$, alors w aurait impliqué que

$$\|\nabla u(t,)\|_{L^\infty} \sim (T-t)^{\frac{-1}{2(q-1)}}.$$

Mais ceci et incompatible avec le Théorème 1.2.5 car $\frac{1}{q-2} > \frac{1}{2(q-1)}$. Autrement dit la vitesse d'explosion du gradient est plus rapide que la vitesse autosimilaire. L'estimation supérieure de la vitesse d'explosion est encore un problème ouvert. Néanmoins pour M assez grand et pour les solutions **croissantes en temps** du problème 1D :

$$\begin{cases} u_t - u_{xx} = |u_x|^p, & x \in (0,1), t > 0 \\ u(t,0) = 0, u(t,1) = M, & t > 0. \end{cases} \tag{1.13}$$

on a

$$c_0\,(T^* - t)^{\frac{-1}{q-2}} \leq \max_{0 \leq x \leq 1} |u_x(t,x)| \leq c_1\,(T^* - t)^{\frac{-1}{q-2}}. \tag{1.14}$$

Dans [124] Z. Zhang a généralisé ce résultat au cas de la diffusion non linéaire pour des solutions classiques croissantes en temps et en espace. On a

$$C_1\,(T^* - t)^{\frac{-1}{(q-p)}} \leq \max_{0 \leq x \leq 1} |u_x(t,x)| \leq C_2\,(T^* - t)^{\frac{-1}{(q-p)}}. \tag{1.15}$$

La preuve de ces deux derniers résultats repose sur l'application du principe du maximum à une fonctionnelle bien choisie et sur le lemme de Hopf appliqué à u_t. Notons qu'on ne sait pas si cette vitesse est la seule possible si on ne suppose pas que la solution est croissante en temps.

Solution de viscosité vs solutions faibles

Le lien entre solutions de viscosité et solutions faibles (au sens des distributions) n'est pas encore très bien compris. Quand ces deux notions sont-elles équivalentes ? En utilisant l'unicité des solutions dans la classe des solutions de viscosité et dans la classe des solutions Lipschitziennes, on peut néanmoins montrer le résultat suivant

Proposition 1.2.2. *On suppose que $q > p - 1 > 1$ et $u_0 \in W^{1,\infty}(\Omega)$. Notons $T_{max}(u_0)$ le temps maximal d'existence de l'unique solution faible u_{Lip}, Lipschitzienne, maximale. Alors la solution de viscosité u_{Vis} (globale) du problème* (1.4) *coïncide avec la solution faible sur* $[0, T_{max}(u_0))$.

En effet prenons $u_0 \in W^{1,\infty}(\Omega)$ avec $\|\nabla u_0\|_\infty \leq M$ et notons $u_{\mathbf{Vis}}$ la solution de viscosité associée et $u_{\mathbf{Lip}}$ la solution faible $u_{\mathbf{Lip}} \in L^\infty_{loc}([0, T_{max}(u_0)); W^{1,\infty}(\Omega))$ associée. Fixons $T \in (0, T_{max}(u_0))$ et posons

$$\bar{T} = \sup\left\{s > 0 \quad \text{tel que} \quad u_{\mathbf{Vis}} = u_{\mathbf{Lip}} \quad \text{dans} \quad [0,s] \times \overline{\Omega}\right\}. \tag{1.16}$$

On sait que $\bar{T} \geq t_0 = t_0(M) > 0$. En effet, pour un temps petit $t_0(M)$, nous disposons d'estimations uniformes en ε de la norme L^∞ des solutions u_ε des problèmes approchés ainsi que des estimations Hölderiennes (locales) de ∇u_ε (voir le chapitre 2). Ces estimations impliquent que (u_ε) est relativement compact dans $C(\overline{\Omega} \times [0, t_0(M)])$. D'une part, en utilisant les résultats de stabilité et de comparaison [43, 10, 85], on conclut que u_ε convergent uniformément vers l'unique solution de viscosité $u_{\mathbf{Vis}}$ du problème (1.4) (voir chapitre 4 pour l'unicité). D'autre part les estimations (suffisamment fortes) sur le gradient des u_ε sur $[0, t_0(M)]$ nous permettent aussi de conclure que les u_ε convergent vers l'unique solution faible Lipschitzienne $u_{\mathbf{Lip}}$ du problème (1.4) (voir chapitre 2). Il s'en suit que $u_{\mathbf{Vis}} = u_{\mathbf{Lip}}$ sur $[0, t_0(M)]$. Supposons que $\bar{T} < T$. Notons $A = \sup\limits_{t \in [0,T]} \|u(t)\|_{W^{1,\infty}}$. Pour tout $\eta \in (0, \bar{T})$ il existe un $\tau(A) > 0$ (indépendant de η) tel que la solution de viscosité et la solution faible associées au problème ci-dessous coïncident (pour la même raison que précédemment).

$$\begin{cases} \partial_t w - \operatorname{div}(|\nabla w|^{p-2}\nabla w) = |\nabla w|^q, & x \in \Omega, t > 0, \\ w(x,t) = u(\bar{T} - \eta, x), & x \in \partial\Omega, t > 0, \\ w(x,0) = u(\bar{T} - \eta, x), & x \in \Omega, \end{cases} \tag{1.17}$$

Par conséquent $u_{\mathbf{Vis}}$ et $u_{\mathbf{Lip}}$ coïncident sur $[0, \bar{T} - \eta + \tau]$. Puisque $\bar{T} - \eta + \tau > \bar{T}$ pour η suffisamment petit, on obtient une contradiction avec la définition de $\bar{T}$.

Signalons également que pour $p = 2$ et $q > 1$, Poretta et Zuazua [97] ont montré récemment que les solutions de viscosité (qui sont globales en temps) redeviennent Lipschitziennes (et vérifieront donc les conditions au bord au sens classique) après un certain temps. Plus précisément pour $g = 0$ et $u_0 \in C(\overline{\Omega})$ (sans condition de signe), ils ont montré qu'il existe des constantes positives λ, K, C (dépendant seulement de q et Ω) tel que les solutions de viscosité du problème (1.4) vérifient

$$\|u(t)\|_{L^\infty} + \|\nabla u(t)\|_{L^\infty} \leq Ce^{-\lambda t}, \quad \text{pour} \quad t \geq K \|u_0\|_{L^\infty}. \tag{1.18}$$

La preuve repose sur des arguments de comparaison et sur l'étude du problème linéarisé.

Le lecteur peut trouver une étude de l'équivalence de la notion de solution de viscosité et solution faible dans [73] pour le cas parabolique et dans les travaux de Ishii [71, 72] pour le cas elliptique. Ces études sont basées sur d'autres méthodes (résultat de comparaison, régularisation par sup et inf-convolution et passage à la limite) et montrent l'équivalence des deux notions pour des sous ou sur-solutions.

Régularité jusqu'au bord des solutions locales

La régularité Hölderienne du gradient jusqu'au bord et pas seulement à l'intérieur du domaine Ω est une question intéressante. Pour certaines données au bord et certains domaines, cette régularité pourrait s'obtenir en prolongeant par réflexion la solution dans un domaine plus grand contenant Ω et en utilisant les résultats de régularité de Friedman et DiBenedetto. Pour un résultat dans ce sens voir l'étude du problème unidimensionnel (chapitre 5).

1.3 Etat de l'art et résultats du chapitre 3 : localisation des points d'explosion.

La localisation et la taille de l'ensemble des points d'explosion des solutions du problème (1.4) est une question assez délicate en général. Dans la littérature on trouve principalement deux types d'explosion du gradient :

1. l'explosion a lieu à l'intérieur du domaine Ω.
2. l'explosion a lieu au bord $\partial\Omega$.

Le premier type est observé par exemple pour les solutions de l'équation :

$$\begin{cases} u_t - u_{xx} = |u|^{p-1}u|u_x|^q, & t > 0, -1 < x < 1, \\ u(t, \pm 1) = A_{\pm}, & t > 0, \\ u(0, x) = u_0(x), & -1 < x < 1. \end{cases} \tag{1.19}$$

avec $q > 2$, $p \geq 1$ $u_0 \in C^1$ et $u_0(-1) = A_- \leq u_0 \leq A_+ = u_0(1)$. L'explosion a lieu si par exemple $\max(A_+, |A_-|) > C > 0$ [3]. Pour d'autres résultats sur l'explosion du gradient à l'intérieur du domaine pour des équations paraboliques quasi-linéaires avec une non-linéarité sur le gradient nous renvoyons le lecteur à [6, 100].

Pour l'équation de Hamilton-Jacobi diffusive (1.4), l'explosion ne peut avoir lieu qu'au bord du domaine. Ce résultat est dû à l'estimation du gradient (1.11) vue précédemment. Par la suite le problème de localisation des points d'explosion offre principalement les possibilités suivantes

- L'ensemble des points d'explosion est le bord tout entier.
- L'ensemble des points d'explosion est un ensemble discret, notamment réduit à un singleton.
- L'ensemble des points d'explosion est de mesure finie et positive (connu sous le nom de "regional blow-up")

L'ensemble des points d'explosion peut éventuellement être un ensemble plus complexe. Précisons ici que l'ensemble des points d'explosion du gradient de u est défini par :

$$\begin{aligned} GBUS(u_0) := \ & \big\{x_0 \in \partial\Omega;\ \nabla u \text{ est non borné dans } (T_{max} - \eta, T_{\max}) \times (\overline{\Omega} \cap B(x_0, \eta)) \\ & \text{pour tout } \eta > 0\big\}. \end{aligned} \tag{1.20}$$

La première possibilité est triviale dans le cas où le domaine Ω est une boule et la donnée initiale u_0 est radiale.

Pour le cas de la diffusion linéaire, Souplet et Li [88] se sont intéressés à la validité de la deuxième possibilité. Ils ont d'abord montré que pour des domaines bornés réguliers $\Omega \subset \mathbb{R}^N$ avec $N \geq 2$, l'ensemble des points d'explosion peut être localisé dans un voisinage arbitraire de n'importe quel point $x_0 \in \partial\Omega$. Ensuite, pour certains types de domaines bidimensionnels, ils ont pu construire des données initiales qui garantissent l'explosion du gradient en un seul point du bord. Ils ont établi le résultat suivant dans le cas $\Omega = B(0,1) \subset \mathbb{R}^2$.

Théorème 1.3.1 (Souplet et Li). *On suppose que $q > 2 = p$ et $g = 0$ dans le problème (1.4). On note $\tilde{u_0}(r,\theta) = u_0(r\cos\theta, r\sin\theta)$. On a le résultat suivant*

(i) Il existe $u_0 \in X_+ := \left\{u_0 \in C^1(\overline{\Omega});\quad u = 0 \text{ sur } \partial\Omega, u_0 \geq 0\right\}$, telle que $T^*(u_0) < \infty$ et

$$\begin{cases} u_0 \text{ est symétrique par rapport à la ligne } y = 0, \\ \dfrac{\partial \tilde{u_0}}{\partial \theta} \leq 0 \text{ dans } B_+ := \{(r,\theta); 0 < r < 1, 0 < \theta < \pi\}, \\ GBUS(u_0) \neq \partial\Omega. \end{cases}$$

(ii) Pour une telle donnée initiale u_0, $GBUS(u_0)$ contient seulement le point $(x,y) = (1,0)$.

Le deuxième type de domaine qu'ils ont considéré sont les domaines symétriques par rapport à l'axe $x = 0$ et avec un bord contenant une portion plate centrée en l'origine (c'est à dire il existe $\rho >$ tel que $(-\rho,\rho) \times \{0\} \subset \partial\Omega$). Dans ce cas ils ont réussi à construire des données initiales qui garantissent que le seul point d'explosion est $(0,0)$.

Le problème d'explosion en un seul point pour des équations paraboliques semi-linéaires a été traité pour la première fois par Weissler [119] pour l'équation de la chaleur unidimensionnelle avec un terme source u^α. Ensuite Friedman et Mcleod [56] ont étendu ce résultat en introduisant une fonctionnelle J de la forme $J = u_x + c(x)u^q$ et en lui appliquant le principe du maximum. Depuis, leur méthode est devenue une des méthode "phare" pour démontrer des résultats d'explosion en un seul point. Néanmoins cette technique est utilisée en majorité pour l'explosion en amplitude et dans le cas unidimensionnel ou pour des solutions radiales.

La méthode de Souplet et Li [88] est une adaptation astucieuse de la technique de Friedman-Mcleod au cas de **la dimension 2** en espace et pour le phénomène nouveau de **l'explosion du gradient**. Dans le cas qui nous intéresse ($q > p > 2$) et pour des domaines localement plats, nous allons adapter la stratégie utilisée dans [88] aux diverses complications apportées par la diffusion non linéaire pour montrer que l'ensemble des points d'explosion du problème (1.4) est un singleton.

Tout d'abord précisons les hypothèses géométriques sur le domaine Ω. On suppose que pour $L_1, L_2 > 0$,

$$\Omega \subset \mathbb{R}^2 \text{ est un domaine borné régulier de classe } C^{2+\epsilon} \text{ pour un } \epsilon \in (0,1); \tag{1.21}$$

$$\Omega \text{ est symétrique par rapport à l'axe } x = 0; \tag{1.22}$$

$$\Omega \subset \{y > 0\} \text{ et } \Omega \text{ contient le rectangle } (-L_1, L_1) \times (0, 2L_2); \tag{1.23}$$

$$\Omega \text{ est convexe dans la direction } x \text{ des abscisses.} \tag{1.24}$$

Dans le problème (1.4) on pose $g = \mu y$ où $\mu > 0$ est une constante et on suppose que la donnée initiale u_0 est dans $\mathcal{V}_\mu$, où

$$\mathcal{V}_\mu := \left\{u_0 \in C^1(\overline{\Omega}),\ u_0 \geq \mu y \ \text{ dans } \Omega,\ u_0 = \mu y \text{ sur } \partial\Omega\right\}.$$

Ensuite, on impose des conditions spécifiques sur la donnée initiale (profil concentré près de l'origine, symétrie et décroissance en la première variable d'espace, croissance par rapport à la deuxième variable) qui se transmettent à la solution u. Grâce à ce profil particulier, nous avons le résultat suivant.

Théorème 1.3.2 (Attouchi et Souplet). *On suppose que dans le problème* (1.4), $q > p > 2$, $g = \mu y$, $u_0 \in \mathcal{V}_\mu$ *et* Ω *vérifie les hypothèses* (1.21)–(1.24). *On note* $\Omega_+ := \Omega \cap \{x > 0\}$. *Alors*

(i) Pour tout $\rho \in (0, L_1)$, *il existe* $\mu_0 = \mu_0(p, q, \Omega, \rho) > 0$ *tel que, pour tout* $\mu \in (0, \mu_0]$, *il existe une donnée initiale* u_0 *dans* $\mathcal{V}_\mu \cap C^2(\overline{\Omega})$ *pour laquelle la solution* u *du problème (1.4) vérifie les propriétés suivantes :*

$$T := T_{max}(u_0) < \infty \text{ et } GBUS(u_0) \subset [-\rho, \rho] \times \{0\}, \tag{1.25}$$

$$u(\cdot, t) \text{ est symétrique par rapport à l'axe } x = 0, \text{ pour tout } t \in (0, T), \tag{1.26}$$

$$u_x \leq 0 \quad \text{dans } \Omega_+ \times (0, T), \tag{1.27}$$

$$u_y \geq \mu/2 \quad \text{dans } \Omega \times (0, T). \tag{1.28}$$

(ii) Pour tout μ *et* u_0 *vérifiant les conditions de (i), on a que*

$$GBUS(u_0) = \{(0, 0)\}.$$

Bien que nous ne réussissions pas à passer outre, la restriction à des solutions qui vérifient (1.28) (d'où le choix de la donnée au bord) semble être d'ordre technique. En effet, à cause du terme non linéaire $|\nabla u|^{p-2}$ dans la partie principale du $\Delta_p u$, les quantités qui apparaissent dans le calcul de l'équation de $J := u_x + c(x)d(y)F(u)$ ne dépendent pas seulement de u et ∇u mais **aussi** de D^2u et u_t. Ne disposant pas d'estimations assez fortes de D^2u, on se sert de l'équation pour ré-écrire les termes qui font apparaître D^2u en fonction de puissances de ∇u dont certaines sont négatives. Ainsi pour pouvoir dériver une équation sur la fonctionnelle J, l'hypothèse (1.28) nous a été cruciale et les quantités qui y apparaissent explosent quand μ tend vers 0 (voir le chapitre 3 pour plus de détail). Ce genre de difficulté n'apparaît pas pour un opérateur de diffusion linéaire.

La preuve est longue et technique, plus encore que dans [88]. Les outils clés de la preuve sont l'estimation locale du gradient, une construction d'une donnée initiale bien préparée qui nous permet de localiser l'ensemble des points d'explosion dans un voisinage de l'origine. Ensuite, pour obtenir les conditions aux bords et initiales pour la fonction auxiliaire J, on a recours au lemme de Hopf classique et une version parabolique du lemme du coin de Serrin dont on donnera une preuve. Bien que le lemme de Serrin soit assez connu dans le cas elliptique [103, 112] notamment pour l'étude de problèmes de symétrie, on ne trouve que de très rares versions paraboliques de ce lemme [101]. Une version parabolique du lemme de Serrin pour une équation de la chaleur dans un rectangle est prouvée dans [88] en utilisant des fonctions à variables séparables en espace. Notre preuve est différente et repose sur une modification de celle utilisée dans [103]. Notons que la propriété (1.28) garantit que la solution u est classique et par conséquent ses dérivées secondes et troisièmes en espace (D^2u, D^3u) sont localement bornées ce qui nous servira pour utiliser les lemmes de Hopf et la version parabolique du lemme de Serrin.

Problèmes ouverts ou avec des réponses partielles

L'étude de la localisation des singularités pour le problème (1.4) est loin d'être achevée. Elle peut être poursuivie dans diverses directions en étudiant un certain nombre de questions intéressantes. La première direction viendrait compléter ce travail sur le caractère discret de l'ensemble des points d'explosion. Peut on étendre les théorèmes 1.3.2 et 1.3.1 à des domaines de dimension $N > 2$ (en utilisant des données initiales symétriques) ? Peut-on s'affranchir de la condition de bord localement plat ? Comment traiter le cas $\mu = 0$ et relaxer l'hypothèse (1.28) ? Ensuite, il serait aussi intéressant de savoir s'il est possible de construire (comme c'est déjà le cas pour certaines équations semi-linéaires [40]) des solutions dont l'ensemble de points d'explosion est discret fini mais non réduit à un singleton (c'est à dire qu'il contient 2, 3 ou n points) et s'il est possible de construire des solutions qui explosent en des points qu'on aura fixé [94].

D'autre part on peut se demander si on peut construire des solutions qui valident la possibilité de "régional blow-up".

1.4 Etat de l'art et résultats du chapitre 4 : prolongement des solutions au delà des singularités

Pour les solutions classiques qui explosent en temps fini il est intéressant de savoir s'il est possible de les étendre au delà du temps maximal d'existence et d'étudier le comportement asymptotique de ces solutions prolongées. Il est alors important de savoir en quel sens on peut le faire. L'apparition d'une singularité en gradient est un obstacle non négligeable pour étendre les solutions classiques. Une première tentative peut être trouvée dans les travaux [60, 52]. Fila et Lieberman [52], ont étudié l'équation (1.4) pour $N = 1$ et $p = 2$ et avec une non-linéarité $F(u_x)$ plus générale. Pour $\Omega = (0, L)$ et $g = 0$, ils ont montré que pour L assez grand et pour certaines données initiales, le gradient de la solution de (1.4) n'explose qu'en $x = 0$. Ensuite, ils ont pu prolonger la solution explosive par une solution classique qui satisfait la condition au bord en $x = L$ mais qui ne la satisfait pas au point où a lieu l'explosion du gradient. Plus précisément la solution prolongée vérifie $U(0,t) > 0$ et $U_x(0,t) = +\infty$ pour $t > T_{max}(u_0)$. Les ingrédients clés de ce résultat sont l'étude du problème stationnaire "singulier" associé à l'équation (1.4), l'équation satisfaite par une fonction auxiliaire $h(u_x)$ et le cadre unidimensionnel. Des résultats sur le comportement en temps grand de la solution prolongée ont été obtenus montrant la convergence vers l'unique état stationnaire "singulier" Φ (i.e $\Phi'(0) = +\infty$ et $\Phi(L) = 0$).

Pour notre équation et pour pouvoir traiter le cas d'une dimension $N \geq 1$, la théorie des solutions de viscosité offre un bon cadre pour la continuation des solutions au delà des singularités et l'étude des propriétés des solutions prolongées. En effet, récemment pour le cas de la diffusion linéaire ($p = 2$), Barles et Da Lio [17] ont introduit la notion de solutions de viscosité **généralisées** qui ont l'avantage d'autoriser la perte des conditions au bord. Généralement, pour ces solutions les conditions au bord sont comprises dans un sens faible comme suit : soit la solution atteint la condition au bord au sens classique, soit l'équation elle même est satisfaite au bord (au sens de viscosité).

Dans ce chapitre on s'intéresse plus particulièrement au problème suivant :

$$\begin{aligned} u_t - \mathrm{div}(|\nabla u|^{p-2}\nabla u) + |\nabla u|^q = &\quad f(x,t), \quad x \in \Omega, t > 0, \\ u(x,t) = &\quad g(x), \quad x \in \partial\Omega, t > 0, \\ u(x,0) = &\quad u_0(x), \quad x \in \overline{\Omega}. \end{aligned} \tag{1.29}$$

Contrairement aux chapitres précédents, on demande **moins de régularité** sur les fonctions u_0 et g. On suppose que u_0, g et f sont des fonctions **continues** et que $u_0(x) = g(x)$ pour $x \in \partial\Omega$.
NB : Le signe de la non-linéarité n'est pas important dans cette partie. Il suffit de changer u en $-u$ et les résultats restent valables à condition de changer sous-solutions par sur-solutions et vice versa.

Dans [17, 16], on peut trouver une étude du lien entre l'explosion du gradient (due à la forte non-linéarité du terme du premier ordre) et l'éventuelle perte des conditions aux bord. Lorsque $q > 1 = (p-1)$, cette étude est faite en utilisant le lien entre l'équation (1.29) et un problème de contrôle stochastique. En effet, dans ce cas, Barles et Da Lio proposent une expression explicite de la solution de viscosité de (1.29) comme étant la fonction-valeur d'un problème de contrôle stochastique de type temps de sortie. Pour le cas elliptique, signalons que B. Kawohl et N. Kutev [76] ont aussi étudié cette problématique en donnant des conditions plus optimales qui garantissent l'explosion du gradient à l'intérieur ou au bord. Le problème de pertes des conditions aux limites est lié aux "couches limites" et il faudrait alors construire des "barrières" pour assurer que celles-ci sont bien satisfaites.

Dans un premier temps nous rappelons la définition d'une solution de viscosité.

Définition 1.4.1 (Sous-solutions, sur-solutions, solutions). *On dit qu'une fonction u : $[0,T] \times \overline{\Omega} \to \mathbb{R}$ est sous-solution de viscosité de (1.29) si u est semi-continue-supérieure (SCS) dans $[0,T] \times \overline{\Omega}$ et si, pour toute fonction-test $\phi \in C^2([0,T] \times \overline{\Omega})$ telle que $u - \phi$ a un maximum local en un point $(t_0, x_0) \in]0,T[\times\Omega$, on a*

$$\phi_t(t_0,x_0) - div(|\nabla\phi|^{p-2}\nabla\phi) + |\nabla\phi(t_0,x_0)|^q \leq f(t_0,x_0).$$

La condition de Dirichlet au bord associée à (3.9.20) *doit être relaxée et comprise au sens de viscosité de la manière suivante : si $(t_0,x_0) \in]0,T] \times \partial\Omega$ alors*

$$min\ (\phi_t - div(|\nabla\phi|^{p-2}\nabla\phi) + |\nabla\phi|^q, u - g) \leq 0.$$

De même la condition initiale est comprise au sens suivant : si $(t_0,x_0) \in \{0\} \times \overline{\Omega}$ alors

$$min\ (\phi_t - div(|\nabla\phi|^{p-2}\nabla\phi) + |\nabla\phi|^q, u - u_0) \leq 0.$$

Symétriquement, on dit qu'une fonction u :$]0,T[\times\Omega$ est sur-solution de viscosité de (1.29) si u est semi-continue-inférieure (SCI) dans $]0,T[\times\Omega$ et si, pour toute fonction-test $\phi \in C^2(]0,T[\times\Omega)$ telle que $u - \phi$ a un minimum local en un point $(t_0,x_0) \in]0,T[\times\Omega$, on a

$$\phi_t(t_0,x_0) - \Delta_p(\phi(t_0,x_0)) + |\nabla\phi(t_0,x_0)|^q \geq f(t_0,x_0),$$

si $(t_0, x_0) \in]0, T] \times \partial\Omega$ *on a*

$$max\ (\phi_t - div(|\nabla\phi|^{p-2}\nabla\phi) + |\nabla\phi|^q, u - g) \geq 0,$$

et si $(t_0, x_0) \in \{0\} \times \overline{\Omega}$ *alors*

$$min\ (\phi_t - div(|\nabla\phi|^{p-2}\nabla\phi) + |\nabla\phi|^q, u - u_0) \leq 0.$$

Enfin, $u : [0, T] \times \overline{\Omega}$ *est solution de viscosité de (1.29) si* u *est sous et sur-solution de (1.29).*

Le sens avec lequel on définit les conditions aux limites se justifie grâce au résultat de stabilité discontinue et la méthode des semi-limites relaxées introduits par Barles et Perthame [19, 13] (qui requièrent seulement une borne L^∞ uniforme pour les solutions u_ε des problèmes régularisés).

Dans [17] un résultat d'existence-unicité d'une solution de viscosité **globale en temps** du problème (1.29) a été prouvé pour le cas du laplacien ($p = 2$). Notre première contribution vient étendre ce résultat pour le cas de la diffusion non linéaire $p > 2$.

Théorème 1.4.1 (Attouchi, Barles). *On suppose que* $q > p > 2$ *et* u_0, g, f *sont continues. Alors il existe une unique solution de viscosité du problème* (1.29)*, qui est définie pour tout temps* $t > 0$.

L'argument clé pour monter le théorème 1.4.1 est un résultat de comparaison fort (i.e un résultat de comparaison pour des sur et sous-solutions discontinues). Notre approche est légèrement différente de celle de [17] puisqu'on utilise l'astuce de la sup-convolution temporelle introduite dans les travaux de [81] et le résultat de régularité ci-dessous pour EDP elliptiques avec une forte non linéarité du gradient. Cette astuce nous permet d'approcher toute sous-solution du problème (1.29) (qui a priori peut n'être que SCS) par une sous-solution continue et qui de ce fait satisfait automatiquement la condition de cône et de pouvoir ainsi terminer la preuve dans l'esprit de [17]. La méthode de Perron [70] nous permettant alors de montrer l'existence. Cette méthode consiste à d'abord construire une sous-solution et une sur-solution qui se comporte bien sur le bord parabolique, on construit alors une sous-solution maximale et on montre que c'est aussi une sur-solution.

Théorème 1.4.2 (Attouchi, Barles : Régularité elliptique des sous-solutions)**.** *Soit l'équation*

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + |\nabla u|^q = f(x) \quad dans \quad \Omega, \tag{1.30}$$

où $q > p \geq 2$ *et* f *est une fonction continue sur* $\overline{\Omega}$ *à valeurs réelles. Toute sous-solution de viscosité de l'équation* (1.30) *est dans* $C^{0,\beta}(\overline{\Omega})$ *avec* $\beta = \dfrac{q-p}{q-p+1}$ *et la norme dans* $C^{0,\beta}$ *ne dépend que de* $p, q, \|f\|_\infty$.

Le théorème 1.4.2 est une généralisation des résultats de I. Capuzzo Dolcetta, F. Leoni et A. Porretta [34] et de sa version revisitée par G. Barles [15] au cas quasi-linéaire. Notre preuve est différente de celle de [34, Théorème 2.11] et s'appuie sur la méthode utilisée dans [15]. Signalons que c'est la croissance forte du terme $|\nabla u|^q$ qui donne la régularité. De plus on a comme conséquence directe que l'on ne peut pas résoudre le problème de Dirichlet au sens classique pour toute donnée au bord g. En effet le théorème 1.4.2 implique que g doit être au moins Hölderienne.

Une autre application du théorème 1.4.2 se trouve dans l'étude du comportement asymptotique de la solution globale de (1.29). Désormais on suppose que $f(x,t) = \tilde{f}(x)$ dans (1.29). Une approche classique pour l'étude du comportement en temps grand de la solution globale réside dans l'introduction du "problème ergodique" associé à (1.29) consistant à trouver un couple (c, u_∞) solution de l'équation

$$- \operatorname{div}\left(|\nabla u_\infty|^{p-2}\nabla u_\infty\right) + |\nabla u_\infty|^q - \tilde{f}(x) = c \quad \text{dans } \Omega, \tag{1.31}$$

associée à la contrainte d'état au bord :

$$- \operatorname{div}\left(|\nabla u_\infty|^{p-2}\nabla u_\infty\right) + |\nabla u_\infty|^q - \tilde{f}(x) \geq c \quad \text{sur } \partial\Omega. \tag{1.32}$$

On a le résultat suivant qui vient étendre celui de [15].

Théorème 1.4.3. *On suppose que $q > p > 2$, Ω est un domaine borné de classe C^2 et $\tilde{f} \in C(\overline{\Omega})$, alors il existe une unique constante c tel que le problème* (1.31)–(1.32) *admet une solution de viscosité $u_\infty \in C(\overline{\Omega})$.*

L'existence de (c, u_∞) nous permet d'avoir le résultat suivant.

Théorème 1.4.4. *On suppose que $q > p > 2$, Ω est un domaine borné de classe C^2, $u_0 \in C(\overline{\Omega})$, $g \in C(\partial\Omega)$ satisfaisant la condition de compatibilité et $\tilde{f} \in C(\overline{\Omega})$. Soit (c, u_∞) une solution de* (1.31)–(1.32) *et u l'unique solution de viscosité de* (1.29)*, alors $u + c^+ t$ est bornée, avec $c^+ = max(c, 0)$. En particulier on a*

$$\lim_{t\to\infty} \frac{u(x,t)}{t} = -c^+$$

uniformément dans $\overline{\Omega}$.

Questions ouvertes ou avec des réponses partielles

Il est naturel de se demander si on peut avoir une description plus précise du comportement en temps grand des solutions globales de l'équation (1.29). Quand le problème stationnaire admet une solution, on s'attend à ce que u converge vers l'unique état stationnaire (voir [85, 110, 25]).

Quand le problème stationnaire n'admet pas de solutions, un résultat classique nous dit que la solution u du problème de Dirichlet (1.29) devrait se comporter comme

$$-ct + u_\infty(x) + o_t(1) \qquad \text{quand} \quad t \to \infty,$$

où u_∞ est une solution du problème ergodique stationnaire qui correspond à une unique constante ergodique c. Ceci a été démontré dans les travaux de Tchamba [113] pour $q > 2 = p$. Cette étude plus précise se base généralement sur deux ingrédients clés : la régularité des solutions et le principe de comparaison fort (qui se réduit au principe du maximum fort pour le cas de la diffusion linéaire $p = 2$). Le problème ergodique n'admet pas en général une unique solution même à une constante additive près rendant l'analyse du comportement asymptotique assez délicate. Mais dans le cas de solutions qui sont Lipschitziennes, l'application d'un principe de comparaison fort étendu aux solutions de viscosité permet de montrer l'unicité à une constante additive près. Pour le cas de la diffusion non-linéaire $p > 2$, la régularité Lipschitzienne des solutions pourrait être obtenue en adaptant les arguments de [34]. Par conséquent, la principale difficulté consiste à démontrer un principe de comparaison fort (ce qui est assez délicat vu le caractère quasi-linéaire de l'équation). Signalons qu'un principe du maximum fort a été démontré pour le p-Laplacien [45]. D'autre part, signalons aussi que la résolubilité du problème stationnaire est capitale pour l'étude du comportement asymptotique des solutions globales (voir chapitre 5). Une autre difficulté liée au caractère quasi-linéaire du p-Laplacien consiste à établir un résultat de comparaison fort pour le problème stationnaire (qui ne satisfait donc pas une condition de monotonie) en utilisant seulement la donnée d'une sous-solution stricte (voir [113] pour le cas $p = 2$). Le problème stationnaire semble n'être résoluble que dans le cas $c \leq 0$ (voir [113, 110]. L'étude du problème ergodique vient donc apporter une précision sur le comportement asymptotique des solutions globales dans le cas de non-existence de solutions stationnaires.

1.5 Etat de l'art et résultats du chapitre 5 : classification des solution globales

Nous revenons ici à l'étude des solutions faibles Lipschitziennes maximales (c.f Théorème 1.2.1). Étant donné que dans l'étude de l'équation (1.4) on a l'alternative suivante :

- solutions globales en temps qui sont uniformément bornées en temps en norme $W^{1,\infty}$ (c'est à dire non explosives)
- solutions globales en temps qui explosent en temps infini (c'est à dire des solutions qui existent pour tout tout $0 < t < \infty$ mais qui vérifient $\lim\limits_{t\to\infty} \|\nabla u(t)\|_\infty = +\infty$)
- solutions qui explosent en temps fini,

il est naturel de s'intéresser à la "classification" des solutions.
Dans ce chapitre on apporte une contribution à cette problématique en étendant le résultat de Arrieta- Rodriguez Bernal-Souplet. On se place dans le cadre unidimensionnel et on considère le problème suivant :

$$\begin{cases} u_t - (|u_x|^{p-2}u_x)_x = |u_x|^q & x \in (0,1), t > 0, \\ u(t,0) = 0, u(t,1) = M & t > 0, \\ u(0,x) = u_0(x) & x \in (0,1). \end{cases} \tag{1.33}$$

Théorème 1.5.1 (Classification des solutions globales). *On suppose que $q > p > 2$ et que $u_0 \in W^{1,\infty}$ satisfait la condition de compatibilité. On pose $M_c = \frac{q-p+1}{q-p}\left(\frac{q-p+1}{p-1}\right)^{\frac{1}{p-1-q}}$. Alors*

(i) Si $0 \leq M < M_c$ alors toutes les solutions globales de (1.33) *sont bornées en norme C^1 et convergent dans $C^1([0,1])$ vers l'unique solution stationnaire.*

(ii) Si $M > M_c$ alors toutes les solutions de (1.33) *explosent en temps fini.*

Ici M_c est la valeur critique pour l'existence d'une solution stationnaire (unique). Notons que le lien entre l'existence et les propriétés des solutions de l'équation stationnaire et le comportement asymptotique des solutions globales est crucial.

La preuve du théorème 1.5.1 s'inspire de celle de [5] et procède par contradiction. Elle se compose de trois étapes. La première étape consiste à construire une fonctionnelle de Lyapunov ayant de bonnes propriétés. Etant donné qu'on est en dimension 1, on peut utiliser la méthode de Zelenyak pour en construire une. Cette méthode s'applique principalement aux EDP uniformément paraboliques. Étant donné le manque de régularité des solutions faibles de (1.33), une des principales difficultés de la preuve du théorème 1.5.1 est d'établir des estimations assez fortes sur les fonctions approchées (u_ε) permettant d'avoir la convergence jusqu'au bord des $(u_\varepsilon)_x$ vers u_x. En effet on construit en fait une fonctionnelle de Lyapunov approchée et on doit avoir assez de compacité pour pouvoir passer à la limite dans les termes non-linéaires en $(u_\varepsilon)_x$. Cette fonctionnelle nous permet de montrer la convergence des solutions globales (éventuellement non bornées dans $W^{1,\infty}$) vers la solution stationnaire. Enfin on utilise le profil de la solution stationnaire et des estimations sur la dérivée pour aboutir à une contradiction.

Questions ouvertes ou avec des réponses partielles

Que se passe-t-il pour $M = M_c$? On sait que toutes les solutions globales doivent exploser en temps fini ou infini. Mais existe-t-il des solutions globales qui explosent en temps infini ? Dans le cas de la diffusion linéaire, Souplet et Vàzquez [108] ont montré que pour $M = M_b$, la solution globale u tend vers l'état stationnaire singulier (unique) noté v_{ss} dés que u_0 est majorée par v_{ss}. Une étude précise et assez technique de la formation des singularités est fournie dans [108]. Notons aussi que dans le cas de la diffusion linéaire si la non linéarité est remplacée par une non-linéarité exponentielle l'alternative d'explosion en temps infini à été montrée dans [125]. D'autre part l'explosion du gradient en temps infini a aussi été observé pour des équations quasi-linéaires de courbure moyenne [6].

Étant donné que la méthode de Zelenyack pour construire des fonctionnelles de Lyapunov est restreinte au cadre unidimensionnel, comment traiter le cas d'une dimension quelconque ? Par ailleurs il serait intéressant d'avoir des estimations uniformes sur les solutions.

Enfin on regroupe ici quelques résultats récents [85, 111] sur le comportement en temps grand de l'équation (1.4) pour d'autres valeur de p et q. On prend $g = 0$ dans (1.4) et on suppose que $p > 2$.

– On suppose que $q < p-1$. Si $N = 1$ ou bien u_0 est radiale, alors il existe une famille d'états stationnaires et l'unique solution de viscosité de (1.4) converge vers l'un d'eux quand $t \to \infty$.
– Pour $p-1 < q \leq p$ et des données initiales quelconques ou $q > p$ et des données initiales **suffisamment petites**, l'unique solution de viscosité de (1.4) converge vers 0 à la vitesse $t^{-\frac{1}{p-2}}$ et $t^{-\frac{1}{p-2}}u(t,x)$ tend vers une fonction ξ solution positive de

$$-\Delta_p \xi - |\nabla f|^{p-1} - \frac{f}{p-2} = 0.$$

1.6 Etat de l'art et résultats du chapitre 6 : un théorème de type Liouville

Dans ce chapitre on s'intéresse à une autre propriété des solutions de l'équation (1.1) qui n'est pas reliée au phénomène d'explosion du gradient mais qui est une application directe d'une estimation, locale en espace du gradient des solutions localement bornées. Plus précisément, on s'intéresse à un théorème de Liouville pour les solutions anciennes ($t < 0$) dans l'espace entier $\mathbb{R}^N$ de l'équation (1.1). Les estimations de gradient s'obtiennent en général via des techniques de type Bernstein [29]. La technique a été introduite par Bernstein (1910) et a été étendue par Serrin dans les années 60 pour étudier certaines propriétés d'EDP elliptiques quasi-linéaires et récemment généralisée par Barles au cadre des solutions de viscosité [14]. Cette technique consiste à appliquer le principe du maximum à l'inconnue $|\nabla v|^2$ où $u = f(v)$. Le choix de f doit être assez judicieux. Nous renvoyons le lecteur aux travaux [22, 110] où différents choix de la fonction f ont permis d'obtenir des estimations de gradient de natures différentes. Signalons aussi que pour les estimations locales, on doit introduire une fonction de troncature bien adaptée. Quand on applique cette méthode aux solutions de l'équation (1.1) on obtient le résultat suivant.

Théorème 1.6.1. *On suppose que $q > p-1 > 1$, $x_0 \in \mathbb{R}^N$ et $R, T > 0$. On pose $Q_{T,R} = B(x_0,R) \times (0,T)$. Soit u une solution faible dans $L^\infty((0,T); W^{1,\infty}(B(x_0,R))$ de*

$$\partial_t u - \Delta_p u = |\nabla u|^q \qquad \text{dans} \quad Q_{T,R}.$$

On suppose que $|u| \leq M$ pour une certaine constante $M \geq 1$. Alors,

$$|\nabla u| \leq C(p,N,q)\left(t^{\frac{-1}{q}} + R^{-1} + R^{\frac{-1}{q-p+1}}\right) M \quad \text{dans} \quad Q_{T,\frac{R}{2}}. \tag{1.34}$$

Le théorème 1.6.1 vient étendre les estimations obtenues dans [110] au cas de la diffusion non-linéaire ($p > 2$). Notons que dans [110] une borne supérieure locale sur u suffit pour avoir l'estimation de gradient. Signalons aussi que, comparativement à l'estimation (1.11), on n'utilise que la norme L^∞ locale de la solution mais ceci a un prix puisque la puissance sur R est moins bonne.

Comme application directe du théorème 1.6.1, on a le résultat de type Liouville suivant.

Théorème 1.6.2. *On suppose que* $q > p - 1 > 1$ *et on pose* $\sigma = \min\left(1, \frac{1}{q-p+1}\right)$*. Soit* $u \in L^\infty_{loc}((-\infty, 0); W^{1,\infty}_{loc}(\mathbb{R}^N))$ *une solution faible de*

$$u_t - \Delta_p u = |\nabla u|^q, x \in \mathbb{R}^N, -\infty < t < 0,$$

satisfaisant

$$|u(x,t)| = o(|x|^\sigma + |t|^{\frac{1}{q}}), \quad \textit{quand} \quad |x|^\sigma + |t|^{\frac{1}{q}} \to \infty. \tag{1.35}$$

Alors u *is constante.*

Ce résultat peut être vu comme une version parabolique du résultat récent de Bidaut-Véron, Véron et Huidobro [30] où une estimation du gradient et un théorème de type Liouville ont été obtenus pour l'équation elliptique associée à (1.1).

Quand $q = p$, la transformation de Hopf-Cole $v = e^{s/(p-1)} - 1$ permet de relier l'équation (1.1) à l'équation

$$|z|^{p-2} z_t = \Delta_p z.$$

Dans ce cas, pour $p < 2$, F. Wang [117] obtient une estimation de gradient similaire à (1.34) pour des solutions bornées supérieurement de l'équation (1.1) dans des variétés Riemanniennes avec une métrique décrite par un flot de Ricci. Pour des résultats de type Liouville pour le p-Laplacien sans le terme de gradient, nous renvoyons le lecteur à [49, 114]. Dans [114] une nouvelle approche basée sur des normes intrinsèques, un argument de régularité et un argument de blow-up est utilisée pour démontrer un théorème de type Liouville.

Questions ouvertes ou avec des réponses partielles

La condition de croissance (1.35) est importante comme le montre l'exemple de la fonction $u := x + t$. Cependant on ne sait pas si elle est optimale.

1.7 Publications liées à la thèse

1. Well-posedness and gradient blow-up estimate near the boundary for a Hamilton-Jacobi equation with degenerate diffusion.
   J. Differential Equations 253 (2012), no. 8, 2474–2492.
   http://dx.doi.org/10.1016/j.jde.2012.07.002
2. Boundedness of global solutions of a p-Laplacian evolution equation with a nonlinear gradient term (à paraître dans Asymptotic Analysis 2014) http://arxiv.org/abs/1209.5023
3. Global Continuation beyond Singularities on the Boundary for a Degenerate Diffusive Hamilton-Jacobi Equation (avec **Guy Barles** soumis en 2013)
   http://hal.archives-ouvertes.fr/hal-00904365
4. Single point gradient blow-up on the boundary for a Hamilton-Jacobi equation with a p-Laplacian diffusion (avec **Philippe Souplet**), soumis 2014 http://arxiv.org/abs/1404.5386
5. Gradient estimate and a Liouville theorem for a p-Laplacian equation with gradient nonlinearity (soumis 2014) http://arxiv.org/pdf/1405.5896.pdf

Chapitre 2

Théorie locale d'existence et profil spatial de l'explosion du gradient

Dans ce chapitre nous établissons une théorie locale en temps, dans la classe naturelle des données initiales lipschitziennes, avec alternative d'explosion sur le gradient, qui fournit un bon cadre pour l'étude des singularités. Nous obtenons aussi une estimation du gradient près du bord donnant le profil spatial de l'explosion. Cette estimation sera utile pour le chapitre suivant.

2.1 Introduction and main results

This chapter is concerned with the existence and qualitative properties of weak solutions of the initial boundary value problem of the p-Laplacian with a nonlinear gradient source term

$$\begin{cases} \partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^q, & x \in \Omega, t > 0, \\ u(x,t) = g(x), & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ of class $C^{2+\alpha}$ for some $\alpha > 0$, $p > 2$ and $q > p-1$. Throughout this chapter we assume that the boundary data $g \geq 0$ is the trace on $\partial\Omega$ of a regular function in $C^2(\overline{\Omega})$, also denoted g, and the initial data u_0 satisfies

$$u_0 \in W^{1,\infty}(\Omega), \qquad u_0 \geq 0, \quad u_0(x) = g(x) \qquad \text{for} \quad x \in \partial\Omega. \tag{2.1.2}$$

We note that, as far as bounded solutions are concerned, there is no loss of generality in assuming $g, u_0 \geq 0$, since the partial differential equation in (2.1.1) is unchanged when adding a constant to u.

When $p = 2$, the differential equation of (2.1.1) is the so-called viscous Hamilton-Jacobi equation and it appears in the physical theory of growth and roughening of surfaces, where it is known as the Kardar-Parisi-Zhang equation ($q = 2$), and has been studied by many authors (see for example [24, 100] and the references therein). It is known that, under certain conditions, $|\nabla u|$ blows up in a finite time $t = T_{max}$ while, by the maximum principle, all solutions are uniformly bounded (cf. [106, 67, 110]). We shall call such phenomenon gradient blow-up (GBU). This is different from the usual blow-up in which the L^∞ norm of the solution tends to infinity as $t \to T_{max}$ (cf. [100]). Sharp results on gradient blow-up analysis, including blow-up rate, blow-up set, blow-up profile and continuation after blow-up have been recently obtained, see e.g. [88, 65, 67, 100, 5, 108] and the references therein.

When $p > 2$, equation (2.1.1) is a degenerate parabolic equation for $|\nabla u| = 0$ and one cannot expect the existence of classical solutions. Weak solutions can be obtained by approximation with solutions of regularized problems. This was done in [126] when the right hand side in (2.1.1) is replaced with a general nonlinearity $f(u, \nabla u, x, t)$. In the case where f depends on ∇u, typically for problem (2.1.1), the results in [126] require the assumption $q \leq p - 1$, in which case a global solution is directly constructed for any initial data. Local-in-time existence results are also given in [126] but they require that f actually does not depend on ∇u. In [36], the existence of a global weak solution for $q > p - 1$ was proved for small data, under the assumption that the mean curvature of $\partial\Omega$ is nonpositive. In the articles [85, 18], problem (2.1.1) was studied in the framework of viscosity solutions, but only in situations where global existence of a $W^{1,\infty}$ solution is guaranteed, namely for $q \leq p$ or for suitably small initial data when $q > p$. On the other hand, when $q > p$, global existence is not expected in general for large initial data. A result in this direction was given in [85, Theorem 5.2], where it was proved that problem (2.1.1) (with $g = 0$) cannot admit a global, Lipschitz continuous, weak solution for large initial data. See [89, 50, 60] and the references therein for earlier counter-examples concerning related quasilinear equations.

Our first goal will be to complete the above results by constructing a unique, maximal in time, $W^{1,\infty}$ solution, without size restriction on the initial data and to establish the blow up alternative in $W^{1,\infty}$ norm. This will enable us to interpret the above mentioned global nonexistence result from [85] appropriately as a gradient blow-up (GBU) result (see Theorem 2.1.4 and Remark 2.4.1 below), and will provide the grounds for the subsequent analysis of the asymptotic behavior of GBU solutions. For the local existence part, we will follow and suitably modify the approximation procedure used in [126].

The main difficulty is to get relevant estimates on the first order derivatives of the approximate solutions in order to pass to the limit in the nonlinear source term. To deal with this difficulty, our main new ingredient with respect to [126] is the construction of suitable barrier functions, in order to get uniform pointwise estimates on the gradients near the boundary for small time. We then use a strong result of DiBenedetto and Friedman [48, 47] on the Hölder regularity of gradients of weak solutions of degenerate parabolic equations and consequently we will use the framework of weak rather than viscosity solutions.

First, let us state the precise definition of solution. Let $Q_T = \Omega \times (0, T)$ and $\partial_p Q_T = \{\partial\Omega \times [0, T]\} \cup \{\overline{\Omega} \times \{0\}\}$, $T > 0$. Throughout this chapter, we will use the following definition of weak solution for (2.1.1).

Definition 2.1.1. *Set $m = \max(p, q)$. A function $u(x, t)$ is called a weak super- (sub-) solution of problem (2.1.1) on Q_T if*

$$u \in C(\overline{\Omega} \times [0, T)) \cap L^m((0, T); W^{1,m}(\Omega)),$$

$$u_t \in L^2((0, T); L^2(\Omega)),$$

$$u(x, 0) \geq\ (\leq)\, u_0(x), u \geq\ (\leq)\, g \quad on \quad \partial\Omega \quad and$$

$$\int\int_{Q_T} u_t\psi + |\nabla u|^{p-2}\nabla u \cdot \nabla\psi\, dx\, dt \geq (\leq) \int\int_{Q_T} |\nabla u|^q \psi\, dx\, dt \qquad (2.1.3)$$

holds for all $\psi \in C^0(\overline{Q_T}) \cap L^p((0, T); W^{1,p}(\Omega))$ such that $\psi \geq 0$, $\psi = 0$ on $\partial\Omega \times (0, T)$. A function u is a weak solution of (2.1.1) *if it is a super-solution and a sub-solution.*

Our first result concerns local existence and uniqueness of weak solutions (see also Section 2 for a comparison principle).

Theorem 2.1.1. *Assume that $q > p - 1 > 1$. Let $M > 0$ and let u_0 satisfy* (2.1.2) *and $\|\nabla u_0\|_\infty \leq M$. Then*

(i) There exist a time $T = T\left(M, p, q, N, \|g\|_{C^2}\right) > 0$ and a weak solution u of (2.1.1) on $[0, T)$, which moreover satisfies $u \in L^\infty_{loc}([0, T); W^{1,\infty}(\Omega))$.

(ii) For any $\mathcal{T} > 0$ the problem (2.1.1) has at most one weak solution u such that $u \in L^\infty_{loc}([0, \mathcal{T}); W^{1,\infty})$.

(iii) There exists a (unique) maximal, weak solution of (2.1.1)*, still denoted by u. Let $T_{max}(u_0)$ be its existence time.*
Then

$$\min_\Omega u_0 \leq u \leq \max_\Omega u_0 \qquad in \quad \Omega \times (0, T_{max}(u_0)) \qquad (2.1.4)$$

and

$$if \quad T_{max}(u_0) < \infty, \quad then \quad \lim_{t \to T_{max}(u_0)} \|\nabla u\|_{L^\infty} = \infty \qquad (gradient\ blow\ up\ GBU).$$

Remark 2.1.1. *Concerning Definition 2.1.1, we note that if $0 < T_1 < T_2 < \infty$ and u is a weak solution on Q_{T_2}, then the restriction of u to Q_{T_1} is a weak solution on Q_{T_1} (this can be easily checked, taking any test function ψ on Q_{T_1}, by extending ψ as $\tilde{\psi}_n(x, t) = \psi(x, T_1)[1 - n(t - T_1)]_+$ for $t \in (T_1, T_2]$ and letting $n \to \infty$). Then, in Theorem 2.1.1(iii), by u being the maximal weak solution of* (2.1.1)*, we mean that u is a weak solution on Q_τ for any $\tau \in (0, T_{max}(u_0))$ but cannot be extended to a weak solution on $Q_{T'}$ for any $T' > T_{max}(u_0)$.*

In what follows, the (unique) solution constructed in Theorem 2.1.1, will be called the maximal weak solution $u \in L^\infty_{loc}\left([0, T_{max}(u_0)); W^{1,\infty}(\Omega)\right)$ or, for short, the weak Lipschitz solution of (2.1.1).

We next establish a precise gradient estimate involving the distance to the boundary. Here and in the rest of the chapter we denote $\delta(x) = \text{dist}(x, \partial\Omega)$.

Theorem 2.1.2. *Let $q > p-1 > 1$. Let $M > 0$ and let u_0 satisfy* (2.1.2) *and* $\|\nabla u_0\|_{L^\infty} \leq M$. *Let u be the unique weak solution of* (2.1.1) *in* $L^\infty_{loc}([0, T_{max}(u_0)); W^{1,\infty}(\Omega))$. *Then*

$$|\nabla u| \leq C_1 \delta^{-1/(q-p+1)}(x) + C_2. \qquad in \quad \Omega \times (0, T_{max}(u_0)). \tag{2.1.5}$$

where $C_1 = C_1(q,p,N) > 0$ *and* $C_2 = C_2(q,p,\Omega,M,\|g\|_{C^2}) > 0$.

This estimate in particular implies that $|\nabla u|$ remains bounded away from the boundary. Therefore, when $T_{max}(u_0) < \infty$, the blow-up may only take place on the boundary and (2.1.5) provides information on the blow-up profile near $\partial\Omega$. Estimate (2.1.5) is sharp in one space dimension, see [7]. Similar results are already available for $p = 2$ and have been established in [110],[5]. For $p > 2$, only global-in-space gradient estimates were available up to now (ie for $\Omega = \mathbb{R}^N$, see [22]). The proof of estimate (2.1.5) is based on similar arguments as for the case $p = 2$, namely Bernstein type arguments, but they are much more technical. Moreover, the proof of (2.1.5) also relies on a regularizing effect for solutions to (2.1.1) which seems to be new and which is stated below.

Theorem 2.1.3. *Assume that $q > p-1 > 1$ and let u be the unique weak Lipschitz solution of problem (2.1.1). Then*

$$u_t \leq \frac{1}{p-2} \frac{\|u_0\|_{L^\infty}}{t} \quad \text{in} \quad \mathcal{D}'(\Omega) \quad \text{a.e. } t > 0. \tag{2.1.6}$$

Let us note that due to the positivity of the source term, this inequality implies the semi-concavity estimate

$$\Delta_p(u) = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \leq \frac{C}{t}, \tag{2.1.7}$$

which was obtained in the case $\Omega = \mathbb{R}^N$ by a different method in [51] and for $q = p$ in [84].

Finally we give the following blow-up result, which is a variant of a global nonexistence result in [85], reinterpreted in terms of GBU in the light of Theorem 2.1.1. Let φ_1 be the first eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary conditions

Theorem 2.1.4. *Assume that $q > p > 2$ and let u be the unique weak Lipschitz solution of (2.1.1). Let $\alpha \geq 1$ satisfy $\dfrac{p-1}{q-p+1} < \alpha < q-1$, then there exists a constant $C = C(q,p,\alpha,\Omega,\|g\|_\infty) > 0$ such that if $\int_\Omega u_0 \varphi_1^\alpha dx \geq C$, then $T_{max}(u_0) < \infty$, i.e. gradient blow-up occurs.*

For results concerning other aspects of equation (2.1.1) and the corresponding Cauchy problem, see e.g. [39, 104, 36, 126, 22] and the references therein. Asymptotic behavior of global solution is investigated in [111, 18, 85, 84, 86, 1, 2] and references therein.

The rest of the chapter is organized as follows : In Section 2.2 we prove the well-posedness of (2.1.1) in $W^{1,\infty}(\Omega)$, as well as the regularizing effect. Section 2.4 is devoted to the proof of Theorem 2.1.2. Finally in section 4 we prove the sufficient blow-up criterion of Theorem 2.1.4.

2.2 Proof of Theorem 2.1.1 and Theorem 2.1.3

2.2.1 Local existence

Consider the following approximate problems for (2.1.1) :

$$\begin{cases} \partial_t u_n - \operatorname{div}\left(\left(|\nabla u_n|^2 + \dfrac{1}{n}\right)^{(p-2)/2} \nabla u_n\right) = \left(|\nabla u_n|^2 + \dfrac{1}{n}\right)^{q/2} - \dfrac{1}{n^{q/2}}, & x \in \Omega, t > 0, \\ u_n(x,t) = g(x), & x \in \partial\Omega, t > 0, \\ u_n(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2.2.1}$$

For each fixed $n \in \mathbb{N}$, problem (2.2.1) is no longer degenerate and the regularity theory of quasilinear parabolic equations [80] provides local-in-time solutions u_n, which are smooth for $t > 0$ and continuous up to $t = 0$.

To find the limit function $u(x,t)$ of the sequence $\{u_n(x,t)\}$, we divide our proof into 5 steps. Recall that there exists $\eta_0 > 0$ small such that, for any $x \in \overline{\Omega}$ with $\delta(x) \le \eta_0$, the point $\widetilde{x} := \mathrm{proj}_{\partial\Omega}(x)$ (the projection of x onto the boundary) is well defined and unique.

STEP 1. There exist a small time $T_0 > 0$, $\eta \in (0, \eta_0)$ and $M_2 > 0$, all independent of n and depending on u_0 through M only, such that

$$\|u_n\|_{L^\infty(Q_{T_0})} \le M_1 := \max\left(\|u_0\|_{L^\infty}, \|g\|_{L^\infty}\right), \tag{2.2.2}$$

and

$$\sup_{\substack{x\in\Omega \\ \delta(x)\le\eta}} \frac{|u_n(x,t) - u_n(\widetilde{x},t)|}{\delta(x)} \le M_2, \qquad 0 < t \le T_0. \tag{2.2.3}$$

Estimate (2.2.2) is a direct consequence of the maximum principle since M_1 is a super solution for any n.

In order to prove estimate (2.2.3), we are going to construct a local barrier function under the exterior sphere condition satisfied by the domain Ω, i.e. for any x near $\partial\Omega$, a supersolution in a neighborhood of x.

Let $\rho > 0$ be such that for all $x \in \partial\Omega$, $\overline{B_\rho(x + \rho\nu_x)} \cap \overline{\Omega} = \{x\}$, where ν_x is the unit outward normal vector on $\partial\Omega$ at x. Fix an arbitrary $x_0 \in \Omega$ such that $\delta(x_0) \le \eta$ where $\eta \in (0, \eta_0)$ will be chosen later. Define $x_1 = \widetilde{x_0} + \rho\nu_{\widetilde{x_0}}$. Without loss of generality we may assume that $x_1 = 0$ and we write $r = |x|$. Let us denote, for $s \ge 0$,

$$a(s) = \left(s + \frac{1}{n}\right)^{(p-2)/2}, \quad \text{and} \qquad \kappa = \frac{2a'(s)s}{a(s)} \in [0, p-2]. \tag{2.2.4}$$

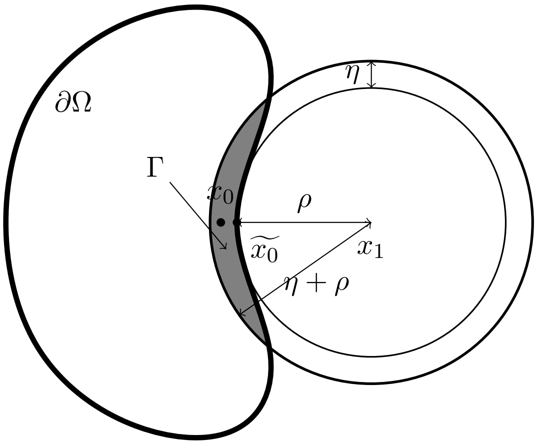

FIGURE 2.1 – Local barrier function

We recall that for a function $\phi(x) = \phi(|x|)$, we have :

$$\begin{aligned}
\nabla\phi(x) &= \phi'(r)\frac{x}{r}, \\
D^2\phi(x) &= \phi''(r)\frac{x\otimes x}{r^2} + \frac{\phi'(r)\mathrm{Id}}{r} - \phi'(r)\frac{x\otimes x}{r^3}, \\
\Delta\phi(x) &= \phi''(r) + \frac{(N-1)\phi'(r)}{r},
\end{aligned} \tag{2.2.5}$$

where Id is the unit matrix and $(x\otimes x)_{ij} = x_i x_j$. The barrier function will have the form

$$\bar{v}(x,t) = \phi(r-\rho) + g(x),$$

where ϕ is a smooth function of one variable which is increasing and **concave**. First let us write

$$\begin{aligned}
\mathrm{div}\left(\left(|\nabla\bar{v}|^2 + \frac{1}{n}\right)^{(p-2)/2}\nabla\bar{v}\right) &= a(|\nabla\bar{v}|^2)\Delta\bar{v} + 2a'(|\nabla\bar{v}|^2)(\nabla\bar{v})^t D^2\bar{v}\nabla\bar{v}, \\
&= a(|\nabla\bar{v}|^2)\left(\Delta\bar{v} + \kappa(|\nabla\bar{v}|^2)\,\frac{(\nabla\bar{v})^t D^2\bar{v}\nabla\bar{v}}{|\nabla\bar{v}|^2}\right).
\end{aligned} \tag{2.2.6}$$

Using (2.2.5), we have

$$\begin{aligned}\left[\Delta\bar{v}+\kappa(|\nabla\bar{v}|^2)\,\frac{(\nabla\bar{v})^tD^2\bar{v}\nabla\bar{v}}{|\nabla\bar{v}|^2}\right] &= \phi''(r-\rho)+\frac{(N-1)\phi'(r-\rho)}{r}+\Delta g\\ &+\kappa(|\nabla\bar{v}|^2)\,\frac{\phi''(r-\rho)(\nabla\bar{v}\cdot x)^2}{r^2|\nabla\bar{v}|^2}+\kappa(|\nabla\bar{v}|^2)\,\frac{\phi'(r-\rho)}{r}\\ &-\kappa(|\nabla\bar{v}|^2)\,\frac{\phi'(r-\rho)(\nabla\bar{v}\cdot x)^2}{r^3|\nabla\bar{v}|^2}+\kappa(|\nabla\bar{v}|^2)\,\frac{(\nabla\bar{v})^tD^2g\nabla\bar{v}}{|\nabla\bar{v}|^2}.\end{aligned}$$

Since $\phi'(r-\rho)\geq 0$, $r\geq\rho$, $\kappa(|\nabla\bar{v}|^2)\geq 0$ and $0\geq\phi''(r-\rho)$, we have

$$\begin{aligned}-\left[\Delta\bar{v}+\kappa(|\nabla\bar{v}|^2)\,\frac{(\nabla\bar{v})^tD^2\bar{v}\nabla\bar{v}}{|\nabla\bar{v}|^2}\right] &\geq -\phi''(r-\rho)-\frac{(N-1+\kappa(|\nabla\bar{v}|^2))}{\rho}\phi'(r-\rho)\\ &-\|\Delta g\|_\infty-\kappa(|\nabla\bar{v}|^2)\,\left\|D^2g\right\|_{L^\infty}. \end{aligned}\tag{2.2.7}$$

On the other hand $|\nabla\bar{v}|=\left|\phi'(r-\rho)\dfrac{x}{r}+\nabla g\right|\leq\phi'(r-\rho)+|\nabla g|\leq 2\phi'(r-\rho)$ provided that

$$\phi'(r-\rho)\geq\|\nabla g\|_{L^\infty}. \tag{2.2.8}$$

In this case we have

$$\left(|\nabla\bar{v}|^2+\frac{1}{n}\right)^{(q-p+2)/2}\leq\left[4(\phi'(r-\rho))^2+1\right]^{(q-p+2)/2}. \tag{2.2.9}$$

We take

$$\phi(s)=s(s+\mu)^{-\beta},\qquad s\geq 0,$$

where $\beta=\beta(q,p)\in(0,1)$ is to be chosen later. We denote $\Gamma:=B(x_1,\rho+\eta)\cap\Omega$ (see figure 2.1). Our aim is to show that $\bar{v}$ is a super-solution in $\Gamma\times(0,T_0)$ where $T_0,\mu>0$ and $\eta\in(0,\eta_0)$ small enough. In the rest of the proof, the constants T_0,η,δ and C will be independent of x_0, n and will depend on the initial data u_0 through M only (and they will depend on the other data p,q,N,Ω and $\|g\|_{C^2}$ without other mention). We calculate

$$\begin{aligned}\phi'(s)&=\left[(1-\beta)s+\mu\right](s+\mu)^{-\beta-1},\\ \phi''(s)&=-\beta\left[(1-\beta)s+2\mu\right](s+\mu)^{-\beta-2}.\end{aligned}$$

We are looking for condition on β and μ such that

$$-\operatorname{div}\left(\left(|\nabla\bar{v}|^2+\frac{1}{n}\right)\nabla\bar{v}\right)\geq\left(|\nabla\bar{v}|^2+\frac{1}{n}\right)^{q/2}-\left(\frac{1}{n}\right)^{q/2}. \tag{2.2.10}$$

Due to (2.2.6), it suffices to have

$$-\left[\Delta\bar{v}+\kappa(|\nabla\bar{v}|^2)\,\frac{(\nabla\bar{v})^tD^2\bar{v}\nabla\bar{v}}{|\nabla\bar{v}|^2}\right]\geq\left(|\nabla\bar{v}|^2+\frac{1}{n}\right)^{\frac{q-p+2}{2}}, \tag{2.2.11}$$

which, by (2.2.7)-(2.2.4)-(2.2.9) reduces to

$$-\phi''(r-\rho)+\frac{(3-N-p)}{\rho}\phi'(r-\rho)\geq\left[4(\phi'(r-\rho))^2+1\right]^{(q-p+2)/2}+(p-2+\sqrt{N})\left\|D^2g\right\|_{L^\infty}. \tag{2.2.12}$$

Using that $\rho < r+\eta+\rho$ and $(3-N-p)<0$, then (2.2.10) holds if

$$(r-\rho+\mu)^{-\beta-2}\left[2\beta\mu+(3-N-p)\frac{(\eta+\mu)^2}{\rho}\right]\geq\left[4(r-\rho+\mu)^{-2\beta}+1\right]^{(q-p+2)/2} + (p-2+\sqrt{N})\left\|D^2g\right\|_{L^\infty}.$$

Assume that η and μ are such that

$$\begin{cases} 4(r-\rho+\mu)^{-2\beta}\geq 4(\eta+\mu)^{-2\beta}\geq 1, \\ 2\beta\mu+\dfrac{(3-N-p)}{\rho}(\eta+\mu)^2\geq\beta\mu, \end{cases} \tag{2.2.13}$$

then to get (2.2.10) it is sufficient to have

$$\beta\mu(r-\rho+\mu)^{-\beta-2}\geq(r-\rho+\mu)^{-\beta(q-p+2)}4^{(q-p+3)}, \tag{2.2.14}$$

and

$$\beta\delta(r-\rho+\mu)^{-\beta-2}\geq 4(p-2+\sqrt{N})\left\|D^2g\right\|_{L^\infty}. \tag{2.2.15}$$

Inequality (2.2.14) holds if we choose $\eta=\mu$, $\beta=\frac{1}{2(q-p+2)}$, and μ satisfying

$$4^{p-q-4}\beta\geq\mu^{(q-p+3)/(2q-2p+4)}.$$

Inequalities (2.2.14)-(2.2.15) and (2.2.8) hold if we choose μ small enough. We have thus shown that if $\eta=\mu$ is small, then $\bar{v}$ is a supersolution on $\Gamma\times(0,T_0)$ for any $T_0>0$.

Now we need to have a control on the parabolic boundary of $\Gamma\times(0,T_0)$ for $T_0>0$ small. For this purpose, we introduce another comparison function

$$\bar{u}(x,t)=(2C^2K^2+2\left\|\nabla g\right\|^2_{L^\infty}+1)^{q/2}t+C(1-e^{-K(r-\rho)})+\left\|g\right\|_{L^\infty}.$$

It is easy to see that if we fix K sufficiently large $\left(K>\dfrac{N+p-3}{\rho}\right)$, then we can find $C=C(p,N,M,\Omega,\left\|g\right\|_{C^2})>0$ sufficiently large so that

$$-\mathrm{div}\left(\left(|\nabla\bar{u}|^2+\frac{1}{n}\right)^{(p-2)/2}\nabla\bar{u}\right)\geq 0\quad\text{in}\quad\Omega.$$

Indeed, since Ω is bounded there exists $R(\Omega)>0$ such that $\Omega\subset B(x_1,R(\Omega))$ and hence $r-\rho\leq R(\Omega)$. Now once $\left(K>\frac{2(N+p-3)}{\rho}\right)$ is fixed using (2.2.7) it is sufficient to require that

$$CKe^{-K(r-\rho)}\left[K-\frac{N+p-3}{\rho}\right]\geq(p-2+\sqrt{N})\left\|D^2g\right\|_{L^\infty},$$

which is satisfied if

$$C \geq \frac{2e^{KR(\Omega)}(p-2+\sqrt{N})\left\|D^2 g\right\|_{L^\infty}}{K^2}.$$

Thus

$$\partial_t \bar{u} - \operatorname{div}\left(\left(|\nabla \bar{u}|^2 + \frac{1}{n}\right)^{(p-2)/2} \nabla \bar{u}\right) \geq \left(|\nabla \bar{u}|^2 + \frac{1}{n}\right)^{q/2} - \left(\frac{1}{n}\right)^{q/2}.$$

Next we can also choose $C > 0$ large enough such that $C(1-e^{-K(r-\rho)}) + \|g\|_{L^\infty} \geq u_0(x)$ in Ω. Since $\bar{u} \geq g$ on $\partial\Omega \subset \left\{x \in \mathbb{R}^N, \quad |x| \geq \rho\right\}$, by the maximum principle we get that for any n, $u_n \leq \bar{u}$ in Q_T. Thus

$$\begin{aligned} u_n(x,t) &\leq (2C^2K^2 + 2\|\nabla g\|_{L^\infty}^2 + 1)^{q/2} t + C(1-e^{-K\eta}) + \|g\|_\infty \\ &\leq 2^{-\beta}\eta^{1-\beta} + g(x) = \bar{v}(x,t) \end{aligned}$$

on $\{x \in \Omega, |x| = \rho + \eta\} \times [0, T_0]$, provided T_0 and $\eta = \mu$ are small enough (depending only on $M, p, q, \Omega, \|g\|_{C^2}$). Next we can choose $\eta = \mu$ small enough such that

$$\begin{aligned} u_0(x) &\leq g(\tilde{x}) + M|x - \tilde{x}| \leq g(\tilde{x}) + M|r-\rho| \\ &\leq g(\tilde{x}) + (r-\rho)\left[(2\eta)^{-\beta} - \|\nabla g\|_{L^\infty}\right] \leq \bar{v}(x,0). \end{aligned}$$

On the other hand $u_n = g \leq \bar{v}$ on $\partial\Omega \times [0, T_0]$. We conclude that $\bar{v}$ is a super solution on $\Gamma \times (0, T_0)$. Similarly $\underline{v} := g - \phi(r-\rho)$ is a sub-solution. Applying the maximum principle we get $\underline{v} \leq u_n \leq \bar{v}$ on $\Gamma \times [0, T_0]$, and hence in particular

$$\frac{|u_n(x_0,t) - u_n(\tilde{x_0},t)|}{|x_0 - \tilde{x_0}|} \leq \sup_{0 \leq s \leq \eta} |\phi'(s)| + \|\nabla g\|_\infty \leq \eta^{-\beta} + \|\nabla g\|_{L^\infty} =: M_2, \quad 0 < t \leq T_0,$$

which yields (2.2.3).

STEP 2. There holds

$$\|\nabla u_n\|_{L^\infty(Q_{T_0})} \leq M_3 := \sup(M, M_2 + \|\nabla g\|_{L^\infty}). \tag{2.2.16}$$

We use a similar argument as in [75, Theorem 5]. Let $h \in \mathbb{R}^N$ satisfy $|h| \leq \eta$. Due to the translation invariance of (2.2.1), if u_n is a classical solution of (2.2.1) in Ω, then the function $u_n^h := u_n(x-h,t)$ is a classical solution of (2.2.1) in $\Omega_h \times (0, T_0)$ where $\Omega_h := \left\{x \in \mathbb{R}^N \mid \ x - h \in \Omega\right\}$. Let $t \in [0, T_0]$ and $x \in \partial(\Omega \cap \Omega_h)$. We may assume for instance $x \in \partial\Omega$, the case $x + h \in \partial\Omega$ being similar. Then using $|\widetilde{y} - \widetilde{z}| \leq |y - z|$ and (2.2.3), we get

$$\begin{aligned} |u_n(x,t) - u_n(x+h,t)| &= |u_n(\widetilde{x},t) - u_n(\widetilde{x+h},t) + u_n(\widetilde{x+h},t) - u_n(x+h,t)| \\ &\leq \|\nabla g\|_\infty |\widetilde{x} - \widetilde{x+h}| + M_2\delta(x+h) \\ &\leq (\|\nabla g\|_\infty + M_2)|h| = M_3|h|. \end{aligned}$$

In particular $u_n(x,t) \le u_n^h(x,t) + M_3|h|$ on $\partial(\Omega\cap\Omega_h)\times[0,T_0]$. Applying the maximum principle, we have $u_n(x,t) \le u_n^h(x,t) + M_3|h|$ on $(\Omega\cap\Omega_h)\times[0,T_0]$. By the same argument $u_n^h(x,t) - M_3|h| \le u_n(x,t)$ on $(\Omega\cap\Omega_h)\times[0,T_0]$, hence $|u_n(x,t) - u_n^h(x,t)| \le M_3|h|$. Since $|h|\le\eta$ is arbitrary, the conclusion follows.

STEP 3. Let $\epsilon > 0$ and set $Q_{T_0,\epsilon} = \{x\in\Omega, \delta(x) > \epsilon\}\times(\epsilon, T_0-\epsilon)$. There exists a constant $M_4 > 0$ independent of n, such that

$$|\nabla u_n(x_1,t_1) - \nabla u_n(x_2,t_2)| \le M_4\left(|x_1-x_2|^\alpha + |t_1-t_2|^{\frac{\alpha}{2}}\right) \tag{2.2.17}$$

for any pair of points $(x_i,t_i)\in Q_{T_0,\epsilon}$, where M_4 and α are positive constants depending only on T_0, M_3 and ϵ. Indeed we know from a result of DiBenedetto and Friedman [48, 47] that if $f\in L^r(\Omega_T)$ for some $r > \dfrac{pN}{p-1}$ then weak solutions of degenerate parabolic equation of the form

$$\partial_t v - \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) = f(x,t) \tag{2.2.18}$$

are of class $C^{1,\alpha}_{loc}(Q_T)$ with Hölder norm depending only on $\|f\|_{L^r}$, $\|\nabla u\|_{L^p}$ and $\|u\|_{L^\infty_t, L^2_x}$.

STEP 4. There exists a constant $M_5 > 0$ independent of n, such that

$$\|\partial_t u_n\|_{L^2(Q_{T_0})} \le M_5. \tag{2.2.19}$$

To see this , multiplying (2.2.1) by $\partial_t u_n$ and integrating over Q_{T_0}, we have

$$\begin{aligned}\int_0^{T_0}\int_\Omega(\partial_t u_n)^2dxdt \le &-\int_0^{T_0}\int_\Omega\left(|\nabla u_n|^2+\frac{1}{n}\right)^{(p-2)/2}\nabla u_n\cdot\nabla(\partial_t u_n)dxdt\\ &+\int_0^{T_0}\int_\Omega\left(|\nabla u_n|^2+\frac{1}{n}\right)^{q/2}\partial_t u_n\,dxdt.\end{aligned}$$

By Hölder's inequality and

$$\begin{aligned}&\int_0^{T_0}\int_\Omega\left(|\nabla u_n|^2+\frac{1}{n}\right)^{(p-2)/2}\nabla u_n\cdot\nabla(\partial_t u_n)dxdt\\ &=\frac{1}{p}\int_\Omega\left(|\nabla u_n(x,T_0)|^2+\frac{1}{n}\right)^{p/2}-\frac{1}{p}\int_\Omega\left(|\nabla u_n(x,0)|^2+\frac{1}{n}\right)^{p/2},\end{aligned}$$

we get

$$\begin{aligned}\int_0^{T_0}\int_\Omega(\partial_t u_n)^2dxdt &\le \frac{2}{p}\int_\Omega\left(|\nabla u_n(x,0)|^2+\frac{1}{n}\right)^{p/2}dx + 2\int_0^{T_0}\int_\Omega\left(|\nabla u_n|^2+\frac{1}{n}\right)^{q}dxdt\\ &\le M'.\end{aligned}$$

for some $M' = M'\big(|\Omega|, M_3, T_0, p, q\big) > 0$.

STEP 5. We recall that by the Arzelá-Ascoli theorem we have

$$W^{1,\infty}(\Omega) \overset{c}{\hookrightarrow} C(\overline{\Omega}) \hookrightarrow L^2(\Omega). \tag{2.2.20}$$

Using (2.2.2)-(2.2.16)-(2.2.19)-(2.2.20) and the compactness theorem in [[105] Corollary 4], we have that $\{u_n\}$ is relatively compact in $C\left([0,T_0];C(\overline{\Omega})\right) = C\left(\overline{\Omega}\times[0,T_0]\right)$. By virtue of (2.2.16)-(2.2.17)-(2.2.19), the Ascoli-Arzelà theorem and the relative compactness of $\{u_n\}$ in $C\left(\overline{\Omega}\times[0,T_0]\right)$, we can find a subsequence, still denoted by $\{u_n\}$ for convenience, such that, for each $\epsilon > 0$,

$$\left.\begin{array}{lll} u_n \to u & \text{in} & C\left(\overline{\Omega}\times[0,T_0]\right), \\ \nabla u_n \to \nabla u & \text{in} & C(Q_{T_0,\epsilon}), \\ \partial_t u_n \to \partial_t u & \text{weakly in} & L^2(Q_{T_0}). \end{array}\right\} \tag{2.2.21}$$

We multiply (2.2.1) by a test function and integrate. Then by the Lebesgue's dominated convergence theorem and (2.2.21) we can pass to the limit and check that u is a weak solution of (2.1.1).

2.2.2 The blow-up alternative

Let us temporarily assume the uniqueness result which will be proved in the next section. The construction of the weak solution as a limit of classical solutions implies the blow-up alternative.

Indeed suppose that the maximal existence time $T_{max}(u_0) < \infty$ and that there exist $\mathcal{M} > 0$ and $t_k \to T_{max}(u_0)$ such that for all k

$$\|\nabla u(t_k)\|_{L^\infty(\Omega)} \leq \mathcal{M}. \tag{2.2.22}$$

Then we can find $\tau = \tau(\mathcal{M}) > 0$ independent of k, such that the problem

$$\begin{cases} \partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^q, & x\in\Omega, t>0, \\ u(x,t) = g(x), & x\in\partial\Omega, t>0, \\ u(x,0) = u(x,t_k), & x\in\Omega, \end{cases} \tag{2.2.23}$$

admits a unique weak solution v_k on $[0,\tau)$. Setting $\tilde{u}(t) = \begin{cases} u(t) & \text{for } t\in[0,t_k) \\ v_k(t-t_k) & \text{for } t\in[t_k,t_k+\tau) \end{cases}$, it is easy to see that we get a weak solution defined on $[0,t_k+\tau)$.

Since for k large enough $t_k+\tau > T_{max}(u_0)$, this contradicts the definition of $T_{max}(u_0)$. Hence $T_{max}(u_0) < \infty \Rightarrow \lim\limits_{t\to T_{max}(u_0)} \|\nabla u\|_{L^\infty(\Omega)} = \infty$.

2.2.3 Uniqueness

In this section we prove the uniqueness of the weak solution. This result will be a consequence of the following comparison principle which, in turns, also guarantees (2.1.4).

Proposition 2.2.1. *Let u, v be respectively, sub-, super-solutions of* (2.1.1)*. Assume that $u, v \in L^\infty\left((0,T); W^{1,\infty}(\Omega)\right)$. Then $u \leq v$ on $\Omega \times (0,T)$.*

The proof of Proposition 2.2.1 is mostly based on the following algebraic lemma from which we can show that the source term can be counterbalanced by the diffusion effect (c.f [32] and [91] for usefull inequalities on the p-Laplacian).

Lemma 2.2.1 (Monotonicity Property)**.** *Let $\sigma > 1$. For all a and $b \in \mathbb{R}^N$:*

$$\left\langle |a|^{\sigma-2}a - |b|^{\sigma-2}b, a-b \right\rangle \geq \frac{4}{\sigma^2}\left| |a|^{(\sigma-2)/2}a - |b|^{(\sigma-2)/2}b \right|^2.$$

Proof of Proposition 2.2.1. We set $w = (u-v)^+$. By definition we have $w = 0$ on $\partial\Omega$. By Remark 2.1.1, for any $\tau \in (0,T)$, using $\psi = w$ as test-function, we have

$$\int_0^\tau \int_\Omega w w_t \, dxdt \leq$$
$$\underbrace{\int_0^\tau \int_{\{w(\cdot,t)>0\}} \left[|\nabla u|^q - |\nabla v|^q \right] w \, dxdt}_{\mathcal{B}} - \underbrace{\int_0^\tau \int_{\{w(\cdot,t)>0\}} \left[|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v \right] \cdot \nabla w \, dxdt}_{\mathcal{H}}.$$

We set $a = \nabla u$ and $b = \nabla v$. We get by lemma 2.2.1

$$\mathcal{H} \geq c(p) \int_0^\tau \int_{\{w(\cdot,t)>0\}} \left| |\nabla u|^{(p-2)/2}\nabla u - |\nabla v|^{(p-2)/2}\nabla v \right|^2 dxdt. \tag{2.2.24}$$

Let's consider the term $\mathcal{B}$. We put $h(s) = s^{\frac{2q}{p}}$ for $s \geq 0$. Given that $q \geq p-1 \geq \dfrac{p}{2}$, we have $h'(s) = \frac{2q}{p} s^{\frac{2q-p}{p}}$. The mean value theorem yields

$$\left| |\nabla u|^q - |\nabla v|^q \right|^2 \leq C h'(\theta)^2 \left| |\nabla u|^{(p-2)/2}\nabla u - |\nabla v|^{(p-2)/2}\nabla v \right|^2,$$

for some $0 \leq \theta \leq \max(|\nabla u|^{\frac{p}{2}}, |\nabla v|^{\frac{p}{2}})$.
Since we assumed $u, v \in L^\infty\left((0,T); W^{1,\infty}(\Omega)\right)$, it follows that

$$\left| \, |\nabla u|^q - |\nabla v|^q \, \right|^2 \leq C \left| |\nabla u|^{(p-2)/2}\nabla u - |\nabla v|^{(p-2)/2}\nabla v \right|^2.$$

On the other hand, the Young inequality implies

$$\mathcal{B} \leq \epsilon \int_0^\tau \int_{\{w(\cdot,t)>0\}} |\, |\nabla u|^q - |\nabla v|^q \,|^2 \, dxdt + C(\epsilon) \int_0^\tau \int_{\{w(\cdot,t)>0\}} w^2 dxdt.$$

Combining these two inequalities, we arrive at

$$\mathcal{B} \leq C\epsilon \int_0^\tau \int_{\{w(\cdot,t)>0\}} \left| |\nabla u|^{(p-2)/2}\nabla u - |\nabla v|^{(p-2)/2}\nabla v \right|^2 dxdt + C(\epsilon) \int_0^\tau \int_{\{w(\cdot,t)>0\}} w^2 dxdt. \tag{2.2.25}$$

Choosing ϵ small enough, we get

$$\int_\Omega w^2(\tau)\,dx \le \int_\Omega w^2(0)\,dx + C(\epsilon)\int_0^\tau\int_\Omega w^2\,dxdt, \quad 0<\tau<T. \tag{2.2.26}$$

The Gronwall lemma implies that for any $t\in(0,T)$

$$\int_\Omega w^2(x,t)\;dx \le e^{Ct}\int_\Omega w(x,0)^2\;dx.$$

We conclude that $w\equiv 0$ almost everywhere.

Remark 2.2.1. *(a) The inequality in lemma 2.2.1 for $\sigma\in(1,2)$ can be deduced from the inequality for $\sigma\ge 2$ in [91] as follows :*

We set $a=|\nabla u|^{\sigma-2}\nabla u$ and $b=|\nabla v|^{\sigma-2}\nabla v$.

$$\begin{aligned}\langle|\nabla u|^{\sigma-2}\nabla u-|\nabla v|^{\sigma-2}\nabla v,\ \nabla u-\nabla v\rangle &= \left\langle a-b, a\,|a|^{\frac{2-\sigma}{\sigma-1}}-b\,|b|^{\frac{2-\sigma}{\sigma-1}}\right\rangle \\ &= \langle a-b,\ a\,|a|^{m-2}-b\,|b|^{m-2}\rangle.\end{aligned} \tag{2.2.27}$$

where $m=\frac{\sigma}{\sigma-1}>2$.

(b) The question of uniqueness was partially open in [111]. The preceding proof of Poposition 2.2.1 can be applied to show uniqueness in the case $p-1\ge q\ge\dfrac{p}{2}$ with $p\ge 2$.

(c) In [4] we have a weaker inequality for $p\in(1,2)$ but it is sufficient to prove uniqueness for the case $q>1$:

$$\left\langle|a|^{p-2}a-|b|^{p-2}b,a-b\right\rangle \ge (p-1)|a-b|^2\left(|a|^p+|b|^p\right)^{\frac{p-2}{p}}.$$

2.2.4 Regularizing effect

We use a technique developed by Zhao for the the p-Laplace equation without source term [127]. The idea is to apply a Stampacchia maximum principle argument to the equation satisfied by $\lambda^\gamma u(x,\lambda t)-u(x,t)$ and then let $\lambda\to 1^+$. Let u be a weak solution of (2.1.1) in $L^\infty_{loc}([0,T);W^{1,\infty}(\Omega))$. Set

$$u_\lambda(x,t)=\lambda^\gamma u(x,\lambda t), \qquad \lambda>1,\gamma=\frac{1}{p-2}.$$

Then u_λ is a weak solution of

$$\begin{cases}\partial_t u_\lambda-\operatorname{div}(|\nabla u_\lambda|^{p-2}\nabla u_\lambda)=\lambda^{-(q-p+1)\gamma}|\nabla u_\lambda|^q, & x\in\Omega, t\in\left(0,\frac{T}{\lambda}\right),\\ u_\lambda(x,t)=\lambda^\gamma g(x), & x\in\partial\Omega, t\in\left(0,\frac{T}{\lambda}\right),\\ u_\lambda(x,0)=\lambda^\gamma u_0(x), & x\in\Omega.\end{cases}$$

We set $v(x,t)=u(x,t)+k$ where $k:=(\lambda^\gamma-1)\left\|u_0\right\|_{L^\infty}$, then v satisfy the same equation as u with $v(x,0)=u_0(x)+k$ and $v(x,t)=g(x)+k$ on $\partial\Omega\times(0,T)$. Given that $\lambda^\gamma u_0(x)=$

$u_0(x) + (\lambda^\gamma - 1)u_0(x) \leq u_0(x) + (\lambda^\gamma - 1)\left\|u_0\right\|_{L^\infty}$ and $\lambda^\gamma g(x) \leq g(x) + (\lambda^\gamma - 1)\left\|g\right\|_{L^\infty}$, we have $u_\lambda(x,0) \leq v(x,0)$ in Ω and $u_\lambda \leq v$ on $\partial\Omega \times \left(0, \frac{T}{\lambda}\right)$. Since $\lambda > 1$ and $q > p-1$, we have $\lambda^{-(q-p+1)\gamma}|\nabla u_\lambda|^q \leq |\nabla u|^q$ and hence u_λ is a sub-solution of the equation. Using proposition 2.2.1, we have $u_\lambda(x,t) \leq v(x,t)$ in $\Omega \times \left(0, \frac{T}{\lambda}\right)$ that is

$$\lambda^\gamma u(\lambda t, x) - u(x,t) \leq (\lambda^\gamma - 1)\sup(\left\|u_0\right\|_{L^\infty}, \left\|u_0\right\|_{L^\infty}). \tag{2.2.28}$$

Dividing (2.2.28) by $(\lambda - 1)$ and letting $\lambda \to 1^+$, we get

$$\gamma u(x,t) + t u_t(x,t) \leq \gamma \left\|u_0\right\|_{L^\infty}.$$

We conclude using the positivity of u.

Remark 2.2.2. *The homogeneity of the operator and the boundedness of u are essential.*

2.3 Gradient estimate : proof of Theorem 2.1.2

The proof of (2.1.5) relies on a modification of the Bernstein technique and the use of a suitable cut-off function. It requires the study of the partial differential equation satisfied by $|\nabla u|^2$. We follow the ideas used in [110] and [22]. Let $x_0 \in \Omega$ be fixed, $0 < t_0 < T < T_{max}(u_0)$, $R > 0$ such that $B(x_0, R) \subset \Omega$ and write $Q_{T,R}^{t_0} = B(x_0,R) \times (t_0, T)$

Let $\alpha \in (0,1)$ and set $R' = \frac{3R}{4}$. We select a cut-off function $\eta \in C^2(\overline{B}(x_0, R'))$, $0 < \eta < 1$, with $\eta(x_0) = 1$ and $\eta = 0$ for $|x - x_0| = R'$, such that

$$\left.\begin{array}{r} |\nabla\eta| \leq CR^{-1}\eta^\alpha \\ |D^2\eta| + \eta^{-1}|\nabla\eta|^2 \leq CR^{-2}\eta^\alpha \end{array}\right\} \quad \text{for} \quad |x - x_0| < R' \tag{2.3.1}$$

with $C = C(\alpha) > 0$ (see [110] for an example of such function).

First let us state the following lemma.

Lemma 2.3.1. *Let u_0, u be as in Theorem 2.1.2. We denote $w = |\nabla u|^2$ and $z = \eta w$. Then at any point $(x_1, t_1) \in Q_{T,R'}^{t_0}$ such that $|\nabla u(x_1,t_1)| > 0$, z is smooth and satisfies the following differential inequality*

$$\mathcal{L}z + Cz^{\frac{2q-p+2}{2}} \leq C\left(\frac{\sup(\left\|u_0\right\|_{L^\infty}, \left\|g\right\|_{L^\infty})}{t_0}\right)^{\frac{2q-p+2}{q}} + CR^{-\frac{2q-p+2}{q-p+1}},$$

where

$$\mathcal{L}z = \partial_t z - \mathcal{A}z - H \cdot \nabla z, \tag{2.3.2}$$

$$\mathcal{A}z = |\nabla u|^{p-2}\Delta z + (p-2)|\nabla u|^{p-4}(\nabla u)^t D^2 z \nabla u, \tag{2.3.3}$$

H is defined by (2.3.7) and $C = C(p,q,N) > 0$.

Proof of lemma 2.3.1 We know that a solution u of (2.1.1) is smooth at points where $|\nabla u| > 0$ [22]. More precisely, we know that $\nabla u \in C^{2,1}$ in a neighborhood of such points and hence we can differentiate the equation. As observed in [22], $w = |\nabla u|^2$ satisfies the following differential equation :

$$\partial_t w - \mathcal{A}w = -2|\nabla u|^{p-2}|D^2u|^2 + H \cdot \nabla w$$

Indeed, for $i = 1, \cdots, N$, put $u_i = \dfrac{\partial u}{\partial x_i}$ and $w_i = \dfrac{\partial w}{\partial x_i}$. Differentiating (2.1.1) in x_i, we have

$$\begin{aligned} &\partial_t u_i - |\nabla u|^{p-2}\Delta u_i - \frac{p-2}{2}|\nabla u|^{p-4}\sum_{j=1}^{N}\frac{\partial w_i}{\partial x_j}u_j - \frac{p-2}{2}|\nabla u|^{p-4}\sum_{j=1}^{N} w_j \frac{\partial u_i}{\partial x_j} \\ &\quad = \frac{q}{2}w^{\frac{q-2}{2}}w_i + \frac{p-2}{2}w^{\frac{p-4}{2}}w_i\Delta u + \frac{(p-2)(p-4)}{4}w^{\frac{p-6}{2}}(\nabla u\cdot\nabla w)w_i. \end{aligned} \tag{2.3.4}$$

Multiplying (2.3.4) by $2u_i$, summing up, and using $\Delta w = 2\nabla u \cdot \nabla(\Delta u) + 2|D^2u|^2$, we deduce that

$$\mathcal{L}w = -2w^{\frac{p-2}{2}}|D^2u|^2, \tag{2.3.5}$$

where

$$\mathcal{L}w := \partial_t w - |\nabla u|^{p-2}\Delta w - (p-2)|\nabla u|^{p-4}(\nabla u)^t D^2 w \nabla u - H\cdot\nabla w, \tag{2.3.6}$$

$$\begin{aligned} H := &\left[(p-2)w^{\frac{p-4}{2}}\Delta u + \frac{(p-2)(p-4)}{2}w^{\frac{p-6}{2}}\nabla u\cdot\nabla w + qw^{\frac{q-2}{2}}\right]\nabla u \\ &+ \frac{p-2}{2}w^{\frac{p-4}{2}}\nabla w. \end{aligned} \tag{2.3.7}$$

Setting $z = \eta w$, we get

$$\mathcal{L}z \;=\; \eta\mathcal{L}w + w\mathcal{L}\eta - 2w^{\frac{p-2}{2}}\nabla\eta\cdot\nabla w - 2(p-2)w^{\frac{p-4}{2}}(\nabla\eta\cdot\nabla u)(\nabla w\cdot\nabla u).$$

Now we shall estimate the different terms. In what follows $\delta_i > 0$ can be chosen arbitrarily small.

- Estimate of $|2w^{\frac{p-2}{2}}\nabla\eta\cdot\nabla w|$.
  Using Young's inequality, we have

$$|2w^{\frac{p-2}{2}}\nabla\eta\cdot\nabla w| \le w^{\frac{p-2}{2}}\left[C\eta^{-1}|\nabla\eta|^2 w + \delta_1\eta|D^2u|^2\right], \tag{2.3.8}$$

  where we used the fact that $\nabla w = 2D^2u\nabla u$.
- Estimate of $|2(p-2)w^{\frac{p-4}{2}}(\nabla\eta\cdot\nabla u)(\nabla w\cdot\nabla u)|$.

$$|2(p-2)w^{\frac{p-4}{2}}(\nabla\eta\cdot\nabla u)(\nabla w\cdot\nabla u)| \le w^{\frac{p-2}{2}}\Big[C\eta^{-1}|\nabla\eta|^2 w + \delta_2\eta|D^2u|^2\Big]. \tag{2.3.9}$$

– Estimate of $|w\, H \cdot \nabla\eta|$.

$$|w\, H \cdot \nabla\eta| \leq \underbrace{w^{\frac{p-2}{2}}\left(C\eta^{-1}|\nabla\eta|^2 w + \delta_3[D^2u|^2\eta\right)}_{(1)} + \underbrace{w^{\frac{p-2}{2}}\left(C\eta^{-1}|\nabla\eta|^2 w + \delta_4[D^2u|^2\eta\right)}_{(2)}$$
$$+ \underbrace{w^{\frac{p-2}{2}}\left(C\eta^{-1}|\nabla\eta|^2 w + \delta_5[D^2u|^2\eta\right)}_{(3)} + Cw^{\frac{q+1}{2}}|\nabla\eta|. \qquad (2.3.10)$$

(1) comes from an estimate based on Young's inequality of $w^{\frac{p-2}{2}}\Delta u(\nabla u \cdot \nabla\eta)$, (2) comes from (2.3.9) and (3) comes from an estimate of $w^{\frac{p-2}{2}}\nabla w \cdot \nabla\eta$.
Finally choosing δ_i such that $-2 + \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 = -1$, we arrive at

$$\mathcal{L}z + \eta w^{\frac{p-2}{2}}|D^2u|^2 \leq \; C(p,q,N)w^{\frac{p}{2}}\left[|D^2\eta| + |\Delta\eta| + \eta^{-1}|\nabla\eta|^2\right] + |\nabla\eta|w^{\frac{q+1}{2}}.$$

Using the properties of the cut-off function η, we get

$$\mathcal{L}z + \eta w^{\frac{p-2}{2}}|D^2u|^2 \;\leq\; C(p,q,N)R^{-2}\eta^{\alpha}w^{\frac{p}{2}} + C(p,q,N)R^{-1}\eta^{\alpha}w^{\frac{q+1}{2}}. \qquad (2.3.11)$$

Using the result of Theorem 2.1.3, we shall estimate $|\nabla u|^{p-2}|D^2u|^2$ in terms of a power of w. For $(x_1,t_1) \in Q^{t_0}_{T,R'}$ such that $|\nabla u(x_1,t_1)| > 0$, we have

$$\begin{aligned}|\nabla u(x_1,t_1)|^q &= \partial_t u(x_1,t_1) - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u(x_1,t_1)\right)\\ &\leq \frac{\sup(\|u_0\|_{L^\infty}, \|g\|_{L^\infty})}{(p-2)t_0} + (p-2+\sqrt{N})|\nabla u|^{p-2}|D^2u(x_1,t_1)|.\end{aligned}$$

Hence

$$\frac{1}{2(p-2+\sqrt{N})^2}|\nabla u(x_1,t_1)|^{2q} \leq \left(\frac{\sup(\|u_0\|_{L^\infty}, \|g\|_{L^\infty})}{(p-2)(p-2+\sqrt{N})t_0}\right)^2 + |\nabla u|^{2p-4}|D^2u(x_1,t_1)|^2.$$

There are two cases :

$$\text{either} \quad \frac{1}{2(p-2+\sqrt{N})^2}|\nabla u(x_1,t_1)|^{2q} \leq 2\left(\frac{\sup(\|u_0\|_{L^\infty}, \|g\|_{L^\infty})}{(p-2)(p-2+\sqrt{N})t_0}\right)^2,$$

$$\text{or} \quad \frac{1}{2(p-2+\sqrt{N})^2}|\nabla u(x_1,t_1)|^{2q-p+2} \leq 2|\nabla u|^{p-2}|D^2u(x_1,t_1)|^2.$$

In both cases we arrive at

$$\frac{1}{C(N,p)}|\nabla u(x_1,t_1)|^{2q-p+2} \leq C(p,q,N)\left(\frac{\sup(\|u_0\|_{L^\infty}, \|g\|_{L^\infty})}{t_0}\right)^{\frac{2q-p+2}{q}}$$
$$+ |\nabla u|^{p-2}|D^2u(x_1,t_1)|^2.$$

Using this inequality, it follows from (2.3.11) that, at (x_1,t_1),

$$\mathcal{L}z + \frac{1}{C(N,p)}\eta|\nabla u|^{2q-p+2} \le C(p,q,N)\left(\frac{\sup(\|u_0\|_{L^\infty}, \|g\|_{L^\infty})}{t_0}\right)^{\frac{2q-p+2}{q}} + CR^{-2}\eta^\alpha w^{\frac{p}{2}} + CR^{-1}\eta^\alpha w^{\frac{q+1}{2}}.$$

We take $\alpha = \dfrac{q+1}{2q-p+2} \in (0,1)$ (since $q > p-1$). Using Young's inequality, we have

$$CR^{-1}\eta^\alpha w^{\frac{q+1}{2}} \le CR^{-\frac{2q-p+2}{q-p+1}} + \frac{1}{4C(N,p)}\eta w^{\frac{2q-p+2}{2}},$$
$$CR^{-2}\eta^\alpha w^{\frac{p}{2}} \le CR^{-\frac{2q-p+2}{q-p+1}} + \frac{1}{4C(N,p)}\eta^{\frac{q+1}{p}} w^{\frac{2q-p+2}{2}}.$$

Using that $\eta \le 1$, we get

$$\mathcal{L}z + \frac{1}{C(N,p)}\eta|\nabla u|^{2q-p+2} \le C(p,q,N)\left(\frac{\sup(\|u_0\|_{L^\infty}, \|g\|_{L^\infty})}{t_0}\right)^{\frac{2q-p+2}{q}} + CR^{-\frac{2q-p+2}{q-p+1}} + \frac{1}{2C(N,p)}\eta|\nabla u|^{2q-p+2}.$$

Hence

$$\mathcal{L}z + \frac{1}{2C(N,p)}z^{\frac{2q-p+2}{2}} \le C(p,q,N)\left(\frac{\sup(\|u_0\|_{L^\infty}, \|g\|_{L^\infty})}{t_0}\right)^{\frac{2q-p+2}{q}} + CR^{-\frac{2q-p+2}{q-p+1}}. \quad (2.3.12)$$

Proof of theorem 2.1.2

First let us note that by the proof of the local existence there exists $t_0 \in \Big(0, T_{max}(u_0)\Big)$ with $t_0 = t_0(M,p,q,N,\|g\|_{C^2})$, such that

$$\sup_{0\le t\le t_0} \|\nabla u\|_{L^\infty} \le C(p,q,\Omega,M,\|g\|_{C^2}). \quad (2.3.13)$$

We also know that ∇u is a locally Hölder continuous function and thus z is a continuous function on $\overline{B(x_0,R')} \times [t_0,T] = \overline{Q}$, for any $T < T_{max(u_0)}$. Therefore, unless $z \equiv 0$ in $\overline{Q}$, z must reach a positive maximum at some point $(x_1,t_1) \in \overline{B(x_0,R')} \times [t_0,T]$. Since $z = 0$ on $\partial B_{R'} \times [t_0,T]$, we deduce that $x_1 \in B_{R'}$. Therefore $\nabla z(x_1,t_1) = 0$ and $D^2 z(x_1,t_1) \le 0$. Now we have either $t_1 = t_0$, or $t_0 < t_1 \le T$. If $t_1 = t_0$, then

$$z(x_1,t_1) \le \|\nabla u(t_0)\|^2_{L^\infty} \le C(p,q,\Omega,M,\|g\|_{C^2}).$$

If $t_0 < t_1 \le T$, we have $\partial_t z(x_1,t_1) \ge 0$ and therefore $\mathcal{L}z \ge 0$. Using (2.3.12) we arrive at

$$\frac{1}{2C(N,p)}z(x_1,t_1)^{\frac{2q-p+2}{2}} \le C(p,q,N)\left(\frac{\sup(\|u_0\|_{L^\infty}, \|g\|_{L^\infty})}{t_0}\right)^{\frac{2q-p+2}{q}} + CR^{-\frac{2q-p+2}{q-p+1}}, \quad (2.3.14)$$

that is

$$\sqrt{z(x_1,t_1)} \le C(p,q,N)\left(\frac{\sup(\|u_0\|_{L^\infty}, \|g\|_{L^\infty})}{t_0}\right)^{\frac{1}{q}} + C(p,q,N)R^{-\frac{1}{q-p+1}}. \qquad (2.3.15)$$

Since $z(x_0,t) \le z(x_1,t_1)$ and $\eta(x_0) = 1$, we get

$$|\nabla u(x_0,t)| \le C(p,q,N)\left(\frac{\sup(\|u_0\|_{L^\infty}, \|g\|_{L^\infty})}{t_0}\right)^{\frac{1}{q}} + C(p,q,N)R^{-\frac{1}{q-p+1}} \quad \text{for} \quad t \in [t_0,T].$$

The proof of (2.1.2) follows by taking $R = \delta(x_0)$, letting $T \to T_{max}(u_0)$ and using (2.3.13).

2.4 Blow-up criterion : proof of Theorem 2.1.4

Assume that $T_{max}(u_0) = \infty$, taking φ_1^α as test-function, we have for any $\tau > 0$

$$\int_0^\tau \int_\Omega u_t\, \varphi_1^\alpha\, dxdt = \int_0^\tau \int_\Omega |\nabla u|^q\, \varphi_1^\alpha\, dxdt - \alpha \int_0^\tau \int_\Omega |\nabla u|^{p-2}\, \varphi_1^{\alpha-1}\, \nabla u \cdot \nabla\varphi_1\, dxdt. \quad (2.4.1)$$

Set $y(t) = \int_\Omega u(t)\, \varphi_1^\alpha\, dx$. Since by definition $u_t \in L^2_{loc}((0,\infty); L^2(\Omega))$, we have $y \in W^{1,1}_{loc}(0,\infty)$ and $y'(t) = \int_\Omega u_t\, \varphi_1^\alpha\, dx$. Differentiating (2.4.1) with respect to τ we have, for a.e. $\tau > 0$

$$y'(\tau) = \int_\Omega |\nabla u(\tau)|^q\, \varphi_1^\alpha\, dx - \alpha \int_\Omega |\nabla u(\tau)|^{p-2}\, \varphi_1^{\alpha-1}\, \nabla u(\tau) \cdot \nabla\varphi_1\, dx. \qquad (2.4.2)$$

Assume that $\alpha > \frac{p-1}{(q-p+1)}$. Since $q > p > 1$ and $\|\nabla\varphi_1\|_{L^\infty} \le C'$, using Hölder and Young inequalities we get :

$$\begin{aligned} \alpha \int_\Omega |\nabla u(\tau)|^{p-2}\, \varphi_1^{\alpha-1}\, \nabla u(\tau) \cdot \nabla\varphi_1\, dx &\le \frac{1}{2}\int_\Omega |\nabla u(\tau)|^q\, \varphi_1^\alpha\, dx + C\int_\Omega \varphi_1^{\alpha - q/(q-p+1)}\, dx \\ &\le \frac{1}{2}\int_\Omega |\nabla u(\tau)|^q \varphi_1^\alpha dx + C. \end{aligned}$$

Here we used the fact that $\int_\Omega \varphi_1^{-l} dx < \infty$ for $l < 1$. Therefore

$$y'(\tau) \ge \frac{1}{2}\int_\Omega |\nabla u(\tau)|^q \varphi_1^\alpha dx - C.$$

Assuming that $\alpha < q - 1$, we get

$$\begin{aligned} \int_\Omega |\nabla u(\tau)|\, dx = \int_\Omega |\nabla u(\tau)|\, \varphi_1^{\frac{\alpha}{q}}\, \varphi_1^{-\frac{\alpha}{q}}\, dx &\le \left(\int_\Omega |\nabla u(\tau)|^q\, \varphi_1^\alpha\, dx\right)^{1/q} \left(\int_\Omega \varphi_1^{\frac{-\alpha}{q-1}}\, dx\right)^{\frac{q-1}{q}} \\ &\le C\left(\int_\Omega |\nabla u(\tau)|^q\, \varphi_1^\alpha\, dx\right)^{1/q}. \end{aligned}$$

On the other hand using that $\int_\Omega |u(\tau)|\,dx \le C\,\|u\|_{L^\infty(\partial\Omega)} + C\int_\Omega |\nabla u(\tau)|\,dx$, we have

$$\int_\Omega u(\tau)\,\varphi_1^\alpha\,dx \le \|\varphi_1^\alpha\|_{L^\infty} \int_\Omega u(\tau)\,dx \le C + C\int_\Omega |\nabla u(\tau)|\,dx.$$

Combining these two inequalities we arrive at

$$\int_\Omega |\nabla u(\tau)|^q\,\varphi_1^\alpha\,dx \ge C\left(\int_\Omega u(\tau)\,\varphi_1^\alpha\,dx\right)^q - C.$$

Finally we get the blow-up inequality

$$y'(\tau) \ge C_1 y(\tau)^q - C_2, \qquad \text{for a.e.} \quad \tau > 0,$$

with $C_1 = C_1(p,q,\Omega) > 0$ and $C_2 = C_2(p,q,\alpha,\Omega,\|g\|_{L^\infty})$.

Remark 2.4.1. *Instead of assuming that $\int_\Omega u_0\phi_1^\alpha dx$ is large in Theorem 2.1.4, it would be sufficient to assume that $\|u_0\|_r$ is large for some $r \in [1,\infty)$. In fact, assuming without loss of generality $r \ge (2q-p)/(q-p)$ and denoting $y(t) = \int_\Omega u^r(t)dx$, the Poincaré and Hölder inequalities can be used in order to prove the blow-up inequality $y' \ge C_1 y^{(q+r-1)/r} - C_2$ (see [85]).*

Chapitre 3

Localisation des points d'explosion

Dans ce travail en collaboration avec Philippe Souplet, on s'intéresse à la localisation des point d'explosion du gradient. On a vu précédemment que l'ensemble des points d'explosion est contenu dans le bord du domaine. Il s'agit ici de montrer que pour des domaines bien choisis et des données initiales bien préparées, l'ensemble des points d'explosion se réduit à un singleton. Pour arriver à cette fin on adaptera la méthode de Souplet et Li aux différentes difficultés apportées par une diffusion non-linéaire.

3.1 Introduction

3.1.1 Problem and main result

This chapter is a contribution to the study of the influence of nonlinear diffusion on the qualitative properties of equations of Hamilton-Jacobi type and, in particular, on the formation of finite-time singularity. More specifically, we consider the following problem

$$\begin{cases} u_t = \Delta_p u + |\nabla u|^q, & (x,y) \in \Omega, \quad t > 0, \\ u(x,y,t) = \mu y, & (x,y) \in \partial\Omega, \quad t > 0, \\ u(x,y,0) = u_0(x,y), & (x,y) \in \Omega, \end{cases} \tag{3.1.1}$$

where Δ_p denotes the p-Laplace operator, $\Delta_p = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$. Throughout this paper, we assume that $\mu \geq 0$ is a constant and that

$$q > p > 2. \tag{3.1.2}$$

For reasons that will appear later, we restrict ourselves to a class of planar domains Ω which satisfy certain geometric properties. We assume that, for some $L_1, L_2 > 0$,

$$\Omega \subset \mathbb{R}^2 \text{ is a smooth bounded domain of class } C^{2+\epsilon} \text{ for some } \epsilon \in (0,1); \tag{3.1.3}$$

$$\Omega \text{ is symmetric with respect to the axis } x = 0; \tag{3.1.4}$$

$$\Omega \subset \{y > 0\} \text{ and } \Omega \text{ contains the rectangle } (-L_1, L_1) \times (0, 2L_2); \tag{3.1.5}$$

$$\Omega \text{ is convex in the } x\text{-direction.} \tag{3.1.6}$$

In particular, by (3.1.5), $\partial\Omega$ has a flat part, centered at the origin $(0,0)$. Note that assumption (3.1.6) is equivalent to the fact that $\Omega \cap \{y = y_0\}$ is a line segment for each y_0.

The initial data u_0 is taken in $\mathcal{V}_\mu$, where

$$\mathcal{V}_\mu := \left\{u_0 \in C^1(\overline{\Omega}),\ u_0 \geq \mu y,\ \partial_y u_0 \geq \mu/2 \text{ in } \Omega \text{ and } \ u_0 = \mu y \ \text{ on } \ \partial\Omega\right\}.$$

We shall use the following notation throughout :

$$\Omega_+ := \{(x,y) \in \Omega;\ x > 0\}.$$

For $T > 0$, set $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$ and $\partial_P Q_T = S_T \cup (\overline{\Omega} \times \{0\})$ its parabolic boundary.

Problem (3.1.1) is well posed locally in time (see Section 2 for details), with blow-up alternative in $W^{1,\infty}$ norm. For brevity, when no confusion arises, the existence time of its maximal solution u will be denoted by

$$T := T_{max}(u_0) \leq \infty.$$

It is known (see [7, 85]) that global nonexistence, i.e. $T < \infty$, occurs for suitably large initial data (more generally, for problem (3.1.1) in an n-dimensional bounded domain with Dirichlet boundary conditions). Note that the condition $q > p$ is sharp, since the solutions are global and bounded in $W^{1,\infty}$ if $1 < q \leq p$ (see [111, 85]). Since it follows easily from the maximum principle that u itself remains uniformly bounded on Q_T, global nonexistence can only occur through *gradient blow-up*, namely

$$\sup_{Q_T} |u| < \infty \quad \text{ and } \quad \lim_{t \to T} \|\nabla u(\cdot,t)\|_\infty = \infty.$$

This is different from the usual blow-up, in which the L^∞ norm of the solution tends to infinity as $t \to T_{max}$, which occurs for equations with zero-order nonlinearities, such as $u_t = \Delta_p u + u^q$ (see [57, 58]). The study of $-L^\infty$ or gradient– blow-up singularities, in particular their location, time and spatial structure is very much of interest for the understanding of the physical problems modelled by such equation, as well as for the mathematical richness that they involve. The L^∞ blow-up for the equation $u_t = \Delta_p u + u^q$ has been extensively studied, both in the case of linear ($p = 2$) and nonlinear ($p > 2$) diffusion ; see respectively the monographs [100] and [102] and the numerous references therein.

As for equation (3.1.1) with $p = q = 2$, it is known as the (deterministic version of the) Kardar-Parisi-Zhang (KPZ) equation, describing the profile of a growing interface in certain physical models (see [74]), where u then represents the height of the interface profile. The case of $p = 2$ and $q \geq 1$ is a more general model which was developed by Krug and Spohn, aiming at studying the effect of the nonlinear gradient term on the properties of solutions (see [78]). Our main interest in this paper is to study the effect of a quasilinear gradient diffusivity on the localization of the singularities.

For the case of linear diffusion $p = 2$, various sufficient conditions for gradient blow-up and global existence were provided and qualitative properties were investigated, such as : nature of the blow-up set, rate and profile of blow-up, maximum existence time and continuation after blow-up, boundedness of global solutions and convergence to a stationary state. We refer for these to the works [5, 65, 108, 110, 100, 88] and the references therein.

The case $p > 2$ is far from being completely understood and fewer works deal with the nonlinear diffusion. The large time behavior of global solutions in bounded or unbounded domains has been studied in [86, 111, 85, 22, 18, 84]. Concerning the asymptotic description of singularities, results on the gradient blow-up rate in one space dimension can be found in [8, 125, 124]. On the other hand, in any space dimension, it is known [7] that gradient blow-up can take place only on the boundary, i.e.

$$GBUS(u_0) \subset \partial\Omega,$$

where the gradient blow-up set is defined by

$$GBUS(u_0) = \Big\{ x_0 \in \overline{\Omega};\ \text{for any } \rho > 0, \sup_{(\overline{\Omega} \cap B_\rho(x_0)) \times (T-\rho, T)} |\nabla u| = \infty \Big\}.$$

Moreover, the following upper bounds for the space profile of the singularity were obtained in [7] :

$$|\nabla u| \leq C\delta^{-\frac{1}{q-p+1}} \quad \text{and} \quad u \leq C\delta^{\frac{q-p}{q-p+1}} \quad \text{in } Q_T, \quad \text{where } \delta(x) = \operatorname{dist}(x, \partial\Omega), \tag{3.1.7}$$

and they are sharp in one space dimension [8]. (Our nondegeneracy lemma 3.7.1 below indicates that they are also sharp in higher dimensions.)

It is easy to see, by considering radially symmetric solutions with Ω being a ball, that $GBUS(u_0)$ can be the whole of $\partial\Omega$. A natural question is then :

Can one produce examples (in more than one space dimension)
when $GBUS(u_0)$ is a proper subset of $\partial\Omega$, especially a single point ?

The goal of this work is to provide an affirmative answer to this question. Our main result is the following.

Theorem 3.1.1. *Assume (3.1.2)–(3.1.6).*

(i) For any $\rho \in (0, L_1)$, there exists $\mu_0 = \mu_0(p, q, \Omega, \rho) > 0$ such that, for any $\mu \in (0, \mu_0]$, there exist initial data u_0 in $\mathcal{V}_\mu \cap C^2(\overline{\Omega})$ such that the corresponding solution u of (3.1.1) enjoys the following properties :

$$T := T_{max}(u_0) < \infty \text{ and } GBUS(u_0) \subset [-\rho, \rho] \times \{0\}, \tag{3.1.8}$$

$$u(\cdot, t) \text{ is symmetric with respect to the line } x = 0, \text{ for all } t \in (0, T), \tag{3.1.9}$$

$$u_x \leq 0 \quad \text{in } \Omega_+ \times (0, T), \tag{3.1.10}$$

$$u_y \geq \mu/2 \quad \text{in } Q_T. \tag{3.1.11}$$

(ii) For any such μ and u_0, we have

$$GBUS(u_0) = \{(0, 0)\}.$$

A class of initial data satisfying the requirements of Theorem 3.1.1 is provided in Lemma 3.6.1 below. We note that, in the semilinear case $p = 2$, a single-point boundary gradient blow-up result was obtained in [88]. Although we follow the same basic strategy, the proof here is considerably more complicated. We point out right away that, in view of property (3.1.11), the equation is not degenerate for the solutions under consideration. However, since the essential goal of this work is to study the effect of nonlinear diffusion on gradient blow-up, what is relevant here are the *large* values of the gradient in the diffusion operator (rather than the issues of loss of higher regularity that would arise from the degenerate nature of the equation near the level $\nabla u = 0$). It is an open question whether or not single-point gradient blow-up can still be proved in the case $\mu = 0$. Actually, the lower bound (3.1.11) on $|\nabla u|$ is crucially used at various points of the proof, which is already very long and involved, due to the presence of the nonlinear – even though nondegenerate – diffusion term (see the next subsection for more details).

Section 3.2 is devoted to local in time well-posedness and regularity results. The proof of Theorem 3.1.1 will be split into several sections, namely sections 3.3–3.7 for assertion (i) and sections 3.8–3.9 for assertion (ii) (the latter uses also section 3.5). Finally, in two appendices, we provide the proofs of some regularity properties and a suitable parabolic version of Serrin's corner lemma. Since the proof is quite long and involved, for the convenience of readers, we now give an outline of the main steps of the proof.

3.1.2 Outline of proof

For sake of clarity, we have divided the proof into a number of intermediate steps, each of which being relatively short (one or two pages, say), except for step (f), which involves long and hard computations. For the convenience of readers, we outline the structure of the proof.

(a) *Preliminary estimates* (Lemmas 3.3.1–3.3.4) : The symmetry in the variable x and the decreasing property for $x > 0$ are basic features in order to expect single-point gradient blow-up. Besides u being bounded, we also have boundedness of u_t. Moreover, for sufficiently small μ and under a suitable assumption on u_0, we show that $u_y \geq \mu/2$. These bounds on u_t and u_y seem necessary in the very long calculations of the key step (f) below. In turn, the positivity of u_y guarantees that solutions are actually classical and that D^2u, D^3u satisfy some bounds which seem also necessary to the argument, especially in view of the application of the Hopf lemma and the Serrin corner lemma in step (g).

(b) *Finite time gradient blow-up for suitably concentrated initial data* (Lemma 3.4.1) : by using a rescaling argument and known blow-up criteria, we show that the solution blows up in finite time provided the initial data is suitably concentrated in a small ball near the origin.

(c) *Local boundary gradient control* (Lemma 3.5.1) : if the gradient remains bounded on the boundary near a given boundary point, then the gradient remains also bounded near that point inside the domain, hence it is not a blow-up point. This is proved by a local Bernstein type argument.

(d) *Localization of the gradient blow-up set* (Lemma 3.5.3) : if an initial data is suitably concentrated near the origin, then the gradient blow-up set is contained in a small neighborhood of the origin. This is proved by constructing comparison functions which provide a control of the gradient on the boundary outside a small neighborhood of the origin, and then applying step (c). One then constructs (Lemma 3.6.1) initial data which also fulfill the assumptions in (a) and (b). This ensures the existence of "well-prepared" initial data and thereby completes the proof of assertion (i) of Theorem 3.1.1.

(e) *Nondegeneracy of gradient blow-up* (Lemma 3.7.1) : if the solution is only "weakly singular" in a neighborhood of a boundary point, then the singularity is removable. [1] More precisely, we show the existence of $m = m(p, q) \in (0, 1)$ such that, for a given point $(x_0, 0)$ on the flat part of $\partial\Omega$, if $u(x, y, t) \leq c(x)y^m$ near $(x_0, 0)$ for t close to T, then $(x_0, 0)$ is not a gradient blow-up point. In view of step (c), it suffices to control the gradient on the boundary near the point $(x_0, 0)$. This is achieved by constructing special comparison functions, taking the form of "regularizing (in time) barriers".

(f) *Verification of a suitable parabolic inequality for an auxiliary function* J, of the form

$$J(x, y, t) = u_x + kxy^{-\gamma}u^{\alpha}.$$

(Proposition 3.8.1). This is the most technical step and gives rise to very long computations. Those computations make use, among many other things, of the singular, Bernstein-type boundary gradient estimate (3.1.7), obtained in [7]. They use the bound on u_t and the lower bound on u_y, obtained at step (a) (it is not clear if the latter could be relaxed [2]).

We note that a similar function J was introduced in [88] to treat the semilinear case $p = 2$. The function in [88] was a 2D-modification of a one-dimensional device from [56],

1. In this context, the term "nondegeneracy" describes a property of finite-time singularities (like in, e.g., Giga and Kohn [61]) and should not be confused with the notion of nondegenerate diffusion mentioned after Theorem 3.1.1.

2. It seems that the constants in some of the key estimates there are nonuniform as $\mu \to 0^+$, which prevents us to argue by a limiting procedure from the case $\mu > 0$.

used there to show single point L^∞ blow-up for radial solutions of equations of the form $u_t - \Delta_p u = u^q$ (for $p = 2$, see also [58] for $p > 2$). Although the ideas are related, the calculations here are considerably harder than in [56, 58, 88].

(g) *Verification of initial-boundary conditions for the auxiliary function J in a small subrectangle* near the origin. This requires a delicate parabolic version of Serrin's corner lemma, which we prove in Appendix 2 (see Proposition 3.11.1).

(h) *Derivation of a weakly singular gradient estimate near the origin and conclusion.* Steps (d) and (e) imply $J \leq 0$ by the maximum principle. By integrating this inequality we obtain an inequality of the form $u(x, y, t) \leq c(x)y^k$ as $y \to 0$, for each small $x \neq 0$ and some $k > m$. In view of step (e), this shows that $(0, 0)$ is the only gradient blow-up point.

3.2 Local well-posedness and regularity

In this section we consider the question of local existence and regularity for problem (3.1.1). Actually, we consider the slightly more general problem

$$
\begin{aligned}
u_t - \Delta_p u &= |\nabla v|^q \quad \text{in } Q_T, && (3.2.1)\\
u &= g \quad \text{on } S_T, && (3.2.2)\\
u(\cdot, 0) &= u_0 \quad \text{in } \Omega, && (3.2.3)
\end{aligned}
$$

where the boundary data g and initial data u_0 satisfy :

$$g \geq 0 \text{ is the trace on } \partial\Omega \text{ of a regular function in } C^{2+\gamma}(\overline{\Omega}) \text{ for some } \gamma \in (0,1) \qquad (3.2.4)$$

and

$$u_0 \in W^{1,\infty}(\Omega), \quad u_0 \geq 0, \quad u_0 = g \text{ on } \partial\Omega. \qquad (3.2.5)$$

A function u is called a weak super- (sub-) solution of problem (3.2.1)–(3.2.3) on Q_T if $u(\cdot, 0) \geq (\leq)\, u_0$ in Ω, $u \geq (\leq)\, g$ on S_T,

$$u \in C(\overline{\Omega} \times [0, T)) \cap L^q(0, T; W^{1,q}(\Omega)), \quad u_t \in L^2(0, T; L^2(\Omega))$$

and the integral inequality

$$\int\int_{Q_T} u_t\psi + |\nabla u|^{p-2}\nabla u \cdot \nabla\psi \, dx\, dt \geq (\leq) \int\int_{Q_T} |\nabla u|^q \psi \, dx\, dt$$

holds for all $\psi \in C^0(\overline{Q_T}) \cap L^p(0, T; W^{1,p}(\Omega))$ such that $\psi \geq 0$ and $\psi = 0$ on S_T. A function u is a weak solution of (3.2.1)–(3.2.3) if it is a super-solution and a sub-solution.

The following result was established in Theorem 2.1.1 (actually in any space dimension).

Theorem 3.2.1. *Assume (3.1.3) and $q > p - 1 > 1$. Let $M_1 > 0$, let u_0, g satisfy* (3.2.4)–(3.2.5) *and $\|\nabla u_0\|_\infty \leq M_1$. Then :*

(i) There exists a time $T_0 = T_0(p, q, \Omega, M_1, \|g\|_{C^2}) > 0$ *and a weak solution* u *of* (3.2.1)–(3.2.3) *on* $[0, T_0)$*, which moreover satisfies* $u \in L^\infty(0, T_0; W^{1,\infty}(\Omega))$*. Furthermore,* ∇u *is locally Hölder continuous in* Q_{T_0}*.*

(ii) For any $\tau > 0$*, the problem* (3.2.1)–(3.2.3) *has at most one weak solution* u *such that* $u \in L^\infty(0, \tau; W^{1,\infty}(\Omega))$*.*

(iii) There exists a (unique) maximal, weak solution of (3.2.1)–(3.2.3) *in* $L^\infty_{loc}([0, T); W^{1,\infty}(\Omega))$*, still denoted by* u*, with existence time denoted by* $T = T_{max}(u_0)$*. Then*

$$\min_{\overline{\Omega}} u_0 \le u \le \max_{\overline{\Omega}} u_0 \quad \text{in } Q_T, \tag{3.2.6}$$

∇u *is locally Hölder continuous in* Q_T *and*

if $T < \infty$*, then* $\lim_{t \to T} \|\nabla u(t)\|_\infty = \infty$ *(gradient blow up, GBU).*

Remark 3.2.1. *We also have a comparison principle for problem* (3.2.1)–(3.2.3)*, cf. [7, Proposition 2.1]. More precisely, if* $v_1, v_2 \in C(\overline{Q}_T)$ *are weak sub-/super-solutions of* (3.2.1)–(3.2.3) *in* Q_T*, then*

$$\sup_{Q_T} (v_1 - v_2) \le \sup_{\partial Q_T} (v_1 - v_2).$$

As one expects, the solution will possess additional regularity if we know that $|\nabla u|$ remains bounded away from 0. This is made precise by the following result, which is a consequence of regularity theory [80, 89] for quasilinear uniformly parabolic equations. However, for completeness, we provide a proof in Appendix 1.

Theorem 3.2.2. *Under the assumptions of Theorem 3.2.1, suppose also that* $\inf_{Q_T} |\nabla u| > 0$*.*

(i) Then u *is a classical solution in* Q_T *and*

$$u \in C^{2+\alpha, 1+\alpha/2}_{loc}(\overline{\Omega} \times (0, T)). \tag{3.2.7}$$

for some $\alpha \in (0, 1)$*.*

(ii) Moreover,

$$\nabla u \in C^{2+\beta, 1+\beta/2}_{loc}(\Omega \times (0, T)). \tag{3.2.8}$$

for some $\beta \in (0, 1)$*.*

(iii) If the boundary conditions in (3.2.2) *depend only on* y*, then*

$$u_x \in C^{2+\beta, 1+\beta/2}_{loc}(\overline{\Omega} \times (0, T)). \tag{3.2.9}$$

for some $\beta \in (0, 1)$*.*

3.3 Preliminary estimates : x-symmetry, lower bound on u_y and bound on u_t

Notation. Throughout the paper, we shall use the summation convention on repeated indices, in expressions of the form $a_{ij}u_{ij}$ or $a_{ij}u_iu_j$. Also, the letter C will denote positive constants which may vary from line to line, and whose dependence will be indicated if necessary.

Lemma 3.3.1. *Let $\mu > 0$ and $u_0 \in \mathcal{V}_\mu$. Assume*

$$u_0 \text{ is symmetric with respect to the line } x = 0, \tag{3.3.1}$$

$$\partial_x u_0 \leq 0 \text{ in } \Omega_+. \tag{3.3.2}$$

Then we have

$$u(x, y, t) \geq \mu y \quad \text{in } Q_T. \tag{3.3.3}$$

and properties (3.1.9)-(3.1.10) are satisfied.

Proof. Property (3.1.9) is a direct consequence of (3.3.1) and the local-in-time uniqueness. Due to the assumption $u_0 \in \mathcal{V}_\mu$, $\underline{v} = \mu y$ is a subsolution of (3.1.1). This implies (3.3.3).

To prove (3.1.10), fix $h > 0$ and let

$$u_\pm = u(x \pm h, y, t) \quad \text{ for } (x, y) \in \Omega_h := \{(x, y) \in \Omega_+;\ (x + h, y) \in \Omega\} \text{ and } t \in (0, T).$$

Owing to (3.1.4) and (3.1.6), we see that $(x - h, y) \in \Omega$ for all $(x, y) \in \Omega_h$, so that u_- is well defined. The functions $u_\pm$ are weak solutions of $(3.1.1)_1$ in $\Omega_h \times (0, T)$. Also $u_+ \leq u_-$ at $t = 0$, due to (3.3.1)-(3.3.2) and (3.1.6).

Let $(x, y) \in \partial\Omega_h$. If $x = 0$, then $u_+(x, y, t) = u(h, y, t) = u(-h, y, t) = u_-(x, y, t)$ by (3.1.9). If $x > 0$, then $(x + h, y) \in \partial\Omega$ as a consequence of $(x, y) \in \partial\Omega_h$ and (3.1.6). So, by (3.3.3), we have

$$u_+(x, y, t) = u(x + h, y, t) = \mu y \leq u_-(x - h, y, t).$$

We deduce from the comparison principle that $u_+ \leq u_-$ in $\Omega_h \times (0, T)$, which implies (3.1.10).

□

Our next lemma provides a useful supersolution of problem (3.1.1).

Lemma 3.3.2. *For $0 < \rho < L_1$, denote $\Sigma_\rho = [-\rho/2, \rho/2] \times \{0\}$ and $\Sigma'_\rho = \partial\Omega \setminus ((-\rho, \rho) \times \{0\})$. There exist $\mu_0 = \mu_0(p, q, \Omega, \rho) > 0$ and a function $U \in C^2(\overline{\Omega})$, depending on p, q, ρ, with the following properties :*

$$U > 0 \quad \text{on } \Omega \cup \Sigma_\rho, \tag{3.3.4}$$

$$U = 0 \quad \text{on } \Sigma'_\rho, \tag{3.3.5}$$

$$|U_y| \leq 1/2 \quad \text{in } \Omega \tag{3.3.6}$$

and, for all $0 < \mu \le \mu_0$, *the function* $\bar U = \mu(y+U)$ *satisfies*

$$-\Delta_p \bar U \ge |\nabla \bar U|^q \quad \text{in } \Omega. \tag{3.3.7}$$

Proof. Fix a nonnegative function $\phi \in C^3(\mathbb{R}^2)$ such that $\phi = 1$ on Σ_ρ and $\phi = 0$ on Σ'_ρ. We shall look for U under the form $U = \varepsilon V$, $\varepsilon > 0$, where $V \in C^2(\overline{\Omega})$ is the classical solution of the linear elliptic problem

$$\begin{cases} -\big[V_{xx} + (p-1)V_{yy}\big] = 1 & \text{in } \Omega, \\ V = \phi & \text{on } \partial\Omega. \end{cases} \tag{3.3.8}$$

Note that $V > 0$ in Ω by the strong maximum principle which, along with the boundary conditions in (3.3.8), will guarantee (3.3.4)-(3.3.5). Let $\bar U = \mu(y+U)$. Assume $0 < \varepsilon < 1/(\|V_x\|_\infty + 2\|V_y\|_\infty)$, which implies (3.3.6), as well as $|1+\varepsilon V_y| \ge 1/2$ and $|\nabla \bar U| \le 2\mu$. To check (3.3.7), we compute :

$$\begin{aligned}
\Delta_p \bar U &= |\nabla \bar U|^{p-2}\Big[\Delta \bar U + (p-2)\frac{\bar U_i \bar U_j \bar U_{ij}}{|\nabla \bar U|^2}\Big] \\
&= \mu\varepsilon |\nabla \bar U|^{p-2}\Big[V_{xx} + V_{yy} + (p-2)\frac{(\varepsilon V_x)^2 V_{xx} + 2(\varepsilon V_x)(1+\varepsilon V_y)V_{xy} + (1+\varepsilon V_y)^2 V_{yy}}{(1+\varepsilon V_y)^2 + (\varepsilon V_x)^2}\Big] \\
&= \mu\varepsilon |\nabla \bar U|^{p-2}\Big[V_{xx} + (p-1)V_{yy} + (p-2)\frac{(\varepsilon V_x)^2 (V_{xx} - V_{yy}) + 2(\varepsilon V_x)(1+\varepsilon V_y)V_{xy}}{(1+\varepsilon V_y)^2 + (\varepsilon V_x)^2}\Big] \\
&\le \mu\varepsilon |\nabla \bar U|^{p-2}\Big[-1 + C\varepsilon^2 + C\varepsilon\Big]
\end{aligned}$$

in Ω, with $C > 0$ depending only on Ω, ρ, p (through V). We may then choose ε depending only on Ω, ρ, p, such that $-\Delta_p \bar U \ge \frac{\mu\varepsilon}{2}|\nabla \bar U|^{p-2}$. Next, since $q > p$, for any $\mu \le \mu_0$ with $\mu_0 = \mu_0(p,q,\Omega,\rho) > 0$ sufficiently small, we have

$$-\Delta_p \bar U \ge \frac{\mu\varepsilon}{2}|\nabla \bar U|^{p-2} \ge \frac{\mu\varepsilon}{2(2\mu)^{q+2-p}}|\nabla \bar U|^q \ge |\nabla \bar U|^q \quad \text{in } \Omega.$$

□

Based on Lemma 3.3.2, we construct a class of solutions such that u_y satisfies a positive lower bound.

Lemma 3.3.3. *Let* $0 < \rho < L_1$ *and let* $\mu_0, \bar U$ *be given by Lemma 3.3.2. Assume that* $0 < \mu \le \mu_0$ *and* $u_0 \in \mathcal{V}_\mu$ *satisfy*

$$u_0(x,y) \le \mu\big(y + c\chi_{(-\rho/2,\rho/2)\times(0,L_2)}\big) \quad \text{in } \Omega, \tag{3.3.9}$$

with $c = c(p,q,\Omega,\rho) > 0$ *sufficiently small.*

(i) Then $u \le \bar U$ *in* Q_T.

(ii) Moreover, using the assumption that

$$\partial_y u_0 \ge \mu/2 \quad \text{in } \Omega, \tag{3.3.10}$$

we have that

$$\partial_y u \ge \mu/2 \quad \text{in } Q_T. \tag{3.3.11}$$

Proof. (i) Let $\bar{U} = \mu(y + U)$ be given by Lemma 3.3.2. From (3.3.4), we know that

$$c := \min_{[-\rho/2,\rho/2]\times[0,L_2]} U > 0.$$

Under assumption (3.3.9), we thus have $u_0 \leq \bar{U}$ in Ω. Since $u = \mu y \leq \bar{U}$ on S_T, we infer from the comparison principle that $u \leq \bar{U}$ in Q_T.

(ii) Set $\delta_0 = \mu/2$, fix $h > 0$ and let $\tilde{\Omega}_h = \{(x, y) \in \Omega;\ (x, y + h) \in \Omega\}$. We observe that

$$u_1 := u(x, y, t) + \delta_0 h \quad \text{and} \quad \mathrm{u}_2 := u(x, y + h, t)$$

are weak solutions of $(3.1.1)_1$ in $\tilde{\Omega}_h \times (0, T)$.

Let $(x, y) \in \partial\tilde{\Omega}_h$. If $(x, y) \in \partial\Omega$, then, by (3.3.3), we have

$$u_2(x, y, t) = u(x, y + h, t) \geq \mu(y + h) = u(x, y, t) + \mu h \geq u_1(x, y, t).$$

Otherwise, we have $(x, y) \in \Omega$ and $(x, y + h) \in \partial\Omega$. So there is a minimal $\tilde{h} \in (0, h]$ such that $(x, y + \tilde{h}) \in \partial\Omega$. By the mean-value inequality, it follows that, for some $\theta \in (0, 1)$,

$$U(x, y) = U(x, y + \tilde{h}) - \tilde{h}U_y(x, y + \theta\tilde{h}) \leq |U_y(x, y + \theta\tilde{h})|h \leq h/2,$$

where we used (3.3.6). Therefore, $\bar{U}(x, y) \leq \mu(y + h/2)$, hence

$$u(x, y + h, t) - u(x, y, t) \geq \mu(y + h) - \bar{U}(x, y) \geq \frac{\mu h}{2}.$$

We have thus proved that

$$u_2 \geq u_1 \quad \text{on } \partial\tilde{\Omega}_h. \tag{3.3.12}$$

On the other hand, using (3.3.10) and the fact that $u_0 = \mu y$ on $\partial\Omega$, it is not difficult to show that $y \mapsto u_0(x, y) - \delta_0 y$ is nondecreasing in Ω. (Note that in case Ω is nonconvex, this is not a mere consequence of (3.3.10) alone). It follows that

$$u(x, y + h, 0) \geq u(x, y, 0) + \delta_0 h \quad \text{in } \tilde{\Omega}_h. \tag{3.3.13}$$

Owing to (3.3.12)-(3.3.13), we may then apply the comparison principle to deduce that $u_2 \geq u_1$ in $\tilde{\Omega}_h \times (0, T)$. Since h is arbitrary, the desired conclusion (3.3.11) follows immediately.

□

Assuming that u_0 is sufficiently regular, we also get an estimate on the time derivative.

Lemma 3.3.4. *Let $\mu \geq 0$ and assume that $u_0 \in \mathcal{V}_\mu \cap C^2(\overline{\Omega})$. Then*

$$|u_t| \leq \tilde{C}_1 := \|\Delta_p u_0 + |\nabla u_0|^q\|_\infty \quad \text{in } Q_T. \tag{3.3.14}$$

Proof. It is easy to see that $v_\pm(x,y,t) := u_0(x,y) \pm \tilde{C}_1 t$ are respectively super- and sub-solution of (3.1.1) in Q_T. The comparison principle implies that

$$u_0(x,y) - \tilde{C}_1 t \le u(x,y,t) \le u_0(x,y) + \tilde{C}_1 t \quad \text{in } Q_T. \tag{3.3.15}$$

Now fix $h \in (0,T)$ and set $w_\pm(x,y,t) := u(x,y,t+h) \pm \tilde{C}_1 h$. By (3.3.15), we have $w_-(x,y,0) \le u_0(x,y) \le w_+(x,y,0)$ and it follows that $w_\pm$ are respectively super- and sub-solution of (3.1.1) in Q_{T-h}. By a further application of the comparison principle, we deduce that

$$|u(x,y,t+h) - u(x,y,t)| \le \tilde{C}_1 h \quad \text{in } Q_{T-h}.$$

Since h is arbitrary, we conclude by dividing by h and sending $h \to 0$.

□

3.4 Finite-time gradient blow-up for concentrated initial data

In this section, by a rescaling argument, we show that the solution of (3.1.1) blows up in finite time provided the initial data is suitably concentrated in a small ball near the origin. For such concentrated initial data, under some additional assumptions, we will show in section 3.6 that the gradient blow-up set is contained in a small neighborhood of the origin.

The following lemma shows in particular that gradient blow-up may occur for initial data of arbitrarily small L^∞-norm (but the $W^{1,\infty}$ norm has to be sufficiently large). We note that we do not assume (3.3.10) here, so that in the proof, we work only with weak solutions (in particular, we cannot use the continuity of ∇u up to the boundary).

Lemma 3.4.1. *Let $\kappa = (q-p)/(q-p+1)$. There exists $C_1 = C_1(p,q) > 0$ such that, if $\varepsilon \in (0, \min(L_1, L_2))$, $\mu \ge 0$ and $u_0 \in \mathcal{V}_\mu$ satisfies*

$$u_0(x,y) \ge C_1 \varepsilon^\kappa \quad \text{in } B_{\varepsilon/3}(0,\varepsilon) \subset \Omega, \tag{3.4.1}$$

then $T_{max}(u_0) < \infty$.

Proof. We denote $B_r = B_r(0,0) \subset \mathbb{R}^2$ for $r > 0$. Fix a radially symmetric function $h \in C_0^\infty(B_1)$, $h \ge 0$, such that $\mathrm{supp}(h) \subset B_{1/3}$ and $\|h\|_\infty = 1$. We consider the following problem

$$\begin{cases} v_t - \Delta_p v = |\nabla v|^q, & x \in B_1, \quad t > 0 \\ v(x,y,t) = 0, & x \in \partial B_1, \quad t > 0 \\ v(x,y,0) = v_0(x,y) := C_1 h(x,y), & x \in B_1. \end{cases} \tag{3.4.2}$$

We know from [85] and chapter 2 that, there exists $C_0 = C_0(p,q) > 0$ such that if $\|v_0\|_{L^1} \ge C_0$, then $T_{max}(v_0) < \infty$. Therefore, if we take $C_1 = C_1(p,q) > 0$ large enough then ∇v blows up in finite time in L^∞ norm.

Next we use the scale invariance of the equation, considering the rescaled functions

$$v_\varepsilon(x,y,t) := \varepsilon^\kappa v\left(\frac{x}{\varepsilon}, \frac{y-\varepsilon}{\varepsilon}, \frac{t}{\varepsilon^{(2q-p)/(q-p+1)}}\right).$$

Pick $\varepsilon \in (0, \min(L_1, L_2))$ and denote $\tilde{B}_\varepsilon = B_\varepsilon(0,\varepsilon)$, which is included in Ω and tangent to $\partial\Omega$ at the origin. Set $T_\varepsilon = \varepsilon^{(2q-p)/(q-p+1)} T_{max}(v_0)$ and $\tilde{T}_\varepsilon = \min\big(T_{max}(u_0), T_\varepsilon\big)$. We shall show that, for each $\tau \in (0, \tilde{T}_\varepsilon)$,

$$\|\nabla v_\varepsilon\|_{L^\infty(Q_\tau)} \leq \max\Big(\|\nabla v_\varepsilon(\cdot,0)\|_\infty, \|\nabla u\|_{L^\infty(Q_\tau)}\Big). \tag{3.4.3}$$

Since gradient blow-up occurs in finite time T_ε for v_ε, this will guarantee $T_{max}(u_0) \leq T_\varepsilon < \infty$.

First observe that v_ε solves (3.4.2) in $\tilde{B}_\varepsilon \times (0, T_\varepsilon)$, with initial data $v_\varepsilon(x,y,0) \leq u_0(x,y)$, due to (3.4.1). It follows from the comparison principle that $v_\varepsilon \leq u$ in $\tilde{B}_\varepsilon \times (0, \tilde{T}_\varepsilon)$. In particular, for each $h \in (0,\varepsilon)$ and $t \in (0, \tilde{T}_\varepsilon)$, since $u(0,0,t) = 0$, we get that

$$\frac{v_\varepsilon(0,h,t)}{h} \leq \frac{u(0,h,t)}{h} \leq \|\nabla u(\cdot,t)\|_\infty. \tag{3.4.4}$$

We shall next show that the first quantity in (3.4.4) can be suitably bounded from below in terms of the sup norm of ∇v_ε.

Fix $h \in (0,\varepsilon)$ and let $\tilde{B}_\varepsilon^h := B_\varepsilon(0, \varepsilon - h)$ Since $v_\varepsilon^h(x,y,t) := v_\varepsilon(x, y+h, t)$ is a solution of (3.4.2) in $\tilde{B}_\varepsilon^h$, it follows from the comparison principle (see Remark 3.2.1) that, for any $0 < \tau < \tilde{T}_\varepsilon$,

$$\begin{aligned}
&\sup_{(\tilde{B}_\varepsilon \cap \tilde{B}_\varepsilon^h)\times(0,\tau)} |v_\varepsilon(x,y+h,t) - v_\varepsilon(x,y,t)| \\
&\leq \max\left(\sup_{\tilde{B}_\varepsilon \cap \tilde{B}_\varepsilon^h} |v_\varepsilon^h(x,y,0) - v_\varepsilon(x,y,0)|, \sup_{\partial(\tilde{B}_\varepsilon \cap \tilde{B}_\varepsilon^h)\times(0,\tau)} |v_\varepsilon^h(x,y,t) - v_\varepsilon(x,y,t)|\right).
\end{aligned} \tag{3.4.5}$$

We claim that, for any $0 < t < T_\varepsilon$,

$$\sup_{\partial(\tilde{B}_\varepsilon \cap \tilde{B}_\varepsilon^h)} |v_\varepsilon^h(x,y,t) - v_\varepsilon(x,y,t)| \leq v_\varepsilon(0,h,t). \tag{3.4.6}$$

First consider the case $(x,y) \in \partial\tilde{B}_\varepsilon$. Then $\|(x,y+h) - (0,\varepsilon)\| \geq \|(0,h) - (0,\varepsilon)\|$, due to

$$x^2 + (y+h-\varepsilon)^2 - (h-\varepsilon)^2 = \varepsilon^2 - (y-\varepsilon)^2 + (y+h-\varepsilon)^2 - (h-\varepsilon)^2 = 2hy \geq 0.$$

Since v_ε is radially symmetric and non-increasing with respect to the point $(0,\varepsilon)$, we deduce that

$$|v_\varepsilon(x,y+h,t) - v_\varepsilon(x,y,t)| = v_\varepsilon(x,y+h,t) \leq v_\varepsilon(0,h,t).$$

Next consider the case $(x,y) \in \partial\tilde{B}_\varepsilon^h$ that is, $(x,y+h) \in \partial\tilde{B}_\varepsilon$. Then $\|(x,y) - (0,\varepsilon)\| \geq \|(0,h) - (0,\varepsilon)\|$, due to

$$x^2 + (y-\varepsilon)^2 - (h-\varepsilon)^2 = \varepsilon^2 - (y+h-\varepsilon)^2 + (y-\varepsilon)^2 - (h-\varepsilon)^2 = 2h(2\varepsilon - h - y) \geq 0.$$

Therefore, $|v_\varepsilon(x,y+h,t)-v_\varepsilon(x,y,t)| = v_\varepsilon(x,y,t) \le v_\varepsilon(0,h,t)$, and the claim (3.4.6) is proved.

Now fix $y \in [0,\varepsilon)$. It follows from (3.4.4)-(3.4.6) that, for each $0 < h < \varepsilon - y$ and $0 < t < \tau < \tilde{T}_\varepsilon$,

$$\frac{|v_\varepsilon(0,y+h,t)-v_\varepsilon(0,y,t)|}{h} \le \max\Big(\|\nabla v_\varepsilon(\cdot,0)\|_\infty, \|\nabla u\|_{L^\infty(Q_\tau)}\Big)$$

hence, letting $h \to 0$,

$$|\partial_y v_\varepsilon(0,y,t)| \le \max\Big(\|\nabla v_\varepsilon(\cdot,0)\|_\infty, \|\nabla u\|_{L^\infty(Q_\tau)}\Big).$$

For each $0 < \tau < \tilde{T}_\varepsilon$, taking supremum over $y \in [0,\varepsilon)$ and $t \in (0,\tau)$ and using the fact that v_ε is radially symmetric, we obtain (3.4.3). This concludes the proof of the lemma.

□

3.5 Local boundary control for the gradient and localization of the gradient blow-up set

For simplicity we shall here assume (3.3.9) and (3.3.10), so as to have the continuity of ∇u up to the boundary (although one might possibly relax this assumption at the expense of additional work).

Lemma 3.5.1. *Let ρ, μ, u_0 be as in Lemma 3.3.3 (ii) and let $(x_0,y_0) \in \partial\Omega$. If there exist $M_0, R > 0$ such that*

$$|\nabla u| \le M_0 \qquad \text{in } (B_R(x_0,y_0) \cap \partial\Omega) \times [0, T_{max}(u_0)), \tag{3.5.1}$$

then (x_0,y_0) is not a gradient blow-up point.

The proof is based on a local Bernstein technique. For $(x_0,y_0) \in \partial\Omega$, $R > 0$ and given $\alpha \in (0,1)$, we may select a cut-off function $\eta \in C^2(\overline{B}_R(x_0,y_0))$, with $0 < \eta \le 1$, such that

$$\eta = 1 \ \text{ on } \overline{B}_{R/2}(x_0,y_0), \qquad \eta = 0 \ \text{ on } \partial B_R(x_0,y_0)$$

and

$$\left.\begin{aligned} |\nabla\eta| &\le CR^{-1}\eta^\alpha \\ |D^2\eta| + \eta^{-1}|\nabla\eta|^2 &\le CR^{-2}\eta^\alpha \end{aligned}\right\} \quad \text{on } \overline{B}_R(x_0,y_0), \tag{3.5.2}$$

where $C = C(\alpha) > 0$ (see e.g. [110] for an example of such function). Also, for $0 < t_0 < \tau < T = T_{max}(u_0)$, we denote

$$Q^{t_0}_{\tau,R} = (B_R(x_0,y_0) \cap \Omega) \times (t_0,\tau).$$

For the proof of Lemma 3.5.1, we then rely on the following lemma from chapter 2 (cf. [7, Lemma 3.1] ; it was used there to derive upper estimates on $|\nabla u|$ away from the boundary).

Lemma 3.5.2. *Let $\mu \geq 0$ and $u_0 \in \mathcal{V}_\mu$. Let $(x_0, y_0) \in \partial\Omega$, $R > 0$, $0 < t_0 < \tau < T$ and choose $\alpha = (q+1)/(2q-p+2)$. Denote $w = |\nabla u|^2$ and $z = \eta w$. Then $z \in C^{2,1}(Q^{t_0}_{\tau,R})$ and satisfies the following differential inequality*

$$\mathcal{L}z + C_2 z^{\frac{2q-p+2}{2}} \leq C_3 \left(\frac{\|u_0\|_\infty}{t_0}\right)^{\frac{2q-p+2}{q}} + C_3 R^{-\frac{2q-p+2}{q-p+1}}, \tag{3.5.3}$$

where $C_i = C_i(p,q) > 0$,

$$\mathcal{L}z = z_t - \bar{\mathcal{A}}z - \bar{H} \cdot \nabla z, \tag{3.5.4}$$

$$\bar{\mathcal{A}}z = |\nabla u|^{p-2}\Delta z + (p-2)|\nabla u|^{p-4}(\nabla u)^t D^2 z \nabla u, \tag{3.5.5}$$

and $\bar{H}$ is defined by

$$\begin{aligned}\bar{H} :=& \left[(p-2)w^{\frac{p-4}{2}}\Delta u + \frac{(p-2)(p-4)}{2} w^{\frac{p-6}{2}} \nabla u \cdot \nabla w + q w^{\frac{q-2}{2}}\right] \nabla u \\ &+ \frac{p-2}{2} w^{\frac{p-4}{2}} \nabla w.\end{aligned} \tag{3.5.6}$$

Proof of Lemma 3.5.1 Let $t_0 = T/2 < \tau < T$ and set

$$M_1 := \sup_{0 \leq t \leq t_0} \|\nabla u\|_{L^\infty} < \infty.$$

By Lemma 3.3.3(ii) and Theorem 3.2.2, we know that ∇u is a continuous function on $\overline{\Omega} \times (0,T)$, hence $z \in C(\overline{Q^{t_0}_{\tau,R}})$. Therefore, unless $z \equiv 0$ in $\overline{Q^{t_0}_{\tau,R}}$, z must reach a positive maximum at some point $(x_1, y_1, t_1) \in \overline{Q^{t_0}_{\tau,R}}$. Since

$$z = 0 \quad \text{on } \left(\partial B_R(x_0,y_0) \cap \overline{\Omega}\right) \times [t_0, \tau], \tag{3.5.7}$$

we deduce that either $(x_1, y_1) \in B_R(x_0,y_0) \cap \Omega$ or $(x_1,y_1) \in B_R(x_0,y_0) \cap \partial\Omega$.

- If $t_1 = t_0$, then

$$z(x_1, y_1, t_1) \leq \|\nabla u(t_0)\|^2_{L^\infty} \leq M_1^2. \tag{3.5.8}$$

- If $t_0 < t_1 \leq \tau$ and $(x_1, y_1) \in B_R(x_0, y_0) \cap \partial\Omega$, then, by (3.5.1),

$$z(x_1, y_1, t_1) \leq M_0^2. \tag{3.5.9}$$

- Next consider the case $t_0 < t_1 \leq \tau$ and $(x_1, y_1) \in B_R(x_0,y_0) \cap \Omega$. Then we have $\nabla z(x_1, y_1, t_1) = 0$, $z_t(x_1,y_1,t_1) \geq 0$ and $D^2 z(x_1,y_1,t_1) \leq 0$, and therefore $\mathcal{L}z \geq 0$. Using (3.5.3) we arrive at

$$C_2 z(x_1,y_1,t_1)^{\frac{2q-p+2}{2}} \leq C_3 \left(\frac{\|u_0\|_\infty}{t_0}\right)^{\frac{2q-p+2}{q}} + C_3 R^{-\frac{2q-p+2}{q-p+1}}$$

that is,

$$\sqrt{z(x_1, y_1, t_1)} \leq C \left(\frac{\|u_0\|_\infty}{t_0} \right)^{\frac{1}{q}} + CR^{-\frac{1}{q-p+1}} =: M_2 > 0. \tag{3.5.10}$$

It follows from (3.5.7)-(3.5.10) that

$$\max_{Q^{t_0}_{\tau,R}} z \leq M_3^2, \qquad \text{with } M_3 = \max\{M_0, M_1, M_2\}.$$

Since $z = |\nabla u|^2$ in $\left(B_{R/2}(x_0, y_0) \cap \overline{\Omega}\right) \times (t_0, \tau)$ and $\tau \in (t_0, T)$ is arbitrary, we get

$$|\nabla u| \leq M_3 \quad \text{in } \left(B_{R/2}(x_0, y_0) \cap \overline{\Omega}\right) \times (t_0, T),$$

and we conclude that (x_0, y_0) is not a gradient blow-up point.

□

By combining Lemmas 3.3.3 and 3.5.1, we can now easily obtain a class of initial data whose possible gradient blow-up set is contained in a small neighborhood of the origin.

Lemma 3.5.3. *Let ρ, μ, u_0 be as in Lemma 3.3.3 (ii). Then $GBUS(u_0) \subset [-\rho, \rho] \times \{0\}$.*

Proof. Denote again $\Sigma_\rho = [-\rho/2, \rho/2] \times \{0\}$ and $\Sigma'_\rho = \partial\Omega \setminus ([-\rho, \rho] \times \{0\})$. In view of Lemma 3.5.1, it suffices to show that

$$\sup_{(x,y)\in\Sigma'_\rho,\ t\in(0,T)} |\nabla u(x, y, t)| < \infty. \tag{3.5.11}$$

But (3.5.11) easily follows from a comparison with the function $\bar{U}$ provided in Lemma 3.3.2. Indeed, under the assumptions of Lemma 3.3.3(i), we already know that $u \leq \bar{U}$ in Q_T. Also, $u = \mu y = \bar{U}$ on $\Sigma'_\rho \times (0, T)$. From this, along with (3.3.3), it follows that

$$\partial_\nu \bar{U} \leq \partial_\nu u \leq \mu\, \partial_\nu y \;\; \text{on } \Sigma'_\rho \times (0, T). \tag{3.5.12}$$

From (3.5.12) and $(3.1.1)_2$, we get

$$|\nabla u|^2 \leq \mu^2 + |\partial_\nu u|^2 \leq C \;\; \text{on } \Sigma'_\rho \times (0, T),$$

hence (3.5.11), and the lemma is proved.

□

3.6 Existence of well-prepared initial data : proof of Theorem 3.1.1(i)

We need to construct initial data meeting the requirements from sections 3–5. This will be achieved in the following lemma. Let us fix an even function $\varphi \in C^\infty(\mathbb{R})$ such that $s\varphi'(s) \le 0$, with

$$\varphi(s) = \begin{cases} 1 & \text{for } |s| \le 1/3 \\ 0 & \text{for } |s| \ge 2/3. \end{cases} \tag{3.6.1}$$

Lemma 3.6.1. *Let $\kappa = (q-p)/(q-p+1)$ and let $C_1 = C_1(p,q) > 0$ be given by Lemma 3.4.1. For $\varepsilon \in (0, \min(L_1, L_2/2))$, define*

$$\psi_\varepsilon(y) = \begin{cases} \varphi\Big(\dfrac{y-\varepsilon}{\varepsilon}\Big) & \textit{for } 0 \le y \le \varepsilon \\ \varphi\Big(\dfrac{y-\varepsilon}{L_2}\Big) & \textit{for } y \ge \varepsilon \end{cases} \tag{3.6.2}$$

and let u_0 be defined by

$$u_0(x,y) = \mu y + C_1 \varepsilon^\kappa \varphi\Big(\frac{x}{\varepsilon}\Big)\psi_\varepsilon(y).$$

Next fix $0 < \rho < L_1$ and let $\mu_0 = \mu_0(p,q,\Omega,\rho) > 0$ and $c = c(p,q,\Omega,\rho) > 0$ be given by Lemmas 3.3.2 and 3.3.3. For any $\mu \in (0,\mu_0]$, there exists $\varepsilon_0 = \varepsilon_0(p,q,\Omega,\mu,\rho) > 0$ such that, for all $\varepsilon \in (0,\varepsilon_0]$, the function $u_0 \in \mathcal{V}_\mu$ and satisfies

$$u_0 \textit{ is symmetric with respect to the line } x = 0, \tag{3.6.3}$$

$$\partial_x u_0 \le 0 \quad \textit{in } \Omega_+, \tag{3.6.4}$$

$$\partial_y u_0 \ge \mu/2 \quad \textit{in } \Omega, \tag{3.6.5}$$

$$u_0(x,y) \le \mu\big(y + c\chi_{(-\rho/2,\rho/2)\times(0,L_2)}\big) \quad \textit{in } \Omega, \tag{3.6.6}$$

$$u_0(x,y) \ge C_1\varepsilon^\kappa \quad \textit{in } B_{\varepsilon/3}(0,\varepsilon) \subset \Omega. \tag{3.6.7}$$

Proof. Assume $\varepsilon \le \min(L_1, L_2/12)$. Then

$$\psi_\varepsilon(y) = 0 \quad \text{for } y \notin [\tfrac{\varepsilon}{3}, \tfrac{3L_2}{4}] \tag{3.6.8}$$

(indeed, $y \ge \frac{3L_2}{4}$ implies $\frac{y-\varepsilon}{L_2} \ge \frac{3}{4} - \frac{1}{12} = \frac{2}{3}$) and therefore $u_0 \in \mathcal{V}_\mu$. Properties (3.6.3)-(3.6.4) are clear by the choice of φ.

To check (3.6.5), we note that

$$\partial_y u_0 = \mu + C_1\varepsilon^\kappa \varphi\Big(\frac{x}{\varepsilon}\Big)\psi'_\varepsilon(y).$$

For $0 \le y \le \varepsilon$, we have $\psi'_\varepsilon(y) \ge 0$, hence $\partial_y u_0 \ge \mu$. Whereas, for $y \ge \varepsilon$, we have

$$\psi'_\varepsilon(y) = L_2^{-1}\varphi'\big((y-\varepsilon)/L_2\big) \ge -L_2^{-1}\|\varphi'\|_\infty,$$

hence

$$\partial_y u_0 \geq \mu - C_1 \varepsilon^\kappa L_2^{-1} \|\varphi'\|_\infty \geq \mu/2$$

whenever $\varepsilon^\kappa \leq \mu L_2/(2C_1\|\varphi'\|_\infty)$.

As for (3.6.6), if $C_1\varepsilon^\kappa \leq \mu c$ and $\varepsilon \leq \rho/2$, it immediately follows from $\varphi, \psi_\varepsilon \leq 1$ and $\operatorname{supp}(\varphi) \subset (-1,1)$. Finally, since $\varphi(x/\varepsilon) = 1$ for $|x| \leq \varepsilon/3$ and $\psi_\varepsilon(y) = 1$ for $|y - \varepsilon| \leq \varepsilon/3$, we have (3.6.7). The lemma is proved.

□

Proof of Theorem 3.1.1(i) Let μ and u_0 be as in Lemma 3.6.1.

- The fact that $T_{max}(u_0) < \infty$ follows from Lemma 3.4.1.
- Next, we have $GBUS(u_0) \subset [-\rho, \rho] \times \{0\}$ as a consequence of Lemma 3.5.3.
- Properties (3.1.9)-(3.1.10) follow from Lemma 3.3.1.
- Finally, property (3.1.11) is a consequence of Lemma 3.3.3(ii).

This proves the assertion.

□

3.7 Nondegeneracy of gradient blow-up points

In this section, we show that if u is only "weakly singular" in a neighborhood of a boundary point $(x_0, 0)$, then the singularity is removable.

Lemma 3.7.1. *Let ρ, μ, u_0 be as in Lemma 3.3.3(ii) and let $x_0 \in (-L_1, L_1)$. There exist $c_0 = c_0(p,q) > 0$ such that, if $u_0 \in \mathcal{V}_\mu$ with $T := T_{max}(u_0) < \infty$ and*

$$u(x,y) \leq c_0 y^{(q-p)/(q-p+1)} \qquad \text{in} \quad (B_R(x_0,0) \cap \Omega) \times [t_0, T), \tag{3.7.1}$$

for some $R > 0$ and $t_0 \in (0,T)$, then $(x_0, 0)$ is not a gradient blow-up point.

Proof. Let $x_0 \in (-L_1, L_1)$. Then for some constants $r \in (0,R)$ and $d \in (0, L_2)$, we have that

$$\omega_1 := \Big\{(x,y) \in \mathbb{R}^2; |x - x_0| < r,\ 0 < y < d\Big\} \subset B_R(x_0, 0) \cap \Omega.$$

Setting $\beta = 1/(q-p+1)$, we define the comparison function

$$v = v(x,y,t) = \varepsilon y V^{-\beta} \qquad \text{in } Q := \overline{\omega_1} \times (t_0, T)$$

with

$$V = y + \eta \left(r^2 - (x - x_0)^2\right)(t - t_0),$$

where $\eta, \varepsilon > 0$ are to be determined later. We compute, in Q,

$$v_t = -\varepsilon\beta\eta y (r^2 - (x-x_0)^2) V^{-\beta-1},$$

$$v_x = 2\varepsilon\beta\eta y(x-x_0)(t-t_0)V^{-\beta-1},$$
$$v_y = \varepsilon V^{-\beta} - \varepsilon\beta y V^{-\beta-1} = \varepsilon V^{-\beta}\Big[1-\beta\frac{y}{V}\Big],$$

$$\begin{aligned} v_{xx} &= 2\varepsilon\beta\eta y(t-t_0)V^{-\beta-1} - 4\varepsilon\beta\eta^2 y(x-x_0)^2(t-t_0)^2(-\beta-1)V^{-\beta-2} \\ &= 2\varepsilon\beta\eta(t-t_0)V^{-\beta-1}\Big[y+2(\beta+1)\eta(x-x_0)^2(t-t_0)\frac{y}{V}\Big], \end{aligned}$$

$$v_{yy} = -2\varepsilon\beta V^{-\beta-1} + \varepsilon\beta(\beta+1)yV^{-\beta-2} = \varepsilon\beta V^{-\beta-1}\Big[-2+(\beta+1)\frac{y}{V}\Big],$$

$$\begin{aligned} v_{xy} &= 2\varepsilon\beta\eta(x-x_0)(t-t_0)V^{-\beta-1} - 2\varepsilon\beta(\beta+1)\eta y(x-x_0)(t-t_0)V^{-\beta-2} \\ &= 2\varepsilon\beta\eta(x-x_0)(t-t_0)V^{-\beta-1}\Big[1-(\beta+1)\frac{y}{V}\Big]. \end{aligned}$$

Noting that $\beta < 1$ and $\dfrac{y}{V} \leq 1$, we see that, in Q,

$$0 \leq v_{xx} \leq 2\varepsilon\beta\eta T\big(d+2(\beta+1)\eta r^2 T\big)V^{-\beta-1}, \qquad |v_{xy}| \leq 2\varepsilon\beta\eta r T V^{-\beta-1}$$

and

$$v_{yy} \leq \varepsilon\beta(\beta-1)V^{-\beta-1} < 0.$$

It follows that

$$\begin{aligned} \Delta_p v &= |\nabla v|^{p-2}\Big[\Delta v + (p-2)\frac{v_i v_j v_{ij}}{|\nabla v|^2}\Big] \leq |\nabla v|^{p-2}\big[(p-1)v_{xx} + v_{yy} + (p-2)|v_{xy}|\big] \\ &\leq \varepsilon\beta|\nabla v|^{p-2}V^{-\beta-1}\Big[2(p-1)\eta T\big(d+2(\beta+1)\eta r^2 T\big) + (\beta-1) + 2(p-2)\eta r T\Big]. \end{aligned}$$

On the other hand, we have

$$|\nabla v| \geq |v_y| \geq \varepsilon(1-\beta)V^{-\beta} \geq \varepsilon(1-\beta)(d+\eta T r^2)^{-\beta},$$

hence

$$v_t \geq -\varepsilon\beta\eta d r^2 V^{-\beta-1} \geq -\varepsilon\beta|\nabla v|^{p-2}V^{-\beta-1}\Big[\eta d r^2\left((1-\beta)\varepsilon\right)^{2-p}(d+\eta T r^2)^{(p-2)\beta}\Big].$$

Therefore,

$$\begin{aligned} v_t - \Delta_p v \geq \varepsilon\beta|\nabla v|^{p-2}V^{-\beta-1} \times \Big[&-\eta d r^2\left((1-\beta)\varepsilon\right)^{2-p}(d+\eta T r^2)^{(p-2)\beta} \\ &-2(p-1)\eta T\big(d+2(\beta+1)\eta r^2 T\big) - 2(p-2)\eta r T + (1-\beta)\Big]. \end{aligned}$$

Since also

$$|v_x| \leq 2\varepsilon\beta\eta r T V^{-\beta}, \quad |v_y| \leq \varepsilon V^{-\beta},$$

if we choose $\eta = \eta(p,q,d,r,T,\varepsilon) > 0$ small enough, we get that, in Q,

$$|\nabla v| \leq 2\varepsilon V^{-\beta}$$

and

$$v_t - \Delta_p v \geq \frac{\varepsilon\beta(1-\beta)}{2}|\nabla v|^{p-2}V^{-\beta-1},$$

hence

$$v_t - \Delta_p v \geq \frac{\varepsilon\beta(1-\beta)}{2}|\nabla v|^{p-2}(2\varepsilon)^{-\frac{\beta+1}{\beta}}|\nabla v|^{\frac{\beta+1}{\beta}} = \frac{\beta(1-\beta)}{4}(2\varepsilon)^{-\frac{1}{\beta}}|\nabla v|^q,$$

due to $\beta = 1/(q-p+2)$. If $\varepsilon = \varepsilon_0(p,q) > 0$ is small enough, we thus obtain

$$v_t - \Delta_p v \geq |\nabla v|^q. \tag{3.7.2}$$

Now we shall check the comparison on the parabolic boundary of $\omega_1 \times (t_0,T)$. On $\omega_1 \times \{t_0\}$, choosing $c_0 = 2^{-\beta}\varepsilon_0$, we have

$$u \leq c_0 y^{1-\beta} = 2^{-\beta}\varepsilon_0 y^{1-\beta} \leq v. \tag{3.7.3}$$

On the lateral boundary part $\{(x,y) \in \mathbb{R}^2; |x-x_0| = r, 0 \leq y \leq d\} \times (t_0,T)$, inequality (3.7.3) holds also. On the surface $\{(x,y) \in \mathbb{R}^2; |x-x_0| \leq r, y = 0\} \subset \partial\Omega$, we have for $t_0 < t < T$,

$$u(.,.,t) = v(.,.,t) = 0$$

Finally, on $\{(x,y) \in \mathbb{R}^2; |x-x_0| \leq r, y = d\} \times (t_0,T)$, assuming in addition that η satisfies $\eta \leq dT^{-1}r^{-2}$, we get

$$u \leq c_0 d^{1-\beta} \leq \varepsilon_0 d(d+\eta r^2 T)^{-\beta} \leq v.$$

Using the comparison principle, we get that

$$u \leq v \qquad \text{in } \omega_1 \times (t_0,T). \tag{3.7.4}$$

This implies that

$$|u_y| \leq \varepsilon\left(\eta(r^2 - |x-x_0|^2)(t-t_0)\right)^{-\beta} \leq M_0$$

on $(B_{r/2}((x_0,0)) \cap \partial\Omega) \times ((t_0+T)/2, T)$ for some constant $M_0 > 0$. Lemma 3.7.1 is then a direct consequence of Lemma 3.5.1.

□

3.8 The auxiliary function J and the proof of single-point gradient blow-up

In all this section, we fix ρ, x_1 with

$$0 < \rho < x_1 < L_1 \tag{3.8.1}$$

and we assume that μ and $u_0 \in \mathcal{V}_\mu \cap C^2(\overline{\Omega})$ satisfy the assumption of Theorem 3.1.1(ii) i.e., the corresponding solution of (3.1.1) fulfills properties (3.1.8)-(3.1.11). We denote as before $T = T_{max}(u_0)$.

We consider the auxiliary function

$$J(x,y,t) := u_x + c(x)d(y)F(u),$$

with

$$\begin{cases} F(u) &=& u^\alpha, & \\ c(x) &=& kx, & k>0, \\ d(y) &=& y^{-\gamma} & \end{cases}$$

and

$$1 < \alpha < 1+q-p, \qquad \gamma = (1-2\sigma)(\alpha-1), \tag{3.8.2}$$

where

$$0 < \sigma < \frac{1}{2(q-p+1)} \tag{3.8.3}$$

is fixed. Letting

$$D = (0,x_1) \times (0,y_1),$$

our goal is to use a comparison principle to prove that

$$J \le 0 \quad \text{in } D \times (T/2, T),$$

provided $\alpha > 1$ is chosen close enough to 1 (hence making $\gamma > 0$ small) and $y_1 \in (0, L_2)$ and $k > 0$ are chosen sufficiently small.

3.8.1 Parabolic inequality for the auxiliary function J

By the regularity of u (see Theorem 3.2.2), we have

$$J \in C^{2,1}(Q_T).$$

A key step is to derive a parabolic inequality for J. To this end, we define the operator

$$\mathcal{P}J := J_t - |\nabla u|^{p-2}\Delta J - (p-2)|\nabla u|^{p-4}\left\langle D^2 J\,\nabla u, \nabla u\right\rangle + \mathcal{H}\cdot\nabla J + \mathcal{A}J, \tag{3.8.4}$$

where the functions $\mathcal{H} = \mathcal{H}(x,y,t)$ and $\mathcal{A} = \mathcal{A}(x,y,t)$ are given by formulae (3.9.9)–(3.9.12) below.

Proposition 3.8.1. *Assume (3.8.1), (3.8.3) and let μ, u_0 satisfy the assumption of Theorem 3.1.1(ii). There exist α, γ satisfying (3.8.2), $y_1 \in (0, L_2)$ and $k_0 > 0$, all depending only on $p, q, \Omega, \mu, \|u_0\|_{C^2}, \sigma$, such that, for any $k \in (0, k_0]$, the function J satisfies*

$$\mathcal{P}J \le 0 \qquad \text{in } D \times (T/2, T). \tag{3.8.5}$$

Moreover,

$$\mathcal{H}, \mathcal{A} \in C(D \times (0,T)) \quad \text{and} \quad \mathcal{A} \in L^\infty(D \times (T/2, \tau)) \quad \text{for each } \tau \in (T/2, T). \tag{3.8.6}$$

The proof of Proposition 3.8.1 is very long and technical. In order not to disrupt the main line of argument, we postpone it to section 9 and now present the rest of the proof of Theorem 3.1.1(ii).

3.8.2 Boundary conditions for the auxiliary function J

The verification of the appropriate boundary and initial conditions for the function J depends on an essential way on the applicability to u_x of the Hopf boundary lemma at the points $(x_1, 0)$ and $(0, y_1)$, up to $t = T$. To this end, besides the nondegeneracy of problem (3.1.1), guaranteed by (3.1.11), we also need the following local regularity lemma, which ensures that D^2u remains bounded up to $t = T$ away from the gradient blow-up set.

Lemma 3.8.1. *Let $\rho \in (0, L_1)$ and let μ, u_0 satisfy the assumption of Theorem 3.1.1(ii). Let $\omega' \subset \omega \subset \Omega$ be such that $dist(\omega', \Omega \setminus \omega) > 0$ and $t_0 \in (0,T)$. If*

$$\sup_{\omega \times (0,T)} |\nabla u| < \infty,$$

then

$$\sup_{\omega' \times (t_0,T)} |D^2 u| < \infty.$$

Proof. Introduce an intermediate domain ω'' with $\omega' \subset \omega'' \subset \omega$, such that $dist(\omega'', \Omega \backslash \omega) > 0$ and $dist(\omega', \Omega \setminus \omega'') > 0$. Write the PDE in (3.1.1) as

$$-\nabla \cdot (|\nabla|^{p-2} \nabla u) = |\nabla u|^q - u_t.$$

Using $|\nabla u| \ge \partial_y u \ge \mu/2$ (cf. (3.1.11)) and the boundedness of u_t in Q_T (cf. Lemma 3.3.4), it follows from the elliptic estimate in [80, Theorem V.5.2] that there exists $\theta \in (0,1)$ such that

$$\|\nabla u(\cdot, t)\|_{C^\theta(\overline{\omega}'')} \le C, \quad t_0/2 \le t < T.$$

The boundedness of u_t in Q_T and the interpolation result in [80, Lemma II.3.1] then guarantee the estimate

$$\|\nabla u(x, y, \cdot)\|_{C^\beta([t_0/2, T))} \le C, \quad (x, y) \in \overline{\omega}',$$

where $\beta = \theta/(1+\theta)$. Therefore $\|\nabla u\|_{C^\beta(\overline{\omega}'\times[t_0/2,T))} \leq C$. The conclusion now follows by applying standard Schauder parabolic estimates to the PDE in (3.1.1), rewritten under the form

$$u_t - a_{ij}u_{ij} = f, \quad \text{where } a_{ij} = |\nabla u|^{p-2}\Big[\delta_{ij} + (p-2)\frac{u_i u_j}{|\nabla u|^2}\Big], \quad f = |\nabla u|^q.$$

Indeed, the matrix $(a_{ij}) = (a_{ij}(x,y,t))$ is uniformly elliptic due to (3.1.11) and

$$\begin{aligned} a_{ij}\xi_i\xi_j &= |\nabla u|^{p-2}|\xi|^2 + (p-2)|\nabla u|^{p-4}|\nabla u\cdot\xi|^2 \\ &\geq |\nabla u|^{p-2}|\xi|^2 \geq (\mu/2)^{p-2}|\xi|^2, \end{aligned} \tag{3.8.7}$$

and there exists $\nu \in (0,1)$ such that $a_{ij}, f \in C^\nu(\overline{\omega}' \times [t_0/2, T'])$, for each $T' < T$, with norm independent of T'.

□

Lemma 3.8.2. *Assume (3.8.1)–(3.8.3), let $y_1 \in (0, L_2)$ and let μ, u_0 satisfy the assumption of Theorem 3.1.1(ii). Then*

$$J \in C(\overline{D} \times (0,T)) \tag{3.8.8}$$

and there exists $k_1 > 0$ (depending in particular on y_1) such that, for any $k \in (0, k_1]$, the function J satisfies

$$J \leq 0 \quad \text{on } \partial D \times (T/2, T). \tag{3.8.9}$$

Proof. Since $u = 0$ for $y = 0$ and $|\nabla u| \leq C(\tau)$ in $\Omega \times [0,\tau]$ for each $\tau < T$, we have

$$u \leq C(\tau)y \qquad \text{in } D \times [0,\tau].$$

Due to $\gamma < \alpha$, we may therefore extend the function $c(x)d(y)F(u)$ continuously to be 0 for $y = 0$. Property (3.8.8) then follows from the regularity of u (see Theorem 3.2.2) and we have

$$J = 0 \quad \text{on } (0, x_1) \times \{0\} \times (T/2, T). \tag{3.8.10}$$

By (3.1.10), we have

$$u_x = 0 \quad \text{on } \{0\} \times (0, y_1) \times (0,T),$$

hence

$$J = 0 \quad \text{on } \{0\} \times (0, y_1) \times (T/2, T). \tag{3.8.11}$$

Next, the function $w = u_x$ is ≤ 0 in $\Omega^+ \times (0,T)$ (cf. (3.1.10)) and satisfies there :

$$w_t = a_{ij}(x,y,t)w_{ij} + B(x,y,t)\cdot\nabla w, \tag{3.8.12}$$

with

$$\begin{aligned} a_{ij}(x,y,t) &= |\nabla u|^{p-2}\left[\delta_{ij}+(p-2)\frac{u_i u_j}{|\nabla u|^2}\right], \\ B(x,y,t) &= (p-2)|\nabla u|^{p-4}\nabla u\Delta u+(p-2)(p-4)|\nabla u|^{p-6}\left\langle D^2u\,\nabla u,\nabla u\right\rangle\nabla u \\ &+ q|\nabla u|^{q-2}\nabla u+2(p-2)|\nabla u|^{p-4}(D^2u\,\nabla u). \end{aligned}$$

Fix $\rho < x_3 < x_2 < x_1$. Since $GBUS \subset [-\rho,\rho]\times\{0\}$, we have

$$|\nabla u| \le C \quad \text{in } \big(\Omega\backslash\{(-x_3,x_3)\times(0,y_1/3)\}\big)\times(0,T).$$

It follows from Lemma 3.8.1 that

$$|D^2u| \le C \quad \text{in } \big(\Omega\backslash\{(-x_2,x_2)\times(0,y_1/2)\}\big)\times(T/4,T),$$

hence

$$|B| \le C \quad \text{in } \big(\Omega\backslash\{(-x_2,x_2)\times(0,y_1/2)\}\big)\times(T/4,T).$$

Moreover, the matrix $A(x,y,t)$ is uniformly elliptic (cf. (3.8.7)). We may thus apply the strong maximum principle and the Hopf boundary point Lemma [98, Theorem 6 p. 174], to get

$$\begin{aligned} u_x &\le -c_1 y \quad \text{on } \{x_1\}\times(0,y_1)\times(T/2,T), \\ u_x &\le -c_1 x \quad \text{on } (0,x_1)\times\{y_1\}\times(T/2,T). \end{aligned}$$

Also, since $x_1 > \rho$ and $u(x,0,t)=0$, we get that, for some $c_2 > 0$,

$$u \le c_2 y \quad \text{on } \{x_1\}\times(0,y_1)\times(T/2,T).$$

Consequently, using $\alpha > \gamma+1$ and (3.2.6), we have for $0 < k \le k_1(y_1)$ sufficiently small

$$\begin{aligned} J(x,y_1,t) &\le -c_1x+kxy_1^{-\gamma}\,\|u_0\|_\infty^\alpha \le 0 \quad \text{on } (0,x_1)\times\{y_1\}\times(T/2,T), \\ J(x_1,y,t) &\le -c_1y+kx_1y^{\alpha-\gamma}c_2^\alpha \le 0 \quad \text{on } \{x_1\}\times(0,y_1)\times(T/2,T). \end{aligned}$$

This, along with (3.8.10)-(3.8.11), proves (3.8.9).

□

3.8.3 Initial conditions for J

Lemma 3.8.3. *Assume (3.8.1)–(3.8.3) and let μ, u_0 satisfy the assumption of Theorem 3.1.1(ii). There exists $k_2 > 0$ such that, for any $k \in (0,k_2]$, the function J satisfies*

$$J(x,y,T/2) \le 0 \quad \textit{in } [0,x_1]\times[0,L_2].$$

The proof relies on a parabolic version of the Serrin corner lemma applied to u_x. This is provided by Proposition 3.11.1, which we state and prove in Appendix 2.

Proof. The function $z = u_x$ satisfies equation (3.8.12). We shall apply Proposition 3.11.1 to this equation, with $\tau_1 = T/4$, $\tau_2 = 3T/4$, $X_1 = x_1$, $Y_1 = L_2$, $\hat{X}_1 = L_1$, $\hat{Y}_1 = 2L_2$. We thus need to check the assumption (3.11.2). Let us denote $\hat{D}_T = (0, \hat{c}bX_1) \times (0, \hat{Y}_1) \times (T/4, 3T/4)$.

For $x = 0$ or $y = 0$, we have $u_x = 0$, hence $a_{12} = a_{21} = (p-2)|\nabla u|^{p-2} u_x u_y = 0$. Due to the regularity of u (cf. (3.2.7)), we deduce that

$$a_{12} + a_{21} \geq -C(x \wedge y) \quad \text{in } \overline{\hat{D}_T}. \tag{3.8.13}$$

On the other hand, for $x = 0$ and $0 < y < \hat{Y}_1$, we have $u_{xy} = u_x = 0$. Also, by (3.2.9), we have $|(u_{xy})_x| = |(u_x)_{yx}| \leq C$ in $\overline{\hat{D}_T}$. Consequently $|u_x| + |u_{xy}| \leq Cx$ in $\overline{\hat{D}_T}$. Using (3.2.7) and (3.1.11), we deduce

$$\begin{aligned} B_1 &= (p-2)|\nabla u|^{p-4}(\Delta u)u_x + (p-2)(p-4)|\nabla u|^{p-6} \left\langle D^2 u\, \nabla u, \nabla u \right\rangle u_x \\ &\quad + q|\nabla u|^{q-2}u_x + 2(p-2)|\nabla u|^{p-4}\big(u_{xx}u_x + u_{xy}u_y\big) \\ &\geq -Cx \quad \text{in } \overline{\hat{D}_T}. \end{aligned} \tag{3.8.14}$$

Next, for $y = 0$ and $0 < x < \hat{X}_1$, we have $u_t = 0$ and $u_x = u_{xx} = 0$. Recalling (3.2.7), we thus have

$$\begin{aligned} (u_y)^q = |\nabla u|^q = u_t - \Delta_p u &= -|\nabla u|^{p-2} \left[\Delta u + (p-2)\frac{\langle D^2 u\, \nabla u, \nabla u\rangle}{|\nabla u|^2}\right] \\ &= -|\nabla u|^{p-2}\left[u_{xx} + u_{yy} + (p-2)\frac{u_{xx}u_x^2 + 2u_{xy}u_xu_y + u_{yy}u_y^2}{u_x^2 + u_y^2}\right] = -(p-1)(u_y)^{p-2}u_{yy}. \end{aligned}$$

It follows that, for $0 < x < \hat{X}_1$ and $t \in [T/4, 3T/4]$,

$$\begin{aligned} B_2(x,0,t) &= (p-2)(p-4)|\nabla u|^{p-6}\big(u_{xx}u_x^2 + 2u_{xy}u_xu_y + u_{yy}u_y^2\big)u_y \\ &\quad + (p-2)|\nabla u|^{p-4}(u_{xx} + u_{yy})u_y + q|\nabla u|^{q-2}u_y \\ &\quad + 2(p-2)|\nabla u|^{p-4}\big(u_{yx}u_x + u_{yy}u_y\big) \\ &= (p-2)(p-4)(u_y)^{p-3}u_{yy} + (p-2)(u_y)^{p-3}u_{yy} \\ &\quad + q(u_y)^{q-1} + 2(p-2)(u_y)^{p-3}u_{yy} \\ &= (p-2)(p-1)(u_y)^{p-3}u_{yy} + q(u_y)^{q-1} = (q+2-p)(u_y)^{q-1} \\ &\geq (q+2-p)(\mu/2)^{q-1} > 0. \end{aligned}$$

Therefore, owing to (3.2.7), there exists $\eta > 0$ such that

$$B_2(x,y,t) \geq 0 \quad \text{on } (0, \hat{X}_1) \times [0, \eta] \times [T/4, 3T/4],$$

which implies

$$B_2 \geq -Cy \quad \text{in } \overline{\hat{D}_T}. \tag{3.8.15}$$

In view of (3.8.13)-(3.8.15), we may thus apply Proposition 3.11.1 to deduce

$$u_x(x,y,T/2) \le -c_3xy \quad \text{in } [0,x_1]\times[0,L_2].$$

Let $C := \|\nabla u(\cdot,T/2)\|_\infty$. Since $\alpha > \gamma+1$, we get that, for $k \in (0,k_2]$ with $k_2 > 0$ small enough,

$$J(x,y,T/2) \le -c_3xy + kxC^\alpha y^{\alpha-\gamma} \le [kC^\alpha L_2^{\alpha-\gamma-1} - c_3]xy \le 0 \quad \text{in } [0,x_1]\times[0,L_2].$$

□

3.8.4 Proof of Theorem 3.1.1(ii)

Let α, γ, y_1, k_0 be given by Proposition 3.8.1 and let k_1, k_2 be given by Lemmas 3.8.2-3.8.3. We take $k = \min(k_0,k_1,k_2)$. By these results and the maximum principle, we have

$$J \le 0 \qquad \text{in } D\times(T/2,T). \tag{3.8.16}$$

Integrating inequality (3.8.16) over $(0,x)$ for $0<x<x_1$, with fixed y, we get that

$$u \le Cx^{-2/(\alpha-1)}y^{1-2\sigma} \qquad \text{in } D\times(T/2,T),$$

where $C = C(\alpha,k,\sigma) > 0$. Using that $1-2\sigma > \dfrac{q-p}{q-p+1}$, it follows from the nondegeneracy property in Lemma 3.7.1 that no point $(x_0,0)$ with $0<|x_0|\le\rho$ can be a gradient blow-up point. In view of (3.1.8), we conclude that $GBUS(u_0) = \{(0,0)\}$. □

3.9 Proof of the main parabolic inequality (Proposition 3.8.1)

The proof is quite technical. For sake of clarity, some of the intermediate calculations will be summarized in Lemma 3.9.1 and 3.9.2 below.

We first compute

$$\begin{aligned} J_t &= u_{xt} + cdF'(u)u_t \\ &= (\Delta_p u)_x + (|\nabla u|^q)_x \underbrace{+cdF'\Delta_p u}_{(0_p)} + \underbrace{cdF'|\nabla u|^q}_{(0_q)}. \end{aligned}$$

and

$$\begin{aligned} (\Delta_p u)_x &= |\nabla u|^{p-2}\Delta(u_x) \\ &+ (p-2)\Delta u|\nabla u|^{p-4}\nabla u\cdot\nabla u_x \\ &+ (p-2)|\nabla u|^{p-4}\left\langle D^2u_x\,\nabla u, \nabla u\right\rangle \\ &+ (p-2)(p-4)|\nabla u|^{p-6}\nabla u\cdot\nabla u_x\left\langle D^2u\,\nabla u,\nabla u\right\rangle \\ &+ 2(p-2)|\nabla u|^{p-4}\left\langle D^2u\,\nabla u,\nabla u_x\right\rangle. \end{aligned}$$

Using that $u_x = J - cdF(u)$, we write

$$\nabla u_x = \nabla J - cdF'\nabla u - F \begin{pmatrix} c'd \\ d'c \end{pmatrix},$$

$$\begin{aligned} D^2 u_x &= D^2 J - cdF' D^2 u - cdF'' \begin{pmatrix} u_x^2 & u_x u_y \\ u_x u_y & u_y^2 \end{pmatrix} \\ &\quad -F(u) \begin{pmatrix} c''d & c'd' \\ c'd' & d''c \end{pmatrix} \\ &\quad -F'(u) \begin{pmatrix} 2c'du_x & cd'u_x + c'du_y \\ cd'u_x + c'du_y & 2cd'u_y \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \Delta u_x &= Trace(D^2 u_x) = \Delta J - cdF'\Delta u - cdF''|\nabla u|^2 - F[c''d + d''c] \\ &\quad -2F'c'dJ + 2F'Fc'cd^2 - 2F'(u)d'cu_y, \end{aligned}$$

$$\begin{aligned} \left\langle D^2 u_x, \nabla u, \nabla u \right\rangle &= \left\langle D^2 J, \nabla u, \nabla u \right\rangle - cdF''|\nabla u|^4 - cdF' \left\langle D^2 u, \nabla u, \nabla u \right\rangle - Fc''du_x^2 \\ &\quad -2Fc'd'u_x u_y - Fcd''u_y^2 \\ &\quad -2F'|\nabla u|^2(cd'u_y + dc'u_x) \end{aligned}$$

and

$$\begin{aligned} \left\langle D^2 u, \nabla u, \nabla u_x \right\rangle &= \left\langle D^2 u, \nabla u, \nabla J \right\rangle - cdF' \left\langle D^2 u, \nabla u, \nabla u \right\rangle \\ &\quad -Fc'd\nabla u \cdot \nabla J + cc'd^2 F'F|\nabla u|^2 + d^2F^2(c')^2 J \\ &\quad -d^2F^2(c')^2 cdF + c'cd'dF^2 u_y - Fd'c(u_x u_{xy} + u_y u_{yy}). \end{aligned}$$

Therefore,

$$
\begin{aligned}
(\Delta_p u)_x \;=\;& |\nabla u|^{p-2}\left[\Delta J - cdF'\Delta u - cdF''|\nabla u|^2 - 2c'dF'J\right.\\
&\left.+2cc'd^2FF' - 2d'cF'u_y - F\left(c''d + d''c\right)\right]\\[2ex]
+\;& (p-2)\Delta u|\nabla u|^{p-4}\left[\nabla u\cdot\nabla J - cdF'|\nabla u|^2 - u_y cd'F - c'dFJ + c'cd^2F^2\right]\\[2ex]
+\;& (p-2)|\nabla u|^{p-4}\left[\left\langle D^2J\nabla u,\nabla u\right\rangle - cdF'\left\langle D^2u\nabla u,\nabla u\right\rangle - cdF''|\nabla u|^4\right.\\
&- 2c'dF'|\nabla u|^2J + 2c'cd^2F'F|\nabla u|^2 - 2cd'F'u_y|\nabla u|^2\\
&\left.- c''dF(u_x)^2 - 2c'd'FJu_y + 2c'cd'dF^2u_y - d''cF(u_y)^2\right]\\[2ex]
+\;& (p-2)(p-4)|\nabla u|^{p-6}\left\langle D^2\, u\nabla u,\nabla u\right\rangle\left[\nabla u\cdot\nabla J\right.\\
&\left.- cdF'|\nabla u|^2 - u_ycd'F - c'dFJ + c'cd^2F^2\right]\\[2ex]
+\;& 2(p-2)|\nabla u|^{p-4}\left[\left\langle D^2u\,\nabla u,\nabla J\right\rangle - cdF'\left\langle D^2u\,\nabla u,\nabla u\right\rangle - c'dF\nabla u\cdot\nabla J\right.\\
&+ c'cd^2F'F|\nabla u|^2 + (c'd)^2F^2J - (c'd)^2F^3cd + c'd'cdF^2u_y - d'cFu_{yy}u_y\\
&\left.- d'cF\nabla J\cdot L + d'dc^2F'Fu_yJ - d'd^2c^3F'F^2u_y + (d'cF)^2J - (d'cF)^2cdF\right],
\end{aligned}
$$

where $L = \begin{pmatrix} 0 \\ u_x \end{pmatrix}$. This can be rewritten as

$$
\begin{aligned}
(\Delta_p u)_x \;=\;& |\nabla u|^{p-2}\Delta J + (p-2)|\nabla u|^{p-4}\left\langle D^2J\,\nabla u,\nabla u\right\rangle + \mathcal{H}_1\cdot\nabla J + \mathcal{A}_1(x,y,t)J\\[1ex]
&\left.\begin{aligned}
&-F|\nabla u|^{p-2}\left[c''d + d''c\right] - (p-2)F|\nabla u|^{p-4}\left[c''du_x^2 + d''cu_y^2\right]\\
&-2(p-2)cdF|\nabla u|^{p-4}\left[(c'dF)^2 + (d'cF)^2\right] - (p-1)cdF''|\nabla u|^p\\
&+4(p-2)F^2c'cd'd|\nabla u|^{p-4}u_y
\end{aligned}\right\}\quad (1)\leq 0\\
&\left.\begin{aligned}
&+(4p-6)c'cd^2F'F|\nabla u|^{p-2} - 2(p-1)cd'F'|\nabla u|^{p-2}u_y\\
&-2(p-2)d'd^2F'F^2c^3|\nabla u|^{p-4}u_y
\end{aligned}\right\}\quad (2)\geq 0\\
&\underbrace{-(p-1)cdF'\Delta_p u}_{(3)}\\
&\underbrace{-2(p-2)d'cF|\nabla u|^{p-4}u_yu_{yy}}_{(4)}\\
&\underbrace{-(p-2)cd'Fu_y\left[|\nabla u|^{p-4}\Delta u + (p-4)|\nabla u|^{p-6}\left\langle D^2\,u\nabla u,\nabla u\right\rangle\right]}_{(5)}\\
&\underbrace{+(p-2)c'cd^2F^2\left[|\nabla u|^{p-4}\Delta u + (p-4)|\nabla u|^{p-6}\left\langle D^2\,u\nabla u,\nabla u\right\rangle\right]}_{(6)},
\end{aligned}
$$

where

$$\begin{aligned}\mathcal{H}_1 \;&:=\; (p-2)\left[|\nabla u|^{p-4}\Delta u+(p-4)|\nabla u|^{p-6}\left\langle D^2\,u\nabla u,\nabla u\right\rangle\right]\nabla u\\ &\quad -2(p-2)cd'F|\nabla u|^{p-4}L\\ &\quad -2(p-2)c'dF|\nabla u|^{p-4}\nabla u+2(p-2)|\nabla u|^{p-4}\left(D^2u,\nabla u\right)\end{aligned}\tag{3.9.1}$$

and

$$\begin{aligned}\mathcal{A}_1 \;&:=\; -2(p-1)F'c'd|\nabla u|^{p-2}\\ &\quad -(p-2)Fc'd\left[|\nabla u|^{p-4}\Delta u+(p-4)|\nabla u|^{p-6}\left\langle D^2\,u\nabla u,\nabla u\right\rangle\right]\\ &\quad +2(p-2)|\nabla u|^{p-4}F^2\left[(c'd)^2+(d'c)^2\right]+2(p-2)d'dF'Fc^2|\nabla u|^{p-4}u_y\\ &\quad -2(p-2)c'd'F|\nabla u|^{p-4}u_y.\end{aligned}\tag{3.9.2}$$

On the other hand, we have

$$\begin{aligned}\left(|\nabla u|^q\right)_x &= q|\nabla u|^{q-2}\nabla u\cdot\nabla u_x\\ &= q|\nabla u|^{q-2}\nabla u\cdot\nabla J-qc'dF|\nabla u|^{q-2}J+(7),\end{aligned}$$

where

$$(7):=\underbrace{-qcdF'|\nabla u|^q}_{(7_-)\le 0}+\underbrace{qcc'd^2F^2|\nabla u|^{q-2}-qcd'F|\nabla u|^{q-2}u_y}_{(7_+)\ge 0}.$$

Setting

$$\mathcal{A}_2:=\mathcal{A}_1-qc'dF|\nabla u|^{q-2},\qquad \mathcal{H}_2:=\mathcal{H}_1+q|\nabla u|^{q-2}\nabla u,\tag{3.9.3}$$

we have thus proved the following lemma.

Lemma 3.9.1. *Define the parabolic operator :*

$$\mathcal{L}J:=J_t-|\nabla u|^{p-2}\Delta J-(p-2)|\nabla u|^{p-4}\left\langle D^2J\,\nabla u,\nabla u\right\rangle-\mathcal{H}_2\cdot\nabla J-\mathcal{A}_2J.$$

Then

$$\mathcal{L}J=(0_p)+(0_q)+(1)+(2)+(3)+(4)+(5)+(6)+(7).\tag{3.9.4}$$

As a significant difficulty as compared with the semilinear case $p=2$, many additional terms appear in the contributions (1), (2), (4)–(6), and especially nonlinear, second order terms in (4)–(6). To proceed further, we need to observe that, among the second derivatives of u, u_{yy} needs a special treatment, since it is not immediately expressed in terms of ∇J unlike u_{xx} and u_{xy}. Namely we shall eliminate u_{yy} by expressing it in terms of u_t, ∇u, u_{xx} and u_{xy} by using the equation. Although this will make the computation even more involved, by producing a lot of additional terms, this seems to be the only way to control the effects of u_{yy}. The bound on u_t given by Lemma 3.3.4 will be helpful in this process.

First we have

$$(3)=-(p-1)cdF'\Delta_pu=\underbrace{-cdF'\Delta_pu}_{-(0_p)}\underbrace{-(p-2)cdF'u_t}_{(3_t)}\underbrace{+(p-2)cdF'|\nabla u|^q}_{(3_q)}.\tag{3.9.5}$$

To deal with (4), we set

$$u_{yy} \;=\; \frac{u_t - |\nabla u|^q - \nabla u_x \cdot M}{w},$$

where

$$M := \begin{pmatrix} |\nabla u|^{p-2} + (p-2)|\nabla u|^{p-4}u_x^2 \\ 2(p-2)|\nabla u|^{p-4}u_x u_y \end{pmatrix} \qquad \text{and} \qquad w = |\nabla u|^{p-2} + (p-2)|\nabla u|^{p-4}u_y^2.$$

Since $u_x = J - cdF$, we get

$$\begin{aligned} \nabla u_x \cdot M \;=\; & -cdF'J\left[|(p-1)\nabla u|^{p-2} + (p-2)|\nabla u|^{p-4}u_y^2\right] - 2(p-2)cd'Fu_y|\nabla u|^{p-4}J \\ & +2(p-2)c^2d'dF^2u_y|\nabla u|^{p-4} + c^2d^2F'F\left[|(p-1)\nabla u|^{p-2} + (p-2)|\nabla u|^{p-4}u_y^2\right] \\ & +\nabla J\cdot M - c'dF\left[|\nabla u|^{p-2} + (p-2)|\nabla u|^{p-4}u_x^2\right]. \end{aligned}$$

It follows that

$$\begin{aligned} u_{yy} \;=\; & \frac{u_t - |\nabla u|^q}{w} - \frac{\nabla J\cdot M}{w} + \frac{cdF'J\left[(p-1)|\nabla u|^{p-2} + (p-2)|\nabla u|^{p-4}u_y^2\right]}{w} \\ & + \frac{2(p-2)cd'Fu_y|\nabla u|^{p-4}J}{w} + \frac{c'dF\left[|\nabla u|^{p-2} + (p-2)|\nabla u|^{p-4}u_x^2\right]}{w} \\ & - \frac{c^2d^2FF'\left[(p-1)|\nabla u|^{p-2} + (p-2)|\nabla u|^{p-4}u_y^2\right]}{w} \\ & - \frac{2(p-2)c^2d'dF^2u_y|\nabla u|^{p-4}}{w}. \end{aligned}$$

Now, to treat the contribution of u_{yy} in (5) and (6), we set $N = \begin{pmatrix} u_x^2 \\ 2u_xu_y \end{pmatrix}$ and rewrite

$$\begin{aligned} |\nabla u|^{p-4}\Delta u + (p-4)|\nabla u|^{p-6}\left\langle D^2u\,\nabla u, \nabla u\right\rangle \;&=\; \frac{\Delta_p u}{|\nabla u|^2} - 2|\nabla u|^{p-6}\left\langle D^2u\,\nabla u, \nabla u\right\rangle \\ &=\; \frac{u_t - |\nabla u|^q}{|\nabla u|^2} - 2|\nabla u|^{p-6}\left[\nabla u_x\cdot N + u_y^2u_{yy}\right]. \end{aligned}$$

We have

$$\begin{aligned} \nabla u_x \cdot N \;=\; & \nabla J\cdot N - cdF'J\left[u_x^2 + 2u_y^2\right] + c^2d^2FF'\left[u_x^2 + 2u_y^2\right] - c'dFu_x^2 \\ & -2cd'FJu_y + 2d'dc^2F^2u_y. \end{aligned}$$

The expression in (4) then becomes

$$
\begin{aligned}
(4) \quad = \quad & \left.\frac{2(p-2)d'cF|\nabla u|^{p-4}u_y M\cdot\nabla J}{w}\right\} (4_{\nabla}) \\
& \left.\begin{aligned}
&-\frac{2(p-2)c^2 d' d F' F|\nabla u|^{p-4}u_y\left[(p-1)|\nabla u|^{p-2}+(p-2)|\nabla u|^{p-4}u_y^2\right]J}{w} \\
&-\frac{4(p-2)^2(d'cF)^2|\nabla u|^{p-4}u_y^2|\nabla u|^{p-4}J}{w}
\end{aligned}\right\} (4_J) \\
& \left.\begin{aligned}
&+\frac{2d'cF(p-2)|\nabla u|^{p-4}|\nabla u|^q u_y}{w} \\
&+\frac{2(p-2)c^3 d' d^2 F' F^2|\nabla u|^{p-4}u_y\left[(p-1)|\nabla u|^{p-2}+(p-2)|\nabla u|^{p-4}u_y^2\right]}{w}
\end{aligned}\right\} (4_-) \leq 0 \\
& \left.\begin{aligned}
&+\frac{4(p-2)^2(d'cF)^2 cdF|\nabla u|^{p-4}u_y^2|\nabla u|^{p-4}}{w} \\
&-\frac{2(p-2)d'dc'cF^2 u_y|\nabla u|^{p-4}\left[|\nabla u|^{p-2}+(p-2)|\nabla u|^{p-4}u_x^2\right]}{w}
\end{aligned}\right\} (4_+) \geq 0 \\
& \left.-\frac{2d'cF(p-2)|\nabla u|^{p-4}u_t u_y}{w}\right\} (4_t).
\end{aligned}
$$

The other two terms can be rewritten as

$$
\begin{aligned}
(5) \quad = \quad & \left. 2cd'F(p-2)u_y|\nabla u|^{p-6}N\cdot\nabla J - \frac{2(p-2)cd'Fu_y^3|\nabla u|^{p-6}M\cdot\nabla J}{w} \right\} (5_\nabla) \\
& \left. \begin{array}{l} -2(p-2)c^2d'dF'Fu_y|\nabla u|^{p-6}\left[u_x^2+2u_y^2\right]J - 4(p-2)(cd'F)^2u_y^2|\nabla u|^{p-6}J \\ +\dfrac{2(p-2)c^2d'dF'Fu_y^2|\nabla u|^{p-6}u_y\left[(p-1)|\nabla u|^{p-2}+(p-2)|\nabla u|^{p-4}u_y^2\right]J}{w} \\ +\dfrac{4(p-2)^2(d'cF)^2u_y^4|\nabla u|^{2p-10}J}{w} \end{array} \right\} (5_J) \\
& \left. \begin{array}{l} +(p-2)cd'Fu_y|\nabla u|^{q-2} + 2(p-2)c^3d'd^2F'F^2u_y|\nabla u|^{p-6}\left[u_x^2+2u_y^2\right] \\ +\dfrac{2(p-2)c'cd'dF^2u_y|\nabla u|^{p-6}u_y^2\left[|\nabla u|^{p-2}+(p-2)|\nabla u|^{p-4}u_x^2\right]}{w} \\ -\dfrac{4(p-2)^2(cd'F)^2cdFu_y^4|\nabla u|^{2p-10}}{w} \end{array} \right\} (5_-) \leq 0 \\
& \left. \begin{array}{l} -2(p-2)c'cd'dF^2u_y|\nabla u|^{p-6}u_x^2 + 4(p-2)(cd'F)^2cdFu_y^2|\nabla u|^{p-6} \\ -\dfrac{2(p-2)cd'Fu_y|\nabla u|^{p-4}|u_y^2|\nabla u|^{q-2}}{w} \\ -\dfrac{2(p-2)c^3d'd^2F'F^2u_y|\nabla u|^{p-6}u_y^2\left[(p-1)|\nabla u|^{p-2}+(p-2)|\nabla u|^{p-4}u_y^2\right]}{w} \end{array} \right\} (5_+) \geq 0 \\
& \left. -\frac{(p-2)cd'Fu_yu_t}{|\nabla u|^2} + \frac{2(p-2)cd'Fu_y^2|\nabla u|^{p-6}u_tu_y}{w} \right\} (5_t)
\end{aligned}
$$

and (noticing that (6) can be obtained from (5) by formally multiplying with $\frac{-c'd^2F}{d'u_y}$)

$$
\begin{aligned}
(6) \quad = \quad & \left.-2(p-2)c'cd^2F^2|\nabla u|^{p-6}N\cdot\nabla J + \frac{2(p-2)c'cd^2F^2u_y^2|\nabla u|^{p-6}M\cdot\nabla J}{w}\right\}(6_\nabla) \\
& \left.\begin{aligned}
&+2(p-2)c'c^2d^3F'F^2|\nabla u|^{p-6}\left[u_x^2+2u_y^2\right]J + 4(p-2)c'c^2d'd^2F^3u_y|\nabla u|^{p-6}J \\
&-\frac{2(p-2)c'c^2d^3F'F^2u_y^2|\nabla u|^{p-6}\left[(p-1)|\nabla u|^{p-2}+(p-2)|\nabla u|^{p-4}u_y^2\right]J}{w} \\
&-\frac{4(p-2)^2c'c^2d'd^2F^3|\nabla u|^{p-6}u_y^3|\nabla u|^{p-4}J}{w}
\end{aligned}\right\}(6_J) \\
& \left.\begin{aligned}
&-(p-2)c'cd^2F^2|\nabla u|^{q-2} - 2(p-2)c'c^3d^4F'F^3|\nabla u|^{p-6}\left[u_x^2+2u_y^2\right] \\
&-\frac{2(p-2)(c')^2cd^3F^3|\nabla u|^{p-6}u_y^2\left[|\nabla u|^{p-2}+(p-2)|\nabla u|^{p-4}u_x^2\right]}{w} \\
&+\frac{4(p-2)^2c'c^3d'd^3F^4u_y^3|\nabla u|^{2p-10}}{w}
\end{aligned}\right\}(6_-)\leq 0 \\
& \left.\begin{aligned}
&+2(p-2)(c')^2cd^3F^3|\nabla u|^{p-6}u_x^2 - 4(p-2)c'c^3d'd^3F^4u_y|\nabla u|^{p-6} \\
&+\frac{2(p-2)c'cd^2F^2|\nabla u|^{p-4}u_y^2|\nabla u|^{q-2}}{w} \\
&+\frac{2(p-2)c'c^3d^4F'F^3|\nabla u|^{p-6}u_y^2\left[(p-1)|\nabla u|^{p-2}+(p-2)|\nabla u|^{p-4}u_y^2\right]}{w}
\end{aligned}\right\}(6_+)\geq 0 \\
& \left.+\frac{(p-2)c'cd^2F^2u_t}{|\nabla u|^2} - \frac{2(p-2)c'cd^2F^2u_y^2|\nabla u|^{p-6}u_t}{w}.\right\}(6_t)
\end{aligned}
$$

We shall now collect and relabel the numerous positive and negative terms that we just obtained, when expanding (1)–(7) in the process of eliminating u_{yy}. A number of positive and negative terms will then be paired together according to certain cancellations. Then, the remaining positive terms, as well as the terms involving u_t, will be eventually controlled by using the negative terms.

Using that $d'\leq 0$ and $F', F'', u_y \geq 0$, we first have positive terms :

$$
\begin{aligned}
(a) &:= -2(p-2)c^3d'd^2F'F^2|\nabla u|^{p-4}u_y \\
(b) &:= -\frac{2(p-2)cd'Fu_y^2|\nabla u|^{p-6}|\nabla u|^q u_y}{w} \\
(c) &:= -\frac{2(p-2)c^3d'd^2F'F^2u_y|\nabla u|^{p-6}u_y^2\left[(p-1)|\nabla u|^{p-2}+(p-2)|\nabla u|^{p-4}u_y^2\right]}{w} \\
(d) &:= +2(p-2)(c')^2cd^3F^3|\nabla u|^{p-6}u_x^2 \\
(e) &:= +\frac{2(p-2)c'c^3d^4F'F^3|\nabla u|^{p-6}u_y^2\left[(p-1)|\nabla u|^{p-2}+(p-2)|\nabla u|^{p-4}u_y^2\right]}{w} \\
(f) &:= \left.-\frac{2(p-2)d'dc'cF^2u_y|\nabla u|^{p-4}\left[|\nabla u|^{p-2}+(p-2)|\nabla u|^{p-4}u_x^2\right]}{w}\right\}(f_1) \\
&\qquad \left.-2(p-2)c'cd'dF^2u_y|\nabla u|^{p-6}u_x^2\right\}(f_2)
\end{aligned}
$$

$$
\begin{aligned}
(g) &:= -4(p-2)c'c^3d'd^3F^4u_y|\nabla u|^{p-6}\\
(h) &:= \underbrace{+\frac{4(p-2)^2(d'cF)^2cdF|\nabla u|^{p-4}u_y^2|\nabla u|^{p-4}}{w}}_{(h_1)} + \underbrace{4(p-2)(cd'F)^2cdFu_y^2|\nabla u|^{p-6}}_{(h_2)}\\
(i) &:= -qd'cF|\nabla u|^{q-2}u_y\\
(j) &:= \underbrace{+qc'cd^2F^2|\nabla u|^{q-2}}_{(j_1)} \underbrace{+\frac{2(p-2)c'cd^2F^2|\nabla u|^{p-4}u_y^2|\nabla u|^{q-2}}{w}}_{(j_2)}\\
(l) &:= -2(p-1)cd'F'|\nabla u|^{p-2}u_y\\
(m) &:= +(4p-6)cc'd^2F'F|\nabla u|^{p-2}.
\end{aligned}
$$

They give rise to the following decompositions :

$$
\left\{
\begin{array}{lcl}
(2) & = & (m)+(l)+(a)\\
(4_+) & = & (f_1)+(h_1)\\
(5_+) & = & (b)+(c)+(h_2)+(f_2)\\
(6_+) & = & (d)+(e)+(g)+(j_2)\\
(7_+) & = & (i)+(j_1).
\end{array}
\right.
\tag{3.9.6}
$$

We next have terms with a negative sign :

$$
\begin{aligned}
(\tilde{a}) &:= +2(p-2)c^3d'd^2F'F^2u_y|\nabla u|^{p-6}\left[u_x^2+2u_y^2\right]\\
(\tilde{b}) &:= +\frac{2d'cF(p-2)|\nabla u|^{p-4}|\nabla u|^q u_y}{w}\\
(\tilde{c}) &:= +\frac{2(p-2)c^3d'd^2F'F^2|\nabla u|^{p-4}u_y\left[(p-1)|\nabla u|^{p-2}+(p-2)|\nabla u|^{p-4}u_y^2\right]}{w}\\
(\tilde{d}) &:= -2(p-2)cdF|\nabla u|^{p-4}(c'dF)^2\\
(\tilde{e}) &:= -2(p-2)c'c^3d^4F'F^3|\nabla u|^{p-6}\left[u_x^2+2u_y^2\right]\\
(\tilde{f}) &:= \left.\frac{2(p-2)c'cd'dF^2u_y|\nabla u|^{p-6}u_y^2\left[|\nabla u|^{p-2}+(p-2)|\nabla u|^{p-4}u_x^2\right]}{w}\right\}(\tilde{f}_1)\\
&\quad \left.+4(p-2)c'cd'dF^2|\nabla u|^{p-4}u_y\right\}(\tilde{f}_2)
\end{aligned}
$$

$$
\begin{aligned}
(\tilde g) &:= +(p-2)d'cF|\nabla u|^{q-2}u_y\\
(\tilde h) &:= \underbrace{-(p-1)cdF''|\nabla u|^p}_{(\tilde h_1)}\underbrace{+(p-1-q)|\nabla u|^q cdF'}_{(\tilde h_2)}\underbrace{-(p-1)cd''F|\nabla u|^{p-4}u_y^2}_{(\tilde h_3)}\\
(\tilde i) &:= \underbrace{-cd''F|\nabla u|^{p-4}u_x^2}_{(\tilde i_1)}\underbrace{-(p-2)c'cd^2F^2|\nabla u|^{q-2}}_{(\tilde i_1)}\underbrace{+\frac{4(p-2)^2c'c^3d'd^3F^4u_y^3|\nabla u|^{2p-10}}{w}}_{(\tilde i_3)}\\
(\tilde j) &:= \underbrace{-2(p-2)(cd'F)^2cdF|\nabla u|^{p-4}}_{(\tilde j_1)}\underbrace{-\frac{4(p-2)^2(cd'F)^2cdFu_y^4|\nabla u|^{2p-10}}{w}}_{(\tilde j_2)}\\
(\tilde l) &:= -\frac{2(p-2)(c')^2cd^3F^3|\nabla u|^{p-6}u_y^2\left[|\nabla u|^{p-2}+(p-2)|\nabla u|^{p-4}u_x^2\right]}{w}.
\end{aligned}
$$

With these terms, we have the following decompositions (using $c''=0$) :

$$
\begin{cases}
(1) = (\tilde d) + (\tilde f_2) + (\tilde h_1) + (\tilde h_3) + (\tilde i_1) + (\tilde j_1)\\
(4_-) = (\tilde b) + (\tilde c)\\
(5_-) = (\tilde a) + (\tilde g) + (\tilde j_2) + (\tilde f_1)\\
(6_-) = (\tilde e) + (\tilde i_2) + (\tilde i_3) + (\tilde l)\\
(7_-) = (\tilde h_2) - (0_q) - (3_q).
\end{cases}
\tag{3.9.7}
$$

It follows from (3.9.4) in Lemma 3.9.1 and (3.9.5)–(3.9.7) that

$$
\begin{aligned}
\mathcal{L}J = \;& \big[(0_q)+(0_p)\big]\\
&+\big[(\tilde d)+(\tilde f_2)+(\tilde h_1)+(\tilde h_3)+(\tilde i_1)+(\tilde j_1)\big]\\
&+\big[(m)+(l)+(a)\big]\\
&+\big[-(0_p)+(3_t)+(3_q)\big]\\
&+\big[(\tilde b)+(\tilde c)\big]+\big[(f_1)+(h_1)\big]+\big[(4_\nabla)+(4_J)+(4_t)\big]\\
&+\big[(\tilde a)+(\tilde g)+(\tilde j_2)+(\tilde f_1)\big]+\big[(b)+(c)+(h_2)+(f_2)\big]+\big[(5_\nabla)+(5_J)+(5_t)\big]\\
&+\big[(\tilde e)+(\tilde i_2)+(\tilde i_3)+(\tilde l)\big]+\big[(d)+(e)+(g)+(j_2)\big]+\big[(6_\nabla)+(6_J)+(6_t)\big]\\
&+\big[(\tilde h_2)-(0_q)-(3_q)\big]+\big[(i)+(j_1)\big].
\end{aligned}
$$

Reordering the terms, we obtain

$$
\begin{aligned}
\mathcal{L}J = \;& (a)+(b)+(c)+(d)+(e)+(f)+(g)+(h)+(i)+(j)+(l)+(m)\\
&+(\tilde a)+(\tilde b)+(\tilde c)+(\tilde d)+(\tilde e)+(\tilde f)+(\tilde g)+(\tilde h)+(\tilde i)+(\tilde j)+(\tilde l)\\
&+\big[(3_t)+(4_t)+(5_t)+(6_t)\big]\\
&+\big[(4_\nabla)+(5_\nabla)+(6_\nabla)\big]+\big[(4_J)+(5_J)+(6_J)\big].
\end{aligned}
\tag{3.9.8}
$$

Collecting the terms with J (reps., ∇J) in (3.9.8), together with those in $\mathcal{A}_2$ (resp., $\mathcal{H}_2$) and using (3.9.3), we define

$$\begin{aligned}\mathcal{H} \;:=\; & \mathcal{H}_1 + q|\nabla u|^{q-2}\nabla u + \frac{2(p-2)d'cF|\nabla u|^{p-4}u_y M}{w} \\ & + 2cd'F(p-2)u_y|\nabla u|^{p-6}N - \frac{2(p-2)cd'Fu_y^3|\nabla u|^{p-6}M}{w} \\ & - 2(p-2)c'cd^2F^2|\nabla u|^{p-6}N + \frac{2(p-2)c'cd^2F^2u_y^2|\nabla u|^{p-6}M}{w}\end{aligned} \tag{3.9.9}$$

and

$$\begin{aligned}\mathcal{A} \;:=\; & \mathcal{A}_1 - qc'dF|\nabla u|^{q-2} \\ & - \frac{2(p-2)c^2d'dF'F|\nabla u|^{p-4}u_y\left[(p-1)|\nabla u|^{p-2} + (p-2)|\nabla u|^{p-4}u_y^2\right]}{w} \\ & - \frac{4(p-2)^2(d'cF)^2|\nabla u|^{p-4}u_y^2|\nabla u|^{p-4}}{w} \\ & - 2(p-2)c^2d'dF'Fu_y|\nabla u|^{p-6}\left[u_x^2 + 2u_y^2\right] - 4(p-2)(cd'F)^2u_y^2|\nabla u|^{p-6} \\ & + \frac{2(p-2)c^2d'dF'Fu_y^2|\nabla u|^{p-6}u_y\left[(p-1)|\nabla u|^{p-2} + (p-2)|\nabla u|^{p-4}u_y^2\right]}{w} \\ & + \frac{4(p-2)^2(d'cF)^2u_y^4|\nabla u|^{2p-10}}{w} \\ & + 2(p-2)c'c^2d^3F'F^2|\nabla u|^{p-6}\left[u_x^2 + 2u_y^2\right] + 4(p-2)c'c^2d'd^2F^3u_y|\nabla u|^{p-6} \\ & - \frac{2(p-2)c'c^2d^3F'F^2u_y^2|\nabla u|^{p-6}\left[(p-1)|\nabla u|^{p-2} + (p-2)|\nabla u|^{p-4}u_y^2\right]}{w} \\ & - \frac{4(p-2)^2c'c^2d'd^2F^3|\nabla u|^{p-6}u_y^3|\nabla u|^{p-4}}{w},\end{aligned} \tag{3.9.10}$$

where we recall that

$$\begin{aligned}\mathcal{H}_1 \;:=\; & (p-2)\left[|\nabla u|^{p-4}\Delta u + (p-4)|\nabla u|^{p-6}\left\langle D^2 u\nabla u, \nabla u\right\rangle\right]\nabla u \\ & -2(p-2)cd'F|\nabla u|^{p-4}L \\ & -2(p-2)c'dF|\nabla u|^{p-4}\nabla u + 2(p-2)|\nabla u|^{p-4}\left(D^2u, \nabla u\right),\end{aligned} \tag{3.9.11}$$

with $L = \begin{pmatrix} 0 \\ u_x \end{pmatrix}$, and

$$\begin{aligned}\mathcal{A}_1 \;:=\; & -2(p-1)F'c'd|\nabla u|^{p-2} \\ & -(p-2)Fc'd\left[|\nabla u|^{p-4}\Delta u + (p-4)|\nabla u|^{p-6}\left\langle D^2 u\nabla u, \nabla u\right\rangle\right] \\ & +2(p-2)|\nabla u|^{p-4}F^2\left[(c'd)^2 + (d'c)^2\right] + 2(p-2)d'dF'Fc^2|\nabla u|^{p-4}u_y \\ & -2(p-2)c'd'F|\nabla u|^{p-4}u_y.\end{aligned} \tag{3.9.12}$$

Finally observing that

$$(a) + (\tilde{a}) \leq 0, \quad (b) + (\tilde{b}) \leq 0, \quad (c) + (\tilde{c}) \leq 0, \quad (d) + (\tilde{d}) \leq 0, \quad (e) + (\tilde{e}) \leq 0$$

and using $(\tilde{f}), (\tilde{i}), (\tilde{j}), (\tilde{l}) \leq 0$, we obtain the following lemma.

Lemma 3.9.2. *Recalling the definition (3.8.4) of the parabolic operator $\mathcal{P}$:*

$$\mathcal{P}J := J_t - |\nabla u|^{p-2}\Delta J - (p-2)|\nabla u|^{p-4}\left\langle D^2 J\,\nabla u, \nabla u\right\rangle - \mathcal{H}\cdot\nabla J - \mathcal{A}(x,y,t)J,$$

we have

$$\begin{aligned}\mathcal{P}J \;\leq\;& (f) + (g) + (h) + (i) + (j) + (l) + (m) + (\tilde{g}) + (\tilde{h}) \\ &+(3_t) + (4_t) + (5_t) + (6_t).\end{aligned}$$

Completion of proof of Proposition 3.8.1 Starting from Lemma 3.9.2, we shall estimate the remaining positive and u_t terms by the key negative terms $(\tilde{g})$ and $(\tilde{h})$, after appropriate choice of the parameters. An essential tool in this step will the Bernstein-type estimates (see [7, Theorem 1.2])

$$|\nabla u| \leq C_0 y^{-1/(q-p+1)} \quad \text{and} \quad u \leq C_0 y^{(q-p)/(q-p+1)} \quad \text{in } [0,x_1]\times(0,L_2]\times(0,T), \tag{3.9.13}$$

where $C_0 = C_0(p,q,\Omega,\mu,\|\nabla u_0\|_\infty) > 0$, and we will use also the lower bound from (3.1.11) :

$$|\nabla u| \geq u_y \geq \delta_0 = \mu/2 > 0. \tag{3.9.14}$$

First, using $w \geq |\nabla u|^{p-2}$, we get

$$(f) \leq -2(p-2)pc'cd'dF^2 u_y|\nabla u|^{p-4}.$$

Assume $y_1 \leq 1$. Due to (3.9.13), we have

$$dF \leq C_0^\alpha y^{-\gamma+\alpha(q-p)/(q-p+1)} \leq C_0^\alpha. \tag{3.9.15}$$

Here we used that $\gamma \leq \alpha - 1$ and hence $\alpha(q-p)/(q-p+1) - \gamma \geq 1 - \alpha/(q-p+1) \geq 0$. Assume $k_0 = k_0(p,q,\Omega,\mu,\|\nabla u_0\|_\infty) > 0$ sufficiently small so that

$$0 < k_0 \leq \frac{|\nabla u|^{q-p+2}}{4pdF} \qquad \text{and} \qquad 0 < k_0^3 \leq \frac{|\nabla u|^{q-p+4}}{8x^2F^3d^3}, \tag{3.9.16}$$

which is possible due to (3.9.14), $x \leq L_1$ and (3.9.15)). We then have

$$\begin{aligned}&(f) + (g) + (\tilde{g}) \\ \leq\;& \underbrace{\frac{p-2}{2}cd'Fu_y}_{\leq 0}\left[\underbrace{|\nabla u|^{q-2} - 8k^3x^2F^3d^3|\nabla u|^{p-6} + |\nabla u|^{q-2} - 4pk|\nabla u|^{p-4}dF}_{\geq 0}\right] \leq 0.\end{aligned}$$

Next, we have

$$\begin{aligned}(h) &\leq 4(p-2)(p-1)(cd'F)^2 cdF u_y^2 |\nabla u|^{p-6}\\ (j) &\leq (q+2(p-2))c'cd^2F^2|\nabla u|^{q-2}\end{aligned}$$

and, owing to Lemma 3.3.4,

$$\begin{aligned}(3_t) &\leq (p-2)cdF'K\\ (4_t)+(5_t) &\leq 5(p-2)c|d'|F\frac{|u_t|u_y}{|\nabla u|^2} \leq 5(p-2)c|d'|F\frac{Ku_y}{|\nabla u|^2}\\ (6_t) &\leq \frac{3(p-2)c'cd^2F^2K}{|\nabla u|^2}.\end{aligned}$$

Here and in the rest of the proof, K denotes a constant depending on $\|u_0\|_{C^2}$, p and q. Consequently

$$\begin{aligned}\mathcal{P}J &\leq (h)+(i)+(j)+(l)+(m)+(3_t)+(4_t)+(5_t)+(6_t)+(\tilde{h})\\ &\leq 4(p-2)(p-1)(cd'F)^2cdFu_y^2|\nabla u|^{p-6}+(q+2(p-2))c'cd^2F^2|\nabla u|^{q-2}\\ &\quad +5(p-2)c|d'|F\frac{Ku_y}{|\nabla u|^2}+2(p-1)c|d'|F'|\nabla u|^{p-2}u_y\\ &\quad +(4p-6)cc'd^2F'F|\nabla u|^{p-2}+q|d'|cF|\nabla u|^{q-2}u_y\\ &\quad +\frac{3(p-2)c'cd^2F^2K}{|\nabla u|^2}+(p-2)cdF'K\\ &\quad -(p-1)cdF''|\nabla u|^p+(p-1-q)|\nabla u|^q cdF'-(p-1)cd''F|\nabla u|^{p-4}u_y^2\end{aligned}$$

hence

$$\begin{aligned}\frac{\mathcal{P}J}{cdF} &\leq -(q-p+1)\alpha\frac{|\nabla u|^q}{u}-(p-1)\alpha(\alpha-1)\frac{|\nabla u|^p}{u^2}-(p-1)\frac{\gamma(\gamma+1)|\nabla u|^{p-4}u_y^2}{y^2}\\ &+\frac{2\gamma\alpha(p-1)}{y}\frac{|\nabla u|^{p-2}u_y}{u}+4(p-2)(p-1)k^2\gamma^2x^2u^{2\alpha}y^{-2\gamma}\frac{|\nabla u|^{p-6}u_y^2}{y^2}\\ &+5(p-2)\gamma K|\nabla u|^{-2}\frac{u_y}{y}+(4p-6)k\alpha\frac{u^{\alpha-1}|\nabla u|^{p-2}}{y^\gamma}+(q+2(p-2))k\frac{u^\alpha|\nabla u|^{q-2}}{y^\gamma}\\ &+q\gamma\frac{|\nabla u|^{q-2}u_y}{y}+(p-2)\frac{\alpha K}{u}+\frac{3(p-2)ky^{-\gamma}u^\alpha K}{|\nabla u|^2}.\end{aligned}\tag{3.9.17}$$

Using Young's inequality, we obtain that

$$\frac{2\gamma\alpha}{y}\frac{|\nabla u|^{p-2}u_y}{u}\leq\alpha(\alpha-1)\frac{|\nabla u|^p}{u^2}+\frac{\alpha\gamma^2}{\alpha-1}\frac{|\nabla u|^{p-4}u_y^2}{y^2},$$

hence

$$\begin{aligned}&\frac{2\gamma\alpha(p-1)}{y}\frac{|\nabla u|^{p-2}u_y}{u}-(p-1)\alpha(\alpha-1)\frac{|\nabla u|^p}{u^2}-(p-1)\frac{\gamma(\gamma+1)|\nabla u|^{p-4}u_y^2}{y^2}\\ &\leq\left(\frac{\alpha\gamma^2}{\alpha-1}-\gamma(\gamma+1)\right)\frac{(p-1)|\nabla u|^{p-4}u_y^2}{y^2}=-\frac{2\gamma\sigma(p-1)|\nabla u|^{p-4}u_y^2}{y^2}.\end{aligned}\tag{3.9.18}$$

By (3.9.13), we have also

$$u|\nabla u|^{q-p} \le C_0^{q-p+1}. \tag{3.9.19}$$

Using again Young's inequality, and (3.9.19), we have

$$q\gamma\frac{|\nabla u|^{q-2}u_y}{y} \le \frac{\sigma\gamma|\nabla u|^{p-4}u_y^2}{2y^2} + \frac{q^2\gamma}{2\sigma}|\nabla u|^{2q-p} \le \frac{\sigma\gamma|\nabla u|^{p-4}u_y^2}{2y^2} + \frac{q^2\gamma C_0^{q-p+1}}{2\sigma}\frac{|\nabla u|^q}{u}, \tag{3.9.20}$$

$$5(p-2)\gamma K|\nabla u|^{-2}\frac{u_y}{y} \le \frac{\sigma\gamma|\nabla u|^{p-4}u_y^2}{2y^2} + \left[\frac{25(p-2)^2\gamma K^2|\nabla u|^{-p-q}u}{2\sigma}\right]\frac{|\nabla u|^q}{u}. \tag{3.9.21}$$

Next, using (3.9.13) and (3.9.14), it follows that

$$\begin{aligned} k\alpha(4p-6)\frac{u^{\alpha-1}|\nabla u|^{p-2}}{y^\gamma} &\le k\alpha(4p-6)C_0^{\alpha-1}y^{(\alpha-1)(2\sigma-\frac{1}{q-p+1})+2}\frac{|\nabla u|^2}{u_y^2}\frac{|\nabla u|^{p-4}u_y^2}{y^2} \\ &\le k\alpha(4p-6)\delta_0^{-2}C_0^{\alpha+1}y^{(\alpha-1)(2\sigma-\frac{1}{q-p+1})+\frac{2(q-p)}{q-p+1}}\frac{|\nabla u|^{p-4}u_y^2}{y^2}, \end{aligned} \tag{3.9.22}$$

$$\begin{aligned} (q+2(p-2))k\frac{u^\alpha|\nabla u|^{q-2}}{y^\gamma} &\le (q+2(p-2))kC_0^{\alpha+q-p}y^{(\alpha-1)(2\sigma-\frac{1}{q-p+1})+2}\frac{|\nabla u|^2}{u_y^2}\frac{|\nabla u|^{p-4}u_y^2}{y^2} \\ &\le (q+2(p-2))k\frac{C_0^{\alpha+q-p+2}}{\delta_0^2}y^{(\alpha-1)(2\sigma-\frac{1}{q-p+1})+\frac{2(q-p)}{q-p+1}}\frac{|\nabla u|^{p-4}u_y^2}{y^2} \end{aligned} \tag{3.9.23}$$

and

$$k^2\gamma^2\frac{x^2u^{2\alpha}}{y^{2\gamma}}\frac{|\nabla u|^{p-6}u_y^2}{y^2} \le k^2\gamma^2\frac{x^2C_0^{2\alpha}y^{-2\gamma+2\alpha(q-p)/(q-p+1)}}{\delta_0^2}\frac{|\nabla u|^{p-4}u_y^2}{y^2}. \tag{3.9.24}$$

Finally, using that $u \ge \mu y$, we have

$$\frac{\alpha K}{u} = \frac{\alpha Ky^2}{u|\nabla u|^{p-4}u_y^2}\frac{|\nabla u|^{p-4}u_y^2}{y^2} \le \frac{\alpha Ky}{\mu\delta_0^{p-2}}\frac{|\nabla u|^{p-4}u_y^2}{y^2}. \tag{3.9.25}$$

Using the bounds $u \le \|u_0\|_\infty$ and (3.9.14) we have

$$\frac{ky^{-\gamma}u^\alpha K}{|\nabla u|^2} \le \frac{k\|u_0\|_\infty^\alpha Ky^{2-\gamma}}{\delta_0^p}\frac{|\nabla u|^{p-4}u_y^2}{y^2}. \tag{3.9.26}$$

Combining (3.9.17)-(3.9.26), we get that

$$\begin{aligned} \frac{\mathcal{P}J}{cdF} \le\ & \frac{|\nabla u|^q}{u}\left(\frac{q^2\gamma C_0^{q-p+1}}{2\sigma} + 25(p-2)^2\gamma\frac{K^2\|u_0\|_\infty}{2\sigma\delta_0^{q+p}} - \alpha(q-p+1)\right) \\ & +\frac{|\nabla u|^{p-4}u_y^2}{y^2}\Big\{k\left((q+2(p-2))C_0^{\alpha+q-p+2}\delta_0^{-2}\right)y^{(\alpha-1)(2\sigma-\frac{1}{q-p+1})+\frac{2(q-p)}{q-p+1}} \\ & +k\left(\alpha(4p-6)\delta_0^{-2}C_0^{\alpha+1}\right)y^{(\alpha-1)(2\sigma-\frac{1}{q-p+1})+\frac{2(q-p)}{q-p+1}} \\ & +4(p-2)(p-1)k^2\gamma^2x^2C_0^{2\alpha}y^{-2\gamma+\frac{2\alpha(q-p)}{q-p+1}}\delta_0^{-2} \\ & +(p-2)\frac{\alpha Ky}{\mu\delta_0^{p-2}} + 3(p-2)\frac{k\|u_0\|_\infty^\alpha Ky^{2-\gamma}}{\delta_0^{p-2}} - (2p-3)\sigma\gamma\Big\}. \end{aligned} \tag{3.9.27}$$

Now, we may choose $\alpha = \alpha(p, q, \Omega, \mu, \|u_0\|_{C^2}, \sigma) > 1$ close enough to 1 in such a way that $\gamma = (\alpha - 1)(1 - 2\sigma)$ is small enough to satisfy

$$\gamma\left[\frac{q^2}{2\sigma}C_0^{q-p+1} + 25(p-2)^2\frac{K^2\|u_0\|_\infty}{2\sigma\delta_0^{q-p}}\right] \leq (q-p+1)\alpha \tag{3.9.28}$$

and

$$(\alpha-1)\left(2\sigma - \frac{1}{q-p+1}\right) + \frac{2(q-p)}{q-p+1} \geq 0, \tag{3.9.29}$$

$$\alpha\frac{q-p}{q-p+1} - (\alpha-1)(1-2\sigma) \geq 0. \tag{3.9.30}$$

Finally, once α is fixed (hence γ is also fixed small), recalling that $y \leq y_1 \leq 1$, $x \leq L_1$ and $\gamma \leq 2$, we take $k_0 = k_0(p, q, \Omega, \mu, \|u_0\|_{C^2}, \sigma) > 0$ possibly smaller, in such a way that

$$\begin{aligned} &k_0\left((q+2(p-2))C_0^{\alpha+q-p+2}\delta_0^{-2} + \alpha(4p-6)\delta_0^{-2}C_0^{\alpha+1}\right) \\ &+k_0^2\left(4(p-2)(p-1)\gamma^2L_1^2C_0^{2\alpha}\delta_0^{-2}\right) + k_0\frac{3(p-2)\|u_0\|_\infty^\alpha K}{\delta_0^{p-2}} \leq \frac{2p-3}{2}\sigma\gamma, \end{aligned} \tag{3.9.31}$$

and next we take $y_1 = y_1(p, q, \Omega, \mu, \|u_0\|_{C^2}, \sigma) > 0$ small enough such that

$$\frac{(p-2)\alpha K y_1}{\mu\delta_0^{p-2}} \leq \frac{2p-3}{2}\sigma\gamma. \tag{3.9.32}$$

Then it follows from (3.9.27) that

$$\mathcal{P}J \leq 0 \qquad \text{in } D \times (T/2, T). \tag{3.9.33}$$

Finally, we need to check (3.8.6). The continuity statement is clear from the definition of $\mathcal{A}, \mathcal{H}$. Let us show that $\mathcal{A}$ is bounded in $D \times (T/2, \tau)$ for each $\tau < T$. For this purpose, let us observe that due to $|\nabla u| \leq C(\tau), u \leq C(\tau)y$ and $\alpha - 1 \geq \gamma$, we have for $y \leq 1$ and $\tau \in (T/2, T)$

$$|F'd| = \alpha u^{\alpha-1}y^{-\gamma} \leq C^{\alpha-1}(\tau)y^{\alpha-1-\gamma} \leq \alpha C^{\alpha-1}(\tau) \tag{3.9.34}$$

$$|Fd'| = \gamma u^\alpha y^{-\gamma-1} \leq \gamma C^\alpha(\tau) \tag{3.9.35}$$

$$|Fd| = u^\alpha y^{-\gamma} \leq C^\alpha(\tau). \tag{3.9.36}$$

We also have by (3.9.13) and (3.9.14) :

$$|\nabla u|^r \leq \begin{cases} C^r(\tau), & \text{if } r > 0, \\ \delta_0^r, & \text{if } r < 0. \end{cases} \tag{3.9.37}$$

The assertion then follows easily from (3.9.10), (3.9.12) and (3.2.7).

□

3.10 APPENDIX 1. Proof of regularity results (Theorem 3.2.2)

Proof of Theorem 3.2.2(i) We assume $\delta_0 := \inf_{Q_T} |\nabla u| > 0$. Fix $0 < \tau < T$ and let $M_\tau = \|\nabla u\|_{L^\infty(Q_\tau)} < \infty$. We pick smooth functions $b = b_\tau$ and $F = F_\tau$ with the following properties :

$$b(s) = s^{(p-2)/2} \quad \text{and} \quad F(s) = s^{q/2} \quad \text{for } \delta_0^2 \le s \le M_\tau^2,$$

$$\inf_{[0,\infty)} b > 0, \qquad b' \ge 0, \qquad b'(s) = 0 \;\; \text{for } s \text{ large enough}, \qquad F \ge 0, \qquad \sup_{[0,\infty)} F < \infty.$$

By the results in [80, Chapter V] (see Remark 3.10.1 below for details), there exists a (unique) classical solution $v = v_\tau \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times (0,\tau)) \cap C(\overline{Q}_\tau)$, for some $\alpha \in (0,1)$, of the problem

$$\begin{aligned} v_t - \nabla \cdot (b(|\nabla v|^2)\nabla v) &= F(|\nabla v|^2) \quad \text{in } Q_\tau \\ v &= g \quad \text{on } S_\tau \\ v(\cdot,0) &= u_0 \quad \text{in } \Omega. \end{aligned}$$

Since v is also a weak solution of (3.2.1)–(3.2.3) in Q_τ, by uniqueness of weak solutions (cf. Theorem 3.2.1(ii)), it follows that $u = v_\tau$ on Q_τ, hence (3.2.7).

□

Remark 3.10.1. *More precisely, in the special case when $u_0 \in C^{2+\alpha}(\overline{\Omega})$ and u_0 satisfies the second order compatibility conditions, the existence of v claimed in the above proof follows from [80, Theorem V.6.1]. In the general case $u_0 \in W^{1,\infty}(\Omega)$, with $u_0 = g$ on $\partial\Omega$, this follows by a standard approximation procedure of u_0 by such smooth $u_{0,n}$. Namely, if v_n denotes the solution originating from $u_{0,n}$, then, by [80, Theorems V.4.1, V.1.1 and V.5.4] respectively, we get uniform a priori estimates for the sequence v_n in the spaces $L^\infty(0,\tau; W^{1,\infty}(\Omega))$, $C^\alpha(\overline{Q}_\tau)$ and $C^{2+\alpha,1+\alpha/2}_{loc}(\overline{\Omega} \times (0,\tau])$ for some $\alpha \in (0,1)$. We may then pass to the limit along a subsequence and obtain a solution with the announced properties.*

In the proof of Theorem 3.2.2(ii)(iii), we shall use the following local regularity lemma. We note that only statement (ii) will be used here. The global version of statement (i) was already proved in Theorem 3.2.2(i). However, we give and prove its local version for completeness, since it was mentioned without proof in [7, p. 2487].

Lemma 3.10.1. *Under the assumptions of Theorem 3.2.1, let u be the (maximal) weak solution of (3.1.1) and let $P_0 = (x_0, y_0, t_0) \in Q_T$. Assume $|\nabla u(P_0)| > 0$. Then :*

(i) for some $\alpha \in (0,1)$, u is a classical $C^{2+\alpha,1+\alpha/2}$-solution on a space-time neighborhood of P_0 ;

(ii) for some $\beta \in (0,1)$, ∇u is $C^{2+\beta,1+\beta/2}$ on a space-time neighborhood of P_0.

Proof. (i) Since, by Theorem 3.2.1(iii), ∇u is continuous in Q_T, there exist $\lambda, \rho, M_2 > 0$ such that

$$\lambda \leq |\nabla u| \leq M_2 \quad \text{in } Q^\rho := B_\rho(x_0, y_0) \times [t_0 - \rho, t_0 + \rho] \subset Q_T. \tag{3.10.1}$$

For any unit vector $\vec{e}$ and $0 < h < \rho/2$, let us introduce the differential quotients

$$D_h u = h^{-1}(\tau_h u - u), \quad \text{where } \tau_h u = u\big((x, y) + h\vec{e}, t\big).$$

We have

$$|\nabla \tau_h u|^q - |\nabla u|^q = d^h(x, y, t) \cdot \nabla(\tau_h u - u) \quad \text{in } Q^{\rho/2},$$

where $|d^h(x, y, t)| \leq C$ independent of h. Next denote $b(s) = s^{(p-2)/2}$ and $a_i(p) = b(|p|^2)p_i$ where $p = (p_1, p_2)$, so that $\Delta_p u = \partial_i(a_i(\nabla u))$. Following [80, p.445], we write

$$a_i(\nabla \tau_h u) - a_i(\nabla u) = \int_0^1 \frac{d}{ds} a_i(s\nabla \tau_h u + (1 - s)\nabla u))\, ds = \tilde{a}_{ij}^h \partial_j(\tau_h u - u),$$

where

$$\tilde{a}_{ij}^h(x, y, t) = \int_0^1 \frac{\partial a_i}{\partial p_j}(s\nabla \tau_h u + (1 - s)\nabla u))\, ds.$$

Subtracting the PDE in (3.1.1) for u and for $\tau_h u$ and dividing by h, we see that $D_h u$ is a local weak solution of

$$\partial_t(D_h u) - \partial_i\big[\tilde{a}_{ij}^h \partial_j(D_h u)\big] = d^h(x, y, t) \cdot \nabla(D_h u) \quad \text{in } Q^{\rho/2}. \tag{3.10.2}$$

Moreover, since $\frac{\partial a_i}{\partial p_j}\xi_i\xi_j = b(|p|^2)|\xi|^2 + 2b'(|p|^2)p_i p_j \xi_i \xi_j \geq b(|p|^2)|\xi|^2 \geq \lambda^{p-2}|\xi|^2$ in $Q^{\rho/2}$ by (3.10.1), we have

$$\tilde{a}_{ij}^h \xi_i \xi_j \geq \lambda^{p-2}|\xi|^2 \quad \text{in } Q^{\rho/2}.$$

We then test (3.10.2) with $\varphi^2 D_h u$, where $\varphi \in C_0^\infty(Q^{\rho/2})$ is a cut-off function such that $\varphi = 1$ on $Q^{\rho/3}$. By integration by parts and some simple manipulations, it is easy to see that

$$\lambda^{p-2} \int_{Q^{\rho/3}} |\nabla D_h u|^2\, dxdydt \leq \int_{Q^{\rho/2}} \tilde{a}_{ij}^h \partial_i(D_h u)\partial_j(D_h u)\varphi^2\, dxdydt \leq C.$$

It follows that $D^2 u \in L^2(Q^{\rho/3})$. Consequently, we obtain that $u \in W_2^{2,1}(Q^{\rho/3})$ and is a local strong solution of equation (3.1.1) written in nondivergence form, i.e. :

$$u_t - a_{ij}u_{ij} = f \;\text{ in } Q^{\rho/3}, \quad \text{where } a_{ij} = |\nabla u|^{p-2}\Big[\delta_{ij} + (p-2)\frac{u_i u_j}{|\nabla u|^2}\Big], \quad f = |\nabla u|^q. \tag{3.10.3}$$

Since, by Theorem 3.2.1(iii), a_{ij}, f are Hölder continuous in $\overline{Q}^{\rho/3}$, it follows from interior Schauder parabolic regularity [80, Theorem III.12.2] that, for some $\alpha \in (0, 1)$,

$$u \in C^{2+\alpha, 1+\alpha/2}(\overline{Q}^{\rho/4}). \tag{3.10.4}$$

(ii) Thanks to (3.10.4), we know that u is a classical solution of (3.10.3) in $Q^{\rho/4}$. Keeping the above notation, for $0 < h < \rho/8$, we then have

$$(D_h u)_t - a_{ij}(D_h u)_{ij} = F_h := D_h f + (D_h a_{ij})(\tau_h u_{ij}) \quad \text{in } Q^{\rho/8}.$$

Moreover, as a consequence of (3.10.4), we have, for $1 < r < \infty$,

$$\|F_h\|_{L^r(Q^{\rho/8})} \leq C\|\nabla f\|_{L^r(Q^{\rho/4})} + \|\nabla A\|_{L^r(Q^{\rho/4})}\|D^2 u\|_{L^\infty(Q^{\rho/4})} \leq C, \quad 0 < h < \rho/8.$$

It thus follows from interior parabolic L^r estimates (see [80, Theorem III.12.2]) that, for $0 < h < \rho/8$,

$$\|D^2 D_h u\|_{L^r(Q^{\rho/16})} + \|\partial_t D_h\|_{L^r(Q^{\rho/16})} \leq C(\rho)\big(\|F_h\|_{L^r(Q^{\rho/8})} + \|D_h u\|_{L^r(Q^{\rho/8})}\big) \leq C.$$

We deduce that $Du_t, D^3 u \in L^r_{loc}(Q_T)$. Then differentiating (3.10.3) in space, we see that the function $z = \partial_\ell u_x$ ($\ell = 1, 2$) is a local strong solution of

$$z_t - a_{ij} z_{ij} = \tilde{f} \quad \text{in } Q^{\rho/16}, \tag{3.10.5}$$

where $\tilde{f} = \partial_\ell f - u_{ij}\partial_\ell a_{ij}$. Since $a_{ij}, \tilde{f}$ are Hölder continuous in $\overline{Q}^{\rho/16}$ due to (3.10.4), it follows from interior Schauder parabolic regularity [80, Theorem III.12.2] that $z \in C^{2+\alpha,1+\alpha/2}(\overline{Q}^{\rho/20})$ for some $\alpha \in (0, 1)$.

□

Proof of Theorem 3.2.2 (continued) (ii) This is a direct consequence of Lemma 3.10.1.

(iii) It follows from (i)(ii) that $v = u_x \in C^{2,1}(Q_T) \cap C(\overline{\Omega} \times (0, T))$ is a classical solution of (3.10.5) in Q_T, where a_{ij} are defined in (3.10.3). Moreover, $v = g_x = 0$ on S_T. Taking $\theta(t)$ a cut-off in time and setting $w = \theta v$, we see that w solves

$$w_t - a_{ij} w_{ij} = \bar{f} := \theta\tilde{f} + \theta_t v \quad \text{in } Q_T, \tag{3.10.6}$$

with 0 initial-boundary conditions. By [89, Theorem 4.28], since $\bar{f}$ is locally Hölder continuous in $\overline{\Omega}\times[0, T)$ due to (i), there exists a solution to this problem in $C^{2+\beta,1+\beta/2}(\overline{\Omega}\times[0, T))$ for some $\beta \in (0, 1)$. Since we have uniqueness in the class $C^{2,1}(Q_T) \cap C(\overline{\Omega} \times [0, T))$ by the maximum principle, the conclusion (3.2.9) follows.

□

3.11 APPENDIX 2. A parabolic version of Serrin's corner lemma

In [88, p. 512], a Serrin corner property in a rectangle was shown for a parabolic equation involving the Laplacian. This was proved by comparison with a suitable product of functions of x, t and y, t. This result and method are no longer sufficient here and we shall establish a result for general nondivergence operators by modifying the original proof of [103] for the elliptic case.

Proposition 3.11.1. *Let $0 < X_1 < \hat{X}_1$, $0 < Y_1 < \hat{Y}_1$, $\hat{D} = (0, \hat{X}_1) \times (0, \hat{Y}_1) \subset \mathbb{R}^2$, $0 < \tau_1 < \tau_2$, $\hat{D}_\tau = \hat{D} \times (\tau_1, \tau_2)$. Let the coefficients $a_{ij} = a_{ij}(x, y, t)$ satisfy*

$$a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{in } \overline{\hat{D}_\tau} \tag{3.11.1}$$

for some $\lambda > 0$ and assume that

$$a_{ij}, B_i \in C(\overline{\hat{D}_\tau}), \quad a_{12} + a_{21} \geq -C(x \wedge y), \quad B_1 \geq -Cx, \quad B_2 \geq -Cy \quad \text{in } \overline{\hat{D}_\tau}. \tag{3.11.2}$$

Let $z \in C^{2,1}(\hat{D}_\tau) \cap C(\overline{\hat{D}_\tau})$ satisfy

$$\mathcal{L}z := z_t - a_{ij}z_{ij} - B_i z_i \leq 0 \ \text{ in } \hat{D}_\tau, \quad z(x, y, t) \leq 0 \text{ in } \overline{\hat{D}_\tau}, \quad z(0, 0, t) = 0. \tag{3.11.3}$$

Then, for each $t_0 \in (\tau_1, \tau_2)$, there exists $c_0 > 0$ such that

$$z \leq -c_0 xy \qquad \text{in } (0, X_1) \times (0, Y_1) \times [t_0, \tau_2). \tag{3.11.4}$$

Proof. Let $a = \min(X_1, Y_1, \frac{t_0 - \tau_1}{2})$ and $\tau_3 = \frac{\tau_1 + t_0}{2}$, so that $\tau_1 < \tau_3 < t_0 < \tau_2$. Fix $t_1 \in [t_0, \tau_2)$ and let

$$K_1 := \left\{(x, y, t);\ x^2 + (a - y)^2 + (t_1 - t)^2 < a^2,\ x > 0,\ t \leq t_1\right\}.$$

Observe that $K_1 \subset \hat{D} \times [\tau_3, t_1]$ and set $K_2 = B\big((0, 0), a/2\big) \times [\tau_3, t_1]$ and $K = K_1 \cap K_2$.

Now set

$$\bar{v}(x, y, t) := e^{-\alpha(x^2 + (y-a)^2 + (t-t_1)^2)} - e^{-\alpha a^2}, \quad v(x, y, t) = e^{-\alpha(x^2 + (y-a)^2 + (t-t_1)^2)},$$

with $\alpha > 0$ to be chosen later on, and define the auxiliary function $h = x\bar{v}$. It is clear that $h > 0$ in K. We compute

$$h_t = -2\alpha x(t - t_1)v, \qquad \nabla h = \begin{pmatrix} \bar{v} - 2\alpha x^2 v \\ -2\alpha x(y - a)v \end{pmatrix},$$

$$D^2h = v \begin{pmatrix} -6\alpha x + 4\alpha^2 x^3 & -2\alpha(y - a) + 4\alpha^2 x^2(y - a) \\ -2\alpha(y - a) + 4\alpha^2 x^2(y - a) & -2\alpha x + 4\alpha^2 x(y - a)^2 \end{pmatrix},$$

$$B(x, y, t) \cdot \nabla h = -2\alpha x v[xB_1 + (y - a)B_2] + B_1\bar{v}.$$

Using that $a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2$, we have

$$\begin{aligned} a_{ij}h_{ij} &= va_{11}(-6\alpha x + 4\alpha^2 x^3) + va_{22}\big(-2\alpha x + 4\alpha^2 x(y-a)^2\big) \\ &\quad + v\,(a_{12}+a_{21})\big(-2\alpha(y-a) + 4x^2\alpha^2(y-a)\big) \\ &= \alpha x v \Bigg[4\alpha\big(a_{11}x^2 + (y-a)x(a_{12}+a_{21}) + (y-a)^2 a_{22}\big) \\ &\quad - 6a_{11} - 2a_{22} - \frac{2(y-a)(a_{12}+a_{21})}{x} \Bigg] \\ &\geq \alpha x v \left[4\alpha\lambda(x^2 + (y-a)^2) - 6a_{11} - 2a_{22} + \frac{2(a-y)(a_{12}+a_{21})}{x} \right], \end{aligned}$$

hence

$$\begin{aligned} \mathcal{L}h \leq \alpha x v \Bigg[&-4\alpha\lambda(x^2+(y-a)^2) + 2(t_1-t) + 6a_{11} + 2a_{22} - \frac{2(a-y)(a_{12}+a_{21})}{x} \\ &+2xB_1 + 2(y-a)B_2 - \frac{B_1}{\alpha x}\big(1 - e^{\alpha(x^2+(y-a)^2+(t-t_1)^2-a^2)}\big) \Bigg]. \end{aligned}$$

On the one hand, on K, we have $y < a/2$, hence $x^2 + (y-a)^2 > a^2/4$. On the other hand, using part of assumptions (3.11.2) along with $0 \leq a - y \leq a$ and $0 \leq 1 - e^{\alpha(x^2+(y-a)^2+(t-t_1)^2-a^2)} \leq 1$ on K, it follows that for $\alpha > 1$ large enough,

$$\begin{aligned} \mathcal{L}h &\leq \alpha x v\left[-\alpha\lambda a^2 + 2(t_1-t) + 6a_{11} + 2a_{22} + 2Ca + 2xB_1 + 2(y-a)B_2 + C\right] \\ &\leq -\frac{\lambda\alpha^2 a^2 x v}{2} < 0 \quad \text{in } K. \end{aligned} \tag{3.11.5}$$

We now set $w = z + \varepsilon h$ where ε is a positive constant to be chosen. By (3.11.3) and (3.11.5), we have

$$\mathcal{L}w < 0 \quad \text{in } K. \tag{3.11.6}$$

Denote $M = \max_{\overline{K}} w \geq 0$. Since $\mathcal{L}$ is (uniformly) parabolic, by the usual proof of the maximum principle, it follows from (3.11.6) that w cannot attain the value M in K (observe that for each $s \in [\tau_3, t_1]$, the section $K \cap \{t = s\}$ is an open, possibly empty, subset of $\mathbb{R}^2$). To show $M = 0$ (for sufficiently small $\varepsilon > 0$), it thus suffices to verify that $w \leq 0$ on $\partial_P K = \partial K \setminus (K \cap \{t = t_1\})$. We have $\partial_P K = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \partial K_1 \cap \overline{K}_2$ and $\Gamma_2 = \partial K_2 \cap K_1$.

On Γ_1 we have either

$$x^2 + (y-a)^2 + (t_1-t)^2 = a^2 \quad \text{or} \quad x = 0,$$

so that $h = 0$ and $z \leq 0$, hence $w \leq 0$. Next observe that on Γ_2 we have

$$x^2 + (y-a)^2 + (t_1-t)^2 < a^2 \quad \text{and} \quad x^2 + y^2 = a^2/4,$$

hence $\tau_1 < \tau_3 \leq t \leq t_1$ and $a/8 < y < a/2$ (in other words, (x, y) is "far" from the corners of $\hat{D}$). Therefore, by the Hopf boundary point lemma [98, Theorem 6 p. 174] and the strong maximum principle, there exists $c > 0$ (independent of t_1), such that $z \leq -cx$ on Γ_2. Choosing $\varepsilon \in (0, c)$, we then have $w \leq -cx + \varepsilon x < 0$ on Γ_2.

We have thus proved that $M = 0$ that is, $w \leq 0$ in K. Letting $\tilde{a} := a/(2\sqrt{2})$ and noting that $\{0 < x \leq y < \tilde{a}\} \times \{t_1\} \subset K$, we get

$$\begin{aligned}
z(x, y, t_1) &\leq -\varepsilon h(x, y, t_1) = -\varepsilon x e^{-\alpha a^2}\big(e^{\alpha(a^2 - x^2 - (y-a)^2)} - 1\big) \\
&\leq -\varepsilon\alpha e^{-\alpha a^2} x\big(a^2 - x^2 - (y-a)^2\big) = -\varepsilon\alpha e^{-\alpha a^2} x\big(2ay - x^2 - y^2\big) \\
&\leq -a\varepsilon\alpha e^{-\alpha a^2} xy \quad \text{ for } 0 < x \leq y < \tilde{a}.
\end{aligned}$$

Now exchanging the roles of x, y and noticing that the assumptions (3.11.2) are symmetric in x, y, the conclusion already obtained guarantees that also $z(x, y, t_1) \leq -a\varepsilon\alpha e^{-\alpha a^2} xy$ for $0 < y \leq x < \tilde{a}$, hence (3.11.4) in $(0, \tilde{a})^2 \times [t_0, \tau_2)$. The extension to the remaining part of the rectangle $(0, X_1) \times (0, Y_1)$ (away from the corner $(0, 0)$) follows from the Hopf boundary lemma and the strong maximum principle.

$\square$

Chapitre 4

Prolongement de la solution au delà des singularités via les solutions de viscosité

Dans ce travail en collaboration avec Guy Barles, nous nous intéressons à la continuation des solutions explosives au delà du temps d'explosion. Pour se faire la théorie des solutions de viscosité offre un bon cadre de travail. Le comportement asymptotique des solutions prolongées est étudié via la considération standard d'un problème ergodique.

4.1 Introduction and Main Results

In this chapter we are interested in the following generalized Dirichlet problem for second-order degenerate parabolic partial differential equations

$$\begin{aligned} u_t - \operatorname{div}\left([Du|^{p-2}Du\right) + |Du|^q &= f(x,t) \quad \text{in } \Omega \times (0,+\infty) && (4.1.1)\\ u(x,0) &= u_0(x) \quad \text{on } \overline{\Omega}, && (4.1.2)\\ u(x,t) &= g(x,t) \quad \text{on } \partial\Omega \times (0,+\infty), && (4.1.3) \end{aligned}$$

where $q > p \geq 2$, u_0 and g are continuous functions satisfying the compatibility condition

$$u_0(x) = g(x,0) \quad \text{on } \partial\Omega \qquad (4.1.4)$$

Most of works devoted to this degenerate diffusive Hamilton-Jacobi equation concerned the case where $\Omega = \mathbb{R}^N$, providing results on well-posedness, gradient estimates and asymptotic behavior of either classical or weak solutions in the sense of distributions (see [24, 1, 100, 22] and the references therein).

Some other works are concerned with the solvability of the Cauchy-Dirichlet problem. They proved that, under suitable assumptions on u_0 and g, there exists a weak solution on some time interval $[0, T_{max}(u_0))$, with the property that its gradient blows up on the boundary $\partial\Omega$ while the solution itself remains bounded. We refer the reader to [7] [76] and [85] for the degenerate parabolic case and to [110] for the uniformly parabolic case. This singularity is a difficulty to extend the solution past $T_{max}(u_0)$. A natural question is then : Can we extend the weak solution past $t = T_{max}(u_0)$ and in which sense ?

Let us mention here that a result in this direction where the continuation beyond gradient blow-up does not satisfy the original boundary conditions was obtained in [52, 53].

Recently, for the linear diffusion case ($p = 2$), Barles and Da Lio [17] showed that such gradient blow-up is related to a loss of boundary condition and address the problem through a viscosity solutions approach. They proved a "Strong Comparison Result" (that is a comparison result between *discontinuous* viscosity sub and supersolutions) which allowed them to obtain the existence of a unique continuous, *global in time viscosity solution* of (4.1.1)–(4.1.3), the Dirichlet boundary condition being understood in the *generalized sense of viscosity solution theory.* They also provided an explicit expression of the solution of (4.1.1)–(4.1.3) in terms of a value function of some exit time control problem, which allows a simple explanation of the losses of boundary condition when it arises.

We recall that the formulation of the generalized Dirichlet boundary condition for (4.1.1)–(4.1.3) in the viscosity sense reads

$$\min\left(u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) + |Du|^q - f(x,t), u - g\right) \leq 0 \quad \text{on } \partial\Omega \times (0, +\infty), \tag{4.1.5}$$

and

$$\max\left(u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) + |Du|^q - f(x,t), u - g\right) \geq 0 \quad \text{on } \partial\Omega \times (0, +\infty). \tag{4.1.6}$$

Our first result mainly extends the investigation of [17] to the degenerate diffusion case $p > 2$.

Theorem 4.1.1. *Assume that $q > p \geq 2$ and that Ω is a bounded domain with a C^2-boundary. For any $u_0 \in C(\overline{\Omega})$, $f \in C\left(\overline{\Omega} \times [0,T]\right)$ and $g \in C\left(\partial\Omega \times [0,T]\right)$ satisfying* (4.1.4)*, there exists a unique continuous solution u of* (4.1.1)–(4.1.3) *which is defined globally in time.*

As it is classical in viscosity solutions theory, the proof of Theorem 4.1.1 relies on a Strong Comparison Result (SCR in short), the existence of the global solution u being an almost immediate consequence of the Perron's method introduced in the context of viscosity solutions by Ishii [70] (see also [44]).

The most important difficulties in the proof of Strong Comparison Results come from the formulation of the boundary condition in the viscosity sense, the discontinuity of the sub and the supersolution to be compared and the strong nonlinearity of the Hamiltonian term $|Du|^q$. A key argument in the proof of the SCR in [17] is the "cone condition" which is useful in the treatment of boundary points. Roughly speaking the "cone condition" holds if at any point $(\tilde{x}, \tilde{t})$ of the boundary $\partial\Omega \times (0, T)$, an usc subsolution u satisfies

$u(\tilde{x},\tilde{t}) = \lim_{k\to\infty} u(x_k,t_k)$ where $\{(x_k,t_k)\}_k$ is a sequence of points of $\Omega \times (0,T)$ with the following properties

$$(x_k,t_k) \to (\tilde{x},\tilde{t}) \quad \text{and} \quad d_{\partial\Omega}(x_k,t_k) \geq b\left(|x_k-\tilde{x}|+|t_k-\tilde{t}|\right),$$

where b is a positive constant.

Our approach is slightly different : instead of directly proving the "cone condition" for any viscosity subsolution of (4.1.1)–(4.1.3) as it was done in [17], we use a combination of a $C^{0,\beta}$ regularity result for subsolutions of stationary problems, strongly inspired by the result of Capuzzo Dolcetta, Leoni and Porretta [34], together with a regularization by a sup-convolution in time. These arguments provide an approximation of the (a priori only usc) subsolution by a continuous subsolution, which automatically satisfies the "cone condition", allowing to borrow the methods of [21] to conclude.

The generalisation of the $C^{0,\beta}$ regularity result of [34] is the following.

Theorem 4.1.2. *If u is a locally bounded, usc viscosity subsolution of*

$$-\,div\left(|Du|^{p-2}Du\right)+|Du|^q \leq C \quad in\ \Omega\,, \tag{4.1.7}$$

where Ω is an open subset of $\mathbb{R}^N$ and C is a positive constant, and if $q > p \geq 2$, then $u \in C^{0,\beta}_{loc}(\Omega)$ with $\beta = \dfrac{q-p}{q-p+1}$.

Moreover, if Ω is a bounded domain with a C^2-boundary, then u is bounded on $\overline{\Omega}$ and it can be extended as a $C^{0,\beta}$-function on $\overline{\Omega}$ and

$$|u(x)-u(y)| \leq M|x-y|^\beta \quad for\ all\ x,y \in \overline{\Omega}, \tag{4.1.8}$$

for some positive constant M depending only on p,q, C and $\partial\Omega$.

The regularity result of [34] was revisited in [15], where an interpretation was given in terms of state-constraint problems together with several possible applications. Our proof will rely on the arguments of [15].

A second motivation where such regularity results are useful, is the asymptotic behavior as $t \to +\infty$ of solutions of the evolution equation. For this purpose, one has first to study the ergodic (or additive eigenvalue) problem

$$-\,\mathrm{div}\left(|Du_\infty|^{p-2}Du_\infty\right)+|Du_\infty|^q-\tilde{f}(x) = c \quad \text{in } \Omega, \tag{4.1.9}$$

associated to a state-constraint boundary condition on $\partial\Omega$

$$-\,\mathrm{div}\left(|Du_\infty|^{p-2}Du_\infty\right)+|Du_\infty|^q-\tilde{f}(x) \geq c \quad \text{on } \partial\Omega. \tag{4.1.10}$$

We recall that, in this type of problems, both the solution u_∞ and the constant c (the ergodic constant) are unknown. First we have the following result.

Theorem 4.1.3. *Assume that Ω is a bounded domain with a C^2-boundary, $\tilde{f} \in C(\overline{\Omega})$ and $q > p \geq 2$, then there exists a unique constant c such that the state-constraints problem* (4.1.9)–(4.1.10) *has a continuous viscosity solution u_∞.*

A typical result that connects the study of the ergodic problem to the large time behavior of the solution u of (4.1.1)–(4.1.3) is the following.

Theorem 4.1.4. *Assume that Ω is a bounded domain with a C^2-boundary, $u_0 \in C(\overline{\Omega})$, $g \in C(\partial\Omega)$ satisfying* (4.1.4) *and assume that $f(x,t) = \tilde{f}(x)$ with $\tilde{f} \in C(\overline{\Omega})$ and $q > p \geq 2$. If (c, u_∞) is the solution of* (4.1.9)–(4.1.10) *and if u is the unique viscosity solution of* (4.1.1)–(4.1.3)*, then $u + c^+ t$ is bounded, where $c^+ = max(c, 0)$. In particular*

$$\lim_{t\to\infty} \frac{u(x,t)}{t} = -c^+$$

uniformly on $\overline{\Omega}$.

The next step in the study of the asymptotic behavior would be to show that $u(x,t) + ct \to u_\infty(x)$ as $t \to \infty$ where u_∞ solves (4.1.9)–(4.1.10). The main difficulty to prove such more precise asymptotic behavior comes from the fact that (4.1.9)–(4.1.10) does not admit a unique solution ((4.1.9)–(4.1.10) is invariant by addition of constants). Such results were obtained recently in [113] for the uniformly elliptic case $p = 2$ through the use of the Strong Comparison Principle (i.e. a result which allows to apply the Strong Maximum Principle to the difference of solutions) and the Lipschitz regularity of u_∞. But, for $p > 2$, such Strong Comparison Principle is not available since the equation is quasilinear and not semilinear. We recall that a Strong Maximum Principle is available for $p > 2$, see [12]. Another difficulty comes from the proof of a strong comparison result for the steady problem in case of an operator that does not fulfill a monotonicity property, even if there exits a strict subsolution. Let us mention the works of [85, 18] for more results on the asymptotic behavior of global solutions.

Finally we point out that it was shown in [20] that the expected asymptotic behavior, namely $u(x,t) + ct \to u_\infty(x)$, is not always true in the $p = 2$-case when the nonlinearity is sub quadratic in Du.

This chapter is organized as follows : in Section 4.2, we present the needed results on viscosity solutions for the stationary and evolution problems we consider ; in particular, we analyze the losses of boundary conditions for subsolutions. In Section 4.3 we prove the Hölder regularity result of Theorem 4.1.2. In Section 4.4 we study the ergodic problem. Section 4.5 is devoted to the proof of Theorem 4.1.1 and the asymptotic behavior of solutions of the evolution equation.

4.2 Preliminaries and Analysis of Boundary Conditions

In this section we collect some preliminary properties of viscosity subsolutions (the boundary conditions being always understood in the viscosity sense) and we also formulate SCR under different forms, some of them being only useful as a step in the proof of the complete regularity result. These results are concerned with either problem (4.1.1)–(4.1.3) or the following two nonlinear elliptic problem

$$-(p-1)|Du|^{p-2} \sum_{\lambda_i(D^2u)>0} \lambda_i(D^2u) + |Du|^q = C \quad \text{in } \Omega, \qquad u = \tilde{g} \quad \text{in } \partial\Omega. \tag{4.2.1}$$

and

$$- \, div \left(|Du|^{p-2} Du \right) + |Du|^q + \lambda u - \tilde{f} = 0 \quad \text{in } \Omega, \qquad u = \tilde{g} \quad \text{in } \partial\Omega. \tag{4.2.2}$$

where $q > p \geq 2$, $C, \lambda \geq 0$, $\tilde{f} \in C(\overline{\Omega})$ and $\tilde{g} \in C(\partial\Omega)$.

From now on, we assume that Ω is a smooth domain with a C^2-boundary. We define the distance from $x \in \overline{\Omega}$ to $\partial\Omega$ by $d_{\partial\Omega}(x) := \text{dist}\,(x, \partial\Omega)$. For $\delta > 0$, we denote by

$$\Omega^\delta := \{x \in \Omega \,|\, d_{\partial\Omega}(x) < \delta\}\,, \tag{4.2.3}$$

$$\Omega_\delta := \{x \in \Omega \,|\, d_{\partial\Omega}(x) > \delta\}\,. \tag{4.2.4}$$

As a consequence of the regularity of $\partial\Omega$, $d_{\partial\Omega}$ is a C^2-function in a neighborhood Ω^δ of the boundary for all $0 < \delta \leq \delta_0$. We denote by d a C^2-function agreeing with $d_{\partial\Omega}$ in Ω^δ such that $|Dd(x)| \leq 1$ in Ω_δ. We also denote by $n(x)$ the C^1-function defined by $n(x) = -Dd(x)$ in Ω^δ; if $x \in \partial\Omega$, then $n(x)$ is just the unit outward normal vector to $\partial\Omega$ at x.

Our first result says that there is no loss of boundary conditions for the subsolutions, namely that the subsolutions satisfy the boundary condition in the classical sense.

Proposition 4.2.1. *Assume that $q > 0$ and $p \geq 2$. We have the following*

i) If u is a bounded, usc subsolution of (4.1.1)–(4.1.3) *on a time interval* $(0, T)$*, then*

$$u \leq g \quad on \quad \partial\Omega \times (0, T). \tag{4.2.5}$$

ii) If u is a bounded, usc subsolution of (4.2.1) *or* (4.2.2) *, then*

$$u \leq \tilde{g} \quad on \quad \partial\Omega. \tag{4.2.6}$$

Proof. We only give the proof for the time dependent problem, the proof for the stationnary problems being similar. We use a result of Da Lio [44, Corollary 6.2]. We denote by $\mathcal{S}^N$ the space of real symmetric $N \times N$ matrices. For $x \in \Omega$, $t \in (0, T)$, $\xi \in \mathbb{R}^N$ and $M \in \mathcal{S}^N$, we define the function F by

$$F(x, t, \xi, M) = -|\xi|^{p-2} Tr(M) - (p-2)|\xi|^{p-4} \langle M\xi, \xi \rangle + |\xi|^q - f(x, t),$$

so that the equation can be written as $u_t + F(x, t, Du, D^2u) = 0$. From [44], we know that, if $u(x_0, t_0) > g(x_0, t_0)$ at some point $(x_0, t_0) \in \partial\Omega \times (0, T)$, then the following conditions hold

$$\liminf_{\substack{(y,t)\to(x_0,t_0)\\ \alpha\downarrow 0}} \left\{ \left[\frac{o(1)}{\alpha} + F\left(y, t, \frac{Dd(y) + o(1)}{\alpha}, -\frac{Dd(y) \otimes Dd(y) + o(1)}{\alpha^2} \right) \right] \right\} \leq 0$$

$$\liminf_{\substack{(y,t)\to(x_0,t_0)\\ \alpha\downarrow 0}} \left\{ \left[\frac{o(1)}{\alpha} + F\left(y, t, \frac{Dd(y) + o(1)}{\alpha}, \frac{D^2d(y) + o(1)}{\alpha} \right) \right] \right\} \leq 0. \tag{4.2.7}$$

But the first condition cannot hold since

$$F\left(y,t,\frac{Dd(y)+o(1)}{\alpha},-\frac{Dd(y)\otimes Dd(y)+o(1)}{\alpha^2}\right)\geq\frac{(p-1)}{\alpha^p}(1+o(1))+\frac{1-o(1)}{\alpha^q}-f(y,t),$$

and the right hand side is going to $+\infty$ as $\alpha\to 0$ since $p\geq 2$, $q>0$ and all terms converge to $+\infty$.

□

Let us point out that the above computation shows that there is no competition between the nonlinear Hamiltonian term and the slow diffusion operator since they both produce positive contribution which prevent any loss of boundary conditions for the subsolution.

Next, we remark that there cannot be loss of initial condition.

Lemma 4.2.1. *Assume that $q>p\geq 2$, $f\in C\left(\overline{\Omega}\times[0,T]\right)$ and $u_0\in C(\overline{\Omega})$, $g\in C\left(\partial\Omega\times[0,T]\right)$ satisfy (4.1.4). Let u and v be respectively a bounded usc viscosity subsolution and a bounded lsc super-solution of (4.1.1)–(4.1.3) then*

$$u(x,0)\leq u_0(x)\leq v(x,0)\quad on\quad \overline{\Omega}. \tag{4.2.8}$$

Proof. Fix $x_0\in\overline{\Omega}$ and define for $\varepsilon>0$ and $C_\varepsilon>0$ the function $\phi_\varepsilon(x,t)$ by

$$\phi_\varepsilon(x,t)=u(x,t)-\frac{|x-x_0|}{\varepsilon^2}-C_\varepsilon t.$$

This function attains a global maximum on $\overline{\Omega}\times[0,T)$ at $x_\varepsilon,t_\varepsilon$. Using the boundedness of u, it is easy to see that, for any $C_\varepsilon>0$, $(x_\varepsilon,t_\varepsilon)\to(x_0,0)$ as $\varepsilon\to 0$. Arguing as in [21], choosing C_ε sufficiently large depending on ε, we are left with $(x_\varepsilon,t_\varepsilon)\in(\partial\Omega\times(0,T))\cup\left(\overline{\Omega}\times\{0\}\right)$ and the two following possibilities

$$\begin{array}{rcl}\text{either}\quad t_\varepsilon=0 & \text{and} & u(x_\varepsilon,0)\leq u_0(x_\varepsilon),\\ \text{or}\quad t_\varepsilon>0, x_\varepsilon\in\partial\Omega & \text{and} & u(x_\varepsilon,t_\varepsilon)\leq g(x_\varepsilon,t_\varepsilon).\end{array}$$

In either case, since $u(x_0,0)\leq\phi_\varepsilon(x_\varepsilon,t_\varepsilon)\leq u(x_\varepsilon,t_\varepsilon)$, we get the desired result for u letting $\varepsilon\to 0$ and using the continuity of u_0 and g. The argument for v is similar.

□

Now we claim that under some assumptions (set out below), a SCR holds for semicontinuous viscosity sub-and supersolutions of (4.1.1)–(4.1.3) or (4.2.1) or (4.2.2). The proof being somehow technical we refer the reader to the appendice for a detailed proof of the following two propostions.

Proposition 4.2.2 (**Parabolic SCR**). *Assume that* $q > p \geq 2$, $f \in C\left(\overline{\Omega} \times [0,T]\right)$ *and* $u_0 \in C(\overline{\Omega})$, $g \in C\left(\partial\Omega \times [0,T]\right)$ *satisfy* (4.1.4). *Let* u *and* v *be respectively a bounded usc viscosity subsolution and a bounded lsc super-solution of* (4.1.1)–(4.1.3), *then* $u \leq v$ *in* $\Omega \times [0,T]$. *Moreover, if we define* $\tilde{u}$ *on* $\overline{\Omega} \times [0,T]$ *by setting*

$$\tilde{u}(x,t) := \begin{cases} \limsup\limits_{\substack{(y,s)\to(x,t)\\(y,s)\in\Omega\times(0,T)}} u(y,s) & \text{for all}\,(x,t) \in \partial\Omega \times (0,T] \\ u(x,t) & \text{otherwise,} \end{cases} \tag{4.2.9}$$

then $\tilde{u}$ *remains an usc subsolution of* (4.1.1)–(4.1.3) *and*

$$\tilde{u} \leq v \quad \text{on } \overline{\Omega} \times [0,T]. \tag{4.2.10}$$

The stationary version of the SCR is used either in the proof of the $C^{0,\beta}$-regularity or for solving the ergodic problem.

Proposition 4.2.3 (**Elliptic SCR**). *Assume that* $q > p \geq 2$, $\tilde{f} \in C\left(\overline{\Omega}\right)$ *and* $\tilde{g} \in C\left(\partial\Omega\right)$.

(i) Let u *and* v *be respectively a bounded usc viscosity subsolution and a bounded lsc super-solution of* (4.2.1). *If* v *is continuous on* $\overline{\Omega}$ *and is a strict supersolution of* (4.2.1), *then*

$$u \leq v \quad \text{on} \quad \overline{\Omega}. \tag{4.2.11}$$

(ii) Let u *and* v *be respectively a bounded usc viscosity subsolution and a bounded lsc super-solution of* (4.2.2). *Assume that either* $\lambda > 0$ *or* $\lambda = 0$ *and* v *is a strict supersolution. We define* $\tilde{u}$ *on* $\overline{\Omega}$ *by setting*

$$\tilde{u}(x) := \begin{cases} \limsup\limits_{\substack{y\to x\\y\in\Omega}} u(y) & \text{for all } x \in \partial\Omega \\ u(x) & \text{otherwise,} \end{cases} \tag{4.2.12}$$

then $\tilde{u}$ *remains an usc subsolution of* (4.1.1)–(4.1.3) *and*

$$\tilde{u} \leq v \quad \text{on} \quad \overline{\Omega}. \tag{4.2.13}$$

4.3 Hölder Regularity of Viscosity Subsolutions for the Degenerate Elliptic Problem

In this section we are going to prove that equation of type (4.1.7) enters into the general framework described in [15] which allows us to state that, if u is a locally bounded, usc viscosity subsolution of (4.1.7), then u is Hölder continuous with exponent $\beta = \dfrac{q-p}{q-p+1}$. The key point is that the strong growth of the first order term balances the degeneracy of the second order term, providing a control on $|Du|$.

Proof of Theorem 4.1.2. If u is a subsolution of (4.1.7), then it is a subsolution in $B_r(x) = \left\{y \in \mathbb{R}^N;\ |y-x| < r\right\}$ of the simpler equation

$$-(p-1)|Du|^{p-2} \sum_{\lambda_i(D^2u)>0} \lambda_i(D^2u) + |Du|^q \leq C.$$

Now we are going to check the required hypotheses in [15].

H1. For $0 < r < 1$, $s \in \mathbb{R}^N$ and $M \in \mathcal{S}^N$, $\mathcal{S}^N$ denoting the space of $N \times N$ real valued symmetric matrices, define the function $G_r(s, M)$ by

$$G_r(s, M) := -(p-1)|s|^{p-2} \sum_{\lambda_i(M)>0} \lambda_i(M) + |s|^q - C.$$

Then, for any $x \in \Omega$ with $d_{\partial\Omega}(x) \geq r$, $G_r(Du, D^2u) \leq 0$ in $B_r(x)$.

H2. There exists a super-solution up to the boundary $w_r \in C\left(\overline{B_r(0)}\right)$ such that $w_r(0) = 0$, $w_r(x) \geq 0$ in $B_r(x)$ and

$$G_r\left(Dw_r, D^2w_r\right) \geq \eta_r > 0 \quad \text{on} \quad \overline{B_r(0)} \setminus \{0\}\,, \tag{4.3.1}$$

for some $\eta_r > 0$.

Despite the construction of the functions w_r is a rather easy adaptation of [15], we reproduce it for the sake of completeness and for the reader's convenience. In order to build w_r, we first build w_1 and then use the scale invariance of the equation. To do so, we borrow arguments from [15]. For $C_1, C_2 > 0$ to be chosen later on and for $\beta = \dfrac{q-p}{q-p+1}$, we consider the function

$$w_1(x) := \frac{C_1}{\beta}|x|^\beta + \frac{C_2}{\beta}\left(d^\beta(0) - d^\beta(x)\right),$$

where $d(x) = 1 - |x|$ on $B_1(0)\backslash B_{1/2}(0)$ and we regularize it in $B_{1/2}(0)$ by changing it into $h(1-|x|)$ where h is a smooth, non-decreasing and concave function such that $h(s)$ is constant for $s \geq 3/4$ and $h(s) = s$ for $s \leq 1/2$. Obviously we have $w_1(0) = 0$, $w_1 \geq 0$ in $\overline{B_1(0)}$ and w_1 is smooth in $\overline{B_1(0)} \setminus \{0\}$.

We first remark that $-(p-1)|Dw_1(x)|^{p-2}\lambda_i(D^2w_1(x)) + |Dw_1(x)|^q$ can be written as

$$|Dw_1(x)|^{p-2}\left[-(p-1)\lambda_i(D^2w_1(x)) + |Dw_1(x)|^{q-p+2}\right]\ .$$

Therefore, in order to prove the claim, we are going to show that, for $C_1, C_2 > 0$ large enough, the bracket is positive and bounded away from 0 and that $|Dw_1(x)|^{p-2}$ remains large.

Computing the derivatives of w_1 in $B_1(0) \setminus \{0\}$, we have

$$\begin{aligned} Dw_1(x) &= C_1|x|^{\beta-2}x - C_2 d^{\beta-1}(x)Dd(x), \\ D^2w_1(x) &= C_1|x|^{\beta-2}Id + (\beta-2)C_1|x|^{\beta-4}x \otimes x \\ &- C_2 d^{\beta-1}(x)D^2d(x) - (\beta-1)C_2 d^{\beta-2}(x)Dd(x) \otimes Dd(x). \end{aligned}$$

Using that $-Dd(x) = \mu(x)x$ for some $\mu(x) \geq 0$ and that $q > p > 2$, we have

$$\begin{aligned}
|Dw_1(x)|^{q-p+2} &= \left(|C_1|x|^{\beta-2}x| + |C_2 d^{\beta-1}(x)Dd(x)|\right)^{q-p+2} \\
&\geq |C_1|x|^{\beta-2}x|^{q-p+2} + |C_2 d^{\beta-1}(x)Dd(x)|^{q-p+2} \\
&= C_1^{q-p+2}|x|^{(\beta-1)(q-p+2)} + C_2^{q-p+2} d^{(\beta-1)(q-p+2)}(x)|Dd(x)|^{q-p+2},
\end{aligned}$$

and

$$|Dw_1(x)|^{p-2} \geq C_1^{p-2}|x|^{(\beta-1)(p-2)} + C_2^{p-2} d^{(\beta-1)(p-2)}(x)|Dd(x)|^{p-2}.$$

Using that $d(x) = h(1-|x|)$, with h being C^2, non-decreasing and concave, $0 < \beta < 1$, we have

$$D^2 w_1(x) \leq C_1|x|^{\beta-2} Id + C_2 d^{\beta-1}(x)\left(\frac{h'}{|x|} Id - h'' \frac{x}{|x|} \otimes \frac{x}{|x|}\right) + (1-\beta)C_2 d^{\beta-2}(x) Dd(x) \otimes Dd(x),$$

and

$$\lambda_i(D^2 w_1(x)) \leq C_1|x|^{\beta-2} + C_2 d^{\beta-1}(x)\left(\frac{h'}{|x|} - h''\right) + (1-\beta)C_2 d^{\beta-2}(x)|Dd(x)|^2.$$

At this point, it is worth noticing that because of the properties of h, the term $\left(\frac{h'}{|x|} - h''\right)$ is bounded.

These properties imply that, we can (almost) consider the two terms (in $|x|$ and in $d(x)$) separately. Since $(\beta-1)(q-p+2) = (\beta-2)$, the $\dfrac{C_1}{\beta}|x|^\beta$ term yields

$$-(p-1)C_1|x|^{\beta-2} + |C_1|x|^{\beta-2}x|^{q-p+2} = |x|^{\beta-2}\left(-(p-1)C_1 + C_1^{q-p+2}\right).$$

By choosing C_1 large enough, we can have for any $K_1 > 0$

$$|x|^{\beta-2}\left(-(p-1)C_1 + C_1^{q-p+2}\right) \geq K_1|x|^{\beta-2} \quad \text{in}\, B_1(0)\setminus\{0\}.$$

On the other hand the $\dfrac{C_2}{\beta}\left(d^\beta(0) - d^\beta(x)\right)$ term yields

$$-(p-1)C_2 d^{\beta-1}(x)\left(\frac{h'}{|x|} - h''\right) + (\beta-1)(p-1)C_2 d^{\beta-2}(x)|Dd(x)|^2 + C_2^{q-p+2}|d^{\beta-1}Dd(x)|^{q-p+2}. \tag{4.3.2}$$

We have to consider two cases : either $|x| \geq \dfrac{1}{2}$ and then $h' = 1$, $h'' = 0$ and $Dd(x) = -\dfrac{x}{|x|}$; hence the above quantity is given by

$$-(p-1)C_2 d^{\beta-1}(x)\left(\frac{1}{|x|}\right) - (p-1)(1-\beta)C_2 d^{\beta-2}(x) + C_2^{q-p+2} d^{(\beta-1)(q-p+2)}.$$

Recalling that $(\beta-1)(q-p+2)=(\beta-2)$, then for C_2 large enough we have for any $K_2>0$

$$-(p-1)C_2 d^{\beta-1}(x)\left(\frac{1}{|x|}\right)-(p-1)(1-\beta)C_2 d^{\beta-2}(x)+C_2^{q-p+2}d^{(\beta-1)(q-p+2)}\geq K_2 d^{\beta-2}(x).$$

Now for $|x|\leq\dfrac{1}{2}$, the quantity (4.3.2) coming from the $d(x)$-term is bounded and can be controlled by the $|x|$-term. Hence, for any constant $C>0$, choosing first C_2 large enough and then C_1 large enough, we have in $\overline{B_1(0)}\setminus\{0\}$

$$\begin{aligned}-(p-1)|Dw_1(x)|^{p-2}\lambda_i(D^2w_1(x))+|Dw_1(x)|^q &\geq |Dw_1(x)|^{p-2}\left(K_1|x|^{\beta-2}+K_2d^{\beta-2}(x)\right)\\ &\geq \left(K_1|x|^{(\beta-1)(p-1)-1}\right)\geq C.\end{aligned}$$

Next we set

$$w_r(x):=r^\beta w_1\left(\frac{x}{r}\right).$$

It is easy to check that for $0<r\leq 1$, $G(Dw_r,D^2w_r)\geq r^{(\beta-1)(p-1)-1}C-C\geq 0$ on $\overline{B_r(0)}\setminus\{0\}$.

H3. Comparison result. Let v be any bounded usc viscosity subsolution of $G_r(Dv,D^2v)\leq 0$ in $B_r(0)\setminus\{0\}$ then

$$v(y)\leq v(x)+r^\beta w_1\left(\frac{y-x}{r}\right). \tag{4.3.3}$$

We use the fact that $v(0)+w_r(x)$ is a **strict** super-solution up to the boundary and that it is a **continuous** function. It follows that the comparison is a direct consequence of Proposition 4.2.3.

Since the hypotheses are satisfied, we can apply Proposition 2.1 of [15] to obtain the $C^{0,\beta}$ regularity of subsolutions, both locally and globally with further assumptions on Ω.

Remark 4.3.1. *As far as the exponent β is concerned, the value is the best one can expect in the assumption of the above theorem (see [34]).*
It is well-known that the degeneracy of the p-Laplacian is an an obstruction to the solvability of the Dirichlet problem in the classical sense. The presence of the strongly non-linear term with $q>p$ is another source of obstruction, even in the uniformly elliptic case since examples of boundary layers can occur [17, 82]. By the previous result, we know that every continuous solution to (4.1.7) *is Hölder continuous up to the boundary. Hence, a necessary condition in order that the solution can attain continuously the boundary data g is the existence of some $C\geq 0$ such that*

$$|g(x)-g(y)|\leq C[x-y|^\beta \quad \textit{for all } x,y\in\partial\Omega, \quad \beta=\frac{q-p}{q-p+1}.$$

For the uniformly elliptic case $p=2$, a more detailed study including several gradient bounds and applications can be found in [82].

As an application of the previous regularity result, we consider the generalized Dirichlet problem consisting in solving (4.2.2).

Theorem 4.3.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary. Assume that $q > p \geq 2$, $\tilde{f} \in C(\overline{\Omega})$, $\tilde{g} \in C(\partial\Omega)$ and $\lambda > 0$. Let u and v be respectively a bounded usc subsolution and a bounded lsc super-solution of* (4.2.2) *with u satisfying for $x \in \partial\Omega$*

$$u(x) = \limsup_{\substack{y \to x \\ y \in \Omega}} \quad u(y).$$

Then, $u \leq v$ on $\overline{\Omega}$. Moreover Problem (4.2.2) *has a unique viscosity solution which belongs to $C^{0,\beta}(\overline{\Omega})$.*

Proof. For the comparison part, Theorem 4.1.2 implies that u is Hölder continuous, hence the comparison $u \leq v$ is a direct consequence of Proposition 4.2.3. Once noticed that $-\Big(\lambda^{-1}||\tilde{f}||_{L^\infty} + ||\tilde{g}||_{L^\infty}\Big)$ and $+\Big(\lambda^{-1}||\tilde{f}||_{L^\infty} + ||\tilde{g}||_{L^\infty}\Big)$ are respectively sub and super-solution, we can apply the Perron's method with the version up to the boundary (see [44]). Since a solution is also a subsolution, the Hölder regularity is a direct consequence of Theorem 4.1.2.

□

4.4 The Ergodic Problem

4.4.1 Existence of the pair (c, u_∞)

In this part we study the existence of a pair $(c, u_\infty) \in \mathbb{R} \times C(\overline{\Omega})$ for which u_∞ is a viscosity solution of the state-constraints problem (4.1.9)-(4.1.10), to gather with the uniqueness of the ergodic constant c. For this purpose, we introduce a λu-term in the equation, as it is classical, with the aim to let λ tend toward 0. This key step is described by the following Lemma.

Lemma 4.4.1. *Let $\tilde{f} \in C(\overline{\Omega})$ and $\beta = \dfrac{q-p}{q-p+1}$. For $0 < \lambda < 1$ and $q > p$, there exists a unique viscosity solution $u_\lambda \in C^{0,\beta}(\overline{\Omega})$ of the state constraint problem*

$$-\,div\,(|Du_\lambda|^{p-2} Du_\lambda) + |Du_\lambda|^q + \lambda u_\lambda = \tilde{f}(x) \quad in\ \Omega, \tag{4.4.1}$$

$$-\,div\,(|Du_\lambda|^{p-2} Du_\lambda) + |Du_\lambda|^q + \lambda u_\lambda \geq \tilde{f}(x) \quad on\ \partial\Omega. \tag{4.4.2}$$

Moreover there exists a constant $\tilde{C} > 0$ such that, for all $0 < \lambda < 1$,

$$|\lambda u_\lambda| \leq \tilde{C} \quad in\ \overline{\Omega}. \tag{4.4.3}$$

Proof. For $R > 0$, we consider the following generalized Dirichlet problem

$$\begin{cases} -\mathrm{div}\,(|Du_{R,\lambda}|^{p-2}Du_{R,\lambda}) + |Du_{R,\lambda}|^q + \lambda u_{R,\lambda} = f(x) & \text{in} \quad \Omega, \\ u_{R,\lambda} = R & \text{in} \quad \partial\Omega. \end{cases} \tag{4.4.4}$$

By Theorem 4.3.1, this problem admits a unique viscosity solution $u_{R,\lambda}$.

Moreover, $u_{R,\lambda}$ satisfies

$$-\lambda^{-1}\|f\|_{L^\infty} \leq u_{R,\lambda} \leq -\frac{M_1}{\beta}d^\beta(x) + \frac{M_2}{\lambda} \quad \text{in} \quad \Omega. \tag{4.4.5}$$

Indeed, on the one hand, it is easy to see that $-\lambda^{-1}\|f\|_{L^\infty}$ is a subsolution. On the other hand, borrowing arguments from [113], we claim that for some $M_1, M_2 > 0$ chosen large enough, $\bar{u}(x) = -\frac{M_1}{\beta}d^\beta(x) + \frac{M_2}{\lambda}$ is a supersolution of (4.4.1)-(4.4.2). Indeed, using that $q(\beta-1) = (p-2)(\beta-1) + (\beta-2)$, we have

$$\begin{aligned} -\mathrm{div}\,(|D\bar{u}|^{p-2}D\bar{u}) + |D\bar{u}|^q + \lambda\bar{u} - \tilde{f}(x) &= M_1^{p-1}|Dd|^{p-2}d^{(p-2)(\beta-1)}\Big[(p-1)(\beta-1)d^{\beta-2}|Dd|^2 \\ &+ d^{\beta-1}\Delta d + (p-2)d^{\beta-1}\left\langle D^2d\,\hat{D}d, \hat{D}d\right\rangle\Big] \\ &+ M_1^q d^{q(\beta-1)}|Dd|^q - \lambda\frac{M_1}{\beta}d^\beta + M_2 - \tilde{f} \\ &= M_1^{p-1}|Dd|^{p-2}d^{q(\beta-1)}\Big[(p-1)(\beta-1)|Dd|^2 + d\Delta d \\ &+ (p-2)d\left\langle D^2d\,\hat{D}d, \hat{D}d\right\rangle + M_1^{q-p+1}|Dd|^{q-p+2}\Big] \\ &- \lambda\frac{M_1}{\beta}d^\beta + M_2 - \tilde{f}. \end{aligned}$$

In Ω^δ where $|Dd| = 1$ and $0 \leq d \leq \delta$, we have

$$\begin{aligned} -\mathrm{div}\,(|D\bar{u}|^{p-2}D\bar{u}) + |D\bar{u}|^q + \lambda\bar{u} - \tilde{f}(x) &= M_1^{p-1}d^{q(\beta-1)}\Big[(p-1)(\beta-1) + d\Delta d \\ &+ (p-2)d\left\langle D^2\;dD\,d, Dd\right\rangle + M_1^{q-p+1} \\ &- \lambda\frac{M_1^{2-p}}{\beta}d^{\beta(2-p)+p}\Big] + M_2 - \tilde{f}. \end{aligned}$$

Taking $M_1 > 1$ and $M_2 > 0$ such that

$$M_1^{q-p+1} \geq (p-1)(1-\beta) + (p-2+\sqrt{N})\delta\left\|D^2d\right\|_{L^\infty} + \frac{\delta^{\beta(2-p)+p}}{\beta} \tag{4.4.6}$$

$$\text{and} \qquad M_2 \geq 2\left\|\tilde{f}\right\|_{L^\infty}, \tag{4.4.7}$$

then we have $-\mathrm{div}\,(|D\bar{u}|^{p-2}D\bar{u}) + |D\bar{u}|^q + \lambda\bar{u} - \tilde{f}(x) \geq 0$ in Ω^δ.

Now in Ω_δ, we have $|Dd| \le 1$ and $\delta \le d(x) \le C(\Omega)$. Using that $0 < \lambda < 1$, then we have

$$\begin{aligned} -\mathrm{div}\,(|D\bar{u}|^{p-2}D\bar{u}) + |D\bar{u}|^q + \lambda\bar{u} - \tilde{f}(x) \ \ge\ & M_1^{p-1}\Big[(p-1)(\beta-1)\left\|d^{(\beta-1)(p-1)-1}\right\|_{L^\infty} \\ & -(p-2+\sqrt{N})\left\|d^{(\beta-1)(p-1)}\right\|_{L^\infty}\left\|D^2 d\right\|_{L^\infty}\Big] \\ & -\frac{M_1}{\beta}\left\|d^\beta\right\|_{L^\infty} + M_2 - \left\|\tilde{f}\right\|_{L^\infty}. \end{aligned}$$

Hence if we take M_1 as in (4.4.6) and M_2 such that

$$\begin{aligned} M_2 \ \ge\ & M_1^{p-1}\Big[(p-1)(1-\beta)\left\|d^{(\beta-1)(p-1)-1}\right\|_{L^\infty} + (p-2+\sqrt{N})\left\|d^{(\beta-1)(p-1)}\right\|_{L^\infty}\left\|D^2 d\right\|_{L^\infty}\Big] \\ & +\frac{M_1}{\beta}\left\|d^\beta\right\|_{L^\infty} + 3\left\|\tilde{f}\right\|_{L^\infty}, \end{aligned} \tag{4.4.8}$$

then the function $\bar{u}$ satisfies the supersolution inequality in Ω_δ. The estimate follows by applying the SCR to $-\lambda^{-1}\|f\|_{L^\infty}$, $u_{R,\lambda}$ and $\bar{u}$.

It is worth pointing out that, if M_2 is as in (4.4.8), then

$$u_{R,\lambda} < R \quad \text{on } \overline{\Omega} \quad \text{for any } R > \frac{M_2}{\lambda}.$$

It follows that $u_{R,\lambda}$ is a viscosity solution of (4.4.1)-(4.4.2) for all $R > \frac{M_2}{\lambda}$. Theorem 4.3.1 implies that $u_\lambda = u_{R,\lambda}$ for $R > \frac{M_2}{\lambda}$.

We have

$$-\max\left(\left\|\tilde{f}\right\|_{L^\infty}, M_2\right) \le \lambda u_\lambda \le \max\left(\left\|\tilde{f}\right\|_{L^\infty}, M_2\right).$$

□

Now we are in position to prove Theorem 4.1.3. Using that $u_\lambda \ge -\lambda^{-1}\|f\|_{L^\infty}$ in $\overline{\Omega}$, we have

$$-\mathrm{div}(|Du_\lambda|^{p-2}Du_\lambda) + |Du_\lambda|^q - \tilde{f} \le \left\|\tilde{f}\right\|_{L^\infty} \quad \text{in } \Omega.$$

Theorem 4.1.2 implies uniform Hölder estimates with respect to λ for the functions u_λ. Consequently if x_0 is an arbitrary point in $\overline{\Omega}$, we get that $w_\lambda := u_\lambda(x) - u_\lambda(x_0)$ is also uniformly bounded in $C^{0,\beta}(\overline{\Omega})$ (recall that Ω is connected).

From (4.4.3), we also know that $\{-\lambda u_\lambda(x_0)\}_\lambda$ is bounded. It follows that, by Ascoli's Theorem, we can extract a uniformly converging subsequence from $\{w_\lambda\}_\lambda$ and we can assume that $\{-\lambda u_\lambda(x_0)\}_\lambda$ converges along the same subsequence. Denoting by u_∞ and c, the limits of $\{w_\lambda\}_\lambda$ and $\{-\lambda u_\lambda(x_0)\}_\lambda$ respectively and taking into account that w_λ solves

$$-\mathrm{div}(|Dw_\lambda|^{p-2}Dw_\lambda) + |Dw_\lambda|^q - \tilde{f}(x) + \lambda w_\lambda = -\lambda u_\lambda(x_0) \quad \text{in } \Omega,$$

we can pass into the limit $\lambda \to 0$ and conclude by the stability result for viscosity solutions that, (c, u_∞) solves the ergodic problem.

Now let (c_1, u^1_∞) and (c_2, u^2_∞) be two solutions of the ergodic problem. If $c_1 < c_2$ or $c_1 > c_2$, we could use Proposition 4.2.3 to obtain either $u^1_\infty \leq u^2_\infty$ or $u^2_\infty \leq u^1_\infty$. But such comparison cannot hold since, for all $k \in \mathbb{R}$, $u^i_\infty + k$ are solutions as well of the ergodic problem, proving the uniqueness of c.

4.5 Proof of Theorem 4.1.1 and Study of the Large Time Behavior

4.5.1 Proof of Theorem 4.1.1

Once one noticed that $u_1(x,t) = t\,\|f\|_{L^\infty} + \|g\|_{L^\infty} + \|u_0\|_{L^\infty}$ and $u_2(x,t) = -t\,\|f\|_{L^\infty} - \|g\|_{L^\infty} - \|u_0\|_{L^\infty}$ are respectively super-solution and subsolution of (4.1.1)–(4.1.3), the existence and uniqueness of a continuous global solution can be obtained by Perron's method, combining classical arguments of [43] (see also [70]), the version up to the boundary of Da Lio [44] and the Strong Comparison Result of the Proposition 4.2.2 on any time interval $[0, T]$.

4.5.2 Large Time Behavior

Let u_∞ be a bounded solution of (4.1.9)–(4.1.10). If $c \leq 0$, then u is uniformly bounded. Indeed, if $C > \|u_\infty\|_{L^\infty} + \|u_0\|_{L^\infty} + \|g\|_{L^\infty}$, then $u_\infty - C$ is a subsolution of (4.1.1)–(4.1.3). On the other hand, if $\bar{x}$ is a point far enough from Ω, then $|x - \bar{x}|^2$ is a super-solution. To see this,it suffices to take $\bar{x}$ such that $B(\bar{x}, R) \cap \overline{\Omega} = \emptyset$ with $R > \max(1, (\|f\|_{L^\infty} + (p-1))^{\frac{1}{q-p+2}})$.

Hence applying the Strong Comparison Result, we have

$$u_\infty(x) - C \leq u(x,t) \leq |x - \bar{x}|^2 + C \quad \text{on} \quad \overline{\Omega} \times (0, +\infty),$$

and therefore

$$\lim_{t \to \infty} \frac{u(x,t)}{t} = 0.$$

If $c > 0$, then $u_\infty - ct + C$ is a supersolution of (4.1.1)–(4.1.3) with state constraint condition on $\partial\Omega$. On the other hand, $u_\infty - ct - C$ is a subsolution of (4.1.1)–(4.1.3) which is below u_0 at $t = 0$ and below g on $\partial\Omega$. Applying the Strong Comparison Result, we have

$$-ct + u_\infty - C \leq u(x,t) \leq u_\infty - ct + C \quad \text{on} \quad \overline{\Omega} \times (0, +\infty).$$

The result follows by dividing by t and then letting $t \to +\infty$.

Appendice : A General Strong Comparison Result

A : Properties of the Regularization by Sup-convolution of Viscosity Subsolutions

To circumvent the lack of smoothness of the viscosity subsolution u, we consider instead the more regular time sup-convolution u^α. Such regularization was first introduced by Lasry and Lions [81], and for $0 < \alpha \leq 1$ and u a bounded usc, viscosity subsolution is defined by

$$u^\alpha(x,t) = \sup_{s\geq 0}\left\{ u(x,s) - \frac{|t-s|^2}{\alpha^2}\right\}. \tag{4.5.1}$$

We have the following useful properties on u^α.

Proposition 4.5.1. *If u is a bounded usc viscosity subsolution u of* (4.1.1)–(4.1.3), *the following properties are true*

i) Set $K = \sqrt{2\,\|u\|_{L^\infty}}$. Then up to $o_\alpha(1)$, u^α is an usc viscosity subsolution of (4.1.1)–(4.1.3) *on $\Omega\times(K\alpha, T-K\alpha)$. Moreover u^α is locally Lipschitz w.r.t to the time variable and*
$$||u^\alpha_t||_{L^\infty} \leq \frac{2K}{\alpha}\,.$$

ii) We have $u^\alpha(x,K\alpha) \leq u_0(x) + o_\alpha(1)$ in $\overline{\Omega}$ and $u^\alpha(x,t) \leq g(x,t) + o_\alpha(1)$ on $\partial\Omega \times (K\alpha, T-K\alpha)$.

iii) u^α is Hölder continuous w.r.t the space variable x on $\overline{\Omega}$ uniformly w.r.t the time for $t > K\alpha$.

Proof. Since $u(x)$ is bounded, the supremum in (4.5.1) is attained at some point $s^*(t)$ which belong to the interval $(t-K\alpha, t+K\alpha)$. Let $\varphi \in C^2\left(\overline{\Omega}\times[K\alpha, T-K\alpha]\right)$ and assume that $u^\alpha - \varphi$ has a local maximum at $(x_0,t_0) \in \Omega\times(K\alpha, T-K\alpha)$. Denote by $s^*(t_0)$ a point such that $u^\alpha(x_0,t_0) = u(x_0, s^*(t_0)) - \dfrac{|t_0 - s^*(t_0)|^2}{\alpha^2}$, then the function

$$\tau \mapsto u(x,\tau) - \varphi(x, \tau - s^*(t_0) + t_0)$$

reaches a local maximum at $(x_0, s^*(t_0))$. Recalling that u is a viscosity subsolution of (4.1.1), we get by definition

$$\varphi_t(x_0,t_0) - \mathrm{div}\left(|D\varphi|^{p-2}D\varphi(x_0,t_0)\right) + |D\varphi(x_0,t_0)|^q \leq f(x_0, s^*(t_0)) \leq f(x_0,t_0) + o_\alpha(1),$$

by using the uniform continuity of f on $\overline{\Omega}\times[0,T]$.

Next, let $h > 0$ small enough, then

$$\begin{aligned} u^\alpha(x,t\pm h) - u^\alpha(x,t) &\geq u(x,s^*(t)) - \frac{|t\pm h - s^*(t)|^2}{\alpha^2} - u(x,s^*(t)) + \frac{|t-s^*(t)|^2}{\alpha^2} \\ &= -\left(\frac{h^2 \pm 2h(t-s^*(t))}{\alpha^2}\right) \geq -\left(\frac{h^2 + 2Kh\alpha}{\alpha^2}\right). \end{aligned}$$

A first estimate of u_t^α (from below) follows by dividing the previous inequality by h and sending $h \to 0$. Exchanging the role of $t+h$ and t provides the estimate from above.

The second assertion comes from the upper semi-continuity of u and the fact that $u(x,0) \le u_0(x)$. Indeed

$$u^\alpha(x, K\alpha) = u(x, s^*(K\alpha)) - \frac{|K\alpha - s^*(K\alpha)|^2}{\alpha^2} \le u(x, s^*(K\alpha))$$

with $s^*(K\alpha) \to 0$ as $\alpha \to 0$. Taking the lim sup we get

$$\limsup_{\alpha\to 0} u^\alpha(x, K\alpha) \le \limsup_{\alpha \to 0} u(x, s^*(K\alpha)) \le u(x,0) \le u_0(x).$$

Similarly, we use the semi-continuity of u and Proposition 4.2.1 to prove that

$$u^\alpha(x,t) \le g(x,t) + o_\alpha(1).$$

The last assertion is a consequence of Theorem 4.1.2 where $C = \|f\|_{L^\infty} + \|u_t^\alpha\|_{L^\infty}$.

□

Let us note that, using the lower semi-continuity of v and $v(x,0) \ge u_0(x)$, we have $v(x, K\alpha) \ge u_0(x) - o_\alpha(1)$. Hence

$$u^\alpha(x, K\alpha) \le v(x, K\alpha) + \omega(\alpha) \quad \text{for all} \quad x \in \overline{\Omega}, \tag{4.5.2}$$

for some $\omega(\alpha)$ satisfying $\lim_{\alpha\to 0} \omega(\alpha) = 0$.

B : Proof of Proposition 4.2.2

In order to prove the SCR, we are going to show that $\tilde{u}^\alpha - v \le \omega(\alpha)$ in $\overline{\Omega} \times [K\alpha, T-K\alpha]$. Inequality (4.2.10) follows by passing to the limit as $\alpha \to 0$. To do so, the continuity of u^α is a key point since it allows to use the arguments of [21, 15].

For the sake of simplicity of notations, we drop the $\tilde{}$ on $\tilde{u}^\alpha$. The key idea is to compare $u_\mu^\alpha := \mu u^\alpha$ and v with $0 < \mu < 1$ close to 1 in order to take care of the difficulty due to the $|Du|^q$ term.

We argue by contradiction assuming that $M^\alpha = \max_{\overline{\Omega}\times[K\alpha, T-K\alpha]} (u^\alpha - v - \omega(\alpha)) > 0$. If μ is sufficiently close to 1 and if $\eta_\alpha > 0$ is a constant small enough, then we have $M_{\mu,\eta}^\alpha = \max_{\overline{\Omega}\times[K\alpha,T-K\alpha]} \big(u_\mu^\alpha - v - \omega(\alpha) - \eta_\alpha(t - K\alpha)\big) > M^\alpha/2$.

We denote by (x_0, t_0) a point of $\overline{\Omega} \times [K\alpha, T-K\alpha]$ such that $M_{\mu,\eta}^\alpha = u_\mu^\alpha(x_0,t_0) - v(x_0,t_0) - \omega(\alpha) - \eta_\alpha(t_0 - K\alpha)$. The existence of (x_0, t_0) is guaranteed by the upper and lower semi-continuity of u^α and v respectively (we drop the dependence of (x_0,t_0) on η_α, α and μ for the sake of simplicity of notations). Since $M_{\eta,\mu}^\alpha > 0$, we necessarily have $t_0 > K\alpha$ in view of (4.5.2). By the Maximum Principle of the "Users guide" [43], we have

$(x_0, t_0) \in \partial\Omega \times (K\alpha, T - K\alpha)$.

Next, using the regularity of the boundary, we can find a C^2-function $\xi : \mathbb{R}^N \to \mathbb{R}^N$ which is equal to $n = -Dd$ in a neighborhood of $\partial\Omega$. Now we consider the auxiliary function $\Phi_\varepsilon : \overline{\Omega} \times \overline{\Omega} \times [K\alpha, T - K\alpha] \times [K\alpha, T - K\alpha] \to \mathbb{R}$ defined by

$$\begin{aligned} \Phi_\varepsilon(z, w, t, s) &= u^\alpha_\mu(z, t) - v(w, s) - \omega(\alpha) - \eta_\alpha(t - K\alpha) - \left|\frac{z - w}{\varepsilon} - \chi\left(\frac{z + w}{2}\right)\right|^4 \\ &\quad - \frac{|t - s|^2}{\varepsilon^2} \end{aligned}$$

Let $(\bar{z}, \overline{w}, \bar{t}, \bar{s})$ be a global maximum point of Φ_ε on $\overline{\Omega}\times\overline{\Omega}\times[K\alpha, T-K\alpha]\times[K\alpha, T-K\alpha]$. For notational simplicity we drop again the dependance of $(\bar{z}, \overline{w}, \bar{t}, \bar{s})$ on ε, μ and η. Using the inequality $\Phi_\varepsilon(\bar{z}, \overline{w}, \bar{t}, \bar{s}) \geq \Phi_\varepsilon(x_0, x_0, t_0, t_0)$ and the boundedness of u^α_μ, v and χ, we have

$$\left|\frac{\bar{z} - \overline{w}}{\varepsilon}\right| \leq C, \quad \left|\frac{\bar{t} - \bar{s}}{\varepsilon}\right| \leq C,$$

for some constant $C > 0$ depending on $\|u\|_{L^\infty}, \|v\|_{L^\infty}$ and α. By the compactness of $\overline{\Omega} \times [K\alpha, T - K\alpha]$, we can assume that $(\bar{z}, \bar{t})$, $(\overline{w}, \bar{s})$ converge to $(\tilde{x}, \tilde{t}) \in \overline{\Omega} \times [K\alpha, T - K\alpha]$. Moreover, using the continuity of u^α, we have

$$\Phi_\varepsilon(\bar{z}, \overline{w}, \bar{t}, \bar{s}) \geq \Phi_\varepsilon(x_0 - \varepsilon\xi(x_0), x_0, t_0, t_0) = M^\alpha_{\mu,\eta} - o_\varepsilon(1), \quad \text{as}\, \varepsilon \to 0,$$

and hence

$$\liminf_{\varepsilon\to 0} \Phi_\varepsilon(\bar{z}, \overline{w}, \bar{t}, \bar{s}) \geq M^\alpha_{\mu,\eta}. \tag{4.5.3}$$

On the other hand, we have also

$$\begin{aligned} \limsup_{\varepsilon\to 0} \Phi_\varepsilon(\bar{z}, \overline{w}, \bar{t}, \bar{s}) &\leq \limsup_{\varepsilon\to 0} \left(u^\alpha_\mu(\bar{z}, \bar{t}) - v(\overline{w}, \bar{s}) - \eta_\alpha(\bar{t} - K\alpha) - \omega(\alpha)\right) \\ &\quad -\liminf_{\varepsilon\to 0} \left|\frac{\bar{z} - \overline{w}}{\varepsilon} - \chi\left(\frac{\bar{z} + \overline{w}}{2}\right)\right|^4 \\ &\quad -\liminf_{\varepsilon\to 0} \frac{|\bar{t} - \bar{s}|^2}{\varepsilon^2} \\ &\leq M^\alpha_{\mu,\eta}. \end{aligned} \tag{4.5.4}$$

Therefore, combining (4.5.3) and (4.5.4) with classic arguments, we have

$$\left|\frac{\bar{z} - \overline{w}}{\varepsilon} - \chi\left(\frac{\bar{z} + \overline{w}}{2}\right)\right|^4 = o_\varepsilon(1), \qquad \frac{|\bar{t} - \bar{s}|^2}{\varepsilon^2} = o_\varepsilon(1), \tag{4.5.5}$$

$$\begin{aligned} u^\alpha_\mu(\bar{z}\bar{t}) - v(\overline{w}, \bar{s}) - \eta_\alpha(\bar{t} - K\alpha) - \omega(\alpha) &\to u^\alpha_\mu(\tilde{x}, \tilde{t}) - v(\tilde{x}, \tilde{t}) - \eta_\alpha(\tilde{t} - K\alpha) - \omega(\alpha) \\ &= M^\alpha_{\mu,\eta} \quad \text{as}\, \varepsilon \to 0. \end{aligned} \tag{4.5.6}$$

It follows that $u_\mu^\alpha(\bar z,\bar t)\to u_\mu^\alpha(\tilde x,\tilde t)$ and $v(\overline{w},\bar s)\to v(\tilde x,\tilde t)$.

Now, recalling the properties of u^α and v at $t=K\alpha$, we have $\bar t,\bar s>K\alpha$ for ε small enough. Next we claim that, for ε small enough the viscosity inequalities hold for u^α and v. This is obviously the case for v if $\overline{w}\in\Omega$. If on the contrary $\overline{w}\in\partial\Omega$, then we necessarily have $v(\overline{w},\bar s)<g(\overline{w},\bar s)$. Indeed if $\overline{w}\in\partial\Omega$ then $\tilde x\in\partial\Omega$. Since there is no loss of boundary conditions for subsolution s as clearly specified in Proposition 4.2.1, we have

$$\mu u^\alpha(\tilde x,\tilde t)\le \mu(g(\tilde x,\tilde t)+\omega(\alpha)).$$

Using that $M^\alpha_{\mu,\eta}>0$, we cannot have $v(\tilde x,\tilde t)\ge g(\tilde x,\tilde t)$ since we would then have

$$\frac{M^\alpha}{2}\le M^\alpha_{\mu,\eta}=\mu u^\alpha(\tilde x,\tilde t)-v(\tilde x,\tilde t)-\eta_\alpha(\tilde t-K\alpha)-\omega(\alpha)\le(\mu-1)(g(\tilde x,\tilde t)+\omega(\alpha)),$$

a contradiction by sending $\mu\to1$.

It follows that, if $\tilde x\in\partial\Omega$, then we have necessarily that

$$v(\tilde x,\tilde t)<g(\tilde x,\tilde t)\quad\text{and}\quad \mu u^\alpha(\tilde x,\tilde t)\le\mu(g(\tilde x,\tilde t)+\omega(\alpha)). \tag{4.5.7}$$

Hence, using that $v(\overline{w},\bar s)\to v(\tilde x,\tilde t)<g(\tilde x,\tilde t)$, we deduce that if $\overline{w}\in\partial\Omega$, then $v(\overline{w},\bar s)<g(\overline{w},\bar s)$ for ε small enough and the viscosity inequality holds also in this case.

On the other hand, from (4.5.5) we get that

$$\bar z=\overline{w}+\varepsilon\chi\left(\frac{\bar z+\overline{w}}{2}\right)+o_\varepsilon(1), \tag{4.5.8}$$

which implies by the smoothness of the domain and the properties of χ that $\bar z$ lies in Ω for ε small enough and hence the viscosity inequality for u^α_μ holds too.

Next, we notice that u^α_μ satisfies

$$\frac{1}{\mu}(u^\alpha_\mu)_t-\frac{1}{\mu^{p-1}}\mathrm{div}\,(|Du^\alpha_\mu|^{p-2}Du^\alpha_\mu)+\frac{1}{\mu^q}|Du^\alpha_\mu|^q\le f+o_\alpha(1)\quad\text{in }\Omega\times(K\alpha,T-K\alpha),$$

and we can also re-write it as

$$\begin{aligned}\mu^{p-2}(u^\alpha_\mu)_t-|Du^\alpha_\mu|^{p-2}\left\{\Delta u^\alpha_\mu+(p-2)(D^2u^\alpha_\mu\widehat{Du^\alpha_\mu},\widehat{Du^\alpha_\mu})-\mu^{p-1-q}|Du^\alpha_\mu|^{q-p+2}\right\}&\\ \le\mu^{p-1}(f+o_\alpha(1)),&\end{aligned}$$

where $\widehat{\xi}=\dfrac{\xi}{|\xi|}$ for $\xi\neq0$ and $\widehat{\xi}=0$ if $\xi\equiv0$.

The Jensen-Ishii's Lemma [43] ensures the existence of X, $Y\in\mathcal{S}^N$, a, $b\in\mathbb{R}$, q_1, $q_2\in\mathbb{R}^N$ such that

$$(a,q_1,X)\in\overline{\mathcal{P}}^{2,1,+}u^\alpha_\mu(\bar z,\bar t),\qquad (b,q_2,Y)\in\overline{\mathcal{P}}^{2,1,-}v(\overline{w},\bar s), \tag{4.5.9}$$

$$a - b \geq \eta_\alpha > 0, \qquad |q_1 - q_2| \leq C\varepsilon(|q_1| \wedge |q_2|), \tag{4.5.10}$$

$$-\frac{o(1)}{\varepsilon^2} I_{2N} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{o(1)}{\varepsilon^2} \begin{pmatrix} I_N & -I_N \\ -I_N & I_N \end{pmatrix} + o(1) I_{2N}. \tag{4.5.11}$$

where $|q_1| \wedge |q_2|$ denotes the minimum of $|q_1|$ and $|q_2|$. Indeed, for (4.5.10), we remark that

$$q_1 = \left(I_N - \frac{\varepsilon}{2} D\chi \left(\frac{\bar{z} + \overline{w}}{2} \right) \right) q \quad \text{and} \quad q_2 = \left(I_N + \frac{\varepsilon}{2} D\chi \left(\frac{\bar{z} + \overline{w}}{2} \right) \right) q\,,$$

with

$$q = \frac{4}{\varepsilon} \left| \frac{z - w}{\varepsilon} - \chi \left(\frac{z + w}{2} \right) \right|^2 \left(\frac{z - w}{\varepsilon} - \chi \left(\frac{z + w}{2} \right) \right)\,,$$

and (4.5.10) is an easy consequence of the boundedness of $D\chi$.

Moreover the viscosity inequalities for u_μ^α and v read

$$\mu^{p-2} a - |q_1|^{p-2} \left\{ \operatorname{tr}\Big(\left[Id + (p-2)\left(\hat{q_1} \otimes \hat{q_1} \right) \right] X \Big) - \mu^{p-1-q} |q_1|^{q-p+2} \right\} \tag{4.5.12}$$
$$\leq \mu^{p-1} (f(\bar{z}, \bar{t}) + o_\alpha(1)),$$

$$b - |q_2|^{p-2} \left\{ \operatorname{tr}\Big(\left[Id + (p-2)(\hat{q_2} \otimes \hat{q_2}) \right] Y \Big) - |q_2|^{q-p+2} \right\} \geq f(\bar{w}, \bar{s}). \tag{4.5.13}$$

In the sequel we fix $\eta_\alpha > 2 o_\alpha(1)$ (recall that the $o_\alpha(1)$ comes from the sup-convolution procedure and is fixed, therefore we can choose in such a way η_α). Since we may have a singularity at $q_1 = 0$ or $q_2 = 0$, we have to consider separately three cases. First we assume that there exists a constant $\gamma > 0$ such that

$$|q_1|, |q_2| \geq \gamma.$$

In this case the matrix $A(\xi) = Id + (p-2)(\hat{\xi} \otimes \hat{\xi})$ is positive definite, so that its matrix square root σ exists and satisfies

$$|\sigma(\xi_1) - \sigma(\xi_2)| \leq c \frac{|\xi_1 - \xi_2|}{|\xi_1| \wedge |\xi_2|}.$$

Combining (4.5.10) with the fact that (4.5.11) implies that $X \leq Y + o_\varepsilon(1)$, we have

$$\operatorname{tr}\left(A(q_1) X \right) - \operatorname{tr}\left(A(q_2) Y \right) \leq \frac{o_\varepsilon(1)}{\varepsilon^2} |\sigma(q_1) - \sigma(q_2)|^2 + o_\varepsilon(1) \leq o_\varepsilon(1), \tag{4.5.14}$$

$$\begin{aligned} |q_2|^{q-p+2} - \mu^{p-1-q} |q_1|^{q-p+2} &= |q_2|^{q-p+2} - |q_1|^{q-p+2} + (1 - \mu^{p-1-q}) |q_1|^{q-p+2} \\ &\leq (q - p + 2) |q_1|^{q-p+1} |q_2 - q_1| + (1 - \mu^{p-1-q}) |q_1|^{q-p+2} \\ &\leq o(\varepsilon) |q_1|^{q-p+2} + (1 - \mu^{p-1-q}) |q_1|^{q-p+2}. \end{aligned} \tag{4.5.15}$$

Multiplying (4.5.12) by $\dfrac{|q_2|^{p-2}}{|q_1|^{p-2}}$ which is of order $1 + O(\varepsilon)$ and subtracting from it (4.5.13), we have

$$\begin{aligned}(1+O(\varepsilon))\mu^{p-2}a-b &\leq |q_2|^{p-2}\left\{o_\varepsilon(1)+o(\varepsilon)|q_1|^{q-p+2}+(1-\mu^{p-1-q})|q_1|^{q-p+2}\right\}\\ &+ \mu^{p-1}(1+O(\varepsilon))(f(\bar{z},\bar{t})+o_\alpha(1))-f(\overline{w},\bar{s}). \qquad (4.5.16)\end{aligned}$$

At this point, we recall that the Lipschitz continuity of u^α implies that $|a| \leq \dfrac{2\mu K}{\alpha}$. On the other hand we remark that, since $1-\mu^{p-1-q}<0$, for fixed μ the term $o(\varepsilon)|q_1|^{q-p+2}$ is controlled by the $(1-\mu^{p-1-q})|q_1|^{q-p+2}$ term.

Now we are going to let $\varepsilon \to 0$: if we assume that q_1, q_2 (which depend on ε) are bounded, we may assume that they converge (we still denote their limits as q_1, q_2 respectively). For μ close enough to 1, we get as $\varepsilon \to 0$

$$0<\eta_\alpha/2 \leq \mu^{p-2}a-b \leq (\mu^{p-1}-1)f(\tilde{x},\tilde{t})+\mu^{p-1}o_\alpha(1)+|q_2|^{p-2}(1-\mu^{p-1-q})|q_1|^{q-p+2}$$

Recalling that $\eta_\alpha > 2o_\alpha(1)$, we get a contradiction when $\mu \to 1$ since the last term of the right-hand side is negative. Of course, we get the same contradiction if (at least for some subsequence) q_1 or $q_2 \to \infty$.

If $q_1, q_2 \neq 0$ but $q_1 \to 0$, $q_2 \to 0$ then, noticing that $\dfrac{|q_2|^{p-2}}{|q_1|^{p-2}}$ is still of order $1+O(\varepsilon)$, we can pass to the limit $\varepsilon \to 0$ in the same way and obtain

$$0<\eta_\alpha/2 \leq \mu^{p-2}a-b \leq (\mu^{p-1}-1)f(\tilde{x},\tilde{t})+\mu^{p-1}o_\alpha(1),$$

and we also get a contradiction.

If $q_1=0$ or $q_2=0$, then necessarily $q_1=q_2=0$ and, by subtracting (4.5.13) from (4.5.12), we have

$$\eta_\alpha/2 \leq \mu^{p-2}a-b \leq \mu^{p-1}(f((\bar{z},\bar{t}))+o_\alpha(1))-f(\overline{w},\bar{s}).$$

We get a contradiction when $\varepsilon \to 0$.

In all cases fixing $\eta_\alpha > 2o_\alpha(1)$ we get a contradiction for ε small enough and μ close to 1 and the conclusion follows.

C : Proof of Proposition 4.2.3

The proof of (i) and (ii) are very similar. Indeed we know by Theorem 4.1.2 that subsolutions of (4.2.2) are Hölder continuous, so we are always in a case where the subsolution or the supersolution are continous. We will only give details of the proof for the $p-$*Laplacian* operator in the case where $\lambda>0$ and $v \in C(\overline{\Omega})$. The other cases are an easy adaptation (the equation (4.2.1) is even easier to study). Since v is assumed to be continuous, we follow the proof of [21] with the same trick as before in order to take care of the strong growth of the gradient term. We argue by contradiction assuming that $M=\max\limits_{\overline{\Omega}}(u-v)>0$ If μ

is sufficiently close to 1, then we have $M_\mu = \max_{\overline{\Omega}} (u_\mu - v) > M/2 > 0$. Since u is usc and v is continuous this maximum is achieved at x_0. We may assume that $x_0 \in \partial\Omega$. We drop the dependence of x_0 on μ.

Next, using the regularity of the boundary, we can find a C^2-function $\xi : \mathbb{R}^N \to \mathbb{R}^N$ which is equal to n in a neighborhood of $\partial\Omega$. Now we consider the test function $\Phi_\varepsilon : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ defined by

$$\Phi_\varepsilon(z, w) = \mu u(z) - u(w) - \left| \frac{z - w}{\varepsilon} + \xi\left(\frac{z + w}{2}\right) \right|^4 .$$

Let (x, y) be a global maximum point of Φ_ε on $\overline{\Omega} \times \overline{\Omega}$. For notational simplicity we drop the dependence of x and y on ε. and μ. Using the boundedness of u and v, it is clear that $x - y = O(\varepsilon)$ and it follows that, along a subsequence, $x, y \to \bar{x} \in \overline{\Omega}$. Since u is lsc and v is continuous, we get that $\limsup_{\varepsilon \to 0} \Phi_\varepsilon(x, y) \leq M_\mu$.

On the other hand we have, using the **continuity** of v, we have

$$\Phi_\varepsilon(x, y) \geq \Phi_\varepsilon(x_0, x_0 - \varepsilon\xi(x_0)) \geq M_\mu + O(\varepsilon).$$

It follows that $\liminf_{\varepsilon \to 0} \Phi_\varepsilon \leq M_\mu$. Hence we get that

$$\Phi_\varepsilon(x, y) \to M_\mu \quad \text{as} \quad \varepsilon \to 0. \tag{4.5.17}$$

Standard arguments allow us to deduce from (4.5.17) that

$$\left| \frac{x - y}{\varepsilon} + \xi\left(\frac{x + y}{2}\right) \right|^4 = o_\varepsilon(1), \tag{4.5.18}$$

$$\mu u(x) - v(y) \to \mu u(\bar{x}) - v(\bar{x}) = M_\mu \quad \text{as} \quad \varepsilon \to 0. \tag{4.5.19}$$

Next we claim that, for ε small enough the viscosity inequalities hold for u and v. This is obviously the case if $y \in \Omega$. Using Proposition 4.2.1 and arguing similarly as in the previous proof, we get that, if $y \in \partial\Omega$ than $v(y) \leq \tilde{g}(y)$ and the viscosity inequality holds also in this case. On the other hand, using (4.5.19), we get that

$$x = y - \varepsilon\xi(y) + o_\varepsilon(1) \tag{4.5.20}$$

which implies by the smoothness of the domain and the properties of ξ that x lies in Ω for ε small enough and hence the viscosity inequality holds for μu.

Using the same arguments as the previous proof, we get that the elements $(q_1, X) \in \overline{\mathcal{J}}^{2,+} u_\mu$ and $(q_2, Y) \in \overline{\mathcal{J}}^{2,-} v$ given by the Jensen-Ishii's Lemma satisfy

$$\begin{aligned} \lambda(1 + O(\varepsilon))\mu^{p-2} u_\mu(x) - v(y) &\leq |q_2|^{p-2} \left\{ o_\varepsilon(1) + o(\varepsilon)|q_1|^{q-p+2} + (1 - \mu^{p-1-q})|q_1|^{q-p+2} \right\} \\ &+ \mu^{p-1}(1 + O(\varepsilon))\tilde{f}(x) - \tilde{f}(y). \end{aligned} \tag{4.5.21}$$

Letting $\varepsilon \to 0$ and then $\mu \to 1$, we get a contradiction. It follows that $\tilde{u} \leq v$ on $\overline{\Omega}$.

Chapitre 5

Classification des solutions globales pour le problème unidimensionnel

Dans ce chapitre on s'intéresse à la classification des solutions globales des solutions du problème 1D. Notre objectif est d'exclure l'existence de solutions globales non bornées en norme $W^{1,\infty}$ (i.e l'explosion en temps infini) pour certaines données au bord et initiales étendant ainsi les résultat connus pour le cas de la diffusion linéaire.

5.1 Introduction and main results

In this chapter we are interested in the asymptotic behavior of global solutions to the following one-dimensional degenerate diffusive Hamilton-Jacobi equation

$$\begin{cases} u_t - (|u_x|^{p-2}u_x)_x = |u_x|^q, & 0 < x < 1, t > 0, \\ u(t,0) = 0, \qquad u(t,1) = M \geq 0, & t > 0, \\ u(0,x) = u_0(x), & 0 < x < 1, \end{cases} \tag{5.1.1}$$

with $q > p > 2$, $M \geq 0$ and suitably regular initial data u_0.

Problem (5.1.1) models a variety of physical phenomena which arise for example in the study of surface growth where a stochastic version of it is known as the Kardar-Parisi-Zhang equation ($p = 2, q = 2$). It has also a mathematical interest through the viscosity approximation of Hamilton-Jacobi type equations from control theory.

Solutions of (5.1.1) exhibit a rich variety of qualitative behaviors, according to the values of $p \geq 2$ and $q \in (0, \infty)$.

If $q \leq p$, it is known that all solutions are global and bounded in $W^{1,\infty}$ norm [80]. For $q \in [p-1, p]$ it was proved in [85] that nonnegative viscosity solutions of (5.1.1) with homogeneous Dirichlet boundary condition decay to 0 and the rate of convergence was also obtained, see also [25] for the semilinear case. Concerning the large time behavior of global weak solutions to (5.1.1) with homogeneous boundary conditions and $q \in (0, p-1)$, it has been shown that there exists a one parameter family of nonnegative steady states, and any solution converges uniformly to one of these stationary solutions (cf. [111, 18, 83]).

For $q > p \geq 2$, the situation is quite different. It is known that for any $M \geq 0$ and suitably large u_0, there exist solutions of (5.1.1) for which the L^∞ norm of the gradient blows up in finite time (the L^∞ norm of the solution remaining bounded) [106, 7], while there exist global and decaying solutions for u_0 sufficiently small [110]. In view of a classification of all solutions of (5.1.1), it is then a natural question to ask whether or not C^1-unbounded global solutions may exist. The question of the boundedness of global solutions of (5.1.1) was initiated for the semilinear case $p = 2$ in [5] and further investigated in [108, 110]. Denoting $M_c := (q-1)^{\frac{q-2}{q-1}}/(q-2)$, the result of [5] says that if $0 \leq M < M_c$, then any global solution of (5.1.1) is bounded in C^1 norm for $t \geq 0$, that is,

$$\sup_{t \geq 0} \quad |u_x(t,.)|_\infty < \infty. \tag{5.1.2}$$

On the other hand, it is known from [108] that some unbounded global solutions do exist if $M = M_c$ and $u_0 \leq U(x)$ where $U(x) := M_c x^{(q-2)/(q-1)}$ is the unique singular steady state. Moreover the precise exponential rates of the gradient blow-up in infinite time was obtained.

Motivated by the results of the papers [5, 108], we modify the method used by Arrieta, Rodriguez-Bernal and Souplet and extend their results on the classification of large time behavior of global solutions to the degenerate parabolic equation case $p > 2$.
From now on, we assume that $q > p > 2$. By a solution of (5.1.1), we mean a weak solution (see Section 2 below for a precise definition and well-posedness results). We recall that weak solutions of (5.1.1) satisfy a comparison principle, hence in particular

$$\min\left\{\min_{[0,1]} u_0, 0\right\} \leq u(t,x) \leq \max\left\{\max_{[0,1]} u_0, M\right\}, \qquad 0 \leq t < T_{max}, \qquad 0 \leq x \leq 1. \tag{5.1.3}$$

Our main result is then the following :

Theorem 5.1.1. *Assume that $q > p > 2$ and u_0 in $W^{1,\infty}(0,1)$, $u_0(0) = 0, u_0(1) = M$. Set $M_b = \frac{q-p+1}{q-p}\left(\frac{q-p+1}{p-1}\right)^{\frac{1}{p-1-q}}$.*

(i) If $0 \leq M < M_b$, then any global weak solution of (5.1.1) *is bounded in C^1 norm. Moreover it converges in $C^1([0,1])$ to the unique steady state.*

(ii) If $M > M_b$, then all weak solutions of (5.1.1) *exhibit gradient blow-up in finite time.*

The proof of Theorem 5.1.1 proceeds by contradiction. It relies on the analysis of steady states and the existence of a Lyapunov functional which enjoys nice properties on any global trajectory of (5.1.1), even if it were unbounded in C^1 norm. The construction of such a nice Lyapunov functional which is handled through the Zelenyak technique, together with the fact that the singularities may only take place near the boundary, allow us to prove the following convergence result : any global solution, even unbounded in $W^{1,\infty}$ must converge in $C([0,1])$ to a stationary solution W of (5.1.1) with $W(0)=0$, $W(1)=M$ (see Proposition 5.3.3). On the other hand, if u were unbounded, then our gradient estimates would imply that $W_x(0)=+\infty$ or $W_x(1)=-\infty$. But such a W does not exist if $M\neq M_b$, leading to a contradiction.

Although the scheme of proof follows that in [5] for $p=2$, we have to face a number of additional technical difficulties, caused by the lack of regularity of solutions. In particular, we have to work at the level of regularized problems, including for the construction of Lyapunov-Zelenyak functional. This, in turn requires good convergence properties and estimates of regularized solutions. For this, we heavily rely on results from our previous work [7] (which concerned the higher dimensional problem as well) and an extension up to the boundary of a result of DiBenedetto-Friedman on the regularity of the derivative of weak solutions of a degenerate parabolic problem (see Proposition 5.2.1).

Remark 5.1.1. *(a) For the critical case $M=M_b$, all solutions must blow up in either finite or infinite time. The existence of global solution which are unbounded in $W^{1,\infty}$ norm (that is infinite time gradient blow-up) should occur for some suitable initial data as it is the case for the corresponding semilinear equation (namely for initial data u_0 below the singular steady state), but this still is an open problem. Moreover, we know from Proposition 5.3.3 that, even in this case the solutions will converge in $C([0,1])\cap C^1_{loc}((0,1])$ to the unique singular steady state.*

(b) Since the technique of Zelenyak to obtain a Lyapunov functional is restricted to the one-dimensional setting, the large time behavior and the boundedness of global solution in $W^{1,\infty}$ norm are still open problems in higher dimension.

Let us mention some results concerning related equations possessing solutions with unbounded gradient. When the nonlinearity is replaced with an exponential one and $p=2$, results on boundedness and existence of infinite time gradient blow-up solutions are obtained in [125, 128]. A phenomenon of infinite time gradient blow-up has been observed for quasilinear equations involving mean curvature type operators [38]. For results on interior gradient blow-up we refer the reader to [3, 6]. Finally for other results concerning existence, asymptotic behavior of global solutions for the corresponding Cauchy problem and a viscosity solution approach see [39, 84, 100] and references therein.

The rest of the chapter is organized as follows. Section 5.2 contains some useful preliminary material, including smoothing properties of solutions and estimate of the derivative u_x. In section 5.3, we employ the technique of Zelenyak [123], along with a trick used in [5], to construct an approximate Lyapunov functional for weak solutions to (5.1.1). Section 5.4 is devoted to the proof of Theorem 5.1.1.

5.2 Preliminary estimates and steady states

5.2.1 Space and time derivative estimates

For $u_0 \in W^{1,\infty}((0,1))$, $u_0(0) = 0, u_0(1) = M$, by a (weak) solution of (5.1.1) on $[0,T]$, we mean a function $u \in C([0,T) \times [0,1]) \cap L^q((0,T); W^{1,q}(0,1))$ such that

$$u_t \in L^2((0,T); L^2(0,1)), \quad u(0,x) = u_0(x), \quad u(t,0) = 0, u(t,1) = M$$

and

$$\int_0^T \int_0^1 u_t\psi + |u_x|^{p-2}u_x \cdot \psi_x \, dx\, dt = \int_0^T \int_0^1 |u_x|^q \psi \, dx\, dt, \tag{5.2.1}$$

holds for all $\psi \in C^0([0,T] \times [0,1]) \cap L^p((0,T); W_0^{1,p}((0,1))$.

It is known (see e.g., [7]) that there exists $T_{max} = T_{max}(u_0) \in (0, \infty]$ such that for each $T \in (0, T_{max})$, (5.1.1) admits a unique solution u such that $u \in L^\infty\big((0,T); W^{1,\infty}(0,1)\big)$. In the rest of this chapter, the maximal weak solution of problem (5.1.1) will refer to this solution.

Now let us state the following result (which will be very useful in the sequel) on the Hölder regularity of the derivative of solutions to a possibly degenerate parabolic problem. This result is an extension up to the boundary (in one space dimension) of an interior estimate of DiBenedetto-Friedman [48] (see the appendix for a proof). The definition of a weak solution of (5.2.2) is the same as in (5.2.1), with $|u_x|^q$ replaced by F and $u \in L^q(0,T; W^{1,q}(0,1))$ replaced by $v \in L^p(0,T; W^{1,p}(0,1))$.

Proposition 5.2.1. *Let $\varepsilon \in [0,1), u_0 \in W^{1,\infty}$, $C > 0$ and $F \in L^r((0,T) \times (0,1))$ for some $r > 2$ with $\|F\|_{L^r((0,T)\times(0,1))} \leq C$. Let v be a weak solution of*

$$\begin{cases} v_t = \Big((|v_x|^2 + \varepsilon^2)^{\frac{p-2}{2}} v_x\Big)_x + F(t,x), & t > 0, \quad x \in (0,1), \\ v(t,0) = 0, \qquad v(t,1) = M, & t > 0, \\ v(0,x) = u_0(x), & \qquad\quad x \in (0,1). \end{cases} \tag{5.2.2}$$

Then, for each $\eta > 0$, $v_x \in C^\alpha([\eta, T-\eta] \times [0,1])$ where $\alpha > 0$ and the Hölder norm of v_x depend only on $C, \|v_x\|_{L^p}$ and $\|v\|_{L_t^\infty, L_x^2}$.

As a direct consequence of this proposition, we get that u is a C^1-function w.r.t. the space variable in $(0,T) \times [0,1]$ and that its derivative u_x is locally Hölder continuous.

In order to describe the asymptotic behavior, we need to collect some preliminary estimates. We first give the following theorem which is of independent interest. It gives a useful regularizing property for local solutions of (5.1.1) as well as a uniform bound on u_t away from $t = 0$ for any space dimension.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{2+\alpha}$ for some $\alpha > 0$. Consider the following problem

$$\begin{cases} \partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^q, & x \in \Omega, t > 0, \\ u(x,t) = g(x), & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{5.2.3}$$

where the boundary data g is the trace on $\partial\Omega$ of a regular function in $C^2(\overline{\Omega})$, also denoted g, and the initial data u_0 satisfies

$$u_0 \in W^{1,\infty}(\Omega), \quad u_0(x) = g(x) \qquad \text{for} \quad x \in \partial\Omega. \tag{5.2.4}$$

Theorem 5.2.1. *Assume that $q > p-1$ and let u be a maximal weak Lipschitz solution of problem (5.2.3). We have the following statements.*

(i) Let $osc(u_0) = \max_{\overline{\Omega}} u_0 - \min_{\overline{\Omega}} u_0$, then

$$u_t \le \frac{1}{p-2}\frac{osc(u_0)}{t} \qquad \text{in} \quad \mathcal{D}'((0,T_{max}) \times \Omega). \tag{5.2.5}$$

(ii) Fix $t_0 \in (0, T_{max})$, then

$$u_t \ge -\left(\frac{q-p+1}{p-2}\right)\sup_{[0,t_0]\times\Omega}|\nabla u|^q - \left(\frac{1}{p-2}\right)\frac{osc(u_0)}{t} \qquad \textit{in} \quad \mathcal{D}'\left((0,t_0)\times\Omega\right). \tag{5.2.6}$$

(iii) Fix $t_0 \in (0, T_{max})$, then there exists $C_1 = C_1\left(p, q, t_0, \sup_{[0,t_0]\times\Omega}|\nabla u|, osc(u_0)\right) > 0$ such that

$$|u_t| \le C_1 \qquad \textit{in} \quad \mathcal{D}'\left((t_0, T_{max})\times\Omega\right). \tag{5.2.7}$$

Proof. The initial data being bounded and the sought-for estimate being invariant by addition of a constant, we may replace u with $u - B$, where $B = \min_{\overline{\Omega}} u_0 \ge \min_{\partial\Omega} g$. Then $u \ge 0$ by the maximum principle. (i) This has been proved in chapter 2.
(ii) Fix $t_0 \in (0, T_{max})$ and let $D = \sup_{[0,t_0]\times\Omega} |\nabla u|$. Set

$$w = u_\lambda := \lambda^\gamma u(\lambda t, x) + tL_\lambda \qquad 1 < \lambda < \frac{T_{max}}{t_0}, \gamma = \frac{1}{p-2}.$$

where

$$L_\lambda = (1 - \lambda^{\gamma(p-1-q)})(\lambda^\gamma D)^q.$$

Since $u \ge 0$, $\lambda \ge 1$ and $\gamma \ge 0$, we have $w \ge u$ on $\left\{\{0\}\times\overline{\Omega}\right\} \cup \{(0, t_0/\lambda)\times\partial\Omega\}$. On the other hand, we have on $(0, t_0/\lambda)\times\Omega$:

$$\partial_t w - \Delta_p w - |\nabla w|^q = (\lambda^{\gamma(p-1-q)} - 1)|\nabla w|^q + L_\lambda = (1 - \lambda^{\gamma(p-1-q)})[(\lambda^\gamma D)^q - |\nabla w|^q] \ge 0$$

Hence, by the comparison principle, we get that $w \ge u$ on $(0, t_0/\lambda)\times\Omega$, that is

$$\lambda^\gamma u(x, \lambda t) - u(x,t) \ge -(\lambda^{\gamma(p-1-q)} - 1)(\lambda^\gamma D)^q \tag{5.2.8}$$

Dividing (5.2.8) by $(\lambda - 1)$ and letting $\lambda \to 1^+$, we get

$$\gamma u + t\,\partial_t u \geq -(q-p+1)\gamma D^q\, t \qquad \text{in} \quad \mathcal{D}'\left((0,t_0)\times\Omega\right).$$

The estimate (5.2.6) follows.
(iii) Fix $t_0 \in (0, T_{max})$. By (5.2.5)-(5.2.6), for $h > 0$ small, we have

$$\|u(t_0/2 + h) - u(t_0/2)\|_\infty \leq C_1 h$$

where $C_1 = \left(\left(\dfrac{q-p+1}{p-2}\right) \sup\limits_{(0,t_0)\times\Omega} |\nabla u|^q + \left(\dfrac{1}{p-2}\right)\dfrac{osc(u_0)}{t_0}\right) > 0.$
Due to the translation invariance of (5.1.1), for $t > t_0/2$, $u(t+h,x)$ is still a solution of (5.1.1). Applying a comparison principle, we obtain that

$$\|u(t+h) - u(t)\|_\infty \leq Ch.$$

Since h is arbitrary small, we conclude that

$$|u_t| \leq C_1 \qquad \text{in} \quad \mathcal{D}'\left((t_0/2, T_{max})\times\Omega\right).$$

□

Thanks to the upper bound of u_t, we derive the following lemma giving lower and upper bounds on u_x, showing that u_x remains bounded away from the boundary.

Lemma 5.2.1. *Let u be a maximal weak Lipschiz solution of (5.1.1). For all $t_0 \in (0, T_{max})$, there exists $C_2 = C_2(t_0, p, osc(u_0), M) > 0$ such that for all $t \in [t_0, T_{max})$ and $0 < x < 1$,*

$$u_x(t,x) \leq \left(\left(\frac{q-p+1}{p-1}x\right)^{\frac{1-p}{q-p+1}} + C_2 x\right)^{\dfrac{1}{p-1}}, \tag{5.2.9}$$

$$u_x(t,1-x) \geq -\left(\left(\frac{q-p+1}{p-1}x\right)^{\frac{1-p}{q-p+1}} + C_2 x\right)^{\dfrac{1}{p-1}}. \tag{5.2.10}$$

Proof. Fix $t \in [t_0, T_{max})$ and let $y(x) = \left(|u_x|^{p-2}u_x(t,x) - C_2 x\right)^+$, where $C_2 = \dfrac{osc(u_0)}{(p-2)t_0}$. On any interval (a,b) with $0 < a < b < 1$ where $y > 0$, the function y satisfies in the classical sense $y' + y^{\frac{q}{p-1}} \leq 0$. Indeed, for each $x \in (a,b)$, we have $|u_x|^{p-2}u_x > C_2 x > C_2 a > 0$ and the function u is smooth at such points since the equation is uniformly parabolic [80]. Using theorem 5.2.1, we get that

$$y' + y^{\frac{q}{p-1}} \leq \big((|u_x|^{p-2}u_x)_x - C_2\big) + |u_x|^q \leq 0. \tag{5.2.11}$$

This implies that $y' < 0$ on (a, b) so that necessarily $a = 0$. Integrating inequality (5.2.11), it follows that $y(x) \leq \left(\frac{q-p+1}{p-1}x\right)^{\frac{1-p}{q-p+1}}$ on $(0, b)$ and $y(0) > 0$. If $y \not\equiv 0$, then we can find $c = c(t) \in (0, 1]$ such that $y > 0$ in $(0, c)$ and $y = 0$ in $[c, 1)$. Therefore we get $y(x) \leq \left(\frac{q-p+1}{p-1}x\right)^{\frac{1-p}{q-p+1}}$ on $(0, 1)$ and (5.2.9) is readily deduced. In the same manner, considering $y(x) = \big(-|u_x|^{p-2}u_x(t, 1-x) - C_2 x\big)^+$, we get (5.2.10).

□

Remark 5.2.1. *Similar gradient estimates in any space dimension are already obtained in chapter 2 using a more technical Bernstein type argument.*

The following corollary is a direct consequence of Lemma 5.2.1 that states that, when gradient blow-up occurs on the boundary, it can only be towards $+\infty$ at $x = 0$ or towards $-\infty$ at $x = 1$.

Corollary 5.2.1. *Let u be a maximal weak Lipschitz solution of (5.1.1) and $t_0 \in (0, T_{max})$. There exists $C_3 = C_3(t_0, p, q, osc(u_0)) > 0$ such that for all $t \in [t_0, T_{max})$,*

$$u_x(t, 0) \geq -C_3 \qquad \textit{and} \qquad u_x(t, 1) \leq C_3. \tag{5.2.12}$$

5.2.2 Steady states

It is a well-known fact that the large-time behavior of evolution equations is closely connected to the existence and properties of the stationary states. In this part we are looking for nonnegative stationary solutions W of (5.1.1), that is weak solution of

$$\begin{cases} (|W_x|^{p-2}W_x)_x + |W_x|^q = 0, & x \in (0, 1), \\ W(0) = 0, \qquad W(1) = M \geq 0. \end{cases} \tag{5.2.13}$$

More precisely, $W \in C([0, 1]) \cap C^1(0, 1)$ is a weak solution of (5.2.13) if $W(0) = 0, W(1) = M$ and W satisfies

$$\int_0^1 \left[\big(|W_x|^{p-2}W_x\big)\,\phi_x - |W_x|^q\,\phi\right]\,dx = 0 \qquad \text{for any} \qquad \phi \in C_c^1(0, 1). \tag{5.2.14}$$

It is not difficult to show that any weak solution in the above sense is actually a classical C^2 solution in $(0, 1)$ (for any $x_0 \in (0, 1)$, consider separately the cases $W_x(x_0) \neq 0$ and $W_x(x_0) = 0$). For small values of $M \geq 0$, problem (5.2.13) admits a unique weak solution $W_M = W_M(x) \in C^2([0, 1])$. Namely, this happens for $0 \leq M < M_b$, where M_b is the critical value,

$$M_b = \frac{q-p+1}{q-p}\left(\frac{q-p+1}{p-1}\right)^{-1/(q-p+1)}.$$

More precisely, for $M = 0$ we have $W_0 = 0$ and for $0 \leq M < M_b$, there exists $k = k(M) \in [0, \infty)$ such that $W_M = M_b \left[(x+k)^{\frac{q-p}{q-p+1}} - k^{\frac{q-p}{q-p+1}}\right]$. On the other hand, there is no steady state if $M > M_b$. In the critical case $M = M_b$, there still exists a steady state $W_{M_b} = U$, given by the explicit formula $U(x) = M_b\, x^{\frac{q-p}{q-p+1}}$. U belongs to $C([0,1]) \cap C^2((0,1])$, but it is singular in the sense that it has infinite derivative on the left-hand boundary, $U_x(0) = \infty$.

5.3 Lyapunov functional and convergence to steady states

Since (5.1.1) is a degenerate problem, we do not have sufficient regularity properties of the trajectories to construct a good smooth Lyapunov functional (which exists for one-dimensional uniformly parabolic equations). Hence we first consider a regularized problem, then the main estimate which plays a key role in the proof of the convergence to steady states will be proved by passing to the limit $\varepsilon \to 0$ in the regularizing parameter.

5.3.1 Approximate problem

Let $\varepsilon \in (0, 1/2)$. We consider the following approximate problems :

$$\begin{cases} (u_\varepsilon)_t = \Big((|(u_\varepsilon)_x|^2 + \varepsilon^2)^{\frac{p-2}{2}} (u_\varepsilon)_x\Big)_x + B_\varepsilon((u_\varepsilon)_x) & (t,x) \in (0,+\infty) \times (0,1), \\ u_\varepsilon(t,0) = 0, \qquad u_\varepsilon(t,1) = M, \qquad t > 0, \\ u_\varepsilon(0,x) = u_0(x), \qquad x \in (0,1), \end{cases} \tag{5.3.1}$$

where $B_\varepsilon(v) = (|v|^2 + \varepsilon^2)^{\frac{q-2}{2}} (|v|^2 + \frac{\varepsilon^2}{p-1})$.
Here we collect some useful properties of the sequence $\{u_\varepsilon\}$ which we will use later on.

Let $u \in L^\infty\big([0,T); W^{1,\infty}((0,1))\big)$ for any $T \in (0, T_{max})$ be the unique, maximal weak solution of problem (5.1.1) and let u_ε be the unique, maximal classical solution of (5.3.1) and $T(u_\varepsilon)$ be its existence time. We have the following proposition.

Proposition 5.3.1. *Let $A > 0$ and assume that $\|u_0\|_{W^{1,\infty}} \leq A$. Then for $0 < T < T_{\max}$ and ε small, we have*

a) $T(u_\varepsilon) > T$, $u_\varepsilon \to u$ in $C([0,T] \times [0,1])$ and $(u_\varepsilon)_x \to u_x$ in $C_{loc}((0,T] \times [0,1])$.

b) $|(u_\varepsilon)_x(t,x)| \leq C = C(A,T)$ on $[0,T] \times [0,1]$ and

$$\int_0^T \int_0^1 (u_\varepsilon)_t^2 \, dxdt \leq \tilde{C}(A,T).$$

Proof. For the convenience of the proof, we shall actually replace the initial data u_0 in the approximate problem (5.3.1) with a sequence $u_{\varepsilon,0} \in W^{1,\infty}((0,1))$, where $u_{\varepsilon,0} \to u_0$ in $W^{1,\infty}((0,1))$, and prove that Proposition 5.3.1 remains true in this more general situation. We know from chapter 2 that there exist a small time $\tilde{\tau} = \tilde{\tau}(A) > 0$ and a subsequence $\{u_{\varepsilon_n}\}$ of $\{u_\varepsilon\}$ such that u_{ε_n} converges in $C([0,\tilde{\tau}]\times[0,1])\cap C^{0,1}_{loc}((0,\tilde{\tau})\times(0,1))$ to a solution $\tilde{u}$ of (5.1.1). This was actually proved for $u_{\varepsilon,0}\equiv u_0$, but an inspection of the proof shows that this is true in the general case. The uniqueness of the solution of (5.1.1) implies that $\tilde{u}=u$ and that the whole sequence converges to u. We recall that u_ε is bounded in $L^\infty([0,\tilde{\tau}];W^{1,\infty}(0,1))$ (see Step 3 of the proof of Theorem 1.1 [7]). The boundary regularity result of Proposition 5.2.1 implies that the convergence of $\{(u_\varepsilon)_x\}$ to u_x holds in $C_{loc}((0,\tilde{\tau})\times[0,1])$ (that is up to the boundary). Now fix $T\in(0,T_{max})$ and let

$$\begin{aligned}\tilde{T} \;:=\; & \sup\{s>0 \quad \text{such that} \quad T(u_\varepsilon)>s \quad \text{for} \quad \varepsilon>0 \quad \text{small and} \\ & u_\varepsilon\to u \quad \text{in} \quad C([0,s]\times[0,1])\cap C^{0,1}_{loc}((0,s)\times[0,1])\}.\end{aligned}$$

We know that $\tilde{T}\geq\tilde{\tau}(A)>0$. Assume that $\tilde{T}<T$. Set $A_1=\sup\limits_{t\in[0,T]}\|u(t)\|_{W^{1,\infty}}<\infty$. For any $\eta\in(0,\tilde{T})$, we have

$$u_\varepsilon(\tilde{T}-\eta)\to u(\tilde{T}-\eta) \qquad \text{in} \quad W^{1,\infty}(0,1). \tag{5.3.2}$$

Thanks to (5.3.2) and the small-time existence and convergence result mentioned at the beginning of the proof, we can find $\tau=\tau(A_1)>0$ (independent of η) and $\varepsilon_0=\varepsilon_0(\eta)>0$ such that the problem

$$\begin{array}{lll} (u^\eta_\varepsilon)_t=\left((|(u^\eta_\varepsilon)_x|^2+\varepsilon^2)^{\frac{p-2}{2}}(u^\eta_\varepsilon)_x\right)_x+B_\varepsilon((u^\eta_\varepsilon)_x) & t>0, & x\in(0,1),\\ u^\eta_\varepsilon(t,0)=0, \qquad u^\eta_\varepsilon(t,1)=M, & t>0, & \\ u^\eta_\varepsilon(0,x)=u_\varepsilon(\tilde{T}-\eta,x), & & x\in(0,1), \end{array} \tag{5.3.3}$$

admits a unique classical solution u^η_ε on $[0,\tau]$. Moreover, we have $u^\eta_\varepsilon\to u(\tilde{T}-\eta+\cdot,\cdot)$ in $C([0,\tau]\times[0,1])\cap C^{0,1}((0,\tau)\times[0,1])$. We can extend the solution u_ε of (5.3.1) on $[0,\tilde{T}-\eta+\tau]$ by setting $u_\varepsilon(t,x)=\begin{cases} u_\varepsilon(t,x) & \text{for} \quad x\in[0,\tilde{T}-\eta],\\ u^\eta_\varepsilon(t,x) & \text{for} \quad x\in[\tilde{T}-\eta,\tilde{T}-\eta+\tau]\end{cases}$.

It follows that $u_\varepsilon\to u$ in $C([0,\tilde{T}-\eta+\tau]\times[0,1])\cap C^{0,1}((0,\tilde{T}-\eta+\tau)\times[0,1])$. Since $\tilde{T}-\eta+\tau>\tilde{T}$ for η small enough, this contradicts the definition of $\tilde{T}$. The second assertion follows from the estimates given in [7, Inequalities 2.16 and 2.19].

□

Let us also note that due to $q>p>2$, we have for ε small enough

$$(p-1)\varepsilon^p\cosh(\varepsilon x)^{p-1}\geq\varepsilon^q\cosh(\varepsilon x)^q$$

(it suffices to take $0<\varepsilon<\cosh(1)^{\frac{p-1-q}{q-p}}$). Hence $\|u_0\|_{L^\infty}+M+2-\cosh(\varepsilon x)$ is a supersolution for problem (5.3.1). It is also easy to see that $-\|u_0\|_{L^\infty}$ is a subsolution. Therefore there exists $K>0$ depending only on $\|u_0\|_{L^\infty}$ such that,

$$\forall\varepsilon\in(0,1/2), \qquad \|u_\varepsilon(t,x)\|_{L^\infty}\leq K. \tag{5.3.4}$$

5.3.2 Construction of the Lyapunov functional

Now we construct a Lyapunov functional for (5.3.1) with the help of the technique developed by Zelenyak [123]. Let $D_K = [-K, K] \times \mathbb{R}$, where K is the constant in (5.3.4). We look for a pair of functions $\Phi_\varepsilon \in C^1(D_K; \mathbb{R})$ and $\Psi_\varepsilon \in C(D_K; (0, \infty))$ with the following property :
For any solution u_ε of (5.3.1) with $|u_\varepsilon| \leq K$, defining

$$\mathcal{L}_\varepsilon(u_\epsilon(t)) = \int_0^1 \Phi_\varepsilon\left(u_\varepsilon(t,x), (u_\varepsilon)_x(t,x)\right) dx,$$

it holds

$$\frac{d}{dt}\mathcal{L}_\varepsilon(u_\varepsilon(t)) = -\int_0^1 \Psi_\varepsilon\left(u_\varepsilon(t,x), (u_\varepsilon)_x(t,x)\right)\, (u_\varepsilon)_t^2(t,x)\, dx.$$

Since $(u_\varepsilon)_t(t,0) = (u_\varepsilon)_t(t,1) = 0$, we have

$$\begin{aligned}\frac{d}{dt}\int_0^1 \Phi_\varepsilon(u_\varepsilon, (u_\varepsilon)_x) dx &= \int_0^1 (u_\varepsilon)_t \cdot (\Phi_\varepsilon)_u\left(u_\varepsilon, (u_\varepsilon)_x\right) + (u_\varepsilon)_{xt} \cdot (\Phi_\varepsilon)_v\left(u_\varepsilon, (u_\varepsilon)_x\right) dx \\ &= \int_0^1 (u_\varepsilon)_t \big[(\Phi_\varepsilon)_u\left(u_\varepsilon, (u_\varepsilon)_x\right) - (u_\varepsilon)_x \cdot (\Phi_\varepsilon)_{uv}\left(u_\varepsilon, (u_\varepsilon)_x\right) - (u_\varepsilon)_{xx} \cdot (\Phi_\varepsilon)_{vv}\left(u_\varepsilon, (u_\varepsilon)_x\right)\big] dx.\end{aligned}$$

So it is natural to require that

$$\begin{aligned}&(\Phi_\varepsilon)_u\left(u_\varepsilon, (u_\varepsilon)_x\right) - (u_\varepsilon)_x \cdot (\Phi_\varepsilon)_{uv}\left(u_\varepsilon, (u_\varepsilon)_x\right) - (u_\varepsilon)_{xx} \cdot (\Phi_\varepsilon)_{vv}\left(u_\varepsilon, (u_\varepsilon)_x\right) \\ &\quad = -\Psi_\varepsilon(u_\varepsilon, (u_\varepsilon)_x) \cdot (u_\varepsilon)_t \\ &\quad = -\Psi_\varepsilon(u_\varepsilon, (u_\varepsilon)_x) \left[(p-1)\left(|(u_\varepsilon)_x|^2 + \varepsilon^2\right)^{\frac{p-4}{2}} \left(|(u_\varepsilon)_x|^2 + \frac{\varepsilon^2}{p-1}\right)(u_\varepsilon)_{xx} \right. \\ &\quad \left. + \left(|(u_\varepsilon)_x|^2 + \varepsilon^2\right)^{\frac{q-2}{2}} \left(|(u_\varepsilon)_x|^2 + \frac{\varepsilon^2}{p-1}\right)\right]\end{aligned}$$

A sufficient condition is

$$(\Phi_\varepsilon)_{vv}(u,v) = (p-1)\Psi_\varepsilon(u,v)\left(v^2 + \varepsilon^2\right)^{\frac{p-4}{2}}\left(v^2 + \frac{\varepsilon^2}{p-1}\right), \tag{5.3.5}$$

$$(\Phi_\varepsilon)_u(u,v) - v(\Phi_\varepsilon)_{uv}(u,v) = -\Psi_\varepsilon(u,v)\left(v^2 + \varepsilon^2\right)^{\frac{q-2}{2}}\left(v^2 + \frac{\varepsilon^2}{p-1}\right), \tag{5.3.6}$$

that is Φ_ε satisfies the differential equation :

$$(\Phi_\varepsilon)_u(u,v) - v(\Phi_\varepsilon)_{uv}(u,v) + \frac{\left(v^2 + \varepsilon^2\right)^{\frac{q-p+2}{2}}}{p-1}(\Phi_\varepsilon)_{vv}(u,v) = 0. \tag{5.3.7}$$

We follow the method used in [5] to find such nice functions. For a given function $\rho_\varepsilon(u,v)$, let us denote

$$H_\varepsilon = (\rho_\varepsilon)_u + \frac{(v^2+\varepsilon^2)^{\frac{q-p+2}{2}}}{p-1}(\rho_\varepsilon)_{vv} - v(\rho_\varepsilon)_{uv}.$$

Here we assume that $\rho_\varepsilon, (\rho_\varepsilon)_u, (\rho_\varepsilon)_v, (\rho_\varepsilon)_{uv}$ are continuous and C^1 in v in D_K, and that $(\rho_\varepsilon)_{vv}$ is continuous in D_K and, except perhaps at $v=0$, C^1 in v.
We want to have $(H_\varepsilon)_v = 0$, so that $H_\varepsilon(u,v) = H_\varepsilon(u,0) = H_\varepsilon(u)$. We compute

$$(H_\varepsilon)_v = \frac{(v^2+\varepsilon^2)^{\frac{q-p+2}{2}}}{p-1}(\rho_\varepsilon)_{vvv} + \left(\frac{q-p+2}{p-1}\right) v \left(v^2+\varepsilon^2\right)^{\frac{q-p}{2}}(\rho_\varepsilon)_{vv} - v(\rho_\varepsilon)_{uvv}.$$

For this, it suffices that $f_\varepsilon = (\rho_\varepsilon)_{vv}$ satisfies the following conditions :

$$\begin{cases} (f_\varepsilon)_u - \left(\dfrac{q-p+2}{p-1}\right)(v^2+\varepsilon^2)^{\frac{q-p}{2}} f_\varepsilon - \dfrac{(v^2+\varepsilon^2)^{\frac{q-p+2}{2}}}{(p-1)v}(f_\varepsilon)_v = 0 \quad |u| \le K, v \neq 0, \\ (f_\varepsilon)_v(u,0) = 0. \end{cases} \tag{5.3.8}$$

Now, the equation (5.3.8) can be solved by the method of characteristics. For any $K>0$ such that $|u| \le K$, one finds that the function defined by

$$f_\varepsilon(u,v) = \left[1 + \left(\frac{q-p}{p-1}\right)\left(v^2+\varepsilon^2\right)^{\frac{q-p}{2}}(K+1-u)\right]^{-\frac{q-p+2}{q-p}} > 0$$

is a solution of (5.3.8) on $[-K,K]\times\mathbb{R}$. Define ρ_ε by

$$\rho_\varepsilon(u,v) = \int_0^v \int_0^z f_\varepsilon(u,s)\ ds\ dz \ge 0,$$

and let then

$$\Phi_\varepsilon(u,v) = \rho_\varepsilon(u,v) - \int_0^u H_\varepsilon(s,0)\ ds + K + 1. \tag{5.3.9}$$

We added the constant $K+1$ to ensure that $\Phi_\varepsilon \ge 0$. In fact, given that $\varepsilon \le 1/2$, $2 < p$ and $0 \le (\rho_\varepsilon)_{vv} \le 1$, we get $(\rho_\varepsilon)_u(s,0) = 0$ and

$$0 \le H_\varepsilon(s,0) = (\rho_\varepsilon)_u(s,0) + \frac{\varepsilon^{q-p+2}}{p-1} f_\varepsilon(s,0) \le 1. \tag{5.3.10}$$

Consequently, using that $|u| \le K$, we get

$$-\int_0^u H_\varepsilon(s,0)\ ds \ge -u \ge -K \quad \text{for} \quad u\in[0,K],$$
$$-\int_0^u H_\varepsilon(s,0)\ ds \ge 0 \quad \text{for} \quad u\in[-K,0].$$

Using the definition of H_ε and the fact that $H_\varepsilon(u,v) = H_\varepsilon(u,0)$, we see that :

$$(\Phi_\varepsilon)_u - v(\Phi_\varepsilon)_{uv}(u,v) + \frac{(v^2+\varepsilon^2)^{\frac{q-p+2}{2}}}{p-1}(\Phi_\varepsilon)_{vv}(u,v) = 0,$$

i.e. Φ_ε satisfies (5.3.7), hence (5.3.5)-(5.3.6) with

$$\Psi_\varepsilon(u,v) = \left(v^2 + \frac{\varepsilon^2}{p-1}\right)^{-1} \frac{(v^2+\varepsilon^2)^{\frac{4-p}{2}}(\rho_\varepsilon)_{vv}}{p-1} \geq \frac{(v^2+\varepsilon^2)^{\frac{2-p}{2}}(\rho_\varepsilon)_{vv}}{p-1} > 0. \tag{5.3.11}$$

It follows that

$$\frac{d}{dt}\mathcal{L}_\varepsilon(u_\varepsilon(t)) = -\int_0^1 \frac{((u_\varepsilon)_x^2+\varepsilon^2)^{\frac{2-p}{2}}(\rho_\varepsilon)_{vv}}{(p-1)\left((u_\varepsilon)_x^2 + \dfrac{\varepsilon^2}{p-1}\right)}(u_\varepsilon)_t^2\,dx = -\int_0^1 \Psi_\varepsilon(u_\varepsilon,(u_\varepsilon)_x)\,(u_\varepsilon)_t^2\,dx.$$

Due to $q > p > 2$, we remark that, $\forall \varepsilon \in (0,1)$, $|u| \leq K$ and $v \in \mathbb{R}$,

$$\Psi_\varepsilon(u,v) \geq \mathcal{A}(v) = \frac{(v^2+1)^{\frac{2-p}{2}}}{p-1}\left[1 + \frac{(q-p)}{(p-1)}\left(v^2+1\right)^{\frac{q-p}{2}}(2K+1)\right]^{-\frac{q-p+2}{q-p}}. \tag{5.3.12}$$

As a consequence of the existence of the approximate Lyapunov functional, we have the following estimate.

Proposition 5.3.2. *Assume that $q > p > 2$ and let u be a global weak solution of* (5.1.1). *Then for any $T > 1$ and $\delta > 0$, There exists $C = C(\|u_0\|_{W^{1,\infty}}, \delta, p, q) > 0$ such that*

$$\int_1^T \int_\delta^{1-\delta} (u_t)^2 dxdt \leq C. \tag{5.3.13}$$

Proof. First let us remark that Lemma 5.2.1 implies that, for any $\delta > 0$,

$$|u_x| \leq C(\delta) \qquad \text{in} \quad [1,\infty)\times[\delta, 1-\delta]. \tag{5.3.14}$$

Now we fix $T > 1$ and $\delta \in (0, 1/2)$. On the one hand, by (5.3.12), we have

$$\begin{aligned}
\int_0^T \int_\delta^{1-\delta} \mathcal{A}\left((u_\varepsilon)_x\right)\cdot(u_\varepsilon)_t^2(t,x)\,dx\,dt &\leq \int_0^T \int_0^1 \Psi_\varepsilon(u_\varepsilon,(u_\varepsilon)_x)\cdot(u_\varepsilon)_t^2(x,t)\,dx\,dt \\
&= \mathcal{L}_\varepsilon(u(0)) - \mathcal{L}_\varepsilon(u_\varepsilon(T)) \\
&\leq \tilde{C}(\|u_0\|_{W^{1,\infty}}).
\end{aligned} \tag{5.3.15}$$

On the other hand, by Proposition 5.3.1, there exists $\varepsilon_0(\delta,T)$ such that, for all $\varepsilon < \varepsilon_0$, $x \in [0,1]$, $t \in [1,T]$,

$$|(u_\varepsilon)_x(t,x) - u_x(t,x)| \leq C(\delta).$$

Then, by (5.3.14), $|(u_\varepsilon)_x| \leq 2C(\delta)$ for $(t,x) \in [1,T]\times[\delta, 1-\delta]$ so that

$$\begin{aligned}\int_0^T\int_\delta^{1-\delta}\mathcal{A}\left((u_\varepsilon)_x\right)\cdot(u_\varepsilon)_t^2(t,x)\,dx\,dt &\geq \int_1^T\int_\delta^{1-\delta}\mathcal{A}\left((u_\varepsilon)_x\right)\cdot(u_\varepsilon)_t^2(t,x)\,dx\,dt\\ &\geq \theta(2C(\delta))\int_1^T\int_\delta^{1-\delta}(u_\varepsilon)_t^2(x,t)\,dx\,dt,\end{aligned}$$

where $\theta(R)=\inf\{\mathcal{A}(v);\quad |v|\leq R\}>0$. Letting $\varepsilon\to 0$ and using a lower semicontinuity argument as well as (5.3.15), we obtain

$$\theta(2C(\delta))\int_1^T\int_\delta^{1-\delta}(u)_t^2(t,x)\,dx\,dt\leq\tilde{C}(\|u_0\|_{W^{1,\infty}}), \tag{5.3.16}$$

The result immediately follows.

□

5.3.3 Convergence to steady states

Proposition 5.3.3. *Let u be a global weak solution of (5.1.1). Then $M\leq M_b$ and $u(t)$ converges in $C([0,1])$ to a steady state of* (5.1.1) *as $t\to\infty$. Moreover the convergence also holds in $C^1([\delta,1-\delta])$ for all $\delta>0$.*

Proof. Assume that u is a global weak solution of (5.1.1). Fix a sequence $(t_k)_{k\in\mathbb{N}}$, $1\leq t_k\to\infty$ and set $w_k(t,x)=u(t+t_k,x)$. By (5.1.3), we know that

$$|u|\leq\max\{\|u_0\|_\infty,M\}\qquad\text{in}\quad[1,\infty)\times[0,1], \tag{5.3.17}$$

Using lemma 5.2.1 we have

$$|u_x|\leq C(\delta),\qquad\text{in}\quad[1,\infty)\times[\delta/2,1-\delta/2]. \tag{5.3.18}$$

Thus applying a result of DiBenedetto-Friedman [48], we have that $\{w_k\}$ and $\{(w_k)_x\}$ are Hölder continuous in $[\delta,T-\delta]\times[\delta,1-\delta]$ with a Hölder norm independent of k. It follows that $\{w_k\}$ and $\{(w_k)_x\}$ are relatively compact in $C\left([\delta,T-\delta]\times[\delta,1-\delta]\right)$ for any $\delta,T>0$. Thus, by the Arzelà-Ascoli theorem and a diagonal procedure, there exist a subsequence $(t_{k_l})_{l\in\mathbb{N}}$ of (t_k) and a function $W\in C\left((0,\infty)\times(0,1)\right)$, $W_x\in C\left((0,\infty)\times(0,1)\right)$ such that for any $\delta,T>0$

$$w_{k_l}\to W\qquad\text{strongly in}\quad C\left([\delta,T-\delta]\times[\delta,1-\delta]\right)\qquad\text{as}\quad l\to\infty. \tag{5.3.19}$$

$$(w_{k_l})_x\to W_x\qquad\text{strongly in}\quad C\left([\delta,T-\delta]\times[\delta,1-\delta]\right)\qquad\text{as}\quad l\to\infty. \tag{5.3.20}$$

and W is a distributional solution of

$$W_t-(|W_x|^{p-2}W_x)_x=|W_x|^q,\qquad t>0,x\in(0,1).$$

Further, using lemma 5.2.1 and $q > p$, we get that for some $r > 1$

$$\|(w_k)_x\|_{L^\infty(1,\infty;L^r(0,1))} \leq C. \tag{5.3.21}$$

Combining (5.3.17) with (5.3.21), we get that, for each fixed $t > 0$, $w_k(t, \cdot)$ is relatively compact in $C([0,1])$. Consequently for any $t > 0$, $W(t, .)$ can be extended to a continuous function on $[0,1]$ satisfying

$$W(t,0) = 0 \qquad\qquad W(t,1) = M.$$

Proposition 5.3.2 implies that

$$\int_0^\infty \int_\delta^{1-\delta} (w_{k_l})_t^2(t,x)\, dx\, dt \to 0, \qquad \text{as} \quad l \to \infty.$$

Since $(w_{k_l})_t \to W_t$ in $\mathcal{D}'((0,\infty)\times(0,1))$ and $\delta \in (0,1)$ is arbitrary, it follows that $W_t \equiv 0$. Thus W is a steady state of (5.1.1) which implies that $M \leq M_b$. Given that the sequence $t_k \to \infty$ is arbitrary and the steady states (for given M) are unique, it follows that the whole solution $u(t)$ converges to W.

□

5.4 Proof of Theorem 5.1.1

5.4.1 GBU profiles and lower bound on u_x

Thanks to (5.2.6) in Theorem 5.2.1, we shall derive the following lemma providing a lower bound on the blow up profile of u_x in case GBU occurs in finite or infinite time near $x = 0$ or 1.

Lemma 5.4.1. *Let u be a global weak solution of* (5.1.1) *and $t_0 > 0$. Let $C_1 > 0$ be the constant given in the estimate* (5.2.7) *of Theorem 5.2.1. There exist $C_4 = C_4(C_1, p, q)$, $C_5 = C_5(p,q) > 0$ with the following property. For all $t \in [t_0, +\infty)$ and $0 \leq y \leq x \leq 1$*

$$\left[|u_x|^{p-2}u_x(t,x) + C_4\right]^{\frac{p-1-q}{p-1}} \leq \left[|u_x|^{p-2}u_x(t,y) + C_4\right]^{\frac{p-1-q}{p-1}} + C_5(x-y), \tag{5.4.1}$$

and

$$\left[|u_x|^{p-2}(-u_x)^+(t,1-x) + C_4\right]^{\frac{p-1-q}{p-1}} \leq \left[|u_x|^{p-2}(-u_x)^+(t,1-y)) + C_4\right]^{\frac{p-1-q}{p-1}} + C_5(x-y). \tag{5.4.2}$$

Proof. Fix $t \in [t_0, T_{max})$. and let $z(x) = |u_x|^{p-2}u_x^+(t,x) + C_1^{\frac{p-1}{q}}$, where C_1 is given by the estimate of $|u_t|$ in Theorem 5.2.1. Using that $|u_t| \leq C_1$ and $|u_x(t)|^q$ is bounded, we get that,

z is Lipschitz in $(0,1)$ with a Lipschitz bound depending on $t, p, q, ||u_0||_\infty$ and $||u_x(t)||_\infty^q$. Moreover the function z satisfies almost everywhere

$$\begin{aligned} z' + z^{q/(p-1)} &= (|u_x|^{p-2}u_x(t,x))_x \, \mathbf{1}_{\{u_x>0\}} + \left(|u_x|^{p-2}u_x^+(t,x) + C_1^{\frac{p-1}{q}} \right)^{\frac{q}{p-1}} \\ &\geq [(|u_x|^{p-2}u_x(t,x))_x + |u_x|^q] \, \mathbf{1}_{\{u_x>0\}} + C_1 \\ &\geq 0. \end{aligned}$$

For $0 \leq y \leq x \leq 1$, an integration yields

$$z(x)^{(p-1-q)/(p-1)} \leq z(y)^{(p-1-q)/(p-1)} + \left(\frac{q-p+1}{p-1} \right)(x-y),$$

that is (5.4.1) with $C_4 = C_1^{\frac{p-1}{q}}$ and $C_5 = \dfrac{q-p+1}{p-1}$. It follows that for $0 \leq y \leq x \leq 1$, we have

$$\left[|u_x|^{p-2}u_x^+(t,x) + C_4 \right]^{\frac{p-1-q}{p-1}} \leq \left[|u_x|^{p-2}u_x^+(t,y) + C_4 \right]^{\frac{p-1-q}{p-1}} + C_5(x-y).$$

The estimate (5.4.2) can be obtained similarly by considering $z(x) = |u_x|^{p-2}(-u_x)^+(t, 1-x) + C_1^{\frac{p-1}{q}}$.

□

Remark 5.4.1. *Lemma 5.4.1 yields in particular a lower bound on the gradient blow-up profile, which complements the upper bounds in* (5.2.9)-(5.2.10)*. Namely, if $x = 0$ is a GBU point (in finite or infinite time), i.e. if $|u_x|$ is unbounded in any neighborhood of T_{max} and* 0*, then*

$$\limsup_{t \to T_{max}} u_x(t,x) \geq C(p,q) x^{-1/(q-p+1)}$$

for all sufficiently small $x > 0$. The analogous estimate holds if $x = 1$ is a GBU point.

Now let us state the following lemma which is a direct consequence of the convergence of u to the steady state.

Lemma 5.4.2. *Let $M \geq 0$ and let u be a global weak solution of* (5.1.1)*. Then it holds*

$$\lim_{t \to +\infty} \left(\max_{[0,1]} u(t,x) \right) = M. \tag{5.4.3}$$

Proof. Since $u(t,1) = M$, we get $\left(\max_{[0,1]} u(t,x) \right) \geq M$. Next, using that $w \to W$ in $C([0,1])$ (see Proposition 5.3.3) and $W \leq M$, it holds that $\forall \varepsilon > 0, \exists t_\varepsilon > 0$ such that if $t > t_\varepsilon$ then

$$u(t,x) \leq w(x) + \varepsilon \leq M + \varepsilon \qquad x \in [0,1].$$

It results that $\max_{[0,1]} u(t,x) \leq M + \varepsilon$ if $t > t_\varepsilon$.

□

Thanks to this property we can rule out infinite time gradient blow-up towards $-\infty$ when $x \to 1$.

Lemma 5.4.3. *Let u be a global weak solution of* (5.1.1). *Then*

$$\inf_{[0,\infty)\times(0,1)} u_x > -\infty. \tag{5.4.4}$$

Proof. We proceed by contradiction. Assume that the lemma is false. Then, by Lemma 5.2.1, there exist a sequence $t_n \to +\infty$ and $x_n \to 0$ such that $u_x(t_n, 1-x_n) \to -\infty$. Fix $\varepsilon > 0$, then for $n \geq n(\varepsilon)$ large enough, we have

$$|u_x|^{p-2}(-u_x)^+(t_n, 1-x_n) \geq \varepsilon^{-(p-1)/(q-p+1)}.$$

Taking $t = t_n$ and $y = x_n$ in (5.4.2), we get that for $n \geq n_0(\varepsilon)$ large enough, we have for $x_n \leq x \leq \varepsilon$

$$\begin{aligned} \left[|u_x|^{p-2}(-u_x)^+(t_n, 1-x) + C_4\right]^{\frac{p-1-q}{p-1}} &\leq \left[|u_x|^{p-2}(-u_x)^+(t_n, 1-x_n) + C_4\right]^{\frac{p-1-q}{p-1}} + C_5 x \\ &\leq (C_5+1)\varepsilon. \end{aligned}$$

This implies that

$$\left[|u_x|^{p-2}(-u_x)^+(t_n, 1-x) \geq ((C_5+1)\varepsilon)^{(1-p)/(q-p+1)} - C_4, \qquad x_n \leq x \leq \varepsilon. \tag{5.4.5}$$

Choosing $\varepsilon = \varepsilon(C_5, C_4) > 0$ small enough, we get that $u_x(t_n, 1-x) \leq -1$ on $[x_n, \varepsilon]$, hence

$$u(t_n, 1-x) \geq u(t_n, 1-x_n) + (x - x_n), \qquad x_n \leq x \leq \varepsilon.$$

Using that $u(t_n, 1-x_n) \to M$ (by Proposition 5.3.3) and recalling Lemma 5.4.2, we end up with a contradiction.

□

Remark 5.4.2. *Thanks to lemma 5.4.3 we deduce that, for the case $M = M_b$, if there exist global solutions with infinite time gradient blow up (we expect that this could occur for some particular initial data), then u_x can only blow up at $x = 0$.*

5.4.2 Completion of the proof of Theorem 5.1.1

Proof of the boundedness of u_x for $M = 0$

Lemma (5.4.3) is sufficient to prove the main theorem in the case $M = 0$. Let u be a global solution of (5.1.1). For $M = 0$, we note that $w(t,x) := u(t, 1-x)$ solves (5.1.1) with $u_0(1-x)$ as initial data. Lemma (5.4.3) implies that u_x and w_x are bounded below on $[0, +\infty) \times (0,1)$, therefore u_x is bounded. See the Subsection 5.4.2 for the proof of the convergence to the steady state in the $W^{1,\infty}(0,1)$ norm. □

Proof of the boundedness of u_x for $0 < M < M_b$

We proceed by contradiction. Assume that u is a global weak solution which is unbounded in $W^{1,\infty}$. We know that when $t \to \infty$, u converges to $W = W_M$ in $C[0,1]$ and in $C^1[\delta, 1-\delta]$ for all $\delta > 0$. Since u_x is unbounded and can only blow up to $+\infty$ at $x = 0$, there exist sequences $t_n \to \infty$, $x_n \to 0$ such that

$$u_x(t_n, x_n) \to +\infty \tag{5.4.6}$$

Taking $t = t_n$ and $y = x_n$ in (5.4.1) and sending $n \to \infty$, we deduce that, for any $x \in (0,1)$

$$\left[|W_x(x)|^{p-2}W_x + C_4\right]^{\frac{p-1-q}{p-1}} \leq C_5\, x.$$

This would imply that

$$|W_x|^{p-2}W_x + C_4 \geq (C_5\, x)^{\frac{1-p}{q-p+1}}\,.$$

Passing to the limit $x \to 0$ we get a contradiction since $W = W_M \in C^1([0,1])$. So all the global solutions are bounded in $W^{1,\infty}$.

Proof of the convergence in C^1 norm for $M \in [0, M_b)$

This follows from the proof of Proposition 5.3.3, with (5.3.14) replaced by the boundedness of u_x on $[0,\infty) \times [0,1]$, and using Proposition 5.2.1 which is an extention of the result in [48].

Proof of Theorem 5.1.1 for $M > M_b$

This is an immediate consequence of Proposition 5.3.3 and the fact that (5.2.13) admits no solution for $M > M_b$. $\square$

Further regularity for global solutions for $0 < M < M_b$

As a consequence of the convergence of global solutions to the steady state in $C^1[(0,1)]$, we have the following proposition which is of independent interest. It gives a result of further regularity of global solutions for large time. It is unknown whether or not such property is true in the case $M = 0$.

Proposition 5.4.1. *Assume that $0 < M < M_b$ and let u be a global weak solution of* (5.1.1). *Then there exist $\tilde{T} > 0$ and $\tilde{\eta} > 0$ such that*

$$u_x \geq \tilde{\eta} \qquad \text{on} \quad [\tilde{T}, +\infty) \times [0,1].$$

Moreover, u becomes a classical solution on $[\tilde{T}, +\infty) \times [0,1]$

Proof. First let us note that there exists $\eta > 0$ such that $(W_M)_x \geq 2\eta > 0$ in $[0,1]$. Next, by Theorem 5.1.1, we know that $u_x \to W_x$ uniformly on $[0,1]$. Hence, there exists $\tilde{T} > 0$ such that

$$u_x(t,x) > (W_M)_x(x) - \eta \geq \eta \qquad \text{for all} \quad x \in [0,1],\, t > \tilde{T}. \tag{5.4.7}$$

The last inequality implies that the differential equation is uniformly parabolic for $(t,x) \in [\tilde{T}, \infty] \times [0,1]$. Hence, by the standard theory (see [80]) we know that $u \in C^{1,2}((\tilde{T},\infty) \times [0,1])$

$\square$

Appendix

Proof of Proposition 5.2.1 on the regularity of the derivative up to the boundary

Let $\varepsilon \in [0,1), u_0 \in W^{1,\infty}, C > 0$ and $F \in L^r((0,T) \times (0,1))$ for some $r > 2$ with $\|F\|_{L^r((0,T)\times(0,1))} \leq C$. Since $v_t \in L^2((0,T) \times (0,1))$ by assumption and $r > 2$, it follows that

$$\left((|v_x|^2 + \varepsilon^2)^{\frac{p-2}{2}} v_x\right)_x \in L^2((0,T) \times (0,1)), \tag{5.4.8}$$

and that the partial differential equation in (5.2.2) is satisfied in the sense of equality of functions in $L^2((0,T) \times (0,1))$.

Next, we define an extension v^* of v to $[-1,2]$ by setting

$$v^*(t,x) = \begin{cases} -v(t,-x) & \text{if} \quad x \in [-1,0) \\ v(t,x) & \text{if} \quad x \in [0,1] \\ 2M - v(t,2-x) & \text{if} \quad x \in (1,2] \end{cases} \tag{5.4.9}$$

We will prove that v^* is a weak solution of the following problem

$$\begin{cases} v^*_t = \left((|v^*_x|^2 + \varepsilon^2)^{\frac{p-2}{2}} v^*_x\right)_x + \tilde{F}(t,x) & t > 0, \quad x \in (-1,2), \\ v^*(t,-1) = -M, \qquad v^*(t,2) = 2M, & t > 0, \\ v^*(0,x) = u^*_0(x), & \qquad\qquad x \in (0,1). \end{cases} \tag{5.4.10}$$

where

$$\tilde{F}(t,x) = \begin{cases} -F(t,-x) & \text{if} \quad x \in [-1,0) \\ +F(t,x) & \text{if} \quad x \in [0,1] \\ -F(t,2-x) & \text{if} \quad x \in (1,2] \end{cases}$$

Indeed, let $\psi \in C^0([0,\tau] \times [-1,2]) \cap L^p((0,\tau); W_0^{1,p}((-1,2))$. Due to (5.4.8), for a.e. $t \in (0,T)$, we have $(|v_x|^2 + \varepsilon^2)^{\frac{p-2}{2}} v_x(t,\cdot) \in W^{1,2}(0,1) \subset C([0,1])$. By elementary distribution theory (jump formula), it readily follows that $(|v^*_x|^2 + \varepsilon^2)^{\frac{p-2}{2}} v^*_x(t,\cdot) \in W^{1,2}(-1,2) \subset C([-1,2])$. For a.e. $t \in (0,T)$, we can thus write :

$$\begin{aligned}
\int_0^1 v_t^*(t,x)\psi(t,x)\,dx &= \int_0^1 \left(\left(|v_x^*|^2+\varepsilon^2\right)^{\frac{p-2}{2}} v_x^*\right)_x (t,x)\psi(t,x)\,dx + \int_0^1 F(t,x)\,\psi(t,x)\,dx \\
&= \left(|v_x^*|^2+\varepsilon^2\right)^{\frac{p-2}{2}} v_x^*(t,1)\psi(t,1) - \left(|v_x^*|^2+\varepsilon^2\right)^{\frac{p-2}{2}} v_x^*(t,0)\psi(t,0) \\
&- \int_0^1 \left(|v_x^*|^2+\varepsilon^2\right)^{\frac{p-2}{2}} v_x^*(t,x)\,\psi_x(t,x)\,dx + \int_0^1 F(t,x)\,\psi(t,x)\,dx.
\end{aligned}$$

Using that $\psi(t,-1)=0$ and $\psi(t,2)=0$, we have

$$\begin{aligned}
\int_{-1}^0 v_t^*(t,x)\,\psi(t,x)\,dx &= \int_{-1}^0 \left(\left(|v_x^*|^2+\varepsilon^2\right)^{\frac{p-2}{2}} v_x^*\right)_x (t,x)\,\psi(t,x)\,dx - \int_{-1}^0 F(t,-x)\,\psi\,dx \\
&= \left(|v_x^*|^2+\varepsilon^2\right)^{\frac{p-2}{2}} v_x^*(t,0)\psi(t,0) - \int_{-1}^0 F(t,-x)\psi(t,x)\,dx \\
&- \int_{-1}^0 \left(|v_x^*|^2+\varepsilon^2\right)^{\frac{p-2}{2}} v_x^*(t,x)\psi_x(t,x)\,dx,
\end{aligned}$$

and

$$\begin{aligned}
\int_1^2 v_t^*(t,x)\psi(t,x)dx = &\int_1^2 \left((|v_x^*|^2+\varepsilon^2)^{\frac{p-2}{2}} v_x^*\right)_x (t,x)\psi(t,x)dx - \int_1^2 F(t,2-x)\psi(t,x)dx \\
&= -\left(|v_x^*|^2+\varepsilon^2\right)^{\frac{p-2}{2}} v_x^*(t,1)\psi(t,1) - \int_1^2 F(t,2-x)\,\psi(t,x)\,dx \\
&- \int_1^2 \left(|v_x^*|^2+\varepsilon^2\right)^{\frac{p-2}{2}} v_x^*(t,x)\,\psi_x(t,x)dx.
\end{aligned}$$

Summing these identities and integrating over $(0,T)$, it follows that

$$\begin{aligned}
\int_0^T\int_{-1}^2 v_t^*\,\psi\,dx\,dt \;&+\; \int_0^T\int_{-1}^2 \left(|v_x^*|^2+\varepsilon^2\right)^{\frac{p-2}{2}} v_x^*(t,x)\,\psi_x(t,x)\,dx\,dt \\
&= \int_0^T\int_{-1}^2 \tilde{F}(t,x)\,\psi\,dx\,dt.
\end{aligned} \tag{5.4.11}$$

Next, since $\left\|\tilde{F}(t,x)\right\|_{L^r((0,T)\times(-1,2))} \le C\,\|F\|_{L^r(0,T)\times(0,1))}$, using a result of DiBenedetto and Friedman (see [48] and [47, chapter 9] for the case $\varepsilon>0$) on Hölder regularity of gradient of some degenerate parabolic problems, we get that, for any $\eta>0$, $v_x^*\in C^\alpha_{loc}([\eta,T-\eta]\times(-1,2))$ where $\alpha>0$ and the norm of v_x^* depend only on $\|F\|_{L^r(0,T)\times(0,1))}$, $\|v_x^*\|_{L^p}$ and $\|v^*\|_{L_t^\infty,L_x^2}$. We get the desired result recalling that $v_x^*=v_x$ on $[0,1]$ and using that $[0,1]\subset(-1,2)$.

Chapitre 6

Utilisation d'une estimation de gradient pour l'obtention d'un résultat de type Liouville

Dans ce chapitre on établit une estimation de gradient locale en espace qui nous permettra de prouver un résultat de type Liouville pour des solutions anciennes du problème (1.1) dans l'espace entier $\mathbb{R}^N$.

6.1 Introduction

In this chapter, we are interested in qualitative properties of solutions of the non-linear degenerate parabolic equation

$$u_t - \Delta_p u = |\nabla u|^q, \tag{6.1.1}$$

where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, $q > p - 1 > 1$.
The kind of result we are going to prove are gradient estimates for local solutions in time-space, and a Liouville type theorem for ancient solutions. In the last years, gradient estimates have played a key role in geometry and PDE since at least the early work of Bernstein. Gradient a priori estimates are fundamental for elliptic and parabolic equations, leading to Harnack inequalities, Liouville theorems, and compactness theorems for both linear and nonlinear PDE. For the corresponding elliptic equation of (6.1.1), gradient estimates were first considered by Lions [92] for the linear diffusion case $p = 2$. These estimates were based upon the Bernstein technique. Recently for the possibly degenerate elliptic equation with $q > p - 1 > 0$, Bidaut-Véron, Huidobro, Véron [30] obtained a priori universal gradient estimate for equations on a domain Ω of $\mathbb{R}$ and they extended their estimates to equations on complete non compact manifolds satisfying a lower bound estimate on the Ricci curvature. These estimates allowed them to derive some Liouville type theorems.

It is natural to look also for parabolic Liouville-type-theorems. In the linear diffusion case $p = 2$ and for $q > 1$, Souplet and Zhang [110] obtained local gradient estimate for locally upper bounded solution of (6.1.1) ($u \leq M$) of the form

$$|\nabla u| \leq C(p, N, q) \left(t^{\frac{-1}{q}} + R^{-1} + R^{\frac{-1}{q-1}}\right)(M + 1 - u) \quad \text{in} \quad B(x_0, R) \times (0, T).$$

Relying on this estimate they proved that, under some growth condition at infinity, ancient solutions in the whole of $\mathbb{R}^N$ are constant. Motivated by their result, we generalize the gradient estimate and Liouville theorem to the case $1 < p - 1 < q$. We also require that the solution is locally lower bounded. Using a Bernstein method, we have the following gradient estimate.

Theorem 6.1.1. *Let $q > p - 1 > 1$, $x_0 \in \mathbb{R}^N$ and $R, T > 0$. We set $Q_{T,R} = B(x_0, R) \times (0, T)$. Let u be a solution in $L^\infty((0, T); W^{1,\infty}(B(x_0, R)))$ of*

$$\partial_t u - \Delta_p u = |\nabla u|^q \qquad \text{in} \quad \mathrm{Q_{T,R}}.$$

Suppose that $|u| \leq M$ for some constant $M \geq 1$. Then,

$$|\nabla u| \leq C(p, N, q) \left(t^{\frac{-1}{q}} + R^{-1} + R^{\frac{-1}{q-p+1}}\right) M \quad \textit{in} \quad Q_{T,\frac{R}{2}}. \tag{6.1.2}$$

For the Cauchy-Dirichlet problem associated to (6.1.1), a gradient estimate involving the $W^{1,\infty}$ norm of the initial data has been obtained in [7, 110]. In Theorem 6.1.1 we only use the local L^∞ norm of the solution but we get a weaker estimate regarding the exponent on the distance to the boundary R.
Recently, for the singular diffusion case $1 < p < 2$ and for $q = p$, F. Wang [117] established gradient estimates similar to (6.1.2) for smooth, upper bounded, local solutions to (6.1.1) on closed manifolds or on complete noncompact Riemannian manifolds evolving under a Ricci flow. These estimates are of the form :

$$\frac{|\nabla u|}{1 - u}(x, t) \leq C(N, p) \left(R^{-1} + t^{\frac{-1}{p}} + K^{\frac{2}{p}} + K\right) \quad \text{in} \quad Q_{T,\frac{R}{2}} \tag{6.1.3}$$

where $K > 0$ is a constant related to the Ricci flow and the sectional curvature of the manifold. These estimates allowed to the author to provide some Harnack inequalities for positive solutions of the following p-Laplace heat equation

$$|z|^{p-2} z_t = \Delta_p z. \tag{6.1.4}$$

The estimates (6.1.3) have been obtained by deriving an equation for $w = |\nabla v|^p$, $v = f^{-1}(-u)$ and $f(s) = e^{s/(p-1)} - 1$. For $q > p > 2$, we take a different auxiliary function f, adapted to the degenerate diffusion case and to the fast growing gradient non-linearity.

As an application of the gradient estimate (6.1.2), we can state the following Liouville theorem for (6.1.1).

Theorem 6.1.2. *Assume that* $q > p - 1 > 1$ *and let* $\sigma = \min\left(1, \frac{1}{q-p+1}\right)$. *Assume that* $u \in L^{\infty}_{loc}((-\infty, 0); W^{1,\infty}_{loc}(\mathbb{R}^N))$ *is a weak solution of*

$$u_t - \Delta_p u = |\nabla u|^q, x \in \mathbb{R}^N, -\infty < t < 0,$$

satisfying

$$|u(x,t)| = o(|x|^{\sigma} + |t|^{\frac{1}{q}}), \quad \text{as} \quad |x|^{\sigma} + |t|^{\frac{1}{q}} \to \infty. \tag{6.1.5}$$

Then u is constant.

Remark 6.1.1. *The growth hypothesis* (6.1.5) *is important (see the example of the function* $u(x,t) = x_1 + t$*). However, we do not know if the exponents are sharp.*

Besides the works mentioned above, there are few other studies on gradient estimates and nonlinear Liouville theorems for a parabolic type equation on noncompact Riemannian manifolds. In this case the proof mostly relies on two types of gradient estimates or a combination of them. These estimates are known as Hamilton gradient estimate (the estimate only involves ∇u and u) [66] and Li-Yau's gradient estimate (the estimate involves $\nabla u, u$ and u_t) [87]. Let us also mention that the linear heat equation on noncompact manifolds was studied by Souplet and Zhang in [109] where they obtained a local gradient estimate related to the elliptic Cheng-Yau estimate and Hamilton's estimate for the heat equation on compact manifolds. A Liouville theorem was also proved in [109]. Hamilton-type gradient estimates were also used in [118, 93, 129]. For $q = p > 1$, a nonlinear analogue of Li-Yau's estimate has been established in [77] for positive solutions of (6.1.1) on compact manifolds with nonnegative Ricci curvature. In [77], the gradient estimate was not used to get Liouville theorems but to obtain an entropy formula. Nevertheless, Liouville theorems should be obtained as a consequence of the obtained gradient estimate.

This chapter is organized as follows : In Section 6.2, we provide the proof of the gradient estimate (6.1.2) and we prove Theorem 6.1.2. In Sections 6.3 we give the proof of a technical auxiliary lemma that appears in the proof of the gradient estimate.

6.2 Bernstein-type gradient estimate

The proof of Theorem 6.1.1 is based on the following technical lemma which is based on a Bernstein method. The most significant difficulty being the choice of the auxiliary function f and the estimates coming from the cut-off argument. Let us mention that for different suitable choice of f, gradient bounds global in space for the Cauchy problem associated to (6.1.1) have been obtained in [22].

First let us make precise that by local weak solution of (6.1.1) we mean a function $u \in C(\mathbb{R}^N \times (0,T)) \cap L^{\infty}_{loc}(0,T; W^{1,\infty}_{loc}(\mathbb{R}^N))$ such that the integral equality

$$\int_{\mathbb{R}^N} (u(x,t)\psi(x,t) - u(x,s)\psi(x,s))\,dx + \int_s^t \int_{\mathbb{R}^N} \left(-u\psi_t + |\nabla u|^{p-2}\nabla u \cdot \nabla\psi\right)\,dx\,d\tau$$
$$= \int_s^t \int_{\mathbb{R}^N} |\nabla u|^q \psi\,dx\,d\tau$$

holds for all $0 < s < t < T$ and for all testing function $\psi \in C_c^\infty(\mathbb{R}^N \times (0,T))$.
Now let $\alpha \in (0,1)$ to be chosen later on. Set $R' = \frac{3R}{4}$. We select a cut-off function $\eta \in C^2(\bar{B}(x_o, R'))$, $0 \le \eta \le 1$, satisfying $\eta = 0$ for $|x - x_0| = R'$ and such that

$$\left.\begin{aligned} |\nabla \eta| &\le C R^{-1} \eta^\alpha \\ |D^2 \eta| + \eta^{-1} |\nabla \eta|^2 &\le C R^{-2} \eta^\alpha \end{aligned}\right\} \text{ for } |x - x_0| < R', \tag{6.2.1}$$

for some $C = C(\alpha) > 0$ (see [110] for the existence of such function).

Lemma 6.2.1. *Assume that u is a local weak solution of* (6.1.1) *and that $|u| \le M$ in $Q_{T,R}$ for some $M > 1$. We consider a C^3 smooth increasing function f satisfying $f'' > 0$, the following differential equation*

$$\left(\frac{f''}{f'}\right)' + (p-1)(1+N)\left(\frac{f''}{f'}\right)^2 = 0 \tag{6.2.2}$$

and mapping $[0,3]$ onto $[-M, M]$. Defining $v = f^{-1}(-u)$, we set $w = |\nabla v|^2$ and $z = \eta w$. Then at any point where $|\nabla u| > 0$, z satisfies the following differential inequality

$$\begin{aligned} \mathcal{L}(z) \le &-2(q-1)(f')^{q-2} f'' w^{\frac{q+2}{2}} \eta + C(p,N)(f')^{p-2} R^{-2} \eta^\alpha w^{\frac{p}{2}} \\ &+ C(p,q) R^{-1} \eta^\alpha \left[w^{\frac{p+1}{2}} (f')^{p-3} f'' + w^{\frac{q+1}{2}} (f')^{q-1} \right] \end{aligned} \tag{6.2.3}$$

where

$$\mathcal{L}(z) := \partial_t z - \mathcal{A} z + \mathcal{H} \cdot \nabla z \tag{6.2.4}$$

with $\mathcal{A}$ is given by (6.3.4) *$\mathcal{H}$ is given by* (6.3.5).

The proof of lemma 6.2.1 is postponed to the the next section.

Proof of Theorem 6.1.1

Let $u \in L^\infty_{loc}((0,\infty); W^{1,\infty}_{loc}(\Omega))$ be a local weak solution of (6.1.1). Since u and ∇u are locally bounded, using the result of Di Benedetto and Friedman [48, 47], we get that ∇u is a locally Hölder continuous function. Thus z is a continuous function on $\overline{B(x_0, R')} \times [0,T] = \overline{Q}$, for any $0 < T$. Therefore, unless $z \equiv 0$ in $\overline{Q}$, z must reach a positive maximum at some point $(\hat{x}, \hat{t}) \in \overline{B(x_0, R')} \times [t_0, T]$. Since $z = 0$ on $\partial B_{R'} \times [0,T]$, we deduce that $\hat{x} \in B_{R'}$. Since $z(\hat{x}, \hat{t}) > 0$, we have that $|\nabla u| = f'(v)|\nabla v| > 0$ and hence we can use Lemma 6.2.1.

Now let us take $f(s) = M(s+1)^\gamma - 2M$ where γ is given by

$$\gamma = \gamma(p,N) = \frac{(p-1)(N+1)+1}{(p-1)(N+1)} \tag{6.2.5}$$

It is easy to see that f satisfies the differential equation (6.2.2) and $f', f'' > 0$ and f maps $\left[0, 3^{\frac{1}{\gamma}} - 1\right]$ onto $[-M, M]$. Let us also note that $\gamma \ge 1$ and $\gamma - 1 \le \dfrac{1}{p-1} \le 1$.

By Lemma 6.2.1 we get that, in a small neighbourhood $\tilde{Q}$ of $(\hat{x}, \hat{t})$, z satisfies

$$\begin{aligned}\mathcal{L}z &\le -2(q-1)(f')^{q-2} f'' w^{\frac{q+2}{2}} \eta + C(p,N,\alpha)(f')^{p-2} R^{-2} \eta^{\alpha} w^{\frac{p}{2}} \\ &+ C(p,q,\alpha) R^{-1} \eta^{\alpha} \left[w^{\frac{p+1}{2}} (f')^{p-3} f'' + w^{\frac{q+1}{2}} (f')^{q-1} \right].\end{aligned}$$

Hence

$$\begin{aligned}(f')^{1-q}\mathcal{L}z &\le -2(q-1)\frac{f''}{f'} w^{\frac{q+2}{2}} \eta + C(p,N,\alpha)(f')^{p-1-q} R^{-2} \eta^{\alpha} w^{\frac{p}{2}} \\ &+ C(p,q,\alpha) R^{-1} \eta^{\alpha} \left[w^{\frac{p+1}{2}} (f')^{p-q-1} \frac{f''}{f'} + w^{\frac{q+1}{2}} \right].\end{aligned}$$

Since $v \in \left[0, (3)^{\frac{1}{\gamma}} - 1\right]$, $\gamma, M \ge 1$, we have $1 \le v + 1 \le (3)^{\frac{1}{\gamma}} \le 3$ and hence

$$\frac{1}{3(p-1)(N+1)} \le \left(\frac{f''}{f'}\right) \le \frac{1}{(p-1)(N+1)} \le 1 \tag{6.2.6}$$

Using (6.2.6) together with the fact that $1 \le M \le f'$ and $p - q - 1 < 0$, we get that

$$\begin{aligned}(f')^{1-q}\mathcal{L}z &\le -\frac{2(q-1)}{3(p-1)(N+1)} w^{\frac{q+2}{2}} \eta + C(N,p,\alpha) R^{-2} \eta^{\alpha} w^{\frac{p}{2}} \\ &+ C(p,q,\alpha) R^{-1} \eta^{\alpha} \left[w^{\frac{p+1}{2}} + w^{\frac{q+1}{2}} \right].\end{aligned}$$

We take $\alpha = \max\left(\frac{q+1}{q+2}, \frac{p+1}{q+2}\right)$. Using the Young's inequality and recalling that $\eta \le 1$, then

- for the conjugate exponents $r_1 = \frac{q+2}{p}$, $s_1 = \frac{q+2}{q-p+2}$ we have that

$$\begin{aligned}C(N,p,\alpha) R^{-2} \eta^{\alpha} w^{\frac{p}{2}} &= \eta^{\frac{p}{q+2}} w^{\frac{p}{2}}\, C(N,p,q,\alpha) \eta^{\alpha - p/(q+2)} R^{-2} \\ &\le \varepsilon_1(N,p,q) \eta w^{\frac{q+2}{2}} + C(N,p,q,\alpha) R^{\frac{-2(q+2)}{q-p+2}},\end{aligned}$$

- for the conjugate exponents $r_2 = \frac{q+2}{p+1}$, $s_2 = \frac{q+2}{q-p+1}$ we have that

$$\begin{aligned}C(N,p,q,\alpha) R^{-1} \eta^{\alpha} w^{\frac{p+1}{2}} &= \eta^{\frac{p+1}{q+2}} w^{\frac{p+1}{2}}\, C(N,p,q,\alpha) R^{-1} \eta^{\alpha - \frac{p+1}{q+2}} \\ &\le \varepsilon_2 \eta w^{\frac{q+2}{2}} + C(N,p,q,\alpha) R^{\frac{-(q+2)}{q-p+1}}\end{aligned}$$

- and finally for the conjugate exponent $r_3 = \frac{q+2}{q+1}$, $s_3 = (q+2)$ we have that

$$\begin{aligned}C(N,p,q,\alpha) R^{-1} \eta^{\alpha} w^{\frac{q+1}{2}} &= \eta^{\frac{q+1}{q+2}} w^{\frac{q+1}{2}}\, C(N,p,q,\alpha) R^{-1} \eta^{\alpha - \frac{q+1}{q+2}} \\ &\le \varepsilon_3 \eta w^{\frac{q+2}{2}} + C(N,p,q,\alpha) R^{-(q+2)}.\end{aligned}$$

Choosing ε_i in such way that $\varepsilon_1+\varepsilon_2+\varepsilon_3=\dfrac{1}{4}\dfrac{2(q-1)}{3(p-1)(N+1)}$, we get that

$$\begin{aligned}(f')^{1-q}\mathcal{L}z \quad &\leq -\frac{(q-1)}{2(p-1)(N+1)}w^{\frac{q+2}{2}}\eta + C(N,p,q,\alpha)R^{\frac{-2(q+2)}{q-p+2}} \\ &+C(N,p,q,\alpha)R^{\frac{-(q+2)}{q-p+1}}+C(N,p,q,\alpha)R^{-(q+2)}.\end{aligned} \tag{6.2.7}$$

Using the fact that

$$\begin{cases} \frac{1}{q-p+1}\leq\frac{2}{q-p+2}\leq 1 & \text{for}\, q\geq p, \\ 1\leq\frac{2}{q-p+2}\leq\frac{1}{q-p+1} & \text{for}\, q\leq p,\end{cases}$$

we have that

$$(f')^{1-q}\mathcal{L}z\leq -\frac{(q-1)}{2(p-1)(N+1)}w^{\frac{q+2}{2}}\eta + C(N,p,q,\alpha)\left[R^{-\frac{q+2}{q-p+1}}+R^{-(q+2)}\right].$$

Setting

$$A=A(R,p,q,N):=C(N,p,q)\left(R^{-\frac{1}{q-p+1}}+R^{-1}\right)^2$$

and using that $(f')^{q-1}\geq M^{q-1}\geq 1$, it follows that

$$\mathcal{L}z\leq -\frac{(q-1)}{4(p-1)(N+1)}z^{\frac{q+2}{2}} \;\text{ in }\left\{(x,t)\in Q_{T,R'};\, z(x,t)\geq A\right\}. \tag{6.2.8}$$

Next for $\lambda=\lambda(q,N,p)>0$ suitably chosen, the function $\psi(t)=\lambda t^{\frac{-2}{q}}$ satisfies

$$\psi'(t)\geq -\frac{(q-1)}{4(p-1)(N+1)}\psi^{\frac{q+2}{2}}.$$

Now for $t_0\in(0,T)$ fixed, we define $\tilde{z}(t):=z(t+t_0,x)-\psi(t)$. It is easy to see that

$$\mathcal{L}\tilde{z}\leq 0 \;\text{ in }\left\{(x,t)\in Q_{T-t_0,R'};\, \tilde{z}(x,t)\geq A\right\}.$$

Since $\tilde{z}(t)\leq 0$ for $t>0$ sufficiently small, we deduce from the maximum principle that $\tilde{z}(t)\leq A$, i.e. $z(x,t+t_0)\leq A+\psi(t)$ in $Q_{T-t_0,R'}$.

Finally using that $z=\eta|\nabla v|^2$, letting t_0 to 0, we get that

$$|\nabla v|\leq C(N,p,q)(A+t^{\frac{-2}{q}})^{1/2}.$$

Using that

$$v+1=\left(2-\frac{u}{M}\right)^{\frac{1}{\gamma}} \qquad \text{with}\left|\frac{u}{M}\right|\leq 1,$$

we get

$$\nabla v=\frac{-1}{\gamma M}\left(2-\frac{u}{M}\right)^{\frac{1-\gamma}{\gamma}}\nabla u.$$

It follows that

$$|\nabla u| \leq M\gamma|\nabla v| \leq C(N,p,q)(A + t^{\frac{-2}{q}})^{1/2}M \quad \text{in } Q_{T,\frac{R}{2}}. \tag{6.2.9}$$

Here we used the fact that $\left(2 - \frac{u}{M}\right)^{\frac{\gamma-1}{\gamma}} \leq 1$.
Hence we have

$$|\nabla u| \leq C(N,p,q)\left(R^{-1} + R^{\frac{-1}{q-p+1}} + t^{\frac{-1}{q}}\right)M \quad \text{in } Q_{T,\frac{R}{2}}.$$

and the proof of Theorem 6.1.1 is complete. □

Proof of Theorem 6.1.2

Fix $x_0 \in \mathbb{R}$ and $t_0 \in (-\infty, 0)$. Take $R \geq 1, T = R^{\sigma q}$ and set $Q = B(0,R) \times (0,T)$. Now we consider the function $U := u(x + x_0, t + t_0 - T)$. Using (6.1.5), we have that $|U| \leq M_R$ in $\overline{Q}$, where

$$M_R := \sup_{B(x_0,R)\times(t_0-T,t_0)} |u| = o(T^{\frac{1}{q}} + R^\sigma) = o(R^\sigma), \quad \text{as} \quad R \to \infty.$$

Applying Theorem 6.1.1 to U in Q, we get that

$$|\nabla u(x_0,t_0)| = |\nabla U(0,T)| \leq C(N,p,q)R^{-\sigma}M_R$$

and the conclusion follows by sending R to $+\infty$. □

6.3 Proof of Lemma 6.2.1

Our proof consists of three steps.

Step 1 : computations

Let f be a C^3-function to be determined. We assume that $f', f'' > 0$. We put $v = f^{-1}(-u)$ and $w = |\nabla v|^2$. By a straightforward computation, we have that v satisfies the following equation

$$\begin{aligned}
\partial_t v &= (f')^{p-2}w^{\frac{p-2}{2}}\left[\Delta v + (p-2)\frac{\langle D^2 v, \nabla v, \nabla v\rangle}{w}\right] + (p-1)(f')^{p-3}f''w^{\frac{p}{2}} - (f')^{q-1}w^{\frac{q}{2}} \\
&=(f')^{p-2}w^{\frac{p-2}{2}}\left[\Delta v + (p-2)\frac{\nabla w \cdot \nabla v}{2w}\right] + (p-1)(f')^{p-3}f''w^{\frac{p}{2}} - (f')^{q-1}w^{\frac{q}{2}}.
\end{aligned} \tag{6.3.1}$$

For $i = 1, ..., N$, we set $v_i = \frac{\partial v}{\partial x_i}$. In a neighbourhood $\tilde{Q} := \omega \times (\tau_1, \tau_2)$ of any point $(\hat{x}, \hat{t}) \in Q_{T,R}$ for which $|\nabla u| = f'(v)|\nabla v| > 0$, the equation is uniformly parabolic and hence differentiating (6.3.1) with respect to x_i, we have

$$\begin{aligned}\partial_t v_i =&(f')^{p-2}w^{\frac{p-2}{2}}\left[\Delta v_i + \frac{p-2}{2}\left(\frac{\nabla w_i\cdot\nabla v+\nabla w\cdot\nabla v_i}{w}-\frac{w_i\nabla w\cdot\nabla v}{w^2}\right)\right]\\ &+(p-2)(f')^{p-3}f''v_iw^{\frac{p-2}{2}}\left[\Delta v+(p-2)\frac{\nabla w\cdot\nabla v}{2w}\right]\\ &+\frac{p-2}{2}(f')^{p-2}w_iw^{\frac{p-4}{2}}\left[\Delta v+(p-2)\frac{\nabla w\cdot\nabla v}{2w}\right]\\ &+(p-1)((f')^{p-3}f'')'v_iw^{\frac{p}{2}}-(q-1)(f')^{q-2}f''v_iw^{\frac{q}{2}}\\ &+\frac{p(p-1)}{2}(f')^{p-3}f''w_iw^{\frac{p-2}{2}}-\frac{q}{2}(f')^{q-1}w_iw^{\frac{q-2}{2}}.\end{aligned} \tag{6.3.2}$$

Here and in all the manuscript, the variable v is omitted in the expression of f', f'', $\left(\frac{f''}{f'}\right)'$, etc. The equalities are understood in a classical sense in $\tilde{Q}$. Multiplying (6.3.2) by $2v_i$, summing over i and using that

$$\left\langle D^2v,\nabla v,\nabla v\right\rangle=\frac{1}{2}\nabla w\cdot\nabla v,\qquad \Delta w=2\nabla v\cdot\nabla\Delta v+2|D^2v|^2,$$

$$\sum_i 2(\nabla v_i\cdot\nabla w)v_i=|\nabla w|^2,\qquad \sum_i(\nabla w_i\cdot\nabla v)v_i=\left\langle D^2w,\nabla v,\nabla v\right\rangle,$$

we get that

$$\begin{aligned}\partial_t w =& |\nabla u|^{p-2}\Delta w+(p-2)|\nabla u|^{p-4}\left\langle D^2w,\nabla u,\nabla u\right\rangle-2|\nabla u|^{p-2}|D^2v|^2\\ &+(p-2)(f')^{p-2}w^{\frac{p-4}{2}}\Delta v\,(\nabla v\cdot\nabla w)+\frac{(p-2)}{2}(f')^{p-2}w^{\frac{p-4}{2}}|\nabla w|^2\\ &+\frac{(p-2)(p-4)}{2}(f')^{p-2}w^{\frac{p-6}{2}}\left(\nabla v\cdot\nabla w\right)^2\\ &-q(f')^{q-1}w^{\frac{q-2}{2}}\nabla w\cdot\nabla v+(p(p-1)+(p-2)^2)(f')^{p-3}f''w^{\frac{p-2}{2}}\nabla w\cdot\nabla v\\ &+2\left[(p-1)((f')^{p-3}f'')'w^{\frac{p+2}{2}}-(q-1)(f')^{q-2}f''w^{\frac{q+2}{2}}+(p-2)(f')^{p-3}f''w^{\frac{p}{2}}\Delta v\right].\end{aligned} \tag{6.3.3}$$

Here, when passing from (6.3.2) to (6.3.3), the terms have been transformed according to

$$\begin{array}{llll} L^1_{t1}\to\tilde{L}^1_{t1}+\tilde{L}^1_{t3}, & L^1_{t2}\to\tilde{L}^1_{t2}, & L^1_{t3}\to\tilde{L}^2_{t2}, & L^1_{t4}+L^3_{t2}\to\tilde{L}^3,\\ L^2_{t1}\to\tilde{L}^5_{t3}, & L^2_{t2}\to\tilde{L}^4_{t3}, & & \\ L^3_{t1}\to\tilde{L}^2_{t1}, & & & \\ L^4_{t1}\to\tilde{L}^5_{t1}, & L^4_{t2}\to\tilde{L}^5_{t2}, & & \\ L^5_{t1}\to\tilde{L}^4_{t2}, & L^5_{t2}\to\tilde{L}^4_{t1}, & & \end{array}$$

(with obvious labeling).
Hence w satisfies

$$\partial_t w-\mathcal{A}(w)-\mathcal{H}\cdot\nabla w=-2|\nabla u|^{p-2}|D^2v|^2+\mathcal{N}(w)$$

where

$$\mathcal{A}(w) = |\nabla u|^{p-2}\Delta w + (p-2)|\nabla u|^{p-4}\left\langle D^2 w, \nabla u, \nabla u\right\rangle, \tag{6.3.4}$$

$$\begin{aligned}\mathcal{H} &= (p-2)(f')^{p-2}w^{\frac{p-4}{2}}\Delta v\nabla v + \frac{(p-2)}{2}(f')^{p-2}w^{\frac{p-4}{2}}\nabla w,\\ &+ \frac{(p-2)(p-4)}{2}(f')^{p-2}w^{\frac{p-6}{2}}\left(\nabla v\cdot\nabla w\right)\nabla v - q(f')^{q-1}w^{\frac{q-2}{2}}\nabla v\\ &+ (p(p-1)+(p-2)^2)(f')^{p-3}f''w^{\frac{p-2}{2}}\nabla v\end{aligned} \tag{6.3.5}$$

$$\begin{aligned}\mathcal{N}(w) &= 2(p-1)((f')^{p-3}f'')'w^{\frac{p+2}{2}} - 2(q-1)(f')^{q-2}f''w^{\frac{q+2}{2}}\\ &+ 2(p-2)(f')^{p-2}\frac{f''}{f'}w^{\frac{p}{2}}\Delta v.\end{aligned} \tag{6.3.6}$$

Step 2 : equation for z and useful estimates

We set $z = \eta w$. Defining the operator

$$\mathcal{L}(z) := \partial_t z - \mathcal{A}(z) - \mathcal{H}\cdot\nabla z,$$

we have that

$$\begin{aligned}\mathcal{L}z &= \eta\mathcal{L}w + w\mathcal{L}\eta - 2|\nabla u|^{p-2}\nabla\eta\cdot\nabla w - 2(p-2)|\nabla u|^{p-4}(\nabla u\cdot\nabla\eta)(\nabla w\cdot\nabla u)\\ &= -2|\nabla u|^{p-2}\nabla\eta\cdot\nabla w - 2(p-2)|\nabla u|^{p-4}(\nabla u\cdot\nabla\eta)(\nabla w\cdot\nabla u)\\ &\quad + \eta\mathcal{N}w + w\mathcal{L}\eta - 2|\nabla u|^{p-2}|D^2v|^2\eta.\end{aligned}$$

Leading estimates

Recalling that f is increasing and that $f'' > 0$, we get the following estimates.

1. Estimate of $\eta\mathcal{N}w$

$$\begin{aligned}\eta\mathcal{N}w \quad &\le 2(p-1)((f')^{p-3}f'')'w^{\frac{p+2}{2}}\eta - 2(q-1)(f')^{q-2}f''w^{\frac{q+2}{2}}\eta\\ &+\tfrac{(f')^{p-2}}{2}w^{\frac{p-2}{2}}|D^2v|^2\eta + 2N(p-1)^2(f')^{p-2}\left(\frac{f''}{f'}\right)^2 w^{\frac{p+2}{2}}\eta.\end{aligned} \tag{6.3.7}$$

Here we used that

$$2(p-2)|\frac{f''}{f'}w\Delta v| \le 2N(p-1)^2w^2\left(\frac{f''}{f'}\right)^2 + \frac{|D^2v|^2}{2}.$$

2. Estimate of $w\mathcal{L}(\eta)$
   – Estimate of $w\mathcal{A}(\eta)$

$$|w\mathcal{A}(\eta)| \le (f')^{p-2}w^{\frac{p}{2}}(\sqrt{N} + (p-2))|D^2\eta|. \tag{6.3.8}$$

– Estimate of $|wH \cdot \nabla\eta|$

$$\begin{aligned}
|wH \cdot \nabla\eta| \ &\leq \underbrace{(f')^{p-2}w^{\frac{p-2}{2}}\left(C_1(p,N,\delta_1)\eta^{-1}|\nabla\eta|^2 w + \delta_1[D^2v|^2\eta\right)}_{1} \\
&+ \underbrace{(f')^{p-2}w^{\frac{p-2}{2}}\left(C_2(p,N,\delta_2)\eta^{-1}|\nabla\eta|^2 w + \delta_2[D^2v|^2\eta\right)}_{2} \\
&+ \underbrace{(f')^{p-2}w^{\frac{p-2}{2}}\left(C_3(p,N,\delta_3)\eta^{-1}|\nabla\eta|^2 w + \delta_3[D^2v|^2\eta\right)}_{3} \\
&+2(p-1)^2(f')^{p-3}f''w^{\frac{p+1}{2}}|\nabla\eta| + q(f')^{q-1}w^{\frac{q+1}{2}}|\nabla\eta|.
\end{aligned} \tag{6.3.9}$$

(1) comes from an estimate via the Young's inequality of $|(p-2)w\Delta v\nabla v \cdot \nabla\eta|$. Recalling that $\nabla w = (2D^2v, \nabla v)$, (2) comes from an estimate of $\left|\frac{(p-2)}{2}w\nabla w \cdot \nabla\eta\right|$ and (3) come from an estimate of $\left|\frac{(p-2)(p-4)}{2}w\left(\nabla v \cdot \nabla w\right)\left(\nabla v \cdot \nabla\eta\right)\right|$.

3. Estimate of $2|\nabla u|^{p-2}|\nabla\eta \cdot \nabla w|$.
   Using the Young inequality, we have

$$2|\nabla u|^{p-2}|\nabla\eta \cdot \nabla w| \leq (f')^{p-2}w^{\frac{p-2}{2}}\left[C_4(p,N,\delta_4)\eta^{-1}|\nabla\eta|^2 w + \delta_4|D^2v|^2\eta\right].$$

4. Estimate of $2(p-2)(\nabla u \cdot \nabla\eta)(\nabla w \cdot \nabla u)$

$$|2(p-2)(\nabla u \cdot \nabla\eta)(\nabla w \cdot \nabla u)| \leq (f')^2 w\left[C_5(N,p,\delta_5)\eta^{-1}|\nabla\eta|^2 w + |D^2v|^2\eta\right].$$

Finally recalling that $\nabla u = f'\nabla v$ and choosing δ_i in such way that $-2+\delta_1+\delta_2+\delta_3+\delta_4+\delta_5 = -1$ and then recalling the properties of the function η, we arrive at

$$\begin{aligned}
\mathcal{L}(z) \leq 2(p-1)\eta &\left[\left((f')^{p-3}f''\right)' w^{\frac{p+2}{2}} + N(p-1)(f')^{p-2}\left(\frac{f''}{f'}\right)^2 w^{\frac{p+2}{2}}\right] \\
&- 2(q-1)(f')^{q-2}f''w^{\frac{q+2}{2}}\eta + C(p,N,\alpha)(f')^{p-2}R^{-2}w^{\frac{p}{2}}\eta^{\alpha} \\
&+ C(p,q,\alpha)\eta^{\alpha}R^{-1}\left[w^{\frac{p+1}{2}}(f')^{p-3}f'' + w^{\frac{q+1}{2}}(f')^{q-1}\right].
\end{aligned}$$

Step 3 : suitable choice for the function f

To get rid of the term

$$\left((f')^{p-3}f''\right)' w^{\frac{p+2}{2}} + N(p-1)(f')^{p-2}\left(\frac{f''}{f'}\right)^2 w^{\frac{p+2}{2}} \tag{6.3.10}$$

$$= (f')^{p-2}w^{\frac{p+2}{2}}\left[\left(\frac{f''}{f'}\right)' + (p-2)\left(\frac{f''}{f'}\right)^2 + (p-1)N\left(\frac{f''}{f'}\right)^2\right] \tag{6.3.11}$$

$$\leq (f')^{p-2}w^{\frac{p+2}{2}}\left[\left(\frac{f''}{f'}\right)' + (p-1)(N+1)\left(\frac{f''}{f'}\right)^2\right]. \tag{6.3.12}$$

we shall take a function f satisfying the following differential equation

$$\left(\frac{f''}{f'}\right)' + (p-1)(1+N)\left(\frac{f''}{f'}\right)^2 = 0. \tag{6.3.13}$$

Hence we get that

$$\begin{aligned}\mathcal{L}(z) \leq & -2(q-1)(f')^{q-2} f'' w^{\frac{q+2}{2}} \eta + C(p,N,\alpha)(f')^{p-2} R^{-2} \eta^{\alpha} w^{\frac{p}{2}} \\ & + C(p,q,\alpha) R^{-1} \eta^{\alpha} \left[w^{\frac{p+1}{2}} (f')^{p-3} f'' + w^{\frac{q+1}{2}} (f')^{q-1} \right].\end{aligned}$$

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Résumé en français
Cette thèse est consacrée à l'étude des propriétés qualitatives de solutions d'une équation d'évolution de type Hamilton-Jacobi avec une diffusion donnée par l'opérateur p-Laplacien. On s'attache principalement à l'étude de l'effet de la diffusion non-linéaire sur le phénomène d'explosion du gradient. Les principales questions qu'on étudie portent sur l'existence locale, régularité, profil spatial d'explosion et la localisation des points d'explosion. En particulier on montre un résultat d'explosion en seul point du bord. Dans le chapitre 4, on utilise une approche de solutions de viscosité pour prolonger la solution explosive au delà des singularités et on étudie son comportement en temps grands. Dans l'avant dernier chapitre on s'intéresse au caractère borné des solutions globales du problème unidimensionnel. Dans le dernier chapitre on démontre une estimation de gradient locale en espace et on l'utilise pour obtenir un résultat de type Liouville. On s'inspire et on compare nos résultats avec les résultats connus pour le cas de la diffusion linéaire.

TITRE en anglais : A qualitative study of a Hamilton-Jacobi equation with a nonlinear diffusion

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RESUME en anglais
This thesis is devoted to the study of qualitative properties of solutions of an evolution equation of Hamilton-Jacobi type with a p-Laplacian diffusion. It is mainly concerned with the study of the effect of the non-linear diffusion on the gradient blow-up phenomenon. The main issues we are studying are: local existence and uniqueness, regularity, spatial profile of gradient blow-up and localization of the singularities. We provide examples where the gradient blow-up set is reduced to a single point. In Chapter 4, a viscosity solution approach is used to extend the blowing-up solutions beyond the singularities and an ergodic problem is also analyzed in order to study their long time behavior. In the penultimate chapter, we address the question of boundedness of global solutions to the one-dimensional problem. In the last chapter we prove a local in space, gradient estimate and we use it to obtain a Liouville-type theorem.

Discipline : Mathématique

MOTS-CLES : Problémes paraboliques-dégénérés, estimations de gradient, explosion en un seul point, solutions de viscosité, comportement asymptotique, Théorème de type Liouville.

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