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Corrections radiatives en gravité quantique à mousse de spins

Un étude du graphe de self énergie dans le modèle EPRL Lorentzien

Soutenue le 22 juillet 2014 devant le jury :

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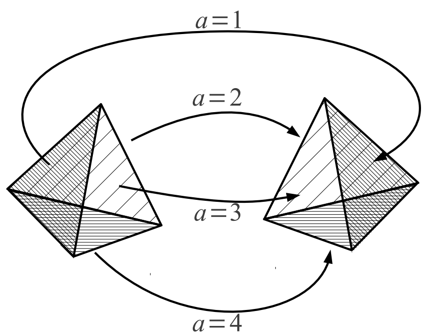
Dissertation

RADIATIVE CORRECTIONS IN SPINFOAM QUANTUM GRAVITY

ALDO RIELLO



A study of the self-energy graph within the Lorentzian EPRL model

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Aldo Riello: *Radiative Corrections in Spinfoam Quantum Gravity,* A study of the self-energy graph within the Lorentzian EPRL model, © July 18, 2014

Alle mie nonne,

Meris e Tilde

PUBLICATIONS

Most of the results presented in this thesis have been the subject of the following publications:

- Christodoulou, Marios, Långvik, Miklos, Riello, Aldo, Röken, Christian, & Rovelli, Carlo, 2013 *Divergences and Orientation in Spinfoams*. Classical and Quantum Gravity 30(5) 055009
- Riello, Aldo, 2013 *Self-Energy of the Lorentzian Engle-Pereira-Rovelli-Livine and Freidel-Krasnov Model of Quantum gravity*. Physical Review D 88(2) 024011
- Riello, Aldo, 2014 *Radiative Corrections to the Lorentzian Engle-Pereira-Rovelli-Livine and Freidel-Krasnov Spinfoam Graviton*. Physical Review D 89(6) 064021

In addition, during my doctorate I co-authored two more articles whose content is not included in this thesis:

- Christodoulou, Marios, Riello, Aldo, & Rovelli, Carlo, 2012 *How to Detect an Anti-Spacetime*. International Journal of Modern Physics D 21(11) 1242014
- Chirco, Goffredo, Haggard, Hal M, & Rovelli, Carlo, 2014 *Spacetime Thermodynamics without Hidden Degrees of Freedom* `arxiv:1401.5262` [submitted]

ABSTRACT

In this thesis, I present the first quantitative study of radiative corrections within the Engle-Pereira-Rovelli-Livine model of quantum gravity. This model is the most advanced proposal we dispose today in the context of Lorentzian four-dimensional background-independent quantum gravity. It can be thought as a realization of the path-integral quantization of general relativity as a sum over geometries. The present study focuses on the properties and geometrical features of the analogue of the self-energy graph within the model, often referred to as the "melon"-graph. In particular, I show that the most dominating contribution to such a graph is characterized by a degree of divergence much smaller than that of closely related topological quantum field theories. Moreover, I work out in detail the dependence of the amplitude from the boundary data, and find that the self-energy graph does not simply induce a wave function renormaliziation. This happens for reasons deeply related to the very foundations of the model. However, it turns out that the amplitude reduces to a wave function renormalzation in the limit of large quantum numbers. After having discussed all of this and its geometrical interpretation in detail, I move on to show the consequences of this calculations on a concrete spinfoam observable: the quantum-metric two-point function. In doing this, I show how the insertion of the self-energy graph in the bulk of the (first-order) spinfoam used in the calculation, has non trivial effects on the correlation function, modifying even its leading order contributions. Most interestingly, this effects do not disappear in the limit of large quantum number. Finally, I discuss the consequences of these calculations for the model itself, and above all I point out and comment the general features which seem to be common to any spinfoam model based on the present model-building schemes.

RÉSUMÉ

Je propose la première étude quantitative des corrections radiatives du modèle EPRL en gravité quantique à mousse de spins. Ce modèle est la proposition la plus élaborée de gravité quantique Lorentzienne 4D dite 'indépendante du fond' ('background independent'). C'est une réalisation, par intégrale de chemin, de la quantification de la Relativité Générale comme somme sur les géométries. L'étude se focalise sur les propriétés et les aspects géométriques de l'analogue du graphe de self-énergie du modèle, connu comme le graphe 'melonique'. Je montre que les contributions dominantes à un tel graphe divergent beaucoup moins que celles de modèles similaires en théorie topologique des champs. De plus, je dérive en détails la dépendance des amplitudes aux données de bords, et montre que ce graphe n'induit pas une renormalisation de la fonction d'onde. Ceci est dû à des raisons reliées aux fondements du modèle. Cependant, il se trouve que l'amplitude se réduit à une telle renormalisation dans la limite de nombres quantiques élevés. Ensuite, je montre les conséquences de ces calculs sur une observable physique : la fonction à deux points de la métrique quantique. Ainsi, je montre comment l'insertion du graphe de self-énergie dans l'intérieur de la mousse de spins utilisée a des effets non-triviaux sur la fonction à deux points, modifiant ses contributions à l'ordre dominant. De façon intéressante, ces effets ne disparaissent pas dans la limite des nombres quantiques élevés. Enfin, je discute les conséquences de ces calculs pour le modèle lui-même, et je souligne et commente les traits généraux qui semblent commun à tout modèle de mousse de spins basé sur le schéma présenté ici.

RÉSUMÉ ÉTENDU

la problématique

Les lois fondamentaux de la nature sont incomplètes à l'heure actuelle. D'un coté la physique des particules élémentaires s'érige sur les piliers de la mécanique quantique, où notre conception de la réalité matérielle est mise à dure épreuve par les principes d'indétermination et de superposition. De l'autre, le dernier bastion, et probablement plus haut sommet, de la physique classique, érigé par le génie de Albert Einstein, décrit la dynamique de l'espace-temps lui même en présence de matière, en expliquant ainsi la vrai nature de la seule force à caractère universelle, c'est à dire de la gravitation. Voici donc le problème de la gravité quantique : comment soumettre le champ gravitationnel aux lois quantiques ? Comment l'espace-temps vient-il influencé par la matière quand elle se trouve dans une superposition d'états ? Comment, et en quel sens, peut lui même se trouver dans une telle superposition ? Et surtout, comment formuler les lois de la mécanique quantique pour l'espace-temps, quand la relativité générale nous prive de toute appuis, c'est à dire quand tout référentiel perd de réalité et dévient pur jauge ?

Le problème de la gravité quantique est ouvert depuis au moins quatre-vingt ans : aussi tôt qu'en 1936, après une attentive analyse du principe d'indétermination d'Heisenberg appliqué au champ gravitationnel au delà du régime perturbative (analysé en grand détaille dans son papier !), Matvei Bronstein conclut [1]

> Ohne eine tiefgehende Umarbeitung der klassischen Begriffe scheint es daher wohl kaum möglich, die Quantentheorie der Gravitation auch auf dieses Gebiet auszudehnen.

Depuis, beaucoup de tentatives se sont succédé pour résoudre ce problème, qui reste néanmoins encore largement ouvert.

la gravité quantique à mousse de spin et le modèle eprl

Dans cette thèse, je poursuis l'analyse d'une des théories qui cherchent de résoudre le problème de la gravité quantique. Cette théorie a vue son origine à la fin des années quatre-vingts, quand, suite à une nouvelle formulation de la relativité générale achevée par Abhay Ashtekar décrivant cette théorie comme une théorie de jauge pour un groupe compact, Ted Jacobson, Carlo Rovelli et Lee Smolin trouvèrent des solutions exactes aux contraints définissants la quantification canonique de la relativité générale. Ces solutions consistaient de quanta de flux du champ gravitationnelle concentrés sur des boucles de Wilson. Malheureusement, ces solutions étaient d'une certaine manière triviale, dans la mesure où le volume associé était nul. Néanmoins, la gravité quantique à boucle était née.

Depuis, la théorie a évolué en empruntant notamment techniques provenant des théories quantique des champs topologiques, développées notamment par Edward Witten et Michael Atiyah toujours vers la fin des années quatre-vingts. Dans la forme actuelle, et dans son formalisme covariant, la théorie prend le nom de gravité quantique à mousse de spin. Le mot « mousse » fait référence aux structures dis-

1. Bronstein, Matvei Quantentheorie schwacher Gravitationsfelder. Physikalische Zeitschrift der Sowjetunion (1936) 9 140-157

crètes qui apparaissent naturellement pendant la quantification, en donnant ainsi une forme concrète à l'intuition de John Wheeler à propos de la nature intime de l'espace-temps quantique. Le mot « spin » vient du rôle prééminent joué dans cette théorie par le groupe $SU(2)$, le résultat étant que les spectre des opérateurs associés à des quantités géométriques sont liée au représentations unitaire irréductibles de ce groupe et donc indexés par des spins. Le fait que la géométrie quantique soit construite à partir de la théorie du moment angulaire fut une intuition de Sir Roger Penrose dans les années soixante-dix et apparaît dans la gravité quantique à boucle comme un résultat d'une quantification canonique.

Dans un modèle de mousse de spin, la théorie de la gravité quantique est définie en donnant une prescription pour calculer des amplitudes de transitions entre états quantiques de la géométrie trois-dimensionnelle. Une telle géométrie quantique trois-dimensionnelle, contenant un nombre fini L de « quanta d'espace » - est décrite par une fonction ψ labellisée par un graphe Γ constitué de L liens orientés et N sommets :

$$\psi_\Gamma \in L_2\left[SU(2)^L//SU(2)^N, d\mu_H\right], \tag{i}$$

$d\mu_H$ étant la mesure de Haar sur $SU(2)$, et le quotient par N copies de $SU(2)$ signifiant que la fonction ψ_Γ est invariante par transformation de jauge aux sommets:

$$\psi_\Gamma(g_{t(\ell)}h_\ell g^{-1}_{s(\ell)}) = \psi_\Gamma(h_\ell). \tag{ii}$$

Ici, $h_\ell, g_n \in SU(2)$ et $s(\ell)$ (respectivement $t(\ell)$) représentent le sommet d'origine et d'arrivé du lien ℓ. La géométrie quantique porté par cet état peut être lit en calculant la valeur moyenne quantique de l'opérateur de métrique (discrète): [2]

$$\hat{g}_{\ell\ell'}(n) := (a_{Pl}\gamma)^2 \vec{J}_\ell \cdot \vec{J}_{\ell'}, \tag{iii}$$

où $\vec{J}_\ell$ est le champ vecteur invariant à droite agissant sur la ℓ-ème copie de $SU(2)$. Dans cette équation, il est définit l'opérateur agissant au sommet n de Γ dans les directions identifiées par le liens ℓ et ℓ'. La partie diagonale de $\hat{g}_{\ell,\ell'}$ est constitué par les opérateurs « aire au carré » des éléments d'espace associés à chaque lien :

$$A_\ell := a_{Pl}\,\gamma\sqrt{\vec{J}_\ell \cdot \vec{J}_\ell} \tag{iv}$$

qui correspond donc à l'opérateur de Casimir de la ℓ-ème copie de $SU(2)$. Par conséquence son spectre est

$$\mathrm{Spec}(A_\ell) = \left\{a_{Pl}\,\gamma\sqrt{j(j+1)},\ j \in \frac{1}{2}\mathbb{N}\right\}, \tag{v}$$

Ici $\gamma \in \mathbb{R}^+$ est un paramètre additionnel de la théorie, connu comme paramètre de Barbero-Immirzi. De même, la partie hors de la diagonale (une fois normalisée) code les angles entre les différentes directions. Le lecteur attentif aura sûrement remarqué le fait que non toute les composantes de $\hat{g}_{\ell\ell'}(n)$ commutent entre eux. Dans cette non-commutativité se cache la quantification de la géométrie.

Un deux-complexe c'est une généralisation du concept de graphe, il contient des sommets, des liens, et de faces

Ensuite, pour donner l'amplitude de transition entre deux état ψ_Γ et $\psi'_{\Gamma'}$, il est nécessaire de définir le deux-complexe Δ^* qui a $\Gamma \sqcup \Gamma'$ comme bord. A chaque Δ^* on associe une amplitude $Z^{\Delta^*}_{\text{modèle}}[\psi_\Gamma, \psi'_{\Gamma'}]$ dépendant de son état de bord qui dépend du modèle de mousse de spin choisi. Enfin, l'amplitude de transition est formellement définie par la limite où le deux complexe est raffiné infiniment :

$$Z[\psi_\Gamma, \psi'_{\Gamma'}] = \lim_{\Delta^* \to \infty} Z^{\Delta^*}_{\text{modèle}}[\psi_\Gamma, \psi'_{\Gamma'}]. \tag{vi}$$

2. $a_{Pl} = \hbar\kappa c = 8\pi G_N c^3$, G_N étant la constant de gravitation universelle de Newton.

Celle-ci n'est pas la seule possibilité pour s'affranchir de la régularisation donnée Δ^* ; autre procédure sont discutée dans la thèse.

Le modèle de mousse de spin faisant sujet de cette thèse est appelé le modèle EPRL, par le nom de ses auteurs (John Engle, Roberto Pereira, Carlo Rovelli et Etera Livine). Il est plus aisément exprimé dans une base de l'espace $L_2\left[SU(2)^L//SU(2)^N; d\mu_H\right]$ nommée base des réseaux de spin. On peut décomposer toute fonction ψ_Γ dans cette base en utilisant le théorème de Peter-Weyl. Sans entrer dans les détails (simples, mais lourdes au niveau de la notation), on peut résumer cette construction en observant que toute fonction de éléments de $SU(2)$ peut être décomposé dans ses représentations unitaire irréductibles, en revanche labellisée par des demi-entiers $j \in \frac{1}{2}\mathbb{N}$, les « spins ». Pour être covariant de Lorentz, ce modèle injecte chaque réseau de spin de $SU(2)$ dans un réseau de spin de $SL(2,\mathbb{C})$ (le double-recouvrement du groupe de Lorentz propre orthochrone $SO(3,1)^{\uparrow}_{+}$, c'est à dire de la partie de $SO(3,1)$ connectée avec l'identité). Pour faire ça chaque vecteur $|j;m\rangle$ dans la j-ème représentation de $SU(2)$, qui apparaissant dans la décomposition de ψ_Γ comme expliqué ci-dessus, identifié avec un vecteur approprié dans une représentation unitaire irréductible de $SL(2,\mathbb{C})$ appropriée (en effet dépendant de la jauge dans laquelle l'injection est accompli). Ce-ci est la signification de la mappe Y_γ :

$$Y_\gamma \quad : \quad |j;m\rangle \mapsto |-\gamma j, j; j, m\rangle\,. \tag{vii}$$

Pour les détails sur la notations voir le corps de la thèse. Ensuite, à chaque face du deux-complexe Δ^* (cet à dire à chaque succession fermée de liens, préalablement identifiée comme face) le modèle EPRL associe l'amplitude « partielle » :

$$w_f^{\mathrm{EPRL}}(j_f;\{g_{ev}\}) = \sum_{m=-j_f}^{j_f} \langle j_f; m| \left(\overleftarrow{\prod_{v\in f}} {Y_\gamma}^\dagger g_{ve'}^{-1} g_{ve} Y_\gamma \right) |j_f; m\rangle \tag{viii}$$

où $g_{ev} \in SL(2,\mathbb{C})$ et v, e, f, labellisent respectivement les sommets, liens et faces de Δ^*. Enfin, l'amplitude de transition est donnée par le produit de ces amplitude partielles intégrés sur tout g_{ev} (avec la mesure de Haar associé à $SL(2,\mathbb{C})$ fixée de jauge à chaque sommet v, voir le corps de la thèse) et sommé sur tout spin $j_f \in \frac{1}{2}\mathbb{N}$ associé à une face ne contenant aucun lien de bord (c'est à dire telle que il n'y existe aucun $e \in f$ tel que $e \subset \Gamma \sqcup \Gamma'$) :

$$Z_{\mathrm{EPRL}}^{\Delta^*} = \sum_{\{j_f,\, f \not\subset \partial\Delta^*\}} (2j_f+1) \prod_{(ev)\not\subset\partial\Delta^*} \int_{SL(2,\mathbb{C})} dg_{ev} \prod_f w_f^{\mathrm{EPRL}}(j_f;\{g_{ev}\})\,. \tag{ix}$$

Une propriété remarquable et non triviale du modèle EPRL (un résultat dû au groupe de Nottingham dirigé par John Barrett), c'est que dans la limite de spin très larges l'amplitude dynamique d'un de ses sommets (pentavalent) est supprimé à moins que son état de bord approche celui d'un quadri-simplex « semi-classique » et dans ce cas l'amplitude est dominée par deux branches caractérisées par une phase égale à plus, ou moins respectivement, l'action de Regge du quadri-simplex classique associé. Ce qui rend ce résultat prometteur c'est le fait que l'action de Regge est connue pour être une discrétisation fiable de l'action de Einstein-Hilbert sur une triangulation donnée.

DIVERGENCES

Cette définition du modèle EPRL n'est pas garanti être mathématiquement solide : en effet deux types de divergences sont présents dans cet formulation.

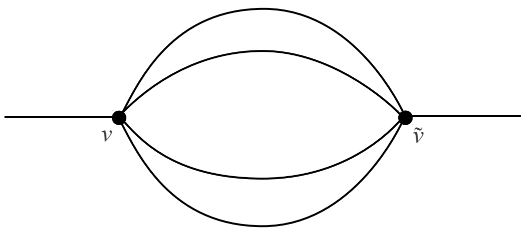

Figure i: Une représentation de l'intérieur de la mousse de spin dite melonique. Elle contient deux sommets internes, six liens internes et six faces internes. Elle a aussi bien deux sommets et quatre faces de bord.

Le premier type de divergence origine dans la présence d'une symétrie de jauge non-compacte et non fixée. En effet, à chaque sommet de $\Delta^* \setminus \partial\Delta^*$ une des intégrations sur les $g_{ev} \in SL(2,\mathbb{C})$ peut être, de manière évidente, toujours réabsorbée dans les autres (voir équation viii). Comme a été montré par deux des auteurs du modèle, cette redondance est aisément curée en éliminant une intégration par sommet de $\Delta^* \setminus \partial\Delta^*$.

Le deuxième type de divergence est beaucoup plus subtil et difficile à la fois à contrôler et à interpréter. Il a son origine dans la somme de équation ix. Plus précisément, ce type de divergence peut-être présent que si le deux-complexe Δ^* est tel que cette somme est non bornée pour au moins une des faces de Δ^*. Ceci est le cas chaque fois que Δ^* contient une « bulle », qui n'est rien d'autre qu'une généralisation (à une dimension plus haute) des boucles des diagrammes de Feynman. Comme telles, C'est sur l'étude de ce type de divergences que ma thèse porte.

En particulier, comme les calculs des amplitude de mousse de spin sont plutôt compliqués et aucune technique standard existe pour leur évaluation, je me suis concentré sur la plus simple des mousses divergentes qui garde au même temps une claire interprétation géométrique. Celle-ci c'est la mousse de spin dite « melonique », qui est schématiquement représentée en figure i.

le melon dans la modèle eprl

Le melon peut facilement être interprété comme la première correction radiative au « collage » de deux quanta d'espace-temps, c'est à dire de deux sommets internes à la mousse de spin. Le processus physique qu'il représente c'est celui d'un quantum d'espace qui se sépare virtuellement en quatre avant de s'identifier avec un autre quantum d'espace. Ceci c'est un processus en tout analogue à la ceci-dite « auto-énergie » en théorie des champs quantique usuelle.

La divergence de la mousse melonique étant due à une somme sur les spins internes qui peuvent devenir arbitrairement larges, on peut se limiter à étudier l'amplitude de cette amplitude dans la limite où le spin internes sont très large par rapport à 1 et aussi par rapport au spins externes (fixés par l'état de bord, ou de manière équivalente par les état « *in* » et « *out* »). Cette limite rend l'étude de l'amplitude de la mousse melonique indépendante de son état de bord portant une simplification considérable du problème. En plus, le fait de considérer une limite de grands spins rend possible l'utilisation des techniques semi-classique discuté ci-dessus après équation ix.

En effet, on se retrouve à considérer le modèle EPRL sur la mousse « réduite » suivante (figure ii):

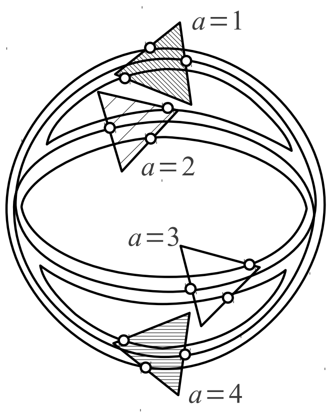

Figure ii: Les faces internes de la mousse de spin melonique. Elles forment une structure dual à une trois-sphère triangulée par deux tétraèdres aux faces identifiées. Les triangles représentent les faces de ces tétraèdres.

Le fait de considérer que le faces internes de la mousse réduit d'une certaine manière la dimensionnalité du problème : les structures qui peuvent être traitées de manière semi-classique sont maintenant deux tétraèdres (un par sommet) dont les faces sont identifiées les unes avec les autres de manière à former une triangulation d'une trois sphère.

Selon les valeurs des spins, dépendant du fait qu'ils respectent ou pas certains inégalités géométriques, la géométrie dominante de ces tétraèdres peut être soit Euclidienne, soit Lorentzienne, ou même dégénérée quand les sommets se trouvent dans de sous-espaces de $\mathbb{R}^{3,1}$ de dimension plus petite que trois. De toute manière la géométrie dominante n'étant déterminée que par les valeurs des spins, la géométrie de deux tétraèdre doit nécessairement être la même pour les deux tétraèdre, et la somme dans l'équation ix résulte naturellement séparée en secteurs associés à ces géométries. Enfin, chaque géométrie contribue avec deux branches d'orientation opposée à chaque sommet.

Dans cette thèse je néglige le secteur dégénéré, pour lequel les technique développées ne fonctionnent pas.

Pour ce qui concerne les autres secteurs, on peut argumenter que la somme est dominée par les géométries caractérisées par des orientations opposées aux deux sommets. Ceci étant le cas, car les deux orientations opposées correspondent à la même action de Regge, mais avec deux signes opposées. Ce qui fait si qu'aucune phase rapidement oscillante soit présente dans la somme en la supprimant.

En outre, les secteurs Euclidien et Lorentzien se distinguent par des différentes relations qu'ils imposent entre les états « *in* » et « *out* ». En particulier, que dans le secteur Euclidien la relation entre ces état ne dépend pas de la valeur particulière de spins interne (au moins dans la limite où ces spins sont grands) évitant ainsi tout phénomène d'interférence lorsque la somme est accompli. Du coup, ce fait nous permet d'estimer l'amplitude de la mousse melonique (pourvu bien sûr que les secteurs dégénéré et Lorentzien soient tout à fait supprimés) :

$$Z^{M}_{EPRL,\Lambda}(j_a, m_a, \tilde{m}_a) \sim \\ \sim \log\left(\frac{\Lambda}{\bar{j}}\right) \int_{SL(2,\mathbb{C})} dK d\tilde{K} \prod_a \langle j_a; m_a | Y_\gamma{}^\dagger K Y_\gamma Y_\gamma{}^\dagger \tilde{K}^{-1} Y_\gamma | j_a; \tilde{m}_a \rangle \,, \tag{x}$$

où Λ est un *cut-off* sur les spins (nécessaire pour régulariser une somme autrement divergente), $\bar{j}$ est l'ordre de grandeur des spins de bord, qui valent $\{j_a\}_{a=1,\dots,4}$ et dont la valeur doit être beaucoup plus petits que celle de Λ ; enfin, les $\{m_a\}_{a=1,\dots,4}$ labellisent l'état de bord dans la base des réseaux de spin.

Donc, on s'aperçoit que la mousse melonique se réduit - dans cette approximation et à une facteur divergent près - à l'introduction de deux sommets bivalents le long du liens liant les deux sommets de la mousse. Ceci est le premier résultat de cette thèse. Sont discussion détaillée est le sujet du chapitre 5.

CORRECTION MELONIQUE AU PROPAGATEUR DU GRAVITON DANS LE MODÈLE EPRL

Au chapitre 6 de cette thèse, je cherche d'appliquer le résultat que je vient de discuter à un observable physique de la théorie. L'observable choisi est le propagateur du graviton calculé pour la première fois dans le modèle EPRL par le groupe de Marseille il y a à peu-près cinq ans. Après avoir fait un résumé assez détaillé de la construction de ce observable dans l'approximation d'un seul sommet d'interaction, je introduit dans la mousse de spin originelle un collage corrigée par une insertion melonique.

Le calcul du graviton se réduisant à un calcul des corrélations semi-classique propagées par la mousse de spin comme s'il se agissait d'une théorie de champs sur réseau (le réseau étant la mousse de spin elle même), le fait d'introduire le melon va corriger le résultat originel par la présence de nouveaux degrés de liberté.

Malheureusement, le calcul de ces corrections nécessite l'inversion d'une matrice (l'Hessian de l'action de la théorie sur réseau calculée sur la solution des équations du mouvement) très large et compliquée, ce qui n'est pas faisable. De toute manière, même dans le travail originel, on s'était rendu compte que le résultat attendu n'est obtenu que dans la limite de petit paramètre de Barbero-Immirzi γ. On peut alors montrer que les corrections melonique disparaissent dans cette même limite à leur ordre dominant. Si ça peut être considéré d'un coté comme un bon résultat car les corrections radiative ne gâchent pas les résultat habituel, de l'autre coté de cette manière très peut vient effectivement appris sur les corrections radiative eux même.

DISCUSSION

Enfin, la thèse se conclue avec une discussion des résultats et de leur possibles interprétations. Entre les autres, je discute l'interprétation qui veut ces divergences comme dues à la présence d'une symétrie par difféomorphismes pas complètement fixé de jauge ; je discute aussi leur relation avec le problème de la platitude, qui est considéré gâcher les propriétés semi-classique du modèle EPRL. En suite, je considère la question de l'apparence de considérations semi-classique dans le calcul de phénomènes profondément quantique comme les divergences radiatives. Enfin, je considère l'hypothèse qu'on puisse interpréter le modèle régularisé par un *cut-off* Λ comme un modèle jouet d'une version du modèle EPRL où la constante cosmologique n'est pas nulle. En effet, en analogie avec la gravité quantique trois-dimensionnelle et le modèle de Turaev-Viro, on s'attend que une constante cosmologique pose une limite supérieure aux représentations de $SU(2)$ admises. Une manière heuristique de comprendre ça, est celle de penser la constante cosmologique comme imposant la présence d'un horizon cosmique qui à son tour pose une limite aux distance, et donc aux aires codées par les spins, admises.

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Marseille
June 30, 2014

Because it is there.

— George Herbert Leigh Mallory, 1923

The motivation for the general program outlined is, of course, a desire ultimately to attack the problem of the role played by gravitation in the quantum domain. No apology will be made for this motivation, although, needless to say, recent experiments have nothing to do with it! In the author's opinion it is sufficient that the problem is *there*, like the alpinist's mountain. Beyond that, however, the historical development of physics teaches a suggestive lesson in this connection, namely, that the existence of any fundamental structure which is far from having been pushed to its logical mathematical conclusions is a situation which may have great potentialities.

— Bryce Seligan DeWitt, 1957

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$(-+++)$ Lorentzian signature in my conventions

$:=$ Defines a symbol on the left-hand-side in terms of the expression on the right-hand-side

$\equiv$ Identifies two expressions term by term

$\doteq$ The abstract quantity on the left-hand-side is realized by a concrete object on the right-hand-side within a particular representation if the abstract space

$\stackrel{!}{=}$ Stands for an equality which must be required, and is not identically satisfied. I also use it in the context of the constraint algebra instead of the more common $\approx$ symbol

$\approx$ Stands for an approximate equality between the two sides (usually up to higher order corrections in some small parameter)

$\sim$ Stands for a qualitative equality (possibly up to proportionality factors of order 1 and higher order terms), or for an order-of-magnitude estimation of an expression

$\succeq$ Stands for a partial order relation, the left-hand-side coming "after" the right-hand-side

c Speed of light, $c \approx 3 \times 10^8$ m s^{-1}. Except when explicitly stated it is set to$c = 1$ by a proper choice of units

κ Reduced Newton constant, equal to $8\pi G_N c^{-4} \approx 2 \times 10^{-43}$ m^{-1}kg^{-1}s^2

a_{Pl} Planck area, equal to $\hbar\kappa c \approx 7 \times 10^{-69}$ m^2

Λ_{CC} Cosmological constant, of the order of 10^{-52}m$^{-2} \approx 10^{-120} {a_{Pl}}^{-1}$

γ The Barbero-Immirzi parameter. *A priori* an element of $\mathbb{C}$, it will be always considered to be real except when differently specified. Because of an unfortunate conventional choice, in most formulas (notably in the definition of the Y_γ-map) $-\gamma_{\text{here}} = \gamma_{\text{literature}}$. I apologize with the reader for this inconvenience

$\mathcal{M}$ A manifold

Δ, Δ^* A triangulation of $\mathcal{M}$, and its dual

$\Gamma := \partial\Delta^*$ The boundary graph

v, e, f Vertices, edges, and faces of Δ^*

n, ℓ Nodes and links of Γ

$s(\ell), t(\ell)$ Soruce and targe of the link ℓ, respectively

$\mu, \nu, \rho, \sigma, \ldots$ Tangential spacetime abstract indices, they range from 0 to the number of spacial dimensions

$a, b, c, d, \ldots$ Tangential space abstract indices, they range from 1 to the number of spacial dimensions

$I, J, K, L, \ldots$ Internal spacetime indices, they range from 0 to the number of spacial dimensions

$i, j, k, l, \ldots$ Internal space indices, they range from 1to the number of spacial dimensions

V^j Vector space carrying the spin-j $SU(2)$ irreducible unitary representation

$V^{\rho,k}$ Vector space carrying the irreducible unitary representation of $SL(2,\mathbb{C})$, labelled by the indices $(\rho, k) \in \mathbb{R} \times \mathbb{Z}$

$d_j := \dim V^j = 2j + 1$ Dimension of the spin-j $SU(2)$ irreducible unitary representation

EPRL model Engle-Pereira-Rovelli-Livine model, from the authors of [89]

Y_γ The so-called "wye" map, the central object of the EPRL model, $Y_\gamma : V^j \to V^{-\gamma j, j}$, $|j; m\rangle \mapsto |-\gamma j, j; j, m\rangle$

A NOT-TOO-TECHNICAL INTRODUCTION

The problem of quantum gravity is "out there" since more than eighty years, and is still waiting for a complete resolution. It stems from the simple observation that the two pillar theories of our physical knowledge are simply incompatible, at least in the form they are usually formulated. The first of these two pillars is quantum mechanics, the theory known to describe with amazing precision all matter and fields, which become somehow granular at the tiniest scales. The second is general relativity, the theory describing the dynamics of spacetime itself, which at the same time pushes and pulls the matter living within it, and is shaped by its flow. Also the theory of general relativity has been verified to a good level of precision at the cosmological and astrophysical level (and it is even necessary for the correct functioning of the everyday technology of the global position system). What is the trouble, then? Quantum mechanics is built upon the notion of unitary implementation of global spacetime symmetries; in particular, the symmetry under translation in time leads to the conservation of energy and hence to the Hamiltonian formulation of the theory. On the other side, within general relativity there is no preferred global notion of "time flow" and any observer perceives time and space differently according to her vicinity to massive objects and to her state of motion. Hence the *conundrum*. These state of facts, however, still does not explain why there is any need of seeking a quantum theory of gravity: discarding our attraction towards a unique set of funding principles which all laws of nature must conform with, is there any other reason why one cannot be satisfied more simply by a theory of quantum fields evolving on a classical spacetime? Writing such a theory is already a difficult task, nonetheless, it is known that it will be not enough. Indeed, how should spacetime respond to the presence of strong quantum fields in some superposed, or highly fluctuating, quantum state? This is a way of formulating the problem of quantum gravity, and the reason why it is necessary to look for a formulation of the theory where spacetime itself undergoes the rules of quantum mechanics, or *vice versa* to look for a quantum mechanical theory respecting all the local symmetries of general relativity. Some tentative models for such a theory exist, and follow many different approaches. I focused on one of them, called spinfoam quantum gravity. It is a covariant, path-integral-like version of loop quantum gravity, in turn giving a fundamental status to some connection variables (the Ashtekar-Barbero connection), rather than to the spacetime metric as it is more commonly done. The interest behind this choice is a formulation of general relativity much closer to that of Yang-Mills gauge theories, describing in particular the force fields present in the standard model of particle physics. More specifically, in this thesis, I study the role of radiative corrections in the spinfoam approach to quantum gravity, by focusing on the appropriate analogue of the self-energy graph. Calculating spinfoam amplitudes is not a straightforward task, and cannot be done exactly within the present models and with the present technology. Numerical analysis is often prohibitive in the case of interesting calculations. Hence, the interest of developing analytical approximation methods which put to the forefront the deep geometrical content of the theory. The interest in calculating radiative corrections is above all that of checking the self-consistency of the model I am dealing with, as well as gaining a better comprehension of the regime in which many spacetime-quanta interact among them, that is a regime which is still poorly understood.

Part I

SPINFOAM QUANTUM GRAVITY

In this first part of the thesis I will briefly review the motivations and the construction of the Engle-Pereira-Rovelli-Livine model of quantum gravity. I start by reviewing different action principles for general relativity: from the classical Einstein-Hilbert one to the Holst-Plebański one, which formulates general relativity as a constrained topological theory of the BF type and is at the basis of the spinfoam models of quantum gravity. I continue by discussing the canonical loop quantization of general relativity, and how it connects to spinfoams. Then, I review the classical BF theory and its spinfoam quantization, which relates directly to three-dimensional quantum gravity and is the starting point of the Engle-Pereira-Rovelli-Livine construction. Such a construction is the last subject of this part.

1 ACTION PRINCIPLES FOR GENERAL RELATIVITY

In its original framework [83, 84, 134, 219], general relativity is - mathematically speaking - a dynamical theory for the Lorentzian metric field $g_{\mu\nu}$ of a given differentiable manifold $\mathcal{M}$. It can be defined via the celebrated Einstein-Hilbert action

$$S_{EH}[g_{\mu\nu}] := \frac{1}{2\kappa}\int_{\mathcal{M}} dx^4\sqrt{-\det g}(R - 2\Lambda_{CC})\,, \tag{1.1}$$

which is the covariant integration of the Ricci scalar curvature $R = R[g]$, supplemented by a cosmological term, proportional to the cosmological constant Λ_{CC}. $\kappa = 8\pi G$ is reduced Newton's constant. Despite the undeniable conceptual elegance of this framework, it has the draw back to be non-polynomial in the fundamental field $g_{\mu\nu}$. Moreover, another drawback of this formulation is given by the difficulties one encounters when trying to couple fermions to the metric gravitational field.

1.1 PALATINI-CARTAN FORMULATIONL

A solution to the difficulties encountered when coupling fermions was found by Hermann Weyl already in 1929. In a famous paper [220], he proposed to use the local orthonormal frames to describe the gravitational field, instead of the metric itself. This allowed him to describe the fermionic matter fields directly in their local inertial frame, where Dirac's special relativistic formalism is available. This formulation has the bonus to put forward in a beautiful way Einstein's equivalence principle. Moreover, once integrated with ideas of Élie Cartan's on the moving frame (*repère mobile*) method [63] and differential forms, as well as ideas of Attilio Palatini's on the first order formulation of gravity [174] (i.e. where metric and connections are regarded as independent variables), also the first problem is solved. Indeed, the action equation 1.1 is equivalent up to some subtleties to

$$S_{PC}[e,\omega] := -\frac{1}{2\kappa}\int_{\mathcal{M}} \frac{1}{2}\epsilon_{IJKL}e^I\wedge e^J\wedge F^{KL}[\omega] - \frac{\Lambda_{CC}}{12}\epsilon_{IJKL}e^I\wedge e^J\wedge e^K\wedge e^L\,. \tag{1.2}$$

The letters "PC" stand for Palatini-Cartan, and this is the name that will used for this action. It is interesting to notice that Cartan considered such a generalization of general relativity from purely mathematical considerations as early as 1922.

From now on, I will neglect the cosmological term, except where explicitly stated.

The cotetrad one-form $e^I \equiv e^I_\mu dx^\mu$ represents Weyl's orthogonal coframe or, equivalently, Cartan's moving coframe. Its defining relation to the metric is

$$g_{\mu\nu} = \eta_{IJ}e^I_\mu e^J_\nu\,, \tag{1.3}$$

where $\eta_{IJ} := \mathrm{diag}(-1,+1,+1,+1)$ is the four-dimensional Minkowski metric. $I, J, \dots$ are internal Lorentzian indices, while $\mu,\nu,\dots$ are abstract space-time indices. The tetrad vector field $e^\mu_I\partial_\mu$ has components such that

Note, e^μ_I is defined only when e^I_μ is invertible as a 4×4 matrix.

$$e^\mu_I e^J_\mu = \delta^I_J \quad\text{and}\quad e^\mu_I e^I_\nu = \delta^\mu_\nu\,. \tag{1.4}$$

It is easily realised that internal and spacetime indices are raised and lowered by the Minkowski metric and the spacetime metric, respectively. Notice that in my notation $e_I := \eta_{IJ} e^J$ is a one-form.

The spin connection one-form $\omega^{IJ} \equiv \omega^{IJ}_\mu dx^\mu$ is what carries information about parallel transportation within this framework. It is antisymmetric in the internal indices $\omega^{IJ} + \omega^{JI} = 0$. Its curvature two-form is

$$F^{IJ}[\omega] := d\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ} \tag{1.5}$$

The spin connection is related to the affine connection $\Gamma^\mu_{\rho\sigma}$ by a so-called "dressing" transformation

$$\Gamma^\mu_{\rho\sigma} = e^\mu_I \left(\omega^I{}_{J\rho} e^J_\sigma + \partial_\rho e^I_\sigma \right) . \tag{1.6}$$

The condition of being antisymmetric in the internal indices for ω, translates into a compatibility condition of the affine connection with the metric (metricity): $\nabla_\mu g_{\rho\sigma} = 0$, the nabla indicating covariant derivatives. It is important to stress that the spin connection is looked at as a variable dynamically independent from the tetrad. In particular, the affine connection equation 1.6 reduces to the usual Levi-Civita one only on shell and in the vacuum, i.e. in absence of spin sources [143**?** , 210]. Indeed, provided that the cotetrad e^I_μ is invertible as a matrix, the vacuum equation of motion of the spin connection is a torsionless condition for the latter:

$$T^I := d_\omega e^I = de^I + \omega^I{}_J \wedge e^J \stackrel{!}{=} 0 . \tag{1.7}$$

Equation 1.6 shows quite explicitly that this condition is equivalent to the symmetry of the affine connection under exchange of its lower indices $\Gamma^\mu_{[\rho\sigma]} \stackrel{!}{=} 0$. It is well known that torsionfreeness and metricity uniquely determine the Levi-Civita connection needed for the definition of the Einstein-Hilbert action.

The relation between the curvature two-form $F^{IJ}[\omega]$ and the Riemann tensor $R^{\mu\nu}{}_{\rho\sigma}$ is

$$R^{\mu\nu}{}_{\rho\sigma} = e^\mu_I e^\nu_J F^{IJ}_{\rho\sigma} . \tag{1.8}$$

A last simple remark about the comparison of the Einstein-Hilbert and Palatini-Cartan actions of general relativity is crucial for what follows in the next chapters. Even when the spin connection is taken to be torsionless, the two actions may differ from one another by a sign. Naively speaking, the same "trick" which helps making the action polynomial in the fundamental fields, that is considering as such the "square-root" e^I_μ of the metric, instead of the metric itself, turns the factor $\sqrt{-\det g} \equiv |\det e|$ which is always positive, into $\det e$, which has no *a priori* fixed sign. This is not an issue at the classical level: since the tetrad (and hence the metric) is asked to be invertible everywhere, the sign of $\det e$ is the same at every point of the manifold and may be required to be in the positive sector by hand (a choice which would be relevant only in presence of matter). However, at the quantum level things are subtler: the tetrad field is now allowed to fluctuate and *a priori* nothing forbids it to tunnel from one sector to the other. This means that in the quantum theory the spacetime can, and in some models actually does, fluctuate from an orientation to the opposite one at every point! How does it come that this fact was not present in the Einstein-Hilbert formulation? Simply, that formulation is completely blind to the orientation of the underlying manifold, and can indeed be applied even to non-orientable ones.

Whether such an orientation-fluctuation property of a theory of a quantum spacetime is desirable or not is currently debated [88, 81, 204]. Personally I take note of the fact that such a feature is *necessary* for the consistency of the quantum dynamics of gravity-related topological theories, such as BF-theory, and postpone the verdict to the time when we will be able to say something about how a classical spacetime emerges from the quantum dynamics. In any case, in models such as the one discussed in this thesis, I do not see any mechanism by which the information about the orientation can be propagated from one spacetime quantum to the next. I will come back to these issues at the end of sections 3.3 and 4.3.

1.2 ASHTEKAR'S NEW VARIABLES

In 1986, Abhay Ashtekar made a breakthrough [12, 13] towards a dramatic simplification of the Hamiltonian approach to general relativity. In this work, he exhibited a new set of spinorial canonical variables, known as Ashtekar variables, such that their constraint algebra is polynomial in the fundamental fields [1] and is found to close provided an appropriate operator ordering is chosen. Moreover, in this formulation, general relativity has the same kinematical phase space as a Yang-Mills theory, opening the way to new hopes for quantization.

Soon afterwards, it was realized that at the basis of Ashtekar's new variable is the Palatini-Cartan formulation of gravity [207, 139, 117, 133]. In particular, in the second of these references it was shown that by considering only the (complex) self-dual part of the spin connection $\omega \mapsto \omega^+$ within the Palatini-Cartan action, one would recast general relativity in an equivalent form naturally described in terms of Ashtekar's variables. In other terms, this can be seen as the result of using the complex Jacobson-Smolin action

$$S_{JS}[e,\omega] := -\frac{1}{2\kappa}\int_{\mathcal{M}} \left[\star(e\wedge e)\right]^{IJ}\wedge F[\omega^+]_{IJ} \tag{1.9}$$

where we introduced the Hodge $\star$ operator on internal indices: $(\star X)^{IJ} := \frac{1}{2}\epsilon^{IJ}{}_{KL}X^{KL}$, and $\star^2 = -1$. Notice that the self-dual part of the spin connection reads $\omega^+ := \frac{1}{2}(1-i\star)\omega$, and that its curvature is nothing but the self-dual part of curvature of the original spin connection $F[\omega^+] := d\omega^+ + \omega^+\wedge\omega^+ \equiv \frac{1}{2}(1-i\star)F[\omega]$.

Ashtekar's reformulation triggered major advances in the quantization of general relativity: as soon as 1988, Ted Jacobson, Lee Smolin and Carlo Rovelli found a large class of solutions to all the constraints of quantum general relativity [2] [140, 197]. The central tool behind this success is the loop representation of the quantum gravitational field [198], based on an appropriate functional transform of the Wilson loops of the Ashtekar connection. Interestingly this was an independent rediscovery of ideas put forward by Rodolfo Gambini and collaborators, almost ten years earlier in the context of Yang-Mills theories. [3]

1.3 REAL ASHTEKAR-BARBERO VARIABLES

Despite many successes, work on the quantum theory of gravity in Ashtekar's variables stumbled against major difficulties related to the reality conditions necessary to recover the original theory from the complexified one. In 1995, Fernando

1. This is true up to a rescaling of the Hamiltonian constraint. See [214] for details.
2. See also the cited reference in the previous footnote.
3. For more on these earlier works, see references 339-341 of [215].

Barbero realized that such difficulties could be bypassed by using real variables in Ashtekar's phase space, known as Ashtekar-Barbero variables, provided one is ready to abandon the particularly simple form of the Hamiltonian constraint [35]. This realization solved various technical problems and hence opened the way to the mathematically rigorous foundations of loop quantum gravity by Abhay Ashtekar, Jerzy Lewandowski, Thomas Thiemann and others (see [215]).

Fernando Barbero presented a whole family of these real variables, which is parametrized by the so-called Barbero-Immirzi parameter $\gamma \in \mathbb{R} \setminus \{0\}$.[4] Choosing $\gamma = i$ reproduces Ashtekar's original self-dual variables leading to a particularly simple Hamiltonian constraint. Analogously, $\gamma = 1$ is the value which drastically simplifies the Hamiltonian constraint of Euclidean general relativity and can be seen as leading to "Wick rotated" variables. However, although in the classical theory all choices of the Barbero-Immirzi parameter are related by a canonical transformation of the phase-space variables, they are *not* related by unitary transformation within the quantum theory [203]. This was first realized by Giorgio Immirzi, who showed that different values of the this parameter lead to inequivalent spectra for the geometrical operators [137]. Most famously, the area spectrum of loop quantum gravity is

Euclidean general relativity, as opposed to Lorentzian, is the theory defined by equation 1.1, where $g_{\mu\nu}$ is positive defined

$$8\pi G\hbar\gamma\sqrt{j(j+1)} \quad \text{with} \quad j \in \frac{1}{2}\mathbb{N}\,. \tag{1.10}$$

As Ashtekar's self-dual variables stem from the Jacobson-Smolin action principle, so Ashtekar-Baerbero variables stem from the Holst action [135, 206]:

$$S_H[e,\omega] := -\frac{1}{2\kappa}\int_{\mathcal{M}} \left[\left(\star + \frac{1}{\gamma}\right)(e \wedge e)\right]^{IJ} \wedge F_{IJ}[\omega]\,. \tag{1.11}$$

Classically, the Holst action is completely equivalent in vacuum to the Palatini-Cartan one (provided the cotetetrad is non-degenerate). Indeed, the equations of motion of the connection in both cases impose its torsionfreeness, and then for a torsion free connection the Holst term in the action (that is $S_H - S_{PC}$) vanishes identically. In presence of matter things are more complicated, and a modification of the action is needed in order to keep its equations of motion equivalent to those in absence of the Holst term [158]. Notice that, in spite of the fact that the Holst term does not contribute to the equations of motion, it would be inexact to say it is a topological term, as the QCD θ-term.

In 2009, it was realized in a paper by Ghanashyam Date, Romesh Kaul, and Sandipan Sengupta [71], that the Hamiltonian analysis of the Holst action is completely equivalent to that obtained by adding to the Palatini-Cartan action a term proportional to the topological Nieh-Yan term. Furthermore, this new action turns out to be universal, i.e. it needs no modification in presence of matter. It reads

$$S_{DKS}[e,\omega] := S_{PC}[e,\omega] + \frac{\gamma^{-1}}{2\kappa}\mathrm{NY}[e,\omega]\,, \tag{1.12}$$

where

$$\mathrm{NY}[e,\omega] := \int_{\mathcal{M}} \left(T^I \wedge T_I - e^I \wedge e^J \wedge F_{IJ}[\omega]\right) = \int_{\mathcal{M}} d\left(T^I \wedge e_I\right) \tag{1.13}$$

with $T^I := d_\omega e^I$ the torsion of the connection (see equation 1.7). Shortly afterwords, the inclusion of the Euler and Pontryagin terms in the Palati-ni-Cartan action was also taken into account [141], obtaining similar results (fermion coupling was not

4. The letter β was used in the original literature.

considered, though). Notice that all three topological terms strongly break CP symmetry. Moreover, together with the cosmological term, they exhaust all possible terms compatible with diffeomorphism and Lorentz invariance generalizing pure gravity in the first order Palatini-Cartan formalism.

1.4 plebański's formulation and BF theories

In 1977, while studying a self-dual formalism for gravity, Jerzy Plebański proposed for the first time a reformulation of general relativity in terms of non-metric structures [182].[5] In the self-dual sector, the variables consisted in a two-form, a connection, and some Lagrange multipliers; analogous variables constituted the anti-self-dual sectors. The two sectors are in a first moment considered as independent, and the Lagrange multipliers require each fundamental two forms to originate from a metric structure. In a modern language these are called the "simplicity constraints". Finally, further constraints require these two metric structures to be one and the same. I.e., borrowing the title of a paper by Michael Reisenberger, they recover "general relativity from 'left-handed area = right-handed area' ".

Putting aside the issue of the self-dual and anti-self dual decomposition of the fields, which by the way resonates perfectly with the formulation in terms of original Ashtekar's variables, I will focus on the idea of formulating general relativity as a constrained theory of a two form. Hence, here is the Holst-Plebański action for the two form B, the connection ω, and the Lagrange multiplier φ:

$$S_{\mathrm{Ple}}[B,\omega,\varphi] := -\frac{1}{2\kappa}\int_{\mathcal{M}} \star B^{IJ}\wedge F_{IJ}[\omega] + \varphi_{IJKL}B^{IJ}\wedge B^{KL}\,. \tag{1.14}$$

The two form B^{IJ} is antisymmetric in its internal indices, and the Lagrange multiplier φ has the following symmetry properties:

$$\varphi_{IJKL} = \varphi_{[IJ]KL} = \varphi_{IJ[KL]} = \varphi_{KLIJ} \quad \text{and} \quad \epsilon^{IJKL}\varphi_{IJKL} = 0\,. \tag{1.15}$$

Thus, variation with respect to φ imposes 20 algebraic equations on the 36 components of B, which are

$$\epsilon^{\mu\nu\rho\sigma}B^{IJ}_{\mu\nu}B^{KL}_{\rho\sigma} \stackrel{!}{=} ||e||\epsilon^{IJKL} \tag{1.16}$$

where $||e||$ is defined by the contraction of the previous equation with ϵ^{IJKL}. The solutions to these equations, nowdays commonly called the simplicity constraints within the "loopy" community, are parametrized by 4×4 matrices e^I_μ, to be clearly interpreted as the cotetrads defining the metric:

$$(I\pm)\ B = \pm e\wedge e \quad \text{and} \quad (II\pm)\ B = \pm\star e\wedge e \tag{1.17}$$

Sector $(I\pm)$ corresponds to the gravitational one, while sector $(II\pm)$ has no propagating degrees of freedom (see discussion about the Holst action in the previous section). While the sign choice reflects the freedom to choose two distinguished (time) orientations, the existence of these two sectors is a direct consequence of the quadratic form of the constraints. I will come back on this issue in the next section.

The Hamiltonian analysis of this action was extensively studied in [58].

5. See the first section of [145] for a brief historical overview of self-dual gravity.

To make contact with the previous section and the Ashtekar-Barbero variables, it is necessary to include a Holst term in the action. [6] The Holst-Plebański action then reads

$$S_{HP}[B,\omega,\varphi] := -\frac{1}{2\kappa}\int_{\mathcal{M}} \left[\left(\star+\frac{1}{\gamma}\right)B\right]^{IJ} \wedge F_{IJ}[\omega] + \varphi_{IJKL} B^{IJ} \wedge B^{KL}\,. \tag{1.18}$$

The Hamiltonian analysis of this action can be found in the recent review by Alejandro Perez [178]. Before analysing the simplicity constraints in more detail, I want to stop for a moment to discuss an important feature of Plebański's formulation, which will be crucial for the formulation of the spinfoam models of quantum gravity.

Ignoring the simplicity constraints, one is left with a four-dimensional field theory. Such a theory is the representative in four-dimensions of a class of topological field theories [20], called BF theories. These were introduced and studied in their classical and quantum forms by Gary Horowitz in 1989 [136], with the goal of generalizing Witten's result on the quantization of three dimensional gravity [223] to any dimension. These are gauge theories with Lie group G living on an n-dimensional manifold, where ω is the $\mathfrak{g}$-valued connection, and B is an $(n-2)$-form with values in the dual of the Lie algebra $\mathfrak{g}^*$. Such theories have a local symmetry group which is so large that no local degrees of freedom are left, and their partition functions are topological invariants of the manifold on which they are defined.

In Plebański's formulation (with or without the Holst term), general relativity is therefore expressed as nothing else than a constrained four dimensional BF-theory with the Lorentz group as the gauge group G. The constraints play the crucial role of unfreezing the local degrees of freedom proper to four-dimensional gravity. See e.g. [110] for the relation between BF theories and gravity at the classical level. [7]

As I will discuss in Chapter 3, BF theories can be quantized exactly and in particular admit a path integral representation which is at the basis of the spinfoam approach to quantum gravity.

1.5 HOLST-PLEBAŃSKI ACTION AND THE SIMPLICITY CONSTRAINTS

As I discussed in the previous section, the simplicity constraints (equation 1.16) are the key ingredient in any formulation of gravity which seeks to start from the phase space of BF theory. Unluckily, from an Hamiltonian point of view their structure is quite involved. I refer the reader to [58] and [178] for the detailed Hamiltonian analysis of the Plebański and Holst-plebański actions, respectively. Here, I will only summarize the main results for the latter theory.

After a 3+1 decomposition of the Holst-Plebański action (equation 1.18) done by choosing a time-like internal direction n^I in the so-called "time gauge", in which

6. Note that the Nieh-Yan formalism is not available within the Plebański formulation, since the torsion cannot be expressed by means of the B fields

7. In this paper it is also discussed a second way to relate gravity to BF theory, which is known as the MacDowell-Mansouri formulation [154]. In this case the tetrad does not arise from a constrained B field, and is directly encoded in a part of the connection. See also [222] for the relation of this theory to Cartan geometries, and [111, 101] for relations with quantum gravity.

$n^I = (1,0,0,0)$, it is easy to realize that the canonical conjugate variables appearing in the action are the 18 pairs

$$\left(\omega_a^{IJ}, \quad \Pi_{IJ}^a := -\frac{1}{2\kappa}\epsilon^{abc}\eta_{IK}\eta_{JL}\left[(\star+\gamma^{-1})B\right]_{bc}^{KL}\right), \tag{1.19}$$

with $a, b, \dots$ are spatial space-time indices. These canonical variables can be re-parametrized as follows

$$\left({}^{\pm}\mathbb{A}_a^i := \frac{1}{2}\epsilon^i{}_{jk}\omega_a^{jk} \pm \gamma\omega_a^{i0}, \quad \mathbb{P}(\gamma;\pm)_i^a := \frac{1}{4}\epsilon_i{}^{jk}\Pi_{jk}^a \pm \frac{1}{2\gamma}\Pi_{0i}^a\right). \tag{1.20}$$

with $i, j, \dots \in \{1,2,3\}$ space-like internal indices.

The primary constraints appearing in the Hamiltonian are subdivided in two sets. The first one is associated to the Lorentz Gauß constraint. Due to the choice of an internal direction n^I, it results broken up into two pieces: a boost part and an $SO(3)$ Gauß constraint (the same type of constraint that would be present in a $SU(2)$ Yang-Mills theory). The second set of constraints is equivalent to the simplicity constraints of equation 1.16. The solution of such constraints can be parametrized in terms of the 9 components of $\mathbb{P}(\gamma,+)$, plus 4 extra parameters which will become the lapse and shift vector, and 3 parameters involved in the choice of the internal direction n^I. Altogether, these are the 16 parameters appearing in the cotetrad e_μ^I. Note that the solution of the simplicity constraints given in this form breaks the original Lorentz invariance, since it reposes on the time-gauge fixing. Also, since $\mathbb{P}(\gamma,-)$ is independent of $\mathbb{P}(\gamma,+)$, the solution requires

$$\mathbb{P}(\gamma,-)_i^a := \frac{1}{4}\epsilon_i{}^{jk}\Pi_{jk}^a - \frac{1}{2\gamma}\Pi_{0i}^a \stackrel{!}{=} 0\,. \tag{1.21}$$

These equations are called the *linear simplicity constraints*, and are the equations on which the construction of the so-called new spinfoam models relies (see Chapter 4, and [103, 89, 115]). As it will be discussed in the next chapter, in the context of spinfoam model building, such a linear form of the simplicity constraints is more advantageous with respect to the quadratic one (equation 1.16), since it automatically selects the gravitational sector out of Plebański's theory (equation 1.17). This was first realized by Laurent Freidel and Kirill Krasnov in [103].

Unfortunately, the Hamiltonian analysis requires still more efforts. Indeed, the stability of the primary constraints under the dynamical evolution gives rise to secondary constraints. Together with the boost part of the Gauß constraint, they essentially require the torsionfreeness of the three dimensional part of the connection ω_a^{ij} with respect to the cotriad e_a^i arisen from the solution of the simplicity constraints:

$$\frac{1}{2}({}^{+}\mathbb{A} + {}^{-}\mathbb{A})_a^i - \Gamma_a^i[e] \stackrel{!}{=} 0\,, \tag{1.22}$$

where Γ_a^i is the unique solution of the torsion-freeness equation

$$\partial_{[a}e_{b]}^i + \epsilon^i{}_{jk}\Gamma_{[a}^j e_{b]}^k = 0\,. \tag{1.23}$$

The secondary constraint of equation 1.22 is actually second class, and is canonically conjugated to equation 1.21. Then, solving these constraints at the level of the action, one finds the usual loop quantum gravity phase space variables and constraints. That is one finds the Ashtekar-Barbero connection and the densitized triad as canonically conjugate variables:

$$\left(A_a^i := {}^{+}\mathbb{A}_a^i \equiv \Gamma_a^i[e] + \gamma K_a^i \quad E_i^a := \kappa\gamma\mathbb{P}(\gamma;+)_a^i\right), \tag{1.24}$$

with

$$\left\{A^i_a(x), E^b_j(y)\right\} = \kappa\gamma\delta^b_a\delta^i_j \quad \text{and} \quad \left\{A^i_a(x), A^j_b(y)\right\} = 0 = \left\{E^a_i(x), E^b_j(y)\right\} . \tag{1.25}$$

Where I introduced

$$K^i_a := \frac{1}{2}({}^+\mathbb{A} - {}^-\mathbb{A})^i_a \quad \text{and} \quad e^i_a := \frac{1}{2}|\det E|^{-1/2}\epsilon_{abc}\epsilon^{ijk}E^b_j E^c_k . \tag{1.26}$$

And the constraints are

$$G^i := \partial_a E^a_i + \epsilon_{ij}{}^k A^j_a E^a_k \stackrel{!}{=} 0 \qquad \text{(Gauß)} \tag{1.27a}$$

$$C_a := E^a_i {}^{(\gamma)}F^i_{ab} - (1+\gamma^2)K^i_a G_i \stackrel{!}{=} 0 \qquad \text{(Vector)} \tag{1.27b}$$

$$H := \frac{E^a_i E^b_j}{\sqrt{|\det E|}}\left[\epsilon^{ij}{}_k {}^{(\gamma)}F^k_{ab} - 2(1+\gamma^2)K^i_{[a}K^j_{b]}\right] \stackrel{!}{=} 0 \qquad \text{(Scalar)} \tag{1.27c}$$

where ${}^{(\gamma)}F^i_{ab} := \partial_a A^i_b - \partial_b A^I_a + \epsilon^i{}_{jk}A^j_a A^k_b$ is the curvature of the Ashte-kar-Barbero connection. (In the previous expressions K^i_a must be interpreted as $A^i_a - \Gamma^i_a[E]$.)

Note how the scalar constraint gets simplified by the choice of Ashte-kar's original self-dual connection variables $\gamma = i$.

A last remark on the Ashtekar-Barbero-connection formulation of general relativity. Classically, the Barbero-Immirzi paramter can take any complex value, leading to Hamiltonian formulations which are completely equivalent to ADM general relativity (see e.g. [215]). However, for the time being, a consistent quantum formulation has only been constructed for real values of γ, the quantum imposition of the reality conditions being the stumbling block for the self-dual formulation. On the other side, from a purely geometrical point of view, Ashtekar's original (anti-)self-dual variables are privileged. Indeed, $\gamma = \pm i$ is the unique value of the Barbero-Immirizi parameter for which the spatial connection A^j_a admits a spacetime interpretation [208]. Even if this remark is only aesthetic in nature, it should, however, be kept in mind when dealing with a covariant approach of a spinfoam type (see also section 3.1.1), in particular whenever a bridge with the canonical theory is sought.

NOTE After completing the writing of this chapter I discovered the existence of an interesting paper reviewing different action principles for classical general relativity. This is [175].

2 FROM LOOP QUANTUM GRAVITY TO SPINFOAMS

2.1 KINEMATICAL HILBERT SPACE OF SPIN NETWORKS

Starting from the very title of Abhay Ashtekar's founding paper in 1986 [12], one of the main purposes in developing the formalism presented at the end of last chapter was that of tackling the quantization of the general theory of relativity. Progress in this direction did not need to wait: as soon as a connection formulation was available, the search of gauge-invariant background-independent solutions of the constraints naturally led Ted Jacobson and Lee Smolin to consider states in the form of Wilson loops of the Ashtekar connection [140].

This was no chance, in fact as stressed for example by Thomas Thiemann [214, 215] the most important feature in the construction of a background independent theory is using natural n-form variables such as a connection and (the dual of) its conjugate electric field, because they can be naturally integrated over n-dimensional submanifolds *without* introducing a metric.

In this spirit, let me introduce holonomies and fluxes as a new set of regularized variables. Within a three-dimensional manifold, to be thought as the initial-data surface, consider a set of oriented paths $\{\ell\}$ each starting at some $s(\ell)$ and ending at some $t(\ell)$, as well as a set of dual oriented surfaces $\{S_\ell\}$. Thanks to these structures, one can build the following set of regularized, smeared, variables out of the Ashtekar-Barbero connection A and its conjugated electric field E (equation 1.24):

$$h_\ell[A] \equiv h_{s(\ell)\to t(\ell)} := \mathbb{P}\exp\int_{s(\ell)}^{t(\ell)} A^i\tau_i \in SU(2) \qquad \text{(holonomy)} \tag{2.1a}$$

$$X_{S_\ell,p}[E,A] := h_{x_\ell\to s(\ell)}\left[\int_S h_{p(x)}(*E^i)(x)\tau_i h^{-1}_{p(x)}\right]h^{-1}_{x_\ell\to s(\ell)} \in \mathfrak{su}(2) \qquad \text{(flux)} \tag{2.1b}$$

where $\mathbb{P}\exp$ is the path-ordered exponential, $\tau^i := -\frac{i}{2}\sigma^i \in \mathfrak{su}(2)$ with σ^i being the Pauli matrices, $*E^i = \frac{1}{2}\epsilon_{abc}E^a_i dx^b \wedge dx^c$, and $\{p(x)\}$ is a set of paths contained within the surface S and going from $x \in S$ to the base point x_ℓ assumed to be the only intersection point of the path ℓ and its dual surface S_ℓ. See e.g. [100] for details on this construction. Note that this particular smearing was chosen to assure natural transformation properties of the new variables under $SU(2)$ gauge transformations:

$$g \triangleright h_\ell = g_{t(\ell)}h_\ell g^{-1}_{s(\ell)} \quad \text{and} \quad g \triangleright X_\ell = g_{s(\ell)}X_\ell g^{-1}_{s(\ell)} \tag{2.2}$$

Given the original Poisson brackets (equation 1.25), one can deduce the algebra fulfilled by these variables. The resulting closed algebra, known as the holonomy-flux algebra, is given by

$$\{h_\ell, h_{\ell'}\} = 0\,, \quad \{X^i_\ell, X^j_{\ell'}\} = \delta_{\ell,\ell'}\epsilon^{ij}{}_k X^k_\ell\,, \quad \text{and} \quad \{X^i_\ell, h_{\ell'}\} = -\delta_{\ell,\ell'}\tau^i h_\ell\,, \tag{2.3}$$

where $X^i := -2\mathrm{Tr}(X\tau^i)$. That is, upon smearing the phase space reduces to $(T^*SU(2))^L$, $L = \#\{\ell\}$ being the number of paths. Note that the fluxes are non-commutative because of the presence of the holonomies of the connection A in the definition of equation 2.1b. These holonomies were also necessary to have the simple transformation properties of equation 2.2. However, flux non-commutativity, though very natural, complicates the construction of the canonical quantum theory. This is why in this context it is most often chosen to work with so-called electric fluxes [215]:

$$E_S := \int_S *E^i . \tag{2.4}$$

Nevertheless, new mathematical tools appeared recently which paved the way toward a non-commutative flux-representation of loop quantum gravity [30]. These are based on a non-commutative Fourier transform mapping (commutative) functions defined on $SU(2)$ onto (non-commutative) functions defined on $\mathfrak{su}(2)$. To my knowledge, most of the results obtained thanks to this new representation in the context of quantum gravity have been within the group field theory setting (e.g. [32, 31, 33, 34]). However, see also [80].

In Schrödinger representation, any function of A acts as a c-number multiplication, and $\hat{X}^i_\ell \doteq -i\hbar\kappa\gamma \left.\frac{\delta}{\delta A^i(x)}\right|_{x=s(\ell)}$, i.e. it acts like a right-invariant vector field. This is equivalent to posing $\hat{X}^i \doteq -i\hbar\{\cdot\, X^i\}$.

The kinematical Hilbert space in the Schrödinger representation of the holonomy-flux algebra is that of square-integrable functions of the holonomies. To explicitly construct it, call Γ the one-complex (here also called simply "graph") composed by L (embedded) links (or paths) intersecting only at their endpoints $(s(\ell), t(\ell))$, and $d\mu_H$ the Haar measure on $SU(2)$. Then,

$$\mathcal{H}''_\Gamma := L_2[SU(2), d\mu_H]^L \tag{2.5}$$

An inspection of equation 2.2, makes clear that the space of $SU(2)$ gauge-invariant states is given by states $\psi_\Gamma(h_\ell) \in \mathcal{H}''_\Gamma$ such that

$$\psi_\Gamma(g_{t(\ell)} h_\ell g^{-1}_{s(\ell)}) = \psi_\Gamma(h_\ell) . \tag{2.6}$$

Since the group $SU(2)$ is compact and $\int d\mu_H = 1$, any such state can be obtained via group averaging at the nodes. Call the space of such states $\mathcal{H}'_\Gamma$. Symbolically,

$$\mathcal{H}'_\Gamma = L_2\left[SU(2)^L // SU(2)^N, d\mu_H\right] \tag{2.7}$$

where N indicates the number of nodes of Γ. This space can be shown to solve (a proper regularization of) the Gauß constraint (equation 1.27a), which at each node n reads

$$G_n := \sum_{\ell \ni n} \tilde{X}_\ell(n) \tag{2.8}$$

where

$$\tilde{X}_\ell(n) = \begin{cases} X_\ell & \text{if } n = s(\ell) \\ -h_\ell^{-1} X_\ell h_\ell & \text{if } n = t(\ell) \end{cases} . \tag{2.9}$$

The solution of the vector constraint (equation 1.27b) is more subtle, and I will not discuss it here. (See [215, 194, 95] for the details.) Roughly speaking, the final result of such an implementation is that of passing from embedded to abstract graphs labelling states and Hilbert spaces. Moreover, in my opinion, most of the ambiguities arising from the solution of the constraints coming from the continuum classical theory may well be irrelevant from the point of view of the fundamental theory which should be kept as simple as possible. Within this perspective, the ambiguities would be just the result of degenerate limiting procedures.

The resulting set of states, which are both gauge- and (spatial-)diffeo-morphism-invariant is that of spin networks. Call it $\mathcal{H}_\Gamma$. In the context of loop quantum gravity, spin network were first introduced by Carlo Rovelli and Lee Smolin in 1995 [201]. [1] More precisely spin networks are a basis of $\mathcal{H}_\Gamma$, obtained by "Fourier transforming" a general state $\psi_\Gamma(h_\ell)$ via Peter-Weyl's theorem into a superposition of functions of spins and intertwiners. See e.g. [194].

By "spins" it is meant the usual label $j \in \frac{1}{2}\mathbb{N}$ of an unitary irreducible representation of $SU(2)$, while "intertwiners" are invariant tensors in the tensor product of such representations. See section 3.3

A last step is needed to conclude the construction of the full kinematical Hilbert space of loop quantum gravity. The goal of this step is to get rid of any specific graph dependence, that is to consistently include all possible graphs. This becomes crucial as soon as one starts dealing with the scalar constraint. In fact, contrary to the Gauß and vector ones, when acting on spin network states it does not preserve their graph structure. It rather acts by attaching "loops" at their nodes [199]. Therefore, it is necessary to deal with a Hilbert space $\mathcal{H}$ which includes all possible graph structures. This can be achieved by means of a limiting procedure, which involves the natural inclusion relation between graphs. It is described in the remainder of this section.

First, observe that the natural inclusion relation induces a partial ordering among the graphs: $\Gamma \succeq \Gamma'$ if Γ contains Γ' as a subgraph, and given any two graphs Γ', Γ'' there always exists some graph Γ such that $\Gamma \succeq \Gamma', \Gamma''$. Then, observe that the same inclusion relation $\Gamma \succeq \Gamma'$ induces also an embedding between the corresponding spin network Hilbert spaces

$$\begin{aligned} p^*_{\Gamma,\Gamma'} : \quad & \mathcal{H}_{\Gamma'} && \hookrightarrow \quad \mathcal{H}_\Gamma \\ & \psi_{\Gamma'}(h_{\ell'}) && \mapsto \quad \psi_\Gamma(h_\ell) := \mathbf{1}(h_{\tilde{\ell}})\psi_{\Gamma'}(h_{\ell'}) \end{aligned} \tag{2.10}$$

by simply introducing a trivial dependence on the holonomies along the links in $\{\tilde{\ell}\} \in \Gamma \setminus \Gamma'$ (here, $\mathbf{1}(h_\ell)$ is the function identically equal to 1). Notice that this map composes nicely $p^*_{\Gamma,\Gamma'} \circ p^*_{\Gamma',\Gamma''} = p^*_{\Gamma,\Gamma''}$ $\forall \Gamma \succeq \Gamma' \succeq \Gamma''$, and preserves the scalar product. At this point, within the union $\cup_\Gamma \mathcal{H}_\Gamma$, first define the equivalence relation $\sim$

$$\psi_{\Gamma'} \sim \phi_{\Gamma''} \quad \Longleftrightarrow \quad \exists \Gamma \succeq \Gamma', \Gamma'' \quad \text{s.t.} \quad p^*_{\Gamma,\Gamma'}\psi_{\Gamma'} = p^*_{\Gamma,\Gamma''}\phi_{\Gamma''}\,, \tag{2.11}$$

and hence define the Ashtekar-Lewandowski kinematical Hilbert space as the projective limit of the kinematical graph Hilbert spaces [16]:

$$\mathcal{H} := \overline{\cup_\Gamma \mathcal{H}_\Gamma / \sim}\,, \tag{2.12}$$

where the closure is defined with respect to the scalar product

$$< \psi_{\Gamma'}, \phi_{\Gamma''} > := < p^*_{\Gamma,\Gamma'}\psi_{\Gamma'}, p^*_{\Gamma,\Gamma''}\phi_{\Gamma'} >_{\mathcal{H}_\Gamma} \;. \tag{2.13}$$

From a physical perspective, the advantage of this construction is that it makes the condition very natural that observables with support on a given graph can be calculated using any finer graph without changing the result. This is called the cylindrical consistency condition. Therefore, from this perspective, the choice of a graph corresponds to the choice of observables which depend only on certain *finite* degrees of freedom, rather than corresponding to a truncation of the theory. This was recently put in evidence by Bianca Dittrich, in her attempt to extend this notion to the spinfoam bulk [79, 81].

"Cylindrical" are called those functionals of the connection which depend only on its holonomies along a finite set of paths. Spin network states are cylyndrical functionals.

1. Spin networks were first introduced by Sir Roger Penrose in the Seventies. Despite the context was different from the present one, spin networks were already meant as a tool to describe some sort of quantum geometry of space [176, 177].

2.2 DISCRETENESS OF AREA AND VOLUME KINEMATIC SPECTRA

It is now the time to explain one of the key results of the quantization program. Starting from the flux operators acting on states in $\mathcal{H}_\Gamma$, one can build (kinematical) geometric operators, such as the area and volume operators [200, 18, 19, 14, 146, 46]. Such operators have the crucial properties of having a discrete spectrum, and in the case of the area operator such a spectrum is particularly simple and is known to have a finite gap. In order to construct the area operator, one has to recall the geometrical meaning of the flux variable E^a_i (equation 1.26) and hence properly regularize the following expression for the area of a surface S

$$A_S[E^a_i] = \int_S \sqrt{E^a_i E^b_j \delta^{ij} n_a n_b} d\sigma^1 d\sigma^2, \tag{2.14}$$

where (σ^1, σ^2) are arbitrary coordinates for the surface S, and $n_a = \frac{\partial x^b}{\partial \sigma^1}\frac{\partial x^c}{\partial \sigma^2}\epsilon_{abc}$ is its co-normal. This can be done and the result for a surface dual to a single link ℓ (and punctured by it non-tangentially) is

$$\hat{A}_\ell = \hbar \sqrt{\hat{X}^i_\ell \hat{X}^j_\ell \delta_{ij}}\,. \tag{2.15}$$

From this expression the spectrum can be immediately computed by using the fact that the algebra of the operators (h_ℓ, X_ℓ) is that of $T^*SU(2)$, and the result is

$$\mathrm{Spec}(A_\ell) = \left\{ a_{Pl}\,\gamma\sqrt{j(j+1)},\ j \in \frac{1}{2}\mathbb{N} \right\}, \tag{2.16}$$

where $a_{Pl} \equiv \hbar\kappa = 8\pi G\hbar$ is the Planck area.

2.3 THE SCALAR CONSTRAINT

At this point there is one single constraint left to be solved. This is the scalar, or Hamiltonian, constraint (equation 1.27c). Solving it would mean completely solving quantum gravity. No surprise this is no easy task, and no one has been able to do this yet. The approaches to its implementation on kinematical states can be split in two main categories: the canonical and covariant ones. In this thesis, I will focus on the covariant spinfoam approach. To be honest and more precise, given the present state of the art, spinfoams should be considered more as an independent approach to quantum gravity. Indeed, no precise map between the canonical construction and the present-day spinfoam models exist. The reason is simple: spinfoams are conceived to bypass some of the difficulties of the canonical implementation of the scalar constraint, and in order to do this they are built by completely different methods. Nevertheless, the final goal is to make the two approaches consistent with one another. [2]

In the context of loop quantum gravity, spinfoams were originally introduced by Michael Reisenberger and Carlo Rovelli in 1997, by means of a formal argument inspired to Feynman's path integral construction [187]. In synthesis, their argument can be stated as follows.

First define the so-called rigging map

$$\eta : \psi \mapsto \Psi := \delta(\hat{H})\psi := \int dt\, e^{-i\hat{H}t}\psi\,, \tag{2.17}$$

2. See e.g. [2, 216] for some recent work on this subject.

which formally implements the Hamiltonian constraint onto kinematical states. Then, using this map, define the scalar product between physical states which solve all the constraints:

$$< \Psi, \Psi' >_{\text{phys}} := < \psi, \delta(\hat{H})\psi' > = \int dt \ < \psi, e^{-i\hat{H}t}\psi' > \ . \tag{2.18}$$

Rigorously implementing these two steps would mean constructing the physical Hilbert space of full non-perturbative loop quantum gravity.

The spinfoam Ansatz can then be introduced as a tentative implementation of the right hand side of equation 2.18, by inserting, in a path-integral-like fashion, resolutions of the identities over spin-network states every ϵ-wide interval of the fictitious time t. From this perspective, spinfoams arise as a perturbative implementation of the canonical scalar constraint. The resulting construction is very similar to a sum over Feynman diagrams, with specific weights associated to vertices, edges and faces of the two complex obtained by evolving "in time" the spin network graph.

As a side remark, it is interesting to notice that the correspondence between canonical theory and spinfoams can be made precise in the context of BF theories, such as three dimensional gravity [166]. This success is due to the particularly simple form taken by the rigging map, which in the end allows the *non*-perturbative implementation of the Hamiltonian constraint without the need of appealing to any limiting procedure.

Anyway, since their first introduction, spinfoams were realized to have a more compelling interpretation than that presented in the previous section. Instead of being interpreted more or less as sorts of Feynman diagrams for the loop quantum gravity dynamics, they should better be seen as a specific lattice regularization of the Misner-Hawking integral over (quantum) four-geometries [162, 130]. This point of view is very close in spirit to other ideas of John Baez, Junichi Iwasaki and Michael Reisenberger himself, who focussed on the surface (or world-sheet) formulation of gravity and gauge theories [21, 186, 138].

In this setting it is also more natural to reinterpret the physical scalar product of equation 2.18 in terms of transition amplitudes between two different quantum states of the three-geometry (and topology). There is indeed a very elegant way to associate particular types of "quantum" three-geometries to the data encoded in a spin network state. Even if hints already appeared in the literature (e.g. [183, 176, 177, 129]), the correspondence was definitely clarified by the work of Andrea Barbieri [36] and immediately afterwards by that of John Baez and John Barrett [24], who introduced the notion of quantum tetrahedra and four-simplices by direct quantization of their classical phase spaces. Since then quantum geometry received contributions from many authors. To my knowledge, the state of the art of this field is given by the understanding of the twistorial structure underlying quantum geometries. For some references see the following series of works by Laurent Freidel, Florian Conrady, Etera Livine, Simone Speziale, Eugenio Bianchi, Pietro Donà, Joannes Tambornino, Wolfgang Wieland and others [70, 105, 108, 109, 45, 153, 152, 120].

These developments lead us to adopt a different viewpoint. In particular, starting from the next chapter, I will review the construction of spinfoam models from ideas of quantum geometry. The canonical setting described so far will not be assumed as a starting point, but rather as an important guide. Therefore, I will reference to it not only whenever an ingredient of the previous analysis will be needed, but also to highlight common and conflicting features between the two approaches.

3 SPINFOAM MODEL OF BF-THEORY

3.1 QUANTUM THREE-GEOMETRIES AND SPIN NETWORKS

At the end of last chapter, I cited a considerable body of work which stemmed from Andrea Barbieri's construction of the quantum tetrahedron. Even if I will not delve into such interesting developments, I want to present at least the basic results about quantum geometry which I will need to introduce spinfoams as tools to study the quantum dynamics of spin networks.

Consider a (flat) convex polyhedron $\mathcal{P}_N$ embedded in $\mathbb{R}^3$, where N is the number of its faces. Its face-vectors $\{\vec{a}_a \equiv a_a \vec{n}_a\}_{a=1,\dots,N}$, defined as the outpointing vectors orthogonal to the faces of the polyhedron and of norm equal to the face areas $\{\vec{a}_a\}$, satisfy the so-called closure condition

$$\sum_{a=1}^{N} a_a \vec{n}_a = \vec{0} \,. \tag{3.1}$$

A beautiful and easy way to understand this property is to think of the total pressure force acted by a fluid onto a virtual polyhedron cut out of the fluid itself. It is a theorem by Hermann Minkowski[1] [161] the fact that the previous equation also *uniquely* determines $\mathcal{P}_N$. Therefore there is a natural correspondence between the convex polyhedra and the (non-planar) polygons, obtained by simply concatenating the vectors $\{\vec{a}_a\}$ one after the other until they close. Note that if the vectors are coplanar, then the corresponding polyhedron is degenerate.

A completely independent work of Michael Kapovich and John Millson [142] showed that there is a natural phase-space structure associate to polygons of fixed side length up tp global rotations.

Putting together these two results, one finds that there is a natural phase space for the shapes of polyhedra up to global rotations, which can therefore be quantized via techniques of geometric quantization [24, 65, 70]. The result of this construction is that one is led to consider invariant subspaces of a tensor product of N irreducible representations of $SU(2)$, which are precisely the definition of the intertwiners sitting at the nodes of spin networks!

In fact, the quantization procedure is most easily understood if one simply postulates that the area vectors $\{\vec{a}_\alpha\}$ are promoted to operators associated with N-copies of $\mathfrak{su}(2)$. More precisely, $\vec{a}_a \mapsto \vec{J}_a$, that is the $\mathfrak{su}(2)$ generator in the a-th copy of the algebra. The Hilbert space on which such an area-vector operator $\vec{J}_a$ acts is given by the spin j_a representation of $SU(2)$ with carrier space V^{j_a}. Hence, quantum (area-)vector states are superpositions of states in V^{j_α}, while quantum polyhedron states are states $|\iota\rangle \in V^{j_1} \otimes \cdots \otimes V^{j_N}$ which satisfy the quantum version of the classical closure condition (equation 3.1):

$$\hat{C}|\iota\rangle \stackrel{!}{=} 0, \quad \text{where} \quad \hat{C} := \sum_a \vec{J}_a \,. \tag{3.2}$$

In the previous formula $\vec{J}_a$ must be understood as the operator $\big(\mathbb{I} \otimes \cdots \otimes \overset{a\text{-th term}}{\vec{J}} \otimes \dots \mathbb{I}\big)$ acting on $V^{j_1} \otimes \cdots \otimes V^{j_N}$.

Since $\hat{C}$ is nothing but the generator of diagonal $SU(2)$ rotations, this constraint

1. See [3] and references therein for a modern approach to Minkowski's theorem.

selects by definition elements

$$|\iota\rangle \in \mathcal{H}_N := \text{Inv}\left(V^{j_1} \otimes \cdots \otimes V^{j_N}\right) , \tag{3.3}$$

i.e. in the space of N-intertwiners between the representations $j_1, \ldots, j_N$.

This construction can be used also to define a discretized metric operator:

$$\hat{g}_{ab} := \vec{J}_a \cdot \vec{J}_b \,. \tag{3.4}$$

In particular, the diagonal elements of this matrix are to be interpreted as the (square) of the areas of the polyhedron faces. These are given by $SU(2)$ Casimir operators, which commute with all the elements of $\hat{g}_{ab}$, and whose spectrum is

$$\sqrt{j(j+1)} \quad \text{with} \quad j \in \frac{1}{2}\mathbb{N}. \tag{3.5}$$

On the other side, the non-diagonal elements (once renormalized by obvious area factors) are to be interpreted as the cosine of the dihedral angles between the normals of the polyhedron faces. Observe that these angles do *not* all commute among them. This is precisely the signature of some "fuzziness" of the quantum geometry. For example, in the case of a quantum tetrahedron, a complete set of geometrical observables is given by the four areas and only one dihedral angle: [2] Heisenberg's uncertainty principle is precluding the observer to measure all six classical phase space variables at the same time.

It is in general very convenient to consider states which have a direct semiclassical interpretation (this also in relation with the geometrical quantization procedure, see e.g. [70]). At this purpose, consider in V^j states of the form

$$|j, \vec{n}\rangle := D^j\left(h(\vec{n})\right)|j;j\rangle \,. \tag{3.6}$$

Here $D^j(h) : V^j \to V^j$ is the matrix corresponding to $h \in SU(2)$ in the j-th representation, and $|j;j\rangle$ is the state of maximal magnetic number in V^j. $\vec{n} \in S_2$ is a three dimensional unit vector, and $h(\vec{n}) \in SU(2)$ is a group element rotating the direction $\vec{z}$ onto the direction $\vec{n}$. [3] Such states are immediately seen to represent a vector of norm j pointing in direction $\vec{n}$:

Notice, $|j, \vec{n}\rangle$ is not an eigenvector of $\vec{J}$, which does not exist, but only of $\vec{J} \cdot \vec{n}$.

$$\langle j, \vec{n}|\, \vec{J}\, |j, \vec{n}\rangle = j\vec{n} \,. \tag{3.7}$$

Moreover, they minimize the uncertainty $|\langle \vec{J}^2\rangle - \langle \vec{J}\rangle^2|$, and in this sense they are also coherent states. Finally, using the orthonormality of the unitary representation matrix elements, it is also immediate to show that these states form an (overcomplete) basis of V^j:

$$\mathbb{I}_j = \sum_m |j;m\rangle\langle j;m| = (2j+1)\int_{S_2} d\vec{n}\; |j, \vec{n}\rangle\langle j, \vec{n}| \,, \tag{3.8}$$

where $d\vec{n}$ is the normalized measure on the unit sphere.

One can use such coherent vector states to build the coherent quantum tetrahedron states known as Livine-Speziale coherent intertwiners. They are defined as [150]

$$\mathcal{H}_N \ni ||j_a, \vec{n}_a\rangle := \int_{SU(2)} dh\; \otimes_a D^{j_a}(h)|j_a, \vec{n}_a\rangle \,, \tag{3.9}$$

2. Here, the closure condition is fundamental.

3. Among all such vectors, by convention $h(\vec{n})$ is chosen to be $h(\vec{n}) := \exp\left(\theta \vec{n}_\perp \cdot \vec{J}\right)$, where $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, and $\vec{n}_\perp := (\sin\phi, -\cos\phi, 0)$.

dh being the Haar measure on $SU(2)$. The $SU(2)$-invariance properties of these intertwiners follow trivially from the translation invariance property of the Haar measure, and therefore such states are immediately seen to satisfy the quantum closure condition of equation 3.2. It is important to stress the fact that they do so for *any* value of the vectors $\{j_a \vec{n}_a\}$, even when they do not sum up to zero. However, as shown in the original reference, their norm is exponentially suppressed with the spin scale whenever this is not the case. This fact can be rephrased by saying that Livine-Speziale intertwiners peak on classical polyhedral geometries.

Finally, another interesting point about Livine-Speziale intertwiners is that they do not only form an overcomplete basis of intertwiners [4] in $\mathcal{H}_N$, but that their "semi-classical" subset does also so (at least for $N = 4$). In fact, as shown in [70], there exist a positive $SU(2)$ invariant measure $\rho(j_i, \vec{n}_i)$ on the space of 4 unit three vectors such that

$$\mathbb{I}_{\mathcal{H}_4} = \prod_a (2j_a + 1) \int_{S_2} \prod_a d\vec{n}_a \; \delta\left(\sum_a j_a \vec{n}_a\right) \; \rho(j_a, \vec{n}_a) \, \| j_a, \vec{n}_a \rangle \langle j_a, \vec{n}_a \| \,. \tag{3.10}$$

3.1.1 *Relation with Spin Networks - Twisted Geometry*

The goal is now to establish a relation between loop quantum gravity spin networks and the quantum-polyhedron construction just described. The first step is to realize the close parallel between the spin-network node Gauß constraint (equation 2.8) and the quantum polyhedra closure condition (equation 3.2). These two equations are very similar, since in both cases they ask that a sum of $SU(2)$ generators annihilates a certain state. Nevertheless, the origin and physical meaning of these two equations is *a priori* very different. However, following the work by Laurent Freidel and Simone Speziale, I will briefly argue here that a precise bridge between the two pictures is actually available. To show that this is not a trivial task, I start by pointing out the differences between the two constructions.

In the quantum polyhedron construction, a crucial role is played by the ambient space dimensionality $n = 3$. This is first of all used to have a correspondence between polygons and polyhedra, since a vector can be identified with a two dimensional surface only in three dimensions. Moreover, applying the same receipt for the quantum tetrahedron with fixed face areas in four dimensions (this time using face bivectors) leads to a one-state, instead than to a one-dimensional, Hilbert space (see [24]). Also, when generalizing the polygon phase space to dimensions higher than three, one is led to consider groups more related to the dimensionality of the ambient space than $SU(2)$ (e.g. [97]). Therefore it is perfectly clear that the results of the previous section are tightly bound to the group of rotations of the space in which the polyhedron and polygon are immersed. Indeed, the closure constraint is most naturally interpreted as stemming from the requirement of invariance of the quantum polyhedron state under rigid rotations in $\mathbb{R}^3$.

The situation is different for loop quantum gravity spin networks, where the closure, or Gauß, constraint is a direct consequence of the $SU(2)$ gauge invariance associated with the Ashtekar-Barbero connection, which is *not* simply associated to local rotations of the three-di-mensional frame. In fact, it is necessary for the Ashtekar-Babero connection to carry information also about the extrinsic geometry of the space in which the spin network is embedded, as exemplified by the formula $A = \Gamma + \gamma K$. The most direct way to understand this is considering the fact that the

4. This fact descends trivially from tensoring equation 3.8.

Ashtekar-Barbero connection is canonically conjugate to the (densitized) triad and must therefore "know" something about its "time evolution", which is on the other hand encoded in the extrinsic curvature K.

At a more superficial level, another difference between the two constructions, is that in the spin network case the objects which are naturally considered are Faraday-Wilson lines of the Ashtekar-Barbero connections, which correspond to extended structures with a totally degenerate (quantum) metric; on the other side, a quantum polyhedron may be interpreted as a flat chunk of quantum three-space endowed with a local quantum metric. Therefore, in order to put the discussion on a more common ground, let me spend a few lines to discuss the spin-intertwiner basis of spin networks. [5]

Given a spin network state $\psi_\Gamma(h_\ell)$, one can decompose it via Peter-Weyl theorem into a superposition of products of Wigner matrices:

$$\psi_\Gamma(h_\ell) = \sum_{\{j_\ell\}} \left\{ \sum_{m_1,n_1=-j_1}^{j_1} \cdots \sum_{m_L,n_L=-j_L}^{j_L} \left[\psi_\Gamma(j_\ell)\right]^{n_1,\ldots,n_L}_{m_1,\ldots,m_L} \prod_\ell D^{j_\ell}(h_\ell)^{m_\ell}{}_{n_\ell} \right\} \tag{3.11}$$

where $L = \#\{\ell\}$ is the number of links of Γ. Now, because of the gauge invariance at the nodes of the spin network (equation 2.6), of which the Gauß constraint is the infinitesimal version, one realizes that $\psi_\Gamma(j_\ell)$ must be a superposition of products of intertwiners situated at the nodes of the graph:

$$\psi_\Gamma(j_\ell) = \sum_{\{v_n\}} \psi_\Gamma(j_\ell, v_n) \prod_n \iota_n^{(v_n, j_{\ell\subset n})} \tag{3.12}$$

where n labels the nodes of Γ, and v_n a basis of intertwiners in $\mathrm{Inv}\left(\bigotimes_{\ell\subset n} V^{j_\ell}\right)$. Symbolically

$$\begin{aligned}\psi_\Gamma(h_\ell) &= \sum_{\{j_\ell, v_n\}} \psi_\Gamma(j_\ell, v_n) \left[\bigotimes_n \iota_n^{(v_n, j_{\ell\subset n})}\right] \cdot \left[\bigotimes_\ell D^{j_\ell}(h_\ell)\right] \\ &\equiv \sum_{\{j_\ell, v_n\}} \psi_\Gamma(j_\ell, v_n) \left\langle h_\ell | \Gamma, j_\ell, v_n \right\rangle . \end{aligned} \tag{3.13}$$

As a final remark, let me underline the fact that one can well choose in the space of intertwiners the Livine-Speziale overcomplete basis. In this case the sums over the labels v_n appearing in the last two equations become integrals, and no other change is required. Assuming this choice, equation 3.13 is then quite naturally read as a superposition of quantum coherent polyhedra living at the nodes of the spin network graph, having faces dual to the links of the graph, and related among them by the holonomies h_ℓ. Note also that this leads to a natural identification of the area vector operators J and the flux operators X (up to a dimensionful factor discussed in the next section).

Therefore there is a discrete quantum geometry naturally associated to every spin network, in such a way that spin network (N-valent) nodes are dual to (N-)polyhedra, and spin network links are dual to polygonal faces "shared" by the two connected polyhedra. I used inverted commas around the word 'shared', in order to stress the following particular property of these quantum geometries: the face dual to a certain link has in general different shapes when observed from the two

5. See footnote 1 of Chapter 2 and the text around it for some historical remarks.

polyhedra sharing this face. This fact is due to the way the shape of the face is reconstructed and in particular to the fact that this reconstruction involves all the data available at a node, and no particular "shape matching" condition is imposed to reciprocally correlate the shapes different polyhedra (intertwiners). However, even if face shapes can be different, their areas must match, since these are determined only by the value of the spin j_ℓ, which is directly associated to the link.

Even if this discussion seems to point toward a definite connection between the discrete quantum geometry and the spin network state picture this is not yet the case: as discussed here above, it seems that the connection involved in the two pictures are different, and that there is a variable which is missing in the discrete quantum geometry picture. To appreciate the latter fact, let me take one step backwards, before the imposition of the closure and Gauß constraints. In the context of spin networks, before such an imposition, one is left with a phase space which factorizes into a tensor product of L copies of $T^*SU(2)$, i.e. one per link (see discussion in Chapter 2). Such a link phase-space has dimension 6. On the quantum discrete geometry side, before imposing the closure constraint, one is left with two area vectors per link, whose only relation is to have the same length whenever they are associated with same link. However, this set of variables cannot even constitute a phase space, since its cardinality is odd, and more precisely equal to 5 (two three vectors with the same length). Therefore there is one missing variable. As discovered by Laurent Freidel and Simone Speziale in their beautiful work [108], it turns out that the missing variable can be identified with a particular instantiation of the extrinsic curvature (called $\xi \in S_1$) which is also the variable canonically conjugate to the face area j_ℓ.

This realization allowed them to develop the notion of twisted geometries, which provide a rigorous connection between the phase space of loop quantum gravity, which is given by spin networks, and that of quantum polyhedra. This line of research, which I only approximatively sketched, can be pushed even further with the introduction of twistors as a description of (an extended Lorentz covariant version) of the previous phase space (see e.g. [109]).

For more details on the relation between the piecewise-flat-geometry and spin-network pictures, see also the work of Laurent Freidel, Marc Geiller, and Johnatan Ziprick [100].

3.1.2 *Reintroducing Physical Units*

When treating loop quantum gravity, I have been careful of keeping track of all the physical units involved. This was possible since the starting point was (some version of) the action of general relativity. In the purely algebraic setting of this section, on the contrary, everything turns out to be expressed in natural units. As a comparison of the area spectrum of equations 2.16 and 3.5 immediately shows, the physical units one has to associate to the quantum area operator are

$$\hat{X}^i \equiv a_{Pl}\, \gamma J^i\,, \quad \text{where } a_{Pl} := \hbar\kappa\,. \tag{3.14}$$

Having clarified this point, in the rest of this thesis I will most often choose appropriate units such that $a_{Pl} = 1$. It will be sometimes important, on the contrary, to track the dependence from the Barbero-Immirzi parameter γ.

3.2 SPINFOAM MODEL OF BF-THEORY

Having at disposal a dictionary between loop quantum gravity spin network states and quantum discrete three-geometries, it is now tempting to try and extend this correspondence to the four-dimensional spacetime. The natural place where to look for such a correspondence is in the evolution of spin networks, i.e. in the spinfoam setting briefly described at the end of last section.

The relation between spin foams and discretized spacetimes was first realized by Fotini Markopulou [156], and later developed by many authors. Among the most important early contributions to the field I cite those of Carlo Rovelli, who as early as 1993 noted the connection between canonical loop quantum gravity and state sums for topological quantum field theories [192]; of John Baez, who first formalized the notion of spinfoams (and called them this way) [22]; of John Barrett and Louis Crane, who proposed one of the first spinfoam models of four-dimensional quantum gravity [39] being inspired at the same time by the state sums for topological quantum field theories [38] and by the construction of the quantum tetrahedron by Andrea Barbieri [36]; and of Laurent Freidel and Kirill Krasnov, who attempted a systematic derivation of spinfoam models from the Lagrangian of BF theory [102].

Clearly, this is just a brief synthesis of a body of results and proposals which I do not even try to review here. I will rather proceed by showing how to easily derive at a formal level a spinfoam model for BF theory. This construction is at the root of virtually all spinfoam models for quantum gravity.

3.2.1 *Quantum* BF-*Theory in the Continuum*

Let $\mathcal{M}$ be a n-dimensional manifold. Let G be a Lie group, whose Lie algebra $\mathfrak{g}$ is endowed with an invariant non-degenerate bilinear form[6] $< \cdot, \cdot >$.Finally, let ω be a connection (sometimes called gauge potential) on a principal G-bundle over $\mathcal{M}$. Finally, let B be a $(n-2)$-form taking values in $\mathfrak{g}$.

The action of BF-theory is defined as

$$S_{BF} := \int_{\mathcal{M}} < B \wedge F[\omega] > , \tag{3.15}$$

where $F[\omega] := d\omega + \omega \wedge \omega \equiv d\omega + \frac{1}{2}[\omega, \omega]$ is the usual two-form field strength associated to ω, familiar from Yang-Mills theory. Note that no metric on $\mathcal{M}$ is needed in order to write the BF action.

The equation of motions of this theory are immediately seen to be given by

$$F = 0 , \quad d_\omega B = 0 . \tag{3.16}$$

where we introduced the covariant derivative $d_\omega := d + [\omega, \cdot]$.

The action is invariant under local Yang-Mills gauge transformations

$$\delta^{YM}_\lambda B = [B, \lambda] , \quad \delta^{YM}_\lambda \omega = d_\omega \lambda , \tag{3.17}$$

6. A generalization to arbitrary Lie groups is possible by considering along the following construction not only the Lie algebra, but also its dual. I will not discuss this more general formulation here. See e.g. [136].

with λ a $\mathfrak{g}$-valued zero-form, as well as under so-called shift-transformations

$$\delta^S_\eta B = d_\omega \eta\,, \quad \delta^S_\eta \omega = 0\,, \tag{3.18}$$

with η a $\mathfrak{g}$-valued $(n-3)$-form. The latter invariance is a direct consequence of the Bianchi identity $d_\omega F[\omega] \equiv 0$. A detailed Hamiltonian analysis (see [58]) shows that these symmetries are generated by first class constraints, respectively

$$G^i = \frac{1}{(n-2)!}\epsilon^{0ab\ldots c}\partial_a B^i_{b\ldots c} \stackrel{!}{=} 0 \quad \text{and} \quad C^i_{ab} = F^i_{ab} \stackrel{!}{=} 0\,, \tag{3.19}$$

and are therefore both gauge symmetries in the proper sense. It is also the case that the constraints are contained in the equations of motion (equation 3.16).

Being written as an integral of an n-form, the action is also manifestly invariant under (orientation-preserving) diffeomorphism. Interestingly, infinitesimal diffeomorphism transfomations can be expressed as a specific linear combination of the gauge and shift transformations and of the equations of motion:

$$\delta^D_\xi B = \delta^{YM}_{i_\xi \omega} B + \delta_{i_\xi B} B + i_\xi(d_\omega B)\,, \quad \delta^D_\xi \omega = \delta^{YM}_{i_\xi \omega}\omega + \delta_{i_\xi B}\omega + i_\xi F\,. \tag{3.20}$$

The classical space of solutions of BF theory is quite simple. On the one side, the equation of motion $F = 0$ says that the connection is flat. On the other side, the equation of motion of the B field says it is curl-less; moreover, we are interested in the space of solutions of the B field only modulo to gauge transformations, that is only up to the curl of an $(n-3)$-form: this is exactly the definition of the $(n-2)$-th cohomology class. [7]

Therefore, the space of gauge inequivalent solutions consists of the gauge inequivalent flat connections together with the elements of the $(n-2)$-th cohomology group with values in $\mathfrak{g}$.

Before concluding this section with a discussion of the quantization of BF theory, let me stress two important facts. First, the previous construction shows that there are no local degrees of freedom within a BF-theory. In fact, according to Poincaré's lemma, all flat connections are *locally* gauge equivalent, and also any curl-free (i.e. closed) form is *locally* equal to the curl of some other form (i.e. it is exact). Therefore locally, all degrees of freedom can be "gauged away". Then, even if the theory is locally trivial, this does not mean that it is globally trivial, too. Indeed, the space of its classical solutions is neither empty, nor necessary zero dimensional. However, this latter fact depends on the topology of the principal bundle in which the connection lives. In any case, the space of solutions of classical BF theory is finite dimensional, and this fact will be crucial in what follows.

Let me now briefly introduce the Hilbert space of quantum BF-theory. Quite generally, the Hilbert space of a given theory can be identified with the space of square integrable functions of a complete set of coordinates for the theory. This set of coordinates is in a precise sense "half" of the phase space. In turn the phase space of a theory can be seen as the set of all its possible initial conditions, which are clearly in one-to-one correspondence with the solutions of its equations of motion. Therefore, if one can show that the space of solutions of a theory is endowed with some natural symplectic structure which turns it into a proper phase space, one can immediately quantize the theory. This is what Gary Horowitz did in the case of BF theory [136], and the resulting Hilbert space is the space of square integral

7. REMARK Actually, things are slightly subtler. Indeed, applying twice the derivative d_ω which appears in the equation of motion for the B field, does not give zero, but rather a term proportional to the field strength F. However, the field strength does vanish on shell of the other equation of motion.

functions on the space of gauge-inequivalent flat connections on Σ.

As a side remark, let me say that to conclude this he needed to deal with manifolds of the form $\mathcal{M} = \Sigma \times \mathbb{R}$: on a technical level this guarantees that the space of solutions (i.e. the will-be phase space of the theory) is even dimensional as it should, while on a more physical level this topological structure constitutes the natural arena for canonical quantization.

To conclude notice that this quantization procedure leads directly to the physical Hilbert space, since it quantizes the theory after having performed a complete symplectic reduction with respect to all gauge-generating constraints. This observation is the starting point of the next section.

3.2.2 *Spin networks and the Spinfoam Quantization of* BF *Theory*

As I discussed in section 1.4, following the work of Jerzy Plebański, general relativity may be formulated as a constrained BF-theory. As we have just seen, it is possible to solve and exactly quantize BF-theory. However, as I stressed at the end of the previous subsection, this quantization was made possible by the complete reduction of gauge and shift symmetries, and by the consequent finite dimensionality of the reduced phase space. However, general relativity does not posses shift symmetry and, as any proper field theory, its phase space is *infinite* dimensional. Therefore, it is of paramount importance to understand how to quantize the non symmetry-reduced phase space of BF theory and how to impose the rest of its constraints at the quantum level.

It turns out that the kinematical Hilbert space of canonical loop quantum gravity (see section 2.1)

$$\mathcal{H} := \overline{\cup_\Gamma \mathcal{H}_\Gamma / \sim} \tag{3.21}$$

provides exactly the common basis of BF theory and general relativity at the pre-constrained quantum level. More precisely, in works by Abhay Ashtekar, Jerzy Lewandowski, Thomas Thiemann, John Baez and others,[8] it was proved that spin network states span, and provide a basis for, the space of square integrable functions of the connection ω. Here, gauge invariance is also very naturally taken into account by the imposition of the Gauß constraint (which has the same form in both theories). See section 1.5 and especially section 3.1.[9]

Disposing of a kinematical gauge-invariant Hilbert space, let me turn to the imposition of BF's quantum dynamics, in the form of the Wheeler-DeWitt equation

$$\text{“}\hat{F}\psi = 0\text{”}\,. \tag{3.22}$$

To impose this constraint, let me introduce a path integral technique, i.e. let me consider the following partition function:

$$Z^{\mathcal{M}}_{BF} = \int \mathcal{D}\omega \int \mathcal{D}B \; e^{i\int_{\mathcal{M}} <B\wedge F[\omega]>}\,. \tag{3.23}$$

8. For references about this body of work, see e.g. section 11.3 of [23].
9. This discussion applies to the formulation of gauge theories of a compact gauge group G. This includes general relativity in Ashtekar's and Ashtekar-Barbero variables. For a generalization to a covariant framework of loop quantum gravity where spin networks for the full gauge group $SL(2,\mathbb{C})$ are taken into account, see [104, 40].

Formally integrating over the B field, this gives quite naturally

$$Z_{BF}^{\mathcal{M}} = \int \mathcal{D}\omega \; \delta(F[\omega])\,, \tag{3.24}$$

i.e. a formal integration over the space of flat connections on the G-bundle over $\mathcal{M}$. To make sense of this ill-defined expression, it is most convenient to introduce a regularization in the form of a discretization of $\mathcal{M}$. For simplicity, from now on, I will consider only discretizations in the form of triangulations.

Thus, consider a triangulation Δ of the n-dimensional manifold $\mathcal{M}$. It is made of a certain number of n-simplices σ_n, glued to one another at boundary $(n-1)$-simplices $\sigma_{(n-1)}$. Now, consider the dual 2-skeleton Δ^* of the triangulation Δ. By definition Δ^* has one vertex v at the center of each n-simplex, one edge e intersecting each $(n-1)$-simplex, and one polygonal face f intersecting each $(n-2)$-simplex. Summarizing:

Δ	Δ^*
σ_n	v
$\sigma_{(n-1)}$	e
$\sigma_{(n-2)}$	f

To avoid confusion, I will often call vertices in Δ^* (respectively edges, and faces), spinfoam vertices (respectively spinfoam edges, and spinfoam faces).

Spinfoam faces will play a particularly important role. In fact, they are the locus where (distributional) curvature most naturally "lives" in the discretized context. This is because curvature is a two form, and is most naturally smeared over 2-surfaces. It is therefore very useful to have a clear picture of this structure.

In three dimensions, a spinfoam face f corresponds to a side of the triangulation Δ. To be more specific, this side is shared by a certain number of triangles (corresponding to spinfoam edges), which in turn are shared by couples of tetrahedra (corresponding to spinfoam vertices) all glued "around" the side itself. Therefore, within the dual 2-skeleton Δ^*, the spinfoam face f can be identified as a particular (closed) sequence of vertices and edges. See figure 3.1.

In four dimensions, nothing changes at the level of Δ^*, where faces are still identified with particular closed sequences of edges and vertices. However, in this case, faces go around one triangle (rather than one side) of the triangulation, which is shared by a series of tetrahedra at the boundary of as many four simplices. See figure figure 3.2.

To conclude with the dictionary, in the case where $\mathcal{M}$ has a boundary $\partial\mathcal{M}$, its triangulation Δ also posses a boundary triangulation $\partial\Delta$. $\partial\Delta$ is indeed a triangulation of $\partial\mathcal{M}$. Consider then the dual 1-skeleton of $\partial\Delta$ and call it Γ, which stands for (boundary) graph. As it will be come clear, it is not a chance that this is the same symbol I used for spin networks. Γ has nodes n dual to $(n-1)$-simplices of $\partial\Delta$, and links ℓ dual to $(n-2)$-simplices of $\partial\Delta$. In particular nodes of Γ lie at the end of an edge which intersect the boundary of the triangulation. Similarly, it is not hard to realize that links are the intersection of (open) faces of Δ^* with the boundary triangulation. It is useful to think of links as "boundary" edges, and of open faces as faces containing a link. Summarizing:

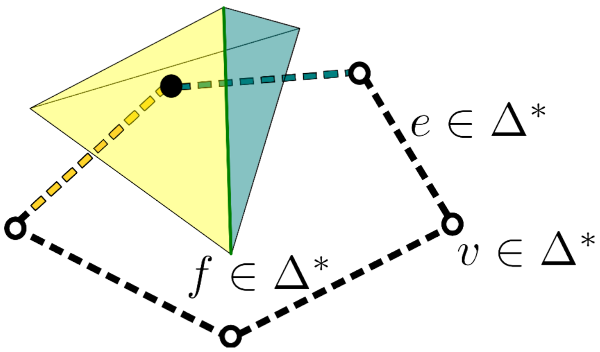


Figure 3.1: A representation of a five-valent spinfoam face $f \in \Delta^*$, in the dual of a four dimensional triangulated manifold. The spinfoam face itself is represented by a sequence of dotted spinfoam edges connecting spinfoam vertices. In green it is drawn the side inΔ (one-simplex) dual to f. It is the side shared by all the five triangles dual the five edges. In the picture two of them are represented in yellow and blue. The four points delimiting the coloured region define the the tetrahedron (three-simplex) dual to the filled vertex.

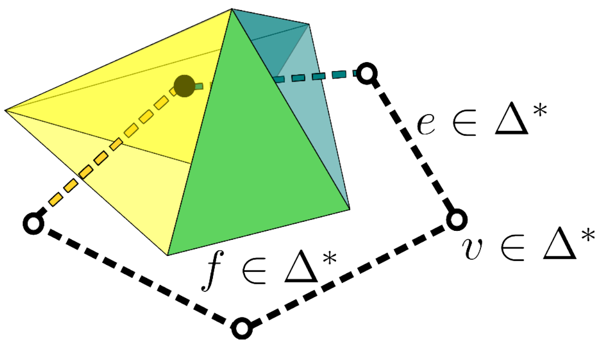


Figure 3.2: A representation of a five-valent spinfoam face $f \in \Delta^*$, in the dual of a four dimensional triangulated manifold. The spinfoam face itself is represented by a sequence of dotted spinfoam edges connecting spinfoam vertices. In green it is drawn the triangle (two-simplex) dual to f. It is the triangle shared by all the five tetrahedra dual the five edges. In the picture two of them are represented in yellow and blue. The five points delimiting the coloured region define the four-simplex dual to the filled vertex.

$\partial\Delta$	Γ
$\sigma_{(n-1)} \subset \partial\sigma_n$	$n \subset \partial e$
$\sigma_{(n-2)} \subset \partial\sigma_{(n-1)}$	$\ell \subset \partial f$

Having all these tools in hand, let me go back to the problem of regularizing the partition function of BF-theory. As already stressed multiple times, the connection ω is naturally smeared over one-dimensional structures, which will be taken to be the edges $e \subset \Delta^*$. Using (non-Abelian) Stokes theorem, it is also clear that this association implies that the curvature must now live on 2-surfaces enclosed by sequences of edges, that is precisely on faces $f \subset \Delta^*$. Supposing that the curvature is essentially constant within a face, and that the parallel transports within a face are small enough to contribute to only higher order terms, one can use the well-known approximation

$$G_f[\omega] := \overleftarrow{\prod_{e\subset f}} g_e \approx \mathbb{I} + a^{\mu\nu} F_{\mu\nu}[\omega](f) \,. \tag{3.25}$$

That is the holonomy of the connection ω around the face f is approximatively given by the identity plus corrections proportional to the curvature $F_{\mu\nu}$ contained within f, and the area bivector $a^{\mu\nu}$ associated to f. Therefore, a good regularization of equation 3.24 is

$$Z^{\Delta}_{BF} := \int_{G^E} \prod_{e\subset\Delta^*} dg_e \prod_{f\subset\Delta^*} \delta\left(G_f\right) \,, \tag{3.26}$$

where the functional integral has been replaced by a finite dimensional integral on $E = \#\{e\}$ copies of the group G, performed with respect to its Haar measure dg. Even if this expression is much more practicable than the previous one, nothing guarantees it is well defined. Indeed, being an integral of a product of many Dirac delta-functions, it is still likely to diverge. I postpone a discussion of this issue to section 4.4.

If the manifold $\mathcal{M}$ has a boundary, the previous expression is readily generalized. The main difference is that now there are also open faces, which contain boundary edges, i.e. links ℓ. As much as edges, links are the natural loci where to smear the connection. They are therefore the natural place where to encode the information about the boundary connection in such a discretized setting. The partition function becomes now a functional of the boundary state:

$$Z^{\Delta}_{BF}[\psi] := \int \prod_{\ell} dh_\ell \; Z^{\Delta}_{BF}[h_\ell]\psi(h_\ell) \,, \tag{3.27}$$

where

$$Z^{\Delta}_{BF}[h_\ell] := \int \prod_{e} dg_e \prod_{f} \delta\left[G_f\left(g_e, h_\ell\right)\right] \,. \tag{3.28}$$

In the last equation, the integration is performed on the bulk holonomies only, and the Wilson loops are calculated, in the case of the open faces, by including the boundary holonomies as if the links were edges. Also, to make the notation less clumpy, I omitted labels of the triangulation and its dual.

This expression for $Z^{\Delta}_{BF}[h_\ell]$ is readily interpreted as the dynamical amplitude of a set of holonomies associated to the boundary of a certain discretized manifold. The easiest way to realize this is to draw the close parallel with Richard Feynman's original path-integral formulation of quantum mechanics [96, 76]: there the coordinates of a particle at the boundary of the time interval $(t_{\text{initial}}, t_{\text{final}})$ are kept fix, while it is summed over all possible intermediate states of the particles. [10]

Also, it should at this point be quite clear that the set of holonomies h_ℓ associated to the boundary graph Γ can be interpreted as the configuration space for the spin network wave functions. Thanks to this interpretation, the previous partition function provides an Ansatz for the kernel of BF-theory physical inner product. This goes as follows. Suppose to be interested in the transition amplitude between states $\psi \in \mathcal{H}_\Gamma$ and $\psi' \in \mathcal{H}_{\Gamma'}$. Hence, consider a triangulated manifold Δ whose dual

10. For extensions of this idea to fields see the work by Robert Oeckl and collaborators (among the other papers, [167, 168, 169]). See also section 6.1.1 for a brief discussion.

2-skeleton Δ^* has the (disjoint) union of Γ and Γ' as boundary: $\partial\Delta^* = \Gamma \sqcup \Gamma'$. This triangulation then provides the following Ansatz for the physical inner product of BF theory:

$$\langle \Psi, \Psi' \rangle^{\Delta}_{\text{phys}} := \langle \psi, Z^{\Delta}_{BF} \psi' \rangle \equiv$$
$$\equiv \int \prod_{\ell \subset \Gamma, \ell' \subset \Gamma'} dh_\ell dh_{\ell'} \; Z^{\Delta}_{BF}[h_\ell, h_{\ell'}] \, \overline{\psi(h_\ell)} \psi'(h_{\ell'}) \,. \tag{3.29}$$

Manifestly, this Ansatz depends on the choice of Δ. I do not discuss this issue in detail here, but leave it for section 4.4. For the moment, let me just note two important facts about it.

First, in the case of topological field theories, like BF theory, the dependence on the regularization Δ should drop off automatically, once issues related to possible divergences as the one mentioned above are taken care of. For example, this Ansatz has been shown to be the correct one [11] in the case of three dimensional gravity, which is a specific realization of BF-theory (see next section), by Karim Noui and Alejandro Perez in [166]. See also the work of Valentin Bonzom and Matteo Smerlak, aiming to generalize this result to arbitrary dimensions and topologies [56, 55].

Second, in order to match the canonical predictions of the theory, no topology change can happen, and therefore Γ and Γ' must be both dual to the same discretized $(n-1)$-dimensional manifold Σ, which must also be the same as any "horizontal" section of the manifold $\mathcal{M}$ dual to Δ. In other words $\mathcal{M} \cong \Sigma \times [0,1]$ (see also the remark at the end of the previous section). In spite of this fact, this formula for the physical scalar product seems to leave more freedom with respect to generalized dynamics which on the contrary allow topology change. One such example would be to consider a sort of Hartle-Hawking no-boundary transition, between the empty non-perturbative Ashtekar-Lewandowski vacuum state Ω and the spin network state ψ. Actually, any transition amplitude can be given in this form. Indeed, define $\Gamma_0 := \Gamma \sqcup \Gamma' = \partial\Delta^*$, and $\psi_0 := \overline{\psi} \otimes \psi' \in \mathcal{H}_{\Gamma_0}$, with the bar meaning complex conjugation, then

The vacuum state Ω, rather than "empty", should be understood as "nothing", in the sense that no-space time at all is present in this state. In fact it annihilates all geometrical operators [15, 17].

$$\langle \psi, Z^{\Delta}_{BF} \psi' \rangle = \langle \Omega, Z^{\Delta}_{BF} \psi_0 \rangle = Z^{\Delta}_{BF}[\psi_0] \,. \tag{3.30}$$

At this purpose, see also John Baez's abstract definition of spinfoam amplitudes in [22, 23].

3.3 THREE DIMENSIONAL EUCLIDEAN QUANTUM GRAVITY

Three dimensional gravity [60] is a specific example of BF-theory. I discuss here its Euclidean version only. It is obtained by considering over the 3-dimensional manifold $\mathcal{M}$, a principal bundle for the group $G = SU(2)$, and by choosing $< \cdot, \cdot >$ as (minus) the $\mathfrak{su}(2)$ Killing form. It is also customary in this context to use the the symbol e (triad) for the B field. Therefore:

The choice of $G = SU(2)$, as opposed to $G = SO(3)$, is dictated by the will of coupling - at some point - spinorial matter to gravity, rather than from considerations about the purely gravitational sector itself. Indeed, the classical theory of the purely gravitational sector is the same for the two choices. However, not so for the quantum theory.

$$S_{3dGR}[\omega, e] = \int_{\mathcal{M}} < e \wedge F[\omega] > = \int_{\mathcal{M}} \delta_{ij} e^i_a F^j_{bc}[\omega] \epsilon^{abc} d^3x \,, \tag{3.31}$$

where $F^j_{bc}[\omega] = \partial_b \omega^j_c - \partial_c \omega^j_b + \epsilon^j{}_{kl} \omega^k_b \omega^l_c$, and where $i, j, \dots \in \{1,2,3\}$ are $\mathfrak{su}(2)$ internal indices, while $a, b, \dots \in \{1,2,3\}$ are space(time) indices. Also, I am using $\tau_i = -i\sigma_i/2$ as a basis for $\mathfrak{su}(2)$, so that $[\tau_i, \tau_j] = \epsilon_{ij}{}^k \tau_k$ and $< \tau_i, \tau_j >= \delta_{ij}$.

11. At least, up to some details of the type just mentioned.

Another way, which is commonly used, to get to the regularized BF partition function of equation 3.26 is that of directly regularize the theory at the level of the action. This goes as follows. Consider a triangulation Δ of $\mathcal{M}$. Integrate the triad on 1-dimensional cells of Δ to obtain $e_f \in \mathfrak{su}(2)$, and build a holonomy $g_e \in SU(2)$ by integrating the connection along the duals of 2-dimensional cells of Δ. This construction is completely analogous to that of section 2.1 (see equations 2.1a and 2.1b). Then, considering the same approximation scheme which led to equation 3.26, the action S_{3dGR} can be approximated by

$$S^{\Delta}_{edGR} = \sum_f \mathrm{Tr}\left(e_f G_f\right), \quad \text{with} \quad G_f := \overleftarrow{\prod}_{e \subset f} g_e\,, \tag{3.32}$$

where the subscripts f and e are spinfoam face and edge indices, respectively, and the trace is understood in the fundamental representation. It is now immediate to write the regularized partition function as

$$Z^{\Delta}_{3dGR} = \int_{SU(2)} \prod_e dg_e \int_{\mathfrak{su}(2)} \prod_f de_f \, e^{i \sum_f \mathrm{Tr}\left(e_f G_f\right)}\,. \tag{3.33}$$

Integrating over the discretized triads (notice that $\mathfrak{su}(2) \sim \mathbb{R}^3$), it is again obtained [12]

$$Z^{\Delta}_{3dGR} = \int_{SU(2)} \prod_e dg_e \prod_f \delta\left(G_f\right)\,. \tag{3.34}$$

As I have already observed, this equation is likely to be just formal, since nothing guarantees that this product of delta functions is well-defined. Nonetheless, let me proceed ignoring this fact at this level. The group delta functions appearing in the previous expression can be decomposed onto a sum of unitary-irreducible-representation characters as

$$\delta(G) = \sum_{j \in \frac{1}{2}\mathbb{N}} d_j \mathrm{Tr}_j(G)\,, \tag{3.35}$$

where $d_j := \dim V^j = 2j+1$ is the dimension of the j-th representation of $SU(2)$, and $\mathrm{Tr}_j(G)$ is a short writing of $\mathrm{Tr}\left[D^j(G)\right]$. Using this identity, the partition function becomes

$$Z^{\Delta}_{3dGR} = \sum_{\{j_f\}} \int_{SU(2)} \prod_e dg_e \prod_f d_{j_f} \mathrm{Tr}_{j_f}\left(G_f\right)\,. \tag{3.36}$$

Notice that this identity has somehow replaced an integral over a set of continuous variables $\{e_f\}$ with a sum over a set of discrete variables $\{j_f\}$. This is a standard feature of the harmonic analysis on a compact space, which in this context is the group $SU(2)$ itself.

The integration over the group elements can also be easily performed now, and traded for another sum over discrete structures. To get to this point, observe that each group element g_e appears exactly three times in the above expression. This is because every spinfoam edge is shared by exactly three faces, or equivalently, every triangle in Δ has exactly three sides. Therefore, isolating the contributions from g_e

12. To be more precise, one obtains the delta function on $SO(3)$ only, and not on the full $SU(2)$. In symbols $\int_{\mathfrak{su}(2)} de \;\; \exp\left[i\mathrm{Tr}(eG)\right] = \delta(G) + \delta(-G)$. This is due to the following simple fact: $\exp\left[i\mathrm{Tr}(eG)\right] = \exp\left[-i \sin\frac{\theta}{2} \vec{n}\vec{e}\right]$, where $e = \vec{e}\vec{\tau}$ and $G = \exp\left(\theta \vec{n}\vec{\tau}\right)$, with $\vec{n}^2 = 1$ and $\theta \in [0, 4\pi)$. Now, integrating over $e \in \mathfrak{su}(2)$, which is equivalent to integrating over $\vec{e} \in \mathbb{R}^3$, one obtains $\sim \delta^{(3)}\left(\sin\frac{\theta}{2}\vec{n}\right)$. Now the issue is that $\sin\frac{\theta}{2}$ vanishes for $\theta = 0$, where $G = \mathbb{I}$, but also for $\theta = 2\pi$, where $G = -\mathbb{I}$. Since the previous derivation of the model is most often considered as just heuristic, this fact is most often ignored. However, see e.g. [107].

in the above formula, and calling the spins of the three faces sharing e, $\{j_1, j_2, j_3\}$, one finds:

$$\int dg_e \, D^{j_1}(g_e) \otimes D^{j_2}(g_e) \otimes D^{j_3}(g_e) = \iota^{(j_1,j_2,j_3)} \otimes \bar{\iota}^{(j_1,j_2,j_3)} . \tag{3.37}$$

Where ι is an intertwiner, and actually the unique intertwiner, between the representations of spins j_1, j_2, and j_3, while $\bar{\iota}$ is its complex conjugate. Now, at each vertex exactly four such intertwiners meet, which get contracted among themselves according to the tetrahedral combinatorics of figure 3.3. The resulting quantity has

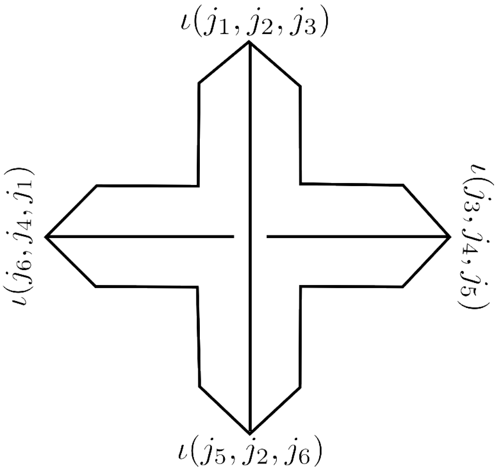


Figure 3.3: the combinatorics of a tetrahedra. The picture shows the 4 three-valent intertwiners representing quantum triangles, and the way they are glued to each other to form a quantum tetrahedron (a 6j symbol). Recall that every line in this graph (a so-called strand) is dual to a side of the tetrahedron.

no free indices and depends only on the values of the 6 spins associated to the 6 faces which meet at a vertex (one per side of the tetrahedron). The final result of this construction is

$$Z^{\Delta}_{3dGR} = \sum_{\{j_f\}} \prod_f (-1)^{2j_f} d_{j_f} \prod_e (-1)^{\sum_{f \supset e} j_f} \prod_v \{6j\}_v \,, \tag{3.38}$$

where $\{6j\}_v$ stands for the 6j-symbol [183] associated to the 6 spins present at each vertex. Notice that the edge sign-factors can be reabsorbed in the vertex amplitude, by defining a new 6-j symbol $\|6j\| := i^{2(j_1+\cdots+j_6)}\{6j\}$. Notice also the presence of a factor $(-1)^{2j}$ in the face weight. As discussed in detail by John Barrett and Ileana Naish-Guzman in [41] (where one can find all the details about this model), the presence of this sign factor has a non-trivial origin and is related to the fact that this state sum model is independent from the choice of an orientation and of a consistent ordering of the structures in Δ^*. (Though, coming from equation 3.36, the presence of this sign reduces to a matter of proper computation.)

The state sum model of equation 3.38 has the name of Ponzano-Regge model,

$$Z^{\Delta}_{3dGR} \equiv Z^{\Delta}_{PR} \,, \tag{3.39}$$

from the names of Giorgio Ponzano and Tullio Regge, who first proposed it in 1968 [183] as a realization of 3-dimensional Euclidean quantum general relativity. Their proposal was based on the observation that in the limit of large spins

$$\{6j\} \overset{j \gg 1}{\sim} \frac{1}{\sqrt{12\pi|\mathrm{Vol}|}} \cos\left[S_R(j) + \frac{\pi}{4}\right] , \tag{3.40}$$

where Vol is the volume of the tetrahedron of side lengths $\left(j_i + \frac{1}{2}\right)$, and $S_R(j)$ is the Regge action for this very same tetrahedron,

$$S_R = \sum_{i=1}^{6} \left(j_i + \frac{1}{2}\right) \Theta_i \,, \tag{3.41}$$

which is in turn known to reproduce the continuous Einstein-Hilbert action when the piecewise-flat discretization on which it is based is suitably refined to a continuum limit. I will come back on the Regge action later, in Chapter 5, where I will also explain the previous notation in detail. For the moment, let me note that because of this property of the $\{6j\}$-symbols, it is possible to claim that equation 3.38 provides the sought non-perturbative definition of Stephen Hawking's integral-over-geometries formulation of quantum general relativity

$$Z_{\text{Hawking}} = \text{``} \int \mathcal{D}g_{\mu\nu} \, e^{iS_{EH}} \text{''} \,. \tag{3.42}$$

Looking at these formulas, probably the biggest surprise is the presence of a cosine in the asymptotics of the 6j symbols, instead of the simple exponential one would expect from the previous derivation. The reason for this mismatch has to be traced back to the Palatini-Cartan formulation of general relativity (see section 1.1), which allows for degenerate metrics as well as for local transitions between different orientations. This cosine will be a crucial ingredient in the analysis of divergences performed in the core chapter of this thesis.

Before moving on, I would like to rewrite the model in a different representation, that of coherent intertwiners. This is in fact a good warm up for what will be a very useful tool in the context of the EPRL-FK model. Hence, let me start once again from equation 3.36, and massage it by doubling the group variables. More concretely, I replace each group element g_e with the product $g_e = g_{v'e} g_{ev}$, and its measure dg_e with $dg_{ev} dg_{ev'}$. In this notation $v = s(e)$ is the source vertex of the edge e, and $v' = t(e)$ its target. Note that g_{ev} is always outgoing from the vertex and is interpreted as the parallel transport from v, i.e. from the centre of the tetrahedron, to the "center" of the edge e, i.e. to the centre of the corresponding face of the tetrahedron. Since the gauge group $SU(2)$ is compact, and its Haar measure is translation invariant and normalized to one, this can be done without altering the expression of Z^{Δ}_{3dGR}. Now, as I have already discussed, each of these group elements appears in the traces associated to three different faces. In each of this traces insert a copy of the resolution of the identity on coherent states (equation 3.8):

$$\mathrm{Tr}_{j_f}\left[\cdots g_{v'e} g_{ev} \cdots\right] = d_{j_f} \int_{S_2} d\vec{n}_{ef} \, \mathrm{Tr}_{j_f}\left[\cdots g_{v'e} | j_f, \vec{n}_{ef}\rangle\langle j_f, \vec{n}_{ef} | g_{ev} \cdots\right] . \tag{3.43}$$

After these insertions, the partition function can be most naturally "re-packed" into wedge contributions. A wedge is the portion of a spinfoam face contained within a single tetrahedron. It is labelled by a face and a vertex (w_{fv}, with $v \subset f$) or, equivalently, by a vertex and a couple of consecutive edges ($w_{e've}$, such that $e, e' \subset f$ and $v \subset e, e'$). Hence:

$$Z^{\Delta}_{PR} = \sum_{\{j_f\}} \int_{SU(2)} \prod_{(ev)} dg_{ev} \int_{S_2} \prod_{(ef)} \underline{d}_{j_f} \vec{n}_{ef} \prod_{f} d_{j_f} \prod_{w} \langle j_f, \vec{n}_{e'f} | g_{e'v} g_{ve} | j_f, \vec{n}_{ef} \rangle \,, \tag{3.44}$$

where I defined the shorthand notation $\int_{S_2} \underline{d}_j \vec{n} := d_j \int_{S_2} d\vec{n}$. Starting from this formula, one can express the partition function in a path-integral form. Indeed, observe that $|j; j\rangle = |\frac{1}{2}; \frac{1}{2}\rangle^{\otimes 2j}$ implies

$$\langle j, \vec{n}' | (g')^{-1} g | j, \vec{n} \rangle = \langle \vec{n}' | (g')^{-1} g | \vec{n} \rangle^{2j} \,, \tag{3.45}$$

where I introduced the shortwriting $|\vec{n}\rangle = |\frac{1}{2}, \vec{n}\rangle$. Hence,

$$Z_{\mathrm{PR}}^{\Delta} = \sum_{\{j_f\}} \int \prod_{(ef)} \underline{d}_{j_f}\vec{n}_{ef} \prod_f d_j \, e^{S_{\mathrm{PR}}}, \tag{3.46}$$

with

$$S_{\mathrm{PR}} := \sum_w j_f \ln\langle \vec{n}_{e'f}| g_{e'v} g_{ve} |\vec{n}_{ef}\rangle^2 \tag{3.47}$$

defining the Ponzano-Regge action.

This last expression is very useful since it allows a direct interpretation of the previously cited result on the asymptotics of the $\{6j\}$ symbols and the relation to the Regge action. Indeed, recall that spins measure physical lengths in Planck units,

In 3-dimensional gravity spins are associated to lengths and not to areas. Consistently, $\hbar\kappa$ has the dimensions of a length.

$$\sqrt{j(j+1)} = \frac{L}{\hbar\kappa}, \tag{3.48}$$

and therefore in the formal limit in which $\hbar \to 0$, at fixed L, the number of quanta becomes very large and $\sqrt{j(j+1)} \approx j \gg 1$. Therefore, in the "semiclassical" limit the Ponzano-Regge partition function is dominated by the critical points of the Ponzano-Regge action. In Chapter 5, the Ponzano-Regge action will be shown to reproduce "on-shell", that is at its critical point, the classical Regge action for the triangulated manifold Δ: the tetrahedral side lengths j are already present in the action, while the logarithm of the on-shell holonomies between the will-be face-normals $\vec{n}_{ef}$ will reproduce the dihedral angles Θ_w.

More precisely, there is an order one contribution which is captured by the $\{6j\}$ asymptotics: $\sqrt{j(j+1)} \approx j + \frac{1}{2}$.

The main subtlety involved in this procedure is that the same side-lengths are associated to two parity-related tetrahedra, and both of them will consequently arise at each vertex. This is the origin of the cosine in the asymptotics of the 6j-symbols. The reader may wonder which is then the origin of the volume factor and of the $\pi/4$ shift. The first is essentially a factor taking into account the width of the Gaussian obtained by developing the Ponzano-Regge action at second order around the critical point,[13] while the origin of the shift is once again the existence of two possible orientations. Indeed, one can rewrite equation 3.40 up to an overall phase as

$$\sqrt{48\pi}\,\{6j\} \sim \frac{e^{iS_R}}{\sqrt{\mathrm{Vol}}} + \frac{e^{-iS_R}}{\sqrt{-\mathrm{Vol}}}, \tag{3.49}$$

where, Vol and S_R are understood to be those of the "positively oriented" tetrahedron.

13. This formula was early guessed by Eugene Wigner, used by Giorgio Ponzano and Tullio Regge, and finally proved by Kalus Schulten and Roy Gordon in 1975 [209]. A more powerful treatment was developed twentyfive years later by Justin Roberts in [191], which is the common reference today. I thank Hal Haggard for clarifying me these historical details.

4 THE EPRL SPINFOAM MODEL OF QUANTUM GRAVITY

4.1 FROM BF-THEORY TO GENERAL RELATIVITY

As discussed in detail in Chapter 1, general relativity can be formulated as a constrained BF-theory. This is referred to as the Plebański formulation. Early attempts to construct spinfoam models based on this formulations were done notably by José Zapata [224], and Michael Reisenberger [187]. Their models, however, had the disadvantage of being quite involved and difficult to analyse. This is also the case for the 1999's model by Laurent Freidel and Kirill Krasnov [1] [102], who derive it with generating functional techniques from a path integral formulation of Plebański gravity. A somewhat different type of model was proposed by John Barrett and Louis Crane in 1998 [39], and short afterwards further clarified by John Baez and John Barrett himself [22, 24], as well as by Roberto De Pietri and Laurent Freidel [72].

The idea behind the Barrett-Crane model is that of reducing BF-theory to general relativity by imposing some version of Plebański's simplicity constraints directly at the *quantum* level. This is often summarized in the motto "first quantize, then constrain". In particular, the idea is that the constraints will reflect onto the state sum model by a restriction on the gauge-group representations which are summed over. Indeed, in a way analogous to the construction of the quantum tetrahedron, under quantization the classical B-field is promoted to a gauge-group generator J^{IJ}, and the constraint involving B are imposed as operator annihilating the states representing the quantum geometry.

The simplicity of this prescription made the model very appealing, as well as its clear geometrical meaning and its similarity to topological state sum models. However, as I will briefly discuss in section 4.1, "first quantizing and then constraining" risks to be a more delicate procedure than it seems at first sight.

To conclude this general introduction, I recall that many versions of the Barrett-Crane models have been proposed, [2] most of which adjusted the spinfoam face and edge weights to obtain some specific properties; however, the model has been in the end basically put aside because of a few undesired properties it has in all its versions. At this purpose, see the discussion section of [33] for a detailed critical review. Anyway, the main concern with the model, apart from the fact that it does not include the Barbero-Immirzi parameter and therefore could not be linked to the canonical approach, was the fact that it posses no degrees of freedom associated to the intertwiners (tetrahedra), but only to the spins (triangular faces) [24]. This issue was related to the fact that within the Barrett-Crane model the simplicity constraints were imposed "too strongly", and finally a few papers came out with models solving this issue. In this respect crucial contributions were given by the work of Etera Livine and Simone Speziale [150, 151], of John Engle, Roberto Pereira and Carlo Rovelli [92, 89], and of Laurent Freidel and Kirill Krasnov [103]. These contributions led to a new-class of spinfoam models, called the EPRL-FK model. Other (different) proposals for a solution of the issues inherent to the Barrett-Crane approach were advanced by Sergei Alexandrov [5], and by Aristide Baratin and

1. Not to be confused with what is nowadays known as the Freidel-Krasnov (FK) model [103], to be discussed later on in this section.
2. See e.g. [194] and reference 14 in [1].

Daniele Oriti [33, 34]. Though interesting, I will not discuss these proposals here.

Interestingly, the Barrett-Crane model had been criticized soon after its appearance on a very different basis. In fact, the papers by Roberto De Pietri and Laurent Freidel [72], by John Baez and collaborators [25], and by John Barrett and Christopher Steele [42], raised questions about the presence, interference, and possible dominance within the state sum of non-gravitational, as well as of geometrically degenerate, sectors. These issues are only partially solved in the new models, and play a crucial role in the analysis of their divergences. For this reason, I will come back on them multiple times in the following chapters.

4.1.1 *Discrete Simplicity Constraints*

In section 1.5, I reviewed the canonical analysis of the continuum simplicity constraints, which turn BF theory into general relativity. However, in the last sections a machinery was developed which relies on the choice of a (piecewise flat) discretization of the manifold. Consequently, to reduce BF spinfoam amplitudes to those of quantum gravity, one needs a discrete version of the simplicity constraints. Describing discrete simplicity constraints is the aim of this section. I proceed by first considering the quadratic version of the constraints, to finally discuss their linear version which is the key to the construction of the new spinfoam models.

The quadratic version of the simplicity constraints was already present in the works by Michael Reisenberger [187] and by John Barrett, Louis Crane and John Baez [39, 24] and was further elucidated (and put into the form presented here) by Roberto De Pietri and Laurent Freidel [72]. On the other side, linear simplicity constraints were first introduced by Laurent Freidel and Kirill Krasnov [103] and then analysed in detail by Steffen Gielen and Daniele Oriti [115].

Let me start from Plebański's original version of the constraints. These are expressed in equation 1.16, which I rewrite here for the reader's ease:

$$\epsilon^{\mu\nu\rho\sigma} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma} \stackrel{!}{=} \|e\| \epsilon^{IJKL} \,. \tag{4.1}$$

Writing a discrete analogue of this equation is not obvious, since there is no free form-index which one can use for smearing. However, provided that $\|e\| \neq 0$, this equation is equivalent to [72]

$$\epsilon_{IJKL} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma} \stackrel{!}{=} \|e\| \epsilon_{\mu\nu\rho\sigma} \,, \tag{4.2}$$

which on the contrary can be easily given a suitable discrete analogue. To see this, one has to first divide the previous equation into three subcases:

$$\epsilon_{IJKL} B^{IJ}_{\mu\nu} B^{KL}_{\mu\nu} \stackrel{!}{=} 0 \qquad \text{(diagonal simplicity)} \tag{4.3a}$$
$$\epsilon_{IJKL} B^{IJ}_{\mu\nu} B^{KL}_{\mu\sigma} \stackrel{!}{=} 0 \qquad \text{(off-diagonal simplicity)} \tag{4.3b}$$
$$\epsilon_{IJKL} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma} \stackrel{!}{=} \|e\| \; \sigma(\mu\nu\rho\sigma) \qquad \text{(volume simplicity)} \tag{4.3c}$$

where in these three equations no summation over Greek spacetime indices is performed, and different letters correspond to different indices. Also $\sigma(\mu\nu\rho\sigma) = \pm 1$ is the parity of the permutation of the four (different) indices. These can be interpreted as relations between triangles within a single four-simplex:

$$\epsilon_{IJKL} B^{IJ}_f B^{KL}_f \stackrel{!}{=} 0 \qquad \text{(diagonal simplicity)} \tag{4.4a}$$

$$\epsilon_{IJKL} B^{IJ}_f B^{KL}_{f'} \stackrel{!}{=} 0 \qquad \text{(off-diagonal simplicity)} \tag{4.4b}$$

$$\epsilon_{IJKL} B^{IJ}_{f_1} B^{KL}_{f'_1} \stackrel{!}{=} \epsilon_{IJKL} B^{IJ}_{f_2} B^{KL}_{f'_2} \qquad \text{(volume simplicity)} \tag{4.4c}$$

where f and f' represent different triangles within the same tetrahedron, while f_1 and f'_1 (and analogously f_2 and f'_2) represent two different triangles sharing vertex 1 (respectively 2) of the same four-simplex. The diagonal simplicity constraints basically require that the bivectors associated to triangles are "simple", i.e. of the form $B = e \wedge e$, while the off-diagonal ones require that the bivectors associated to triangles in the same tetrahedron span at most a three dimensional space. Finally the volume simplicity constraints require that the volume of a four simplex is well defined, i.e. it is the same when calculated choosing different (and suitable) couples of triangles.

In the canonical analysis of the simplicity constraints, there are primary and secondary constraints. The secondary constraints assure that the primary ones are conserved by the evolution. At the discrete level, these are expected to involve the four-simplex structure, and not just that of tetrahedra. As argued by Etera Livine and Simone Speziale in [151], the volume simplicity constraint can exactly be interpreted this way. Indeed, one can show that the imposition of the diagonal and off-diagonal simplicity constraints at *all* tetrahedra, plus the closure relation (Gauß constraint) at each tetrahedron (e)

$$\sum_{f \supset e} B^{IJ}_f \stackrel{!}{=} 0 \qquad \text{(closure)} \tag{4.5}$$

imply the volume simplicity constraint.

Therefore, one can argue that the imposition of the diagonal and off-diagonal simplicity constraints at every tetrahedron (i.e. at every "time") together with the closure constraint, automatically implies the volume simplicity. Consequently, volume simplicity can be safely neglected as an additional condition, and one can equivalently use the following set of constraints

$$\epsilon_{IJKL} B^{IJ}_f B^{KL}_f \stackrel{!}{=} 0 \qquad \text{(diagonal simplicity)} \tag{4.6a}$$

$$\epsilon_{IJKL} B^{IJ}_f B^{KL}_{f'} \stackrel{!}{=} 0 \qquad \text{(off-diagonal simplicity)} \tag{4.6b}$$

$$\sum_{f \supset e} B^{IJ}_f \stackrel{!}{=} 0 \qquad \text{(closure)} \tag{4.6c}$$

At this level, the main drawback of this set of constraints is that it is quadratic in the B field. Therefore, it determines such field only up to a set of discrete symmetries. Most importantly, two different sectors solve them (equation 1.17):

$$(\mathrm{I}\pm)\ B = \pm e \wedge e \quad \text{and} \quad (\mathrm{II}\pm)\ B = \pm \star e \wedge e\,. \tag{4.7}$$

There is a way to reduce this proliferation of classically independent sectors, which can however be related by quantum fluctuations around degenerate solutions. This can be achieved by considering a set of linear simplicity constraints, which have the same geometrical meaning as the quadratic ones, but are slightly stronger (see [103]). In contrast with the quadratic constraints, they require the introduction of an (internal) direction n^I_e at each tetrahedron (e). This direction is taken to be timelike in order to deal with spacelike geometrical structures only (the spacelike case has not been worked out yet). Note that the choice $n^I = (1, 0, 0, 0)$ is actually analogous to the choice of the time gauge in the analysis of section 1.5. This choice will be implicit in the following.

The linear set of constraints reads as follows

$$(n_e)_I B_f^{IJ} \stackrel{!}{=} 0 \qquad \text{(linear simplicity)} \tag{4.8a}$$

$$\sum_{f \supset e} B_f^{IJ} \stackrel{!}{=} 0 \qquad \text{(closure)} \tag{4.8b}$$

where the diagonal and cross-diagonal constraints are both taken into account by the first equation, which at the same time imposes the simplicity of the bivector and the fact that two bivectors associated to triangles in the same tetrahedron span at most a three dimensional space.

Another advantage of this set of constraints is that it easily allows the inclusion of the Barbero-Immirzi parameter into the quantum version of the constraints. In fact, in its presence, it is not the B field which upon quantization becomes the (Lorentz) algebra generator, rather the connection's conjugate momentum $\Pi = -\frac{1}{2\kappa}(\star + \gamma^{-1})B$ (see equation 1.19). Then, inverting this relation, one finds the following simplicity constraints in terms of the momentum Π:

$$(n_e)_I \left[(\gamma \star - 1)\Pi_f\right]^{IJ} \stackrel{!}{=} 0 \qquad \text{(linear simplicity)} \tag{4.9a}$$

$$\sum_{f \supset e} \Pi_f^{IJ} \stackrel{!}{=} 0 \qquad \text{(closure)} \tag{4.9b}$$

Before moving on to the quantum version of these equations, notice that this version of the simplicity constraint is nothing but the direct discretization of that obtained in the canonical analysis of the Holst-Plebański action, equation 1.21. As already discussed in section 1.5, within the classical continuum theory, this constraint is canonically conjugated to the torsionless condition for the spatial part of the spin connection (see equation 1.18), and together form a second class pair. However, I will not enter the set of issues related to the second class constraints. This is actually one of the main open debates in the spinfoam community.

Coming back to quantization, it is enough to substitute Π_{IJ} with the Lorentz Lie algebra generator J_{IJ}. For notational convenience, when working in the time gauge, I will split it into its spatial part (rotation generator) $J_i := \frac{1}{2}\epsilon_i{}^{jk} J_{jk}$, and its spacetime part (boost generator) $K_i := J_{0i}$. Therefore, upon quantization (in the time gauge), the previous set of constraints is easily seen to be equivalent to

$$\overrightarrow{LSC}_f := \gamma \vec{J}_f + \vec{K}_f \stackrel{!}{=} 0 \qquad \text{(linear simplicity)} \tag{4.10a}$$

$$\sum_{f \supset e} \vec{J}_f \stackrel{!}{=} 0 \stackrel{!}{=} \sum_{f \supset e} \vec{K}_f \qquad \text{(closure)} \tag{4.10b}$$

This set of constraints has to be imposed in the next subsection on the boundary states of the spin foam model. To be consistent with the discussion preceding equation 4.5, it will actually be imposed at the boundary of every tetrahedron.

4.1.2 *Imposing Simplicity Constraints*

In this subsection, I will discuss the imposition of the simplicity constraints *à la* EPRL, that is following the imposition procedure proposed by Johnatan Engle, Roberto Pereira, Carlo Rovelli and Etera Livine in a series of papers [92, 93, 90, 89]. A closely related model, which however exists only for the Euclidean theory, is the one proposed by Laurent Freidel and Kirill Krasnov [103]. Both models owe a lot to the work of Etera Livine and Simone Speziale [150, 151].

Living aside the closure constraint for a moment, one can focus on the imposition of the simplicity constraints at one triangle at the time. In analogy to the three dimensional case, the state of a triangle is in this context represented by a state in a unitary irreducible representation of the internal symmetry group, i.e. of $SL(2,\mathbb{C})$. The principal series of the unitary irreducible representations [3] of $SL(2,\mathbb{C})$ is labelled by a continuous index $\rho \in \mathbb{R}$ and by a discrete positive [4] one $k \in \frac{1}{2}\mathbb{N}$. These labels can be read out of the values the two Casimir operators of the Lorentz algebra take in each representations $V^{\rho,k}$:

$$C_1 := \frac{1}{2} J_{IJ} J^{IJ} = \vec{J}^2 - \vec{K}^2 \qquad \text{such that } C_1\big|_{V^{\rho,k}} = (k^2 - \rho^2 - 1)\,\mathrm{id} \tag{4.11a}$$

$$C_2 := \frac{1}{2} (\star J)_{IJ} J^{IJ} = \left[\vec{K}\cdot\vec{J} + \vec{J}\vec{K}\right] \qquad \text{such that } C_2\big|_{V^{\rho,k}} = 2k\rho\,\mathrm{id} \tag{4.11b}$$

Furthermore, the representation space $V^{\rho,k}$ decomposes naturally in subspaces diagonalizing the action of the $SU(2)$ subgroup generated by $\vec{J}$:

$$V^{\rho,k} = \bigoplus_{j\in k+\mathbb{N}} V^{\rho,k}_j\,, \quad \text{with} \quad V^{\rho,k}_j \cong V^j. \tag{4.12}$$

Accordingly, a state in this representation is written as

$$|\psi^{\rho,k}\rangle = \sum_{j,m} \psi^{\rho,k}_{j,m} |\rho,k;j,m\rangle \in V^{\rho,k}\,, \tag{4.13}$$

where $|\rho,k;j,m\rangle \in V^{\rho,k}_j \subset V^{\rho,k}$.

Having the states at disposal, one just has to impose the linear simplicity constraints on them as operator equations. The first guess would be to impose them "strongly":

$$\overrightarrow{LSC}|\psi^{\rho,k}\rangle \stackrel{!}{=} 0\,. \qquad \text{[wrong!]} \tag{4.14}$$

However this would be wrong. The problem with this equation is that the three components of the linear simplicity constraints do not form a closed algebra:

$$\left[LSC^i_f, LSC^j_{f'}\right] = \delta_{f\,f'} \epsilon^{ij}{}_k \left(2\,LSC^k_f - \frac{\gamma^2+1}{\gamma^2} J^k_f\right) \tag{4.15}$$

and therefore it would be inconsistent to impose them as above. Therefore, the strategy is to impose their commuting ("first class") component strongly, and its non-commuting one ("second class") in some appropriate way I will discuss in a moment.

The "first class" part of the simplicity constraints is actually given by the quadratic diagonal simplicity constraint, which in its quantum version is proportional to:

$$\frac{1}{2}\left[\star(\gamma\star - 1)J\right]_{IJ}\left[(\gamma\star - 1)J\right]^{IJ} = (1-\gamma^2)C_2 + 2\gamma C_1 \stackrel{!}{=} 0\,. \tag{4.16}$$

3. See the book by Werner Rühl [205].

4. Non positive values of k are admissible, however the representations (ρ,k) and $(-\rho,-k)$ are unitary equivalent.

These equation is clearly seen to put a constraint on the allowed representations for the boundary states. Indeed, the previous condition within $V^{\rho,k}$ becomes (twice)

$$(1-\gamma^2)k\rho+\gamma(k^2-\rho^2-1)\overset{!}{=}0\,. \tag{4.17}$$

which has no real solution for $k=0$,[5] and gives

$$\rho^{\pm}=\frac{1}{2}\left[\left(\frac{1}{\gamma}-\gamma\right)k\pm\sqrt{\left(\frac{1}{\gamma}+\gamma\right)^2k^2-4}\right]\quad \text{for } k\geqslant\frac{1}{2}\,. \tag{4.18}$$

where the choice of the $\pm$ sign gives rise to two branches of the solution. At this purpose note that the Plbański action is invariant under the transformation $\gamma\mapsto -1/\gamma$ (up to a redefinition of the Newton constant). For values of $k^2\gg 1$, or for values of the Barbero-Immirzi parameter either $|\gamma|\gg 1$ or $|\gamma|^{-1}\gg 1$, this formula gives approximate result

$$\rho\approx\begin{cases}\gamma^{-1}k\\ -\gamma k\end{cases} \tag{4.19}$$

However, this is not the way it is usually taken in the construction of the model. In fact, this solution of the diagonal simplicity constraint is quite involved and can lead to difficulties when one tries to further impose the off-diagonal part of the constraints (even in their linear form). Even worse, since $k=0$ is not a solution, and since $j\geqslant k$, the boundary states of this model could not have the spin zero representation in their decomposition. This evidently clashes with the cylindrical consistency requirement imposed on the (canonical) loop quantum gravity states, one would like to reproduce.

In the literature [89, 178], it is therefore argued that the term $-\gamma$ appearing in equation 4.17 is somewhat spurious and should not be considered. The usual argument says that it is due to some poorly chosen operator ordering. Nevertheless, in my opinion this argument does not look very solid, also on the basis of the fact that it seems hard to produce such a constant by commuting operators $\vec{J}$ and $\vec{K}$. In spite of this fact, the simplicity of the EPRL model comes out definitely in favour of the proposal of neglecting such a constant term. To my taste, a better argument would be to simply say that it is our geometric intuition in terms of simple bivectors which ceases to be meaningful at the verge of the Planck scale, and it rather "emerges" for quantum numbers large with respect to 1.

Whichever the justification, the resulting condition on the representation labels prescribed by the EPRL model is

$$\rho=\begin{cases}\frac{1}{\gamma}k\\ -\gamma k\end{cases} \tag{4.20}$$

where both branches of solutions are here taken into account.

To deal with the non-diagonal simplicity constraints, one then considers again their linear version, and imposes it with a master constraint technique [90, 178]. This demands of selecting the states with minimal eigenvalue for the master constraint

5. Note also, that this solution would not even be part of the complementary series of unitary irreducible representations of $SL(2,\mathbb{C})$: to belong to this series one must have $k=0$ and $-1<i\rho<0$, while in this case $i\rho=-1$.

operator $\mathbf{M}_f$, the latter consisting in the sum of the squared components of the linear simplicity constraints:

$$\mathbf{M}_f := \overrightarrow{\mathrm{LSC}}_f \cdot \overrightarrow{\mathrm{LSC}}_f = (1+\gamma^2)\vec{J}^2 + \gamma C_2 - C_1\,. \tag{4.21}$$

Fortunately, this operator is diagonalized by the states $|\rho, k; j, m\rangle \in V_j^{\rho,k}$:

$$\mathbf{M}|\rho, k; j, m\rangle = \left[(1+\gamma^2)j(j+1) + 2\gamma k\rho + \rho^2 + 1 - k^2\right]|\rho, k; j, m\rangle\,. \tag{4.22}$$

Taking into account the imposition of the diagonal simplicity constraints (equation 4.20), these eigenvalues in the two branches take the respective values

$$\begin{cases} (1+\gamma^2)\left[j(j+1) + \frac{1}{\gamma^2}k^2\right] + 1 \\ \\ (1+\gamma^2)\left[j(j+1) - k^2\right] + 1 \end{cases} \tag{4.23}$$

Now, notice that since $j \geqslant k$, the previous quantities are always positive. Therefore the smallest eigenvalue of the master constraint is obtained in the space $V_j^{\rho,k}$ with minimal spin j, i.e. $j = k$, and in the second branch, i.e. that for which $\rho = -\gamma k$.

To conclude, the EPRL model is defined by restricting the representation labels to

$$(\rho, k) \stackrel{!}{=} (-\gamma k, k)\,, \tag{4.24}$$

and by requiring the states to have minimal spin number

$$j = k\,. \tag{4.25}$$

NOTE I apologize with the reader for having used an unconventional convention for the Barbero-Immirzi paramter, which led me to define the EPRL model via the equation $\rho = -\gamma j$. Usually, the signed used in this relation is the opposite one, therefore, in most formulas, $-\gamma_{\text{here}} = \gamma_{\text{everywhere else}}$. This is notably the case in the definition of the Y_γ-map.

4.1.3 *Dupuis-Livine Map: EPRL Spin Networks are* $SU(2)$ *Spin Networks*

At the end of the last section, I showed how the EPRL Ansatz for the geometrical states which solve the simplicity constraints, forces them to be constructed out of a very limited family of representations and of special states therein, which in time gauge are states of minimal spin. In fact, reversing the logic, one can equivalently argue that it is the $SU(2)$ data plus the choice of a gauge (most conveniently, the time gauge) which completely determine the covariant form of the geometrical states. This latter version is the one most responded in the common parlance, since one sees the mapping of $SU(2)$ representations into $SL(2,\mathbb{C})$ ones as a particular embedding which appropriately parallels the embedding of a three dimensional tetrahedral geometry into a four dimensional space. Note that also this purely geometrical step requires the choice of some time normal, i.e. of a gauge.

More formally, the $SU(2)$ "triangle" states $|j; m\rangle \in V^j$ studied in the context of the quantum three-geometries (section 3.1), are mapped into some $SL(2,\mathbb{C})$ states via the so-called "wye"-map Y_γ:

$$\begin{array}{rccl} Y_\gamma : & V^j & \to & V^{-\gamma j, j} \\ & |j; m\rangle & \mapsto & Y_\gamma|j; m\rangle := |-\gamma j, j; j, m\rangle \end{array} \tag{4.26}$$

These states have the following property

$$\langle j;m|{Y_\gamma}^\dagger\left(LSC^i\right)Y_\gamma|j;m\rangle \approx 0 \tag{4.27}$$

where the equality holds up to terms of order $O(1/j)$. This property can be used as a defining property of the EPRL states (see [89, 90]). Actually, as shown by You Ding and Carlo Rovelli [75], if one wanted the previous equation to hold exactly, one should have shifted the EPRL relation $\rho = -\gamma k$ to $\rho = -\gamma(k+1)$. In the latter case, the master constraint would have selected states with $j = k$, too. Since the EPRL representations $\rho = -\gamma k$ are also the result of some guess, at this level there seems to be no preference between these two alternatives.

In order to talk about spin network states, one needs to consider functions of holonomies. However, the Lorentzian spin network functions cannot be arbitrary in order to encode the geometricity properties required by the simplicity constraints. Indeed, the representations appearing in their decomposition must be of the EPRL type. Following exactly the same logic as in the previous paragraph, it is immediate to realize that the EPRL spin networks must be uniquely related to $SU(2)$ spin networks. The formal, but intuitive, idea is that the EPRL spin networks are obtained by mapping an $SU(2)$ spin network $\psi(h)$ into

$$\text{“}\psi(h) \mapsto \psi\left({Y_\gamma}^\dagger g Y_\gamma\right)\text{”}. \tag{4.28}$$

This is concretely realized by the following embedding which maps square-integrable functions on $SU(2)$ into continuous functions of $SL(2,\mathbb{C})$:

$$\begin{array}{rrcl} K_\gamma: & L_2\Big[SU(2)\Big] & \to & C\Big[SL(2,\mathbb{C})\Big] \\ & \psi(h) & \mapsto & (Y_\gamma\psi)(g) := \int_{SU(2)} dh K_\gamma(g,h)\psi(h) \end{array} \tag{4.29}$$

where the integral kernel is

$$K_\gamma(g,h) := \int_{SU(2)} dk \sum_j d_j^2 \mathrm{Tr}_j(hk)\mathrm{Tr}_{-\gamma j,j}(kg)\,. \tag{4.30}$$

Using Schur's orthogonality relations and Peter-Weyl's theorem, the action of this map can also be written in a way which somehow justifies the writing of equation 4.28:

$$\psi(h) = \sum_j \sum_{m,n} \psi^{(j)\,m}{}_n D^j(h)^n{}_m \overset{K_\gamma}{\longmapsto} \left(K_\gamma\psi\right)(g) = \sum_j \sum_{m,n} \psi^{(j)\,m}{}_n D^{-\gamma j,j}(g)^{jn}{}_{jm}\,. \tag{4.31}$$

This is sometimes called the Dupuis-Livine map, and is actually a particular realization of a more general lift of $SU(2)$ spin networks to covariant (projected) [6] spin networks studied by Maïté Dupuis and Etera Livine in [82].

The last step one has to make to be really dealing with spin networks is that of implementing gauge invariance. This is required by the Lorentzian closure constraint of equation 4.10b. This can be most easily done by direct group averaging at every

6. Projected spin networks are generalized covariant versions of the usual spin networks. In particular, they contain extra information at every node about the choice of the time normal. They arise naturally in the canonical loop quantization of gravity when the gauge is kept arbitrary and not fixed to the "time gauge". See among others the works by Etera Livine, Sergei Alexandrov, and Marc Geiller, Marc Lachièze-Rey and Karim Noui [147, 7, 148, 5, 113, 112].

node of the spin network. Combined with the previous step, this defines the map f_γ from $SU(2)$ spin networks into the EPRL $SL(2,\mathbb{C})$ spin networks:[7]

$$\left(f_\gamma \psi\right)(g_\ell) := \int_{SL(2,\mathbb{C})} \prod_n dk_n \left(K_\gamma \psi\right)\left(k_{t(\ell)} g_\ell k^{-1}_{s(\ell)}\right). \tag{4.32}$$

The reader may be worried about a possible issue concerning this formula, i.e. with the fact that $SL(2,\mathbb{C})$ is noncompact and therefore integrating over too many copies of it may easily lead to divergences. Indeed, the previous definition is only formal at this stage; nonetheless, I will make it precise in the next subsection.

To conclude with this subject, let me mention that Sergei Akexandrov raised a possibly relevant point about this averaging [5, 4, 6]: he claims that the previous averaging is probably too strong, since it completely erases the information about the time normals to the nodes which are on the other hand needed to fully bridge with the $SU(2)$ spin networks. More recently, Aristide Baratin and Daniele Oriti constructed two spinfoam models (with and without the Barbero-Immirizi paramter) for Euclidean four dimensional quantum gravity where this issue does not arise [33, 34]. Interestingly, similar conclusions were also drawn by Steffen Gielen and Daniele Oriti after having analysed a (classical) linear version of the volume simplicity constraint [115]. However, see also the following article by Carlo Rovelli and Simone Speziale for a counter argument [202].

In order to complete the Hilbert space structure, one needs to provide the EPRL spin network states with a Hermitian scalar product. It is clear from equation 4.24 that the EPRL spin network states are built using a zero measure set of unitary irreducible representations of $SL(2,\mathbb{C})$. For this reason one immediately sees that their Hilbert space cannot be a subspace of $L_2\left[SL(2,\mathbb{C})^L//SL(2,\mathbb{C})^N, d\mu_H\right]$, i.e. it cannot be a subspace of the Hilbert space of $SL(2,\mathbb{C})$ BF-theory. Nonetheless, from the previous discussion another natural choice emerges, that is to use the one-to-one map between EPRL spin networks and $SU(2)$ spin networks and endow the first with the scalar product natural for the second:

$$< f_\gamma \psi_\Gamma, f_\gamma \psi'_\Gamma >_{\text{EPRL}} = < \psi_\Gamma, \psi'_\Gamma >_{\mathcal{H}_\Gamma}, \tag{4.33}$$

where the notation of section 2.1 has been used. For more details and subtler issues on this choice, I refer to the analysis performed by Maïté Dupuis and Etera Livine in [82].

4.2 EPRL DYNAMICS

As I discussed in section 4.1.1, the simplicity constraints are not preserved by the BF dynamics, therefore either one deals with the secondary constraints, or one imposes the simplicity constraints at each step of the evolution. The latter strategy is the one adopted in the Barrett-Crane and EPRL models[8] [92].

7. Here, I am committing a slight abuse of notation: the map K_γ is extended in the obvious way to functions of many $SU(2)$ variables, i.e. by tensoring many copies of its kernel and dealing with each of them separately.

8. This strategy was criticized by many researchers, in particular by Sergei Alexandrov, Marc Geiller, Karim Noui, Aristide Baratin, Daniele Oriti. See also the Internation Loop Quantum Gravity Seminar, where also Simone Speziale contributed to the discussion. To be more honest, one should say that virtually any researcher in spinfoams would be happy to see these constraints properly implemented. Nevertheless, the problem is that dealing with the secondary second class constraints is technically very difficult, and for the moment no practically valuable model taking them fully into account has been proposed. [114].

Concretely, given a four-dimensional manifold $\mathcal{M}$ and one of its triangulations Δ, one imposes the simplicity constraints at every tetrahedron of the triangulation and endows with the BF dynamics the spinfoam vertices (that is the four-simplices in Δ). For simplicity, let me start from the amplitude of a single four-simplex.

S_3 is triangulated by five tetrahedra arranged as on the boundary of a four- simplex, exactly as S_2 is triangulated by four triangles arranged as on the boundary of a tetra- hedron. These constitute the simplest (non-dege- nerate) triangulations of these spaces

ONE VERTEX DYNAMICS In this case the boundary state is encoded in the Γ_5 spin network graph, which is dual to the triangulation of a three-sphere by five tetrahedra, and the bulk is composed by a single vertex. I will name the external tetrahedra (nodes) by letters $a, b, c, \dots \in \{1, \dots, 5\}$. The same letters label the internal spinfoam (half-)edges, and the related holonomies, in the obvious way. Boundary holonomies, between tetrahedra are labelled by couple of nodes (ab). Holonomies at the edges are taken outgoing from the vertex, while boundary holonomies between tetrahedra are oriented from a to b with $a > b$, and indicated h_{ba} (I am using leftward composition). See figure 4.1. Now, according to the discussion of the last subsec-

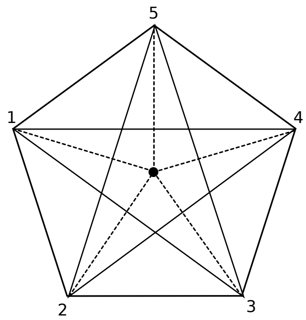


Figure 4.1: The Γ_5 graph, in solid lines, and the five spinfoam (half-)edges connecting its nodes to a spinfoam vertex. Γ_5 is dual to the three-sphere bounding the four-simplex dual to the spinfoam vertex.

tion, an EPRL boundary state is obtained by mapping an "ordinary" $SU(2)$ spin network state ψ_{Γ_5} into an $SL(2,\mathbb{C})$ one via the f_γ map (equation 4.32). Then, the EPRL-partition function for the boundary of a single four simplex is by definition the same as the BF one. However, in what follows, I will just consider the $SU(2)$ spin network state ψ_{Γ_5} to be the EPRL boundary state. Then, with this in mind:

$$\begin{aligned} Z^{\sigma_4}_{\mathrm{EPRL}}[\psi_{\Gamma_5}] &:= Z^{\sigma_4}_{\mathrm{BF}}\left[(f_\gamma \psi_{\Gamma_5})\right] \\ &= \int_{SL(2,\mathbb{C})} \prod_{a>b} dh_{ba} \int_{SL(2,\mathbb{C})} \prod_{a} dg_a \\ &\qquad \prod_{a>b} \delta\left(g_a g_b^{-1} h_{ba}\right) \left(f_\gamma \psi_{\Gamma_5}\right)(h_{ba}) \end{aligned} \tag{4.34}$$

where I used equations 3.27 and 3.30.

The previous expression is clearly only formal, since there are definitely too many integrations over the non-compact $SL(2,\mathbb{C})$ space. Indeed, since $(f_\gamma \psi_{\Gamma_5})$ is $SL(2,\mathbb{C})$-gauge invariant by construction, one can neglect the integrations of the $\{g_a\}$ straight away, obtaining:

$$Z^{\sigma_4}_{\mathrm{EPRL}}[\psi_{\Gamma_5}] = \left(f_\gamma \psi_{\Gamma_5}\right)(\mathbb{I}), \tag{4.35}$$

which simply reflects the fact that the BF dynamics imposes flatness up to gauge (see section 3.2.1).

Despite being very close to be well defined, this expression still needs a regularization. Indeed, it is easily realized that following the definition of equation 4.32, it contains one redundant integration over $SL(2,\mathbb{C})$, which can be safely neglected. I will indicate this with a "prime":

$$\left(f_\gamma \psi_{\Gamma_5}\right)(\mathbb{I}) = \int_{SL(2,\mathbb{C})} {\prod_a}' dg_a \, \left(K_\gamma \psi_{\Gamma_5}\right)\left(g_b g_a^{-1}\right). \tag{4.36}$$

The fact that in this form the vertex amplitude is well-defined and finite was shown by John Engle and Roberto Pereira in [91].

It is useful, for the following, to rewrite this last equation in a coherent state basis:

$$\begin{aligned}\left(f_\gamma \psi_{\Gamma_5}\right)(\mathbb{I}) = \sum_{\{j_{ab}\}} \int_{SL(2,\mathbb{C})} {\prod_a}' dg_a \int_{S_2} \prod_{a,b} \underline{d}_{j_{ab}} \vec{n}_{ab} \Big[\psi_{\Gamma_5}^{\{j_{ab}\}}(\{\vec{n}_{ab}\}) \times \\ \times \prod_{a>b} \langle j_{ab}, \vec{n}_{ba} | Y_\gamma{}^\dagger g_b g_a^{-1} Y_\gamma | j_{ab}, \vec{n}_{ab} \rangle \Big].\end{aligned} \tag{4.37}$$

GLUING OF VERTEX AMPLITUDES At this point, to fully define the EPRL dynamics, one needs to glue all vertex amplitudes to one another. The gluing of two four-simplices happens at a tetrahedron, that is at one spinfoam edge. This means that one needs to sum over all possible states of this tetrahedron, in such a way to form a resolution of the identity. This strategy has been already used - even if with the reversed logic - in the context of three-dimensional quantum gravity in section 3.3. I will not repeat the whole construction again, but just notice that the present case is completely analogue to the three-dimensional one, except for the fact that holonomies are in $SL(2,\mathbb{C})$ and act on the spin network states mediated by the Y_γ-map. Therefore, using coherent states notation it is easy to convince one-self that the EPRL amplitude for a triangulated manifold Δ, without boundary is

$$\begin{aligned}Z^{\Delta}_{\mathrm{EPRL}} = \sum_{\{j_f\}} \int_{SL(2,\mathbb{C})} {\prod_{(ev)}}' dg_{ev} \int_{S_2} \prod_{(ef)} \underline{d}_{j_f} \vec{n}_{ef} \prod_f d_{j_f} \times \\ \times \prod_w \langle j_f, \vec{n}_{e'f} | Y_\gamma{}^\dagger g_{e'v} g_{ve} Y_\gamma | j_f, \vec{n}_{ef} \rangle.\end{aligned} \tag{4.38}$$

A few remarks are in order. First of all, this equation is very similar to equation 3.44. Indeed, the structure of the three-dimensional quantum-gravity partition function was intentionally reproduced here: there are still amplitudes imposing approximate flatness up to gauge within each wedge (but not necessarily around each face). However, the Y_γ-map is here crucially implementing the geometricity of each four-simplex boundary state. This was "automatic" in three-dimensions, but it is not in four, as the whole discussion about the simplicity constraints was aimed to show.

Furthermore, the generalization of the previous formula to a spinfoam with boundary is straightforward: the boundary (coherent) intertwiners, rather than being summed over to form closure relations, are weighted according to the particular boundary state, as in equation 4.37.

Another important detail is the face weight $d_j = (2j+1)$ appearing in equation 4.38. This is the same as in the three-dimensional case. It did not arise naturally from the previous construction, but it has been chosen to guarantee a composition property of the spinfoam amplitude. This consists in requiring that the amplitude of a spinfoam Δ interpolating from ψ to ψ''', is the same as the composition of the amplitudes of two spinfoams Δ' and Δ'' such that $\Delta = \Delta' \sqcup \Delta''$ glued via an intermediate state ψ' which is summed over. Probably, this is most clearly expressed in symbols:

$$Z^{\Delta}[\psi, \psi''] = \sum_{\psi' \text{ colourings}} Z^{\Delta'}[\psi, \psi'] \cdot Z^{\Delta''}[\psi', \psi'']\,. \tag{4.39}$$

This argument is reminiscent of Michael Atiyah's multiplicative axiom [20], and had been already proposed as one of the spinfoam defining axioms by John Baez [22, 23] (from his point of view it was necessary to have a category with spin networks as objects and spinfoams as morphisms associative under composition). In the context of the EPRL model, the argument was used by Eugenio Bianchi, Daniele Regoli and Carlo Rovelli [50] to fix the face weights as here above. It is now easy to understand why the $SU(2)$ face weights are the relevant ones: this is because the boundary states of the EPRL model are essentially $SU(2)$ spin network states endowed with the natural $SU(2)$ scalar product (see the discussion end of the previous subsection).

Nevertheless, the previous argument is not always considered convincing enough, and in the literature it is possible to find other face weights choices. The most common one is the $SL(2,\mathbb{C})$-BF face weight, that is $(k^2 + \rho^2) \mapsto (1+\gamma^2)j^2$, once the simplicity constraints are taken into account (see e.g. [178]). Since the choice of the face weights is crucial for the convergence properties of the model (at this purpose see the discussion of John Baez and collaborators on the many versions of the Barrett-Crane model [25]), I will most often leave this weight free, indicating it

$$\mu(j_f)\,. \tag{4.40}$$

Notice that it is let depend on the face spin only.

4.3 EPRL SEMICLASSICAL LIMIT AND FLATNESS PROBLEM

Since the first proposal by Giorgio Ponzano e Tullio Regge of their model for three-dimensional Euclidean quantum gravity in1968 [183, 221], the development of spinfoam models has always been intertwined with the study of their semiclassical properties. Indeed, the spinfoam vertex amplitude is expected to reproduce the exponential of the Einstein-Hilbert action (with boundaries) of a flat four simplex. More specifically, the idea is to reproduce in the semiclassical large-spin regime some version of quantum Regge calculus. Classical Regge calculus, devised by Tullio Regge in 1961 [185], is a well defined discretization of general relativity [37]. Given a simplicial triangulation of the manifold Δ of interest, the only fundamental variables are the physical (as opposed to coordinate) lengths of the "bones" (sides) of the discretization.

This expectation is supported by results on the large-spin asymptotics of the EPRL model and of its Euclidean companion, the Fridel-Krasnov (FK) model. These are the result of the work of many people, and most notably of that of Florian Conrady and Laurent Freidel [69, 68], and of the Nottingham group led by John Barrett [41, 40]. Muxin Han with Mingyi Zhang and Thomas Krajewski extended the study to a general triangulation [128, 127, 126]. Technically, after having chosen an appropriate boundary state $\psi^0_{\Gamma_5}$, one rewrites the integrand of equation 4.37 in a way suitable for a stationary phase approximation, and then studies the critical point equations to show that they define a geometrical four simplex. Finally one reinserts the solutions to these equations into the integrand and evaluates it at its critical point. The result can be summarized by saying that the EPRL vertex amplitude of equation 4.37 (for an appropriate boundary state and in the large spin limit) is dominated by the sum of two terms, each of which is proportional to the exponential of i times the classical Regge action of the four simplex, taken with the two possible orientations. In formulas,

$$\left(f_\gamma \psi^0_{\Gamma_5}\right)(\mathbb{I}) \sim N_+ \exp\left[iS_R(j)\right] + N_- \exp\left[-iS_R(j)\right] , \tag{4.41}$$

where $N_\pm$ are functions of the spins and

$$S_R(j) = \sum_{a>b} \gamma j_{ab} \Theta_{ab}(j) \,. \tag{4.42}$$

Here $\Theta_{ab}(j)$ is the value of the four-dimensional (Lorentzian) dihedral angle between the tetrahedra a and b on the boundary of the four-simplex. They depend on the values of the ten spins only. A few remarks are in order.

First, notice that the presence of the two orientations has the same origin as in the Ponzano-Regge case and I refer to section 3.3 for more details on this. Then, notice also that in this expression the fundamental variables are the spins, which (in the large spin-regime) are proportional to the areas via the Barbero-Immirzi parameter γ and the factor $\hbar\kappa$ (equation 1.10):

$$a_{ab} \approx \hbar\kappa\gamma j_{ab} \quad \text{with} \quad j_{ab} \gg 1 \,. \tag{4.43}$$

Hence, reintroducing physical units:

$$S_R(a) = \frac{1}{\hbar\kappa} \sum_{a>b} a_{ab} \Theta_{ab}(a) \,. \tag{4.44}$$

Numerically, this is exactly the classical Regge action of the four-simplex. However, in this case the fundamental variables are the areas, and not the bone lengths. This fact as important consequences: when varying the action with respect to such variables, instead of finding the discretized Einstein field equation as in the case of Regge calculus, one finds a flatness constraint. Therefore, one can argue that the sum over the bulk spins is dominated by flat solutions. This argument was first advanced by Valentin Bonzom in [52], where he also showed how this issue, called the "flatness problem", is related to an *a priori* implementation of the closure constraints (see also the discussion after equation 4.32). Recently new arguments supporting the flatness problem have been presented by Frank Hellmann and Wojciech Kaminski [131, 132]. They showed that in the large spin regime, the dominating contribution to the spinfoam amplitude are (under some technical hypothesis) characterized by a discrete set of (distributional) curvatures at each spinfoam face:

$$\gamma\Theta_f \equiv 0 \mod 2\pi \,, \tag{4.45}$$

where $\Theta_f := \sum_{w\subset f} \Theta_w$. Θ_w being the same as Θ_{ab} for a general wedge w. Claudio Perini [180] revisited this result, and suggested the possibility that a "flipped" semiclassical limit is the right one, in which spins are large and the Barbero-Immirzi parameter is taken to be small, so to effectively neutralize the previous constraint. This limit had already appeared in the literature a few times before, in the context of loop quantum cosmology and of the spinfoam graviton propagator. I will discuss it in more detail in Chapter 6. Let me also mention that a more careful analysis of the large-spin limit has been very recently performed by Muxin Han [125, 124], who showed that the previous flatness result does not prevent to recover a meaningful semiclassical limit, but it shows that it must be sought in an appropriate, more sophisticated, regime in which the number of simplicies contributing to the discretization of some region grows as the supported curvature per simplex decreases.

4.4 SPINFOAM CONTINUUM LIMIT AND SPINFOAM DIVERGENCES

In this chapter, I reviewed the construction of the EPRL spinfoam model. In principle, it allows to calculate the transition amplitude between spin network states encoding the boundary (quantum) geometry of a certain manifold $\mathcal{M}$. This requires the choice of a triangulation.[9] Thus a question naturally arises: *which is the status of the triangulation Δ used to calculate the spinfoam amplitude?* This issue is far from being settled, and this thesis wants to offer some causes of reflection about it.

The most general and universal observation one can advance at this purpose is that either the model is for some reason independent on the chosen triangulation or such triangulation must be made disappear in some proper limit, which involves a large number of building blocks.

Triangulation independence is typical of topological theories, which give the same amplitude as soon as two different triangulations define the same topological manifold. I will come back on this in a little while. Anyway, triangulation independence is not a property of the EPRL model (and it is not a property of the Barrett-Crane and the Freidel-Krasnov models, either). Therefore, one is most likely led to consider finer and finer triangulations in order to define the EPRL model independently of any choice of triangulations.

This procedure has analogues in both common treatments of quantum field theories, which are the lattice regularization and the Feynman diagram perturbative approach. In the lattice case, one considers finer and finer regularizations in a fine tuned limit where the size and number of the cells scale inversely with respect to one another; this corresponds to a spacetime continuum limit. In the case of Feynamn diagrams, one sums over more and more complicated processes involving more and more complicated graphs; in this case it is the interaction process which is taken to be continuous instead of concentrated at particular point-like spacetime locations.

In spinfoam gravity, however, these clear pictures become somehow fuzzier. Interpreted as lattice theories, they define the dynamics *of* the lattice, rather than on the lattice. While, interpreted as Feynamn diagrams it is not at all clear in which parameter one is expanding. Anyway, a detailed analysis of these issues would take me too far from the main line of the discussion. Therefore, I leave it as well as a discussion of some of the strategies proposed for getting rid of the triangulation dependence to the end of the thesis (Chapter 7). For the moment, let me discuss a point which is very general and largely independent of the aforementioned strategies. This is the problem of divergences.

In feynmanology, divergences arise when the graph is complicated enough to let some internal momenta unconstrained. This happens in presence of loops, and is due to the integration over arbitrarily large momenta, which forces on us the interpretation that such divergences are UV, i.e. high energy or short distance, in origin. An almost perfect analogue of this happens in spinfoams.[10] Indeed, a triangula-

9. More generally of a cellular decomposition. One can even argue that the abstract two-complexes are the fundamental structures, from which four-manifolds only arise in some circumstances and limits (e.g. [22]). However, a counterargument - at least in the case of BF theory - is due to Valentin Bonzom and Matteo Smerlak, who showed that in order to properly gauge fix BF shift symmetry one needs to have access to cells of all dimensions [56, 55]. I will not pursue this discussion now, but stick to the definition of the model via triangulations.

10. I will not consider divergences due to low momenta, since they have no analogue in spinfoam. I will not consider divergences due to the sum over different graphs, either. In spinfoams, their relevance is scheme dependent.

tion which is "complicated enough" can give rise to unconstrained sums over spins, which being unbounded can give rise to divergences. However, the structures at the origin of divergences are not closed loops, but what Alejandro Perez and Carlo Rovelli named "bubbles" [179]. These, are higher dimensional analogues of loops, and correspond to closed surfaces (collections of spinfoam faces) in the dual triangulation Δ^*. This shift in the dimensionality is due to the fact that the unbounded variables one is summing over are the spins, which are associated to faces. [11]

Note that the divergences are due to large value of the spins, which in analogy to quantum field theoretical divergences correspond to high momenta on the group manifold. However, this manifold is not the physical one, it is not spacetime. In fact, from a spacetime perspective, these same divergences are most directly interpreted as associated to *large* geometries. To realize this, it is enough to think at the spectrum of the area operator, which is (approximately) proportional to the spin quantum number (see discussions in sections 3.3 and 2.2). Therefore, there is some sort of overturning in the UV-IR interpretation of divergences. At this purpose, note that the small-distance divergences are naturally cut-off by spinfoams thanks to the presence of an area-gap. This is somehow similar to the absence of infra-red divergences in massive quantum field theories.

An immediate mathematical consequence of this fact is that in the study of spinfoam bubble divergences semiclassical, that is large-spin, techniques may turn very useful. An explicit (and to my knowledge the first) example of this fact will be the subject of the central chapters of this thesis. At the light of the discussion of the previous section, an obvious advantage is that semiclassical techniques have received a lot of attention from the spinfoam community. A prototypical example, which I already recalled in multiple times, is the proposal of the first spinfoam model by Giorgio Ponzano and Tullio Regge in 1968, since it was in the first place rooted in the study of the semiclassical (large-spin) properties of certain coefficient of angular momentum recombination theory.

Let me take one step backwards, and reconsider the meaning of divergences from a different point of view. As I have already said, divergences are associated to closed surfaces in the dual triangulation Δ^*. These surfaces are in turn dual to some structures in the direct triangulation Δ. However, to which structure depends on the dimensions of the triangulated manifold. Let me start from the well understood three-dimensional case. Here, in the simplest case, [12] the closed surface has the topology of a sphere, and is dual to a vertex (zero-simplex) in the direct triangulation Δ. The sides (one-simplices) of the triangulation which start at this vertex are each dual to one of spinfoam faces composing the closed surface in Δ^* which is at the root of the divergence. The spin associated to each of these faces has to be interpreted as the length of the corresponding side. Changing the values of these spins (subject to constraints) heuristically corresponds to "moving the triangulation vertex around" leaving all the other untouched. It is then very tempting to interpret this invariance as a residual diffeomorphism symmetry which survived the discretization process. As first understood by Laurent Freidel and David Louapre

11. Notice that there is also some analogue of spin "conservation" at spinfoam edges, even if it is less rigid than the particle momentum conservation at Feynamn graph vertices. In three-dimensional gravity this is exactly given by the triangular inequalities among the three spins meeting at the edges; mathematically, this arises by the conditions for the existence of Clebsch-Gordan coefficients. In four-dimensional gravity, there is an analogue of this, always related to the existence of intertwiners among representations.

12. If the Δ is the triangulation of a manifold, and not of a pseudo-manifold or more singular objects, then the following statement is completely general. Again, see [56, 55] for the most general case.

[106, 107], this statement can actually be made precise.[13] It turns out that the relevant residual symmetry is rather BF shift symmetry, which is nevertheless related to diffeomorphisms as discussed in section 3.2.1. This is a gauge symmetry and can be properly fixed by adapting the Fadeev-Popov technique.[14] The result is a finite and well-defined partition function for the Ponzano-Regge model. It is interesting to note that the same formal division by the infinite gauge volume had been already proposed by Giorgio Ponzano and Tullio Regge themselves in their original paper, with the only - relevant - distinction that they introduced this factor as a necessary tool to have a (formally) invariant model under refinement of the triangulation.

Anyway, this new view on spinfoam divergences as stemming from unfixed gauge symmetries dramatically changes their interpretation and meaning. Once associated with a gauge symmetry they are not anymore an unwanted feature of a flawed description of the physical process, but rather the "good" signature of a discretization and quantization procedure which is preserving as many symmetries as possible.

Unluckily, it is not clear how this beautiful picture, which perfectly fits the spinfoam models of BF-theory, should generalize to spinfoam models of quantum four-dimensional gravity. In these models shift symmetry must be broken, and generally speaking also diffeomorphism invariance is also known not to be fully preserved under discretization (see e.g. Bianca Dittrich's [77] and references therein). Nevertheless, linearized gauge symmetries should still be present around flat solutions, and these might "pile up" to give rise to more general symmetries when a large number of building blocks is considered. This point of view and many other related issues have been investigated in the last years by Bianca Dittrich and collaborators, and a fairly large literature can now be found on the subject. Going into this would lead me quite far from the main line of discussion. However, as a reference, I point out to the interested reader the following articles [77, 27, 26, 28, 29, 78, 79, 81, 53]. It is instructive to note also the similarity of this regime and the one studied by Muxin Han and discussed at the end of the previous section.

Apart from the meaning and physical origin of the divergences, there is a common expectation in the spinfoam community. That is, the divergences originating in the sum over spins are naturally regularized once the cosmological term in the Einstein-Hilbert action is taken into account. For example, the cosmological term is used in dynamical triangulations to control the proliferating of simplices [8]. At a more abstract level, a modification of the Ponzano-Regge model is known which has no divergences, defines a finite topological invariant of three-manifolds, and has a vertex amplitude which is asymptotically equal to the exponential of the Regge action of a tetrahedron with the cosmological term included. This is the so called Turaev-Viro model, devised by Vladimir Turaev and Oleg Viro in 1992 [218] and based on a q-deformation of the $\{6j\}$-symbols used in the Ponzano-Regge model. The fact that the deformation parameter q is related to the cosmological constant in the asymptotic limit was shown by Shun'ya Mizoguchi and Tsukasa Tada in [164] (see also the work by Yuka Taylor and Christopher Woodward [212, 213]). Talking about this would open an enormous subject, and since it is quite disconnected from the rest of the thesis I will not go into it.[15] However, what is still relevant for the

13. See also the article [31] by Aristide Baratin, Florian Girelli, and Daniele Oriti , where they developed an algebraic language which allows to "see" this symmetry in the spinfoam amplitude quite explicitly.

14. See also [56, 55] for the generalization to arbitrary dimension and topology.

15. Efforts in dealing with quantum deformed spinfoam models has a long history. Analogues to the Turaev-Viro models in four-dimensions were worked out in the 90s by Louis Crane and David Yetter, and further studied by John Barrett and Louis Crane himself. More recently there has been furhter efforts

forthcoming discussion is the following heuristic interpretation of the finiteness of the quantum deformed models: the presence of a cosmological term classically implies the presence of cosmological horizons, whose distance can be interpreted as a maximal allowed distance. Therefore, the cosmological constant naturally puts a bound on large geometries. On the quantum side, this translates as a natural cut-off in the representation labels (or equivalently on the allowed Lie-algebra element norms). This phenomenon is commonly observed in the representation series of quantum groups, such as $SU(2)_q$, which is bounded by a maximal spin which depends on the value of q. Thus, starting from this observation, one can heuristically reinterpret the study of the divergences regularized via a (very large) cut-off on the spins as a toy model for the study of the most contributing graphs in the presence of a (very small) bare cosmological constant.

Wrapping up and concluding, there is no precise and definite expectation on how divergences should behave in a spinfoam model of quantum gravity. Moreover, the models which are on the market have not yet been studied in great detail from this point of view. One reason for this is simply that computing transition amplitudes with models like the EPRL one is quite involved, and no standard technique is available at the moment. Nonetheless, a lot of work is now being done trying to renormalize [16] simpler classes of spinfoam models, from both a lattice and field theoretic perspective. Therefore progress is hopefully about to come. In the work presented in this thesis, I focussed on a particular model (EPRL), in order to develop new computational techniques and above all to better understand the general and specific properties of a four-dimensional Lorentzian spinfoam model with the simplicity constraints implemented. To accomplish this, I studied the behaviour of the simplest diverging graph admitting a geometrical interpretation, i.e. of the so-called "melon graph". To do so, I introduced a rigid cut-off on the representation spins, and investigated the dominant behaviour in the limit of a very large cut-off. In the end, it has also been interesting to speculate about the cut-off as a toy model of a cosmological constant. The study of the melon graph is the core of this thesis, and the subject of its next part.

in the same direction by Karim Noui and Philippe Roche, by Muxin Han and collaborators, as well as by Winston Fairbairn and Catherine Meusburger. (See section 11.8 and 11.9 of [23] for these and many other early references, and also section 12.5 of [178] for references to more recent work on the subject.) Ongoing efforts are also those by Florian Girelli and Maïté Dupuis, in collaboration with Valentin Bonzom and Etera Livine.

16. Here, I am using the word "renormalization" in a very broad and unspecific way, which encompasses *a priori* very different approaches.

Part II

THE EPRL MELON GRAPH

This is the central part of the thesis, where the main results are presented. Here, I study the simplest example of a radiative correction in the EPRL model of quantum gravity as well as some of its consequences. In particular, what I study is the spinfoam analogue of the self-energy graph of quantum field theory, called the "melon"-graph. I describe its structure and meaning, and above all I detail the (approximate) calculation of its amplitude. A lot of attention is put into the geometrical interpretation of this amplitude and to the fact that it does not reduce to a simple wave function or mass renormalization. Then, I turn the attention to the consequences these radiative corrections may have on observable quantities, such as metric two-point functions. I show that in spite of the fact that the melon graph "trivializes" in the regime of interest, its presence has important effects also at the leading orders of the appropriate expansion.

5 THE EPRL MELON GRAPH

This chapter is extensively based on my paper [189].

5.1 A BRIEF SUMMARY OF THE TECHNICAL RESULT

For the convenience of the reader, in this first section I summarize the main results of the chapter, skipping as much as possible the various technicalities.

The so called melon graph is the simplest diverging spinfoam graph which endowed with a clear geometrical interpretation: it represent a chunk of space dynamically "splitting" into four and then "collapsing" back into a single one again. Geometrically, it is dual [1] to the triangulation of a three-ball by two four-simplices sharing four out of five of their boundary tetrahedra. This triangulation contains a single inner (triangulation) vertex. The summation over spins appearing in the formula for the spinfoam amplitudes implements the summation over all possible bulk geometries by letting this point explore the four-dimensional space in which it is embedded. Within the EPRL spinfoam model, the amplitude turns out to be infinite, and is regularized by a cut-off over the spins. The cut-off can be thought as bounding the space accessible to the inner point.

The divergences being generated by the large spins, one can use the relation the large-spin regime bares with the semiclassical limit. What "goes semiclassical", however, is not the full four-geometry, but an auxiliary three-geometry that can be read out of the internal-face structure of the melon graph. The relevant geometrical configuration is given by two tetrahedra with all their faces identified. The side lengths of these tetrahedra are given by the six spins associated to the six internal faces of the graph. At fixed large spins, each spinfoam-vertex amplitude tends to the cosine of the Regge action for the unique classical geometry identified by the six spin values, if it exists, and is suppressed otherwise. The spins being shared by the two faces the geometries at the two vertices is the same, up to orientation reversal. This freedom in the choice of the orientation is implemented in the aforementioned cosine.

Depending on the values of the spins, three possible types of geometries are given, according to whether the classical tetrahedron closes in Euclidean three dimensional space, Lorentzian two-plus-one dimensional space, or whether it is degenerate. The degenerate case has been mostly ignored. The reason is that the amplitude of this sector is not tractable with the techniques employed for the non-degenerate ones.

In each of the two non-degenerate sectors, two further subcases are given: the orientations of the geometry of the two tetrahedra (one per spinfoam vertex) can be taken as agreeing or as disagreeing. In the first, case the total action associated to the full configuration is twice the Regge-action of a single tetrahedron. In the second case, the total action is simply zero. When summing over all the possible values of the spins, the sectors where the action is non-zero are highly oscillating and are argued to be sub-dominant. Thus, one is left with the sectors associated to

1. This interpretation is valid only "on-shell". More about this will be said later in this chapter.

couples of congruent, orientation-disagreeing, Euclidean and Lorentzian tetrahedra.

Nevertheless, these two latter sectors behave very differently with respect to how they weight the in- and out-states of the melon graph (which I had up to this point practically discarded). In the Lorentzian case, this weighting is highly dependent on the internal-face spins. On the contrary, in the Euclidean case, it reduces to an overall scale which can be factored out in order to highlight the components of the in-out transfer matrix. I also suggested that, because of this reason, it is the Euclidean ("anti"-oriented) sector which dominates the amplitude (provided the degenerate sector is not dominating the whole amplitude). In this case the in-out transfer matrix (or S-matrix) is proportional to the operator

$$\mathbb{T}_\gamma{}^2 , \tag{5.1}$$

where

$$\begin{array}{rccc} \mathbb{T}_\gamma : & \bigotimes_a V^{j_a} & \to & \bigotimes_a V^{j_a} \\ & \bigotimes_a |j_a, n_a\rangle & \mapsto & \int_{SL(2,\mathbb{C})} dK \bigotimes_a Y_\gamma{}^\dagger K Y_\gamma |j_a, n_a\rangle \end{array} . \tag{5.2}$$

This operator is *not* a projector. However, it tends to the projector $\mathbb{P}$ onto the $SU(2)$-invariant subspace of $\bigotimes_a V^{j_a}$ in the limit in which $j_a \gg 1$. To be more precise,$\mathbb{T}_\gamma \to \mathbb{P}$ only up-to a rescaling depending on the spins $\{j_a\}$. This is a result by Jacek Puchta.

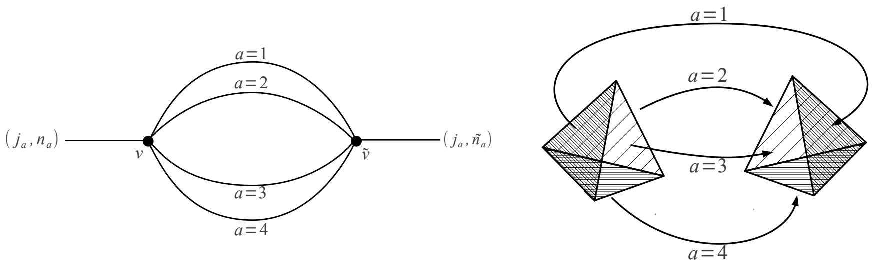
The proportionality constant between the melon graph transfer matrix and the $\mathbb{T}_\gamma{}^2$ depends in particular on the chosen face-weight for the EPRL spinfoam model. If this face weight $\mu(j)$ is such that $\mu(j \gg 1) \sim j^{\bar{\mu}}$, then this constant scales in the spin cut-off Λ as

$$\Lambda^{6(\bar{\mu}-1)} . \tag{5.3}$$

The two most common choices of $\mu(j)$ are $\mu(j) = d_j(2j+1)$, the $SU(2)$ face weight consistent with spinfoam composition, and $\mu(j) = (1+\gamma^2)j^2$, coming from $SL(2,\mathbb{C})$ BF-theory. The first choice leads to a logarithmic divergence $\log(\Lambda/\bar{j})$, where $\bar{j}$ is the scale of the external-face spins $\{j_a\}$. The second one leads to a power law divergence in Λ^6.

5.2 THE MELON GRAPH

The melon graph is the graph representing the spinfoam dual to the triangulation of a four-ball by two four simplices. Geometrically, this is a degenerate triangulation, even though topologically it is perfectly regular. In it, all but two tetrahedra bounding the two four-simplices are identified two by two. This corresponds to the fact that the two vertices of the melon graph share four edges (figure 5.1a). The remaining two (half) edges are dual to the two boundary tetrahedra, which triangulate a three-sphere as shown in figure 5.3b. A spacetime way of thinking to this graph is to imagine that a single tetrahedron (the boundary tetrahedron on the left of the graph, say) evolves into four tetrahedra (at the centre of the graph), and finally re-collapse into a single tetrahedron (the boundary one on the right of the graph). Finally, the melon graph can be reinterpreted as the first topologically nonsingular contribution (in a vertex expansion) to the renormalization of the "gluing function" between two tetrahedra belonging to different four-simplices. In this interpretation, the bare gluing would be the simple identification (up to an $SU(2)$ gauge).



(a) The melon graph. (b) The boundary three-sphere.

Figure 5.1

Notation is as follows: the two spinfoam vertices are dubbed v and $\tilde{v}$, the four internal edges are labelled by indices $a, b, \dots \in \{1, \dots, 4\}$, six internal faces labelled by couples of edges ab, and four external faces, each labelled by the unique internal edge it goes through. Consequently, spins of the internal faces are $j_{ab} = j_{ba}$, while spins of the external ones are j_a. Without loss of generality, I can choose to deal with a coherent-state basis for the boundary state. Thus, boundary intertwiners are $\int dh \bigotimes_a |j_a, n_a\rangle$ and $\int d\tilde{h} \bigotimes_a |j_a, \tilde{n}_a\rangle$ (tilded quantities are always associated to the right vertex in the picture, untilded to the left one). Note that from now on I will omit the vector symbol (arrow) above the unit vectors $n_a, \tilde{n}_a, \dots$ when no risk of confusion arises.[2] It will be also useful to introduce resolution of the identity onto coherent states at each edge in each internal face. This is done via the identity $\mathbb{I}_{j_{ab}} = \int \underline{d}_{j_{ab}} m_{ab}\, |j_{ab}, m_{ab}\rangle\langle j_{ab}, m_{ab}|$. Notice that since there are two edges per face $m_{ab} \neq m_{ba}$, and m_{ab} in this case labels a quantity within face ab sitting on edge a. Similarly, I will only write m_a for those states appearing in the resolution of the identity inserted along the external face a, at the edge a. Finally, the group elements associated to the internal half edges g_{ve} and $g_{\tilde{v}e}$ are noted g_a and $\tilde{g}_a$, respectively. The last two group elements associated to the external half edges are noted simply g and $\tilde{g}$. See figure 5.2 for a summary of the notation.

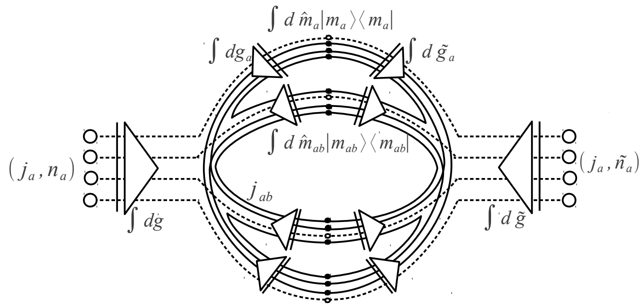


Figure 5.2: The melon graph with all the useful notation highlighted. Here, solid lines represent internal faces, dashed line external ones. Dots represent coherent states insertions. Triangles represent group elements. The bars on one side of the triangles represent Y_γ-map insertions. Integration over the $\mathbb{CP}^1$ variables are not shown.

2. Note that the usual magnetic number basis is distinguished by the use of a semicolon: $|j; m\rangle$.

In the rest of this chapter I will focus on the computation of the amplitude of this graph. It is expected to be divergent. Intuitively this divergence is due to the freedom of "moving around" the triangulation vertex at the centre of the four-ball (however, there is no clue that this is a gauge symmetry for the model, not even in some particular regimes). More algebraically, this type of graph is expected to be divergent because it posses many faces with many edges in common. Moreover these faces are the shortest possible (excluding geometrically and topologically degenerate configurations). In other words, this graph is part of a family of graphs (the "melonic graphs") which maximize the number of summations over the spins at fixed number of vertices. This fact makes this family intuitively, at least, the most diverging one.

Actually, there exists a class of theories, called coloured tensor models, for which the previous statement can be made precise. They are relevant to the present discussion because they are at the basis of the only known renormalizable group field theories, which in turn are a special sort of quantum field theory generating spinfoam amplitudes as Feynamn amplitudes. Although I am not going into this discussion here, let me mention the fact that melonic graphs (consisting of iterated insertions of the melon graph in itself) drive the $1/N$ expansion and renormalization flow of such theories and their study is crucial for the coloured tensor models and group field theories. Hence, this fact constitutes an extra motivation for the study of the EPRL melon graph. [3]

5.3 MELON GRAPH AMPLITUDE IN PATH-INTEGRAL FORM

As already mentioned in the case of the Ponzano-Regge model (section 3.3), it is very useful to introduce a path integral representation of the EPRL model. In fact, this representation is quite powerful when dealing with the large-spin regime, that is the regime expected to be relevant in the study of spinfoam divergences (see discussion of section 4.4).

Any formula I gave concerning the EPRL model has been somewhat implicit up to now. In order to have an explicit formula for the EPRL amplitude, one needs to explicitly write the $SL(2,\mathbb{C})$ Wigner matrices

$$D^{(-\gamma j,j)}(g)^{j,m}{}_{j,m'} = \langle -\gamma j, j; j, m | g | -\gamma j, j; j, m' \rangle. \tag{5.4}$$

In turn, to do this it is needed an explicit form of the $(\rho, k) = (-\gamma j, j)$-th unitary representation of $SL(2,\mathbb{C})$. In turn, to have this one needs to specify the space $V^{-\gamma j,j}$, the action of $SL(2,\mathbb{C})$ on it, and the inner product $\langle -\gamma j, j; j, m | -\gamma j, j; j', m' \rangle$. As explained, for example, in the book by Werner Rühl [205], this can be generally given in terms of particular square-integrable functions of two complex variables (z^0, z^1), homogeneous of degree $(-1 + i\rho + k; -1 + i\rho - k)$. I.e. in terms of particular functions $F \in L_2[\mathbb{C}^2]$, which satisfy for any $\lambda \in \mathbb{C} \setminus \{0\}$

$$F(\lambda z^\alpha) := \lambda^{-1+i\rho+k} \bar{\lambda}^{-1+i\rho-k} F(z^\alpha), \tag{5.5}$$

3. Tensor models and group field theories are a field in fast expansion. This can be in a large part attributed to the developments of new techniques which followed the introduction of "colours" in their formulation. The colour extra label - not present in the usual spinfoam paradigm - allows to keep track of the combinatorial properties of the generated triangulations and to disallow undesired too-degenerate configurations. It was introduced with these purpose by Razavan Gurau, and it was soon realized to be an incredibly powerful bookkeeping tool. Soon afterwards, proofs of the existence of a $1/N$-expansion and phase transitions for tensor models, as well as renormalizable group field theories appeared. I refer the interested reader to the review by Razvan Gurau and Jimmy Ryan [119] on the coloured tensor models, to Sylvain Carrozza's Ph.D. thesis [61] for the renormalizability of group field theories. For relations to quantum gravity, see the review by Laurent Freidel [98] and Daniele Oriti [171].

where z^α is a two-component spinor ($\alpha = 0, 1$), and the bar stands for complex conjugation. Now, given a general spinor z^α, build the $SU(2)$ matrix

$$u(z) := \frac{1}{\sqrt{\langle z, z\rangle}} \begin{pmatrix} z^0 & -\bar{z}^1 \\ z^1 & \bar{z}^0 \end{pmatrix} \equiv \frac{1}{\sqrt{\langle z, z\rangle}} (z, \mathcal{J} z), \tag{5.6}$$

where we introduced $\mathcal{J} : (z^0, z^1)^T \mapsto (-\bar{z}^1, \bar{z}^0)^T$, and the Hermitian scalar product in $\mathbb{C}^2$: $\langle z, w\rangle := \delta_{\dot{\alpha}\alpha} \bar{z}^{\dot{\alpha}} w^\alpha$. Notice that $\mathcal{J}^{-1} = -\mathcal{J}$. The restriction of the canonical basis $F^{(\rho,k)}_{j,m}(z) = \langle z|\rho, k; j, m\rangle \in V^{\rho,k}$ to normalized spinors (z such that $\langle z, z\rangle = 1$) is given by

$$F^{(\rho,k)}_{j,m}(z)\Big|_{\langle z,z\rangle=1} := \sqrt{\frac{2j+1}{\pi}} D^j\left(u(z)\right)^m{}_k , \tag{5.7}$$

where $D^j(u)^m{}_k$ is the usual $SU(2)$ Wigner matrix of $u \in SU(2)$. It is now straightforward to extend such a basis to non-normalized spinors via the homogeneity property (equation 5.5):

$$F^{(\rho,k)}_{j,m}(z) = \sqrt{\frac{2j+1}{\pi}} \langle z, z\rangle^{i\rho-1} D^j\left(u(z)\right)^m{}_k . \tag{5.8}$$

The last missing ingredients are the action of $SL(2, \mathbb{C})$ on the basis $f^{(\rho,k)}_{j,m}$ and the inner product in $V^{\rho,k}$. The first is given by

$$\begin{aligned}\left(g \triangleright F^{(\rho,k)}_{j',m'}\right)(z) &= F^{(\rho,k)}_{j',m'}(g^T z) \\ &=: D^{(\rho,k)}(g)^{j,m}{}_{j',m'} F^{(\rho,k)}_{j,m}(z),\end{aligned} \tag{5.9}$$

where g^T is the transposed of g in the fundamental representation of $SL(2, \mathbb{C})$ acting on $z \in \mathbb{C}^2$. And the inner product, by

$$\langle \rho, k; j, m|\rho, k; j', m'\rangle := \int_{\mathbb{CP}^1} \Omega_2[z]\, \overline{F^{(\rho,k)}_{j,m}(z)} F^{(\rho,k)}_{j',m'}(z) = \delta_{j,j'}\delta_{m,m'} , \tag{5.10}$$

where $\Omega_2[z] := \frac{i}{2}(z_0 dz_1 - z_1 dz_0) \wedge (\bar{z}_0 d\bar{z}_1 - \bar{z}_1 d\bar{z}_0)$ is the standard invariant two form on $\mathbb{C}^2 \setminus \{0\}$.

As constructed in section 4.2, the EPRL wedge amplitudes are of the form $\langle j, m|Y_\gamma{}^\dagger g Y_\gamma|j, m'\rangle = \langle -\gamma j, j; j, m|g| - \gamma j, j; j, m'\rangle$. These can be re-expressed more conveniently in the coherent-state basis: [4]

Note that the action of an $SU(2)$ element commute with the EPRL Y_γ-map.

$$\begin{aligned}\langle j, \vec{n}|Y_\gamma{}^\dagger g^{-1} g' Y_\gamma|j, \vec{n}'\rangle &= \langle j; j|Y_\gamma{}^\dagger h(\vec{n})^\dagger g^{-1} g' h(\vec{n}') Y_\gamma|j; j\rangle \\ &= \int_{\mathbb{CP}^1} \Omega_2[z]\, \overline{F^{(-\gamma j,j)}_{j,j}\left(\left[g h(\vec{n})\right]^T z\right)} F^{(-\gamma j,j)}_{j,j}\left(\left[g' h(\vec{n}')\right]^T z\right) \\ &= \frac{d_j}{\pi} \int_{\mathbb{CP}^1} \Omega_2^{g,g'}[z] \exp\left[S_w(z, g, g', \vec{n}, \vec{n}', j)\right] ,\end{aligned} \tag{5.11}$$

where we introduced the measure

$$\Omega_2^{g,g'}[z] := \frac{\Omega_2[z]}{\langle g^\dagger \bar{z}, g^\dagger \bar{z}\rangle \langle g'^\dagger \bar{z}, g'^\dagger \bar{z}\rangle} \tag{5.12}$$

and, in particular, the wedge action

$$S_w(z, g, g', \vec{n}, \vec{n}', j) := j \ln \frac{\langle \vec{n}, g^\dagger \bar{z}\rangle^2 \langle g'^\dagger \bar{z}, \vec{n}'\rangle^2}{\langle g^\dagger \bar{z}, g^\dagger \bar{z}\rangle \langle g'^\dagger \bar{z}, g'^\dagger \bar{z}\rangle} - i\gamma j \ln \frac{\langle g'^\dagger \bar{z}, g'^\dagger \bar{z}\rangle}{\langle g^\dagger \bar{z}, g^\dagger \bar{z}\rangle} . \tag{5.13}$$

4. To obtain the second equality, the fact the (ρ, k) are unitary representations was used.

To obtain the last step of equation 5.11, we should be compared with the wedge action of the Ponzano-Regge model equation 3.47.

For ease of notation, I will do from now on, a slight change of variables:

$$z_{ab} \mapsto \bar{z}_{ab}. \tag{5.14}$$

The only consequence is a global change of sign due to the measure on $\mathbb{CP}^1$: $\Omega_2[z] = -\Omega_2[\bar{z}]$, which has no consequence for the following calculation. This change of sign will consequently be reabsorbed in the integration measure and forgotten.

The needed tools being introduced, it is now immediate to write an explicit formula for the EPRL melon graph amplitude Z^{Λ}_{M}. This is done by specifying the general formula of equation 4.38 to the melon graph M. In order to have a well defined, and not just formal, amplitude a rigid cut-off $\Lambda \gg 1$ on the spins is inserted. Then, one has

$$\begin{aligned}
Z^{M}_{\mathrm{EPRL},\Lambda}(j_a, n_a, \tilde{n}_a) &= \sum_{\{j_{ab}<\Lambda\}} w_M(j_a, n_a, \tilde{n}_a; j_{ab}) \\
w_M(j_a, n_a, \tilde{n}_a; j_{ab}) &:= \int \mathcal{D}g\mathcal{D}z\mathcal{D}m \left[\prod_a \int_{S_2} \underline{d}_{j_a} m_a \langle j_a, n_a| Y_\gamma{}^\dagger g_a Y_\gamma |j_a, m_a\rangle \times \right. \\
&\qquad\qquad \left. \times \langle j_a, m_a| Y_\gamma{}^\dagger \tilde{g}_a^{-1} Y_\gamma |j_a, \tilde{n}_a\rangle \right] \times \\
&\qquad\qquad \times \left[\prod_{a<b} \mu(j_{ab})\, e^{S_{ab}}\right],
\end{aligned} \tag{5.15}$$

where a shorthand notation for the integrals has been introduced,

$$\begin{aligned}
\int \mathcal{D}g\mathcal{D}z\mathcal{D}m := &\left[\int_{SL(2,\mathbb{C})} dg d\tilde{g} \prod_a \int_{SL(2,\mathbb{C})} dg_a d\tilde{g}_a \; \delta(g)\delta(\tilde{g})\right] \\
&\left[\int_{\mathbb{CP}^1} \left(\frac{d_{j_{ab}}}{\pi}\right)^2 \Omega_2^{g_a, g_b}[z_{ab}] \Omega_2^{\tilde{g}_a, \tilde{g}_b}[\bar{z}_{ab}]\right] \\
&\left[\prod_{a<b} \int_{S_2} \underline{d}_{j_{ab}} m_{ab} \underline{d}_{j_{ab}} m_{ba}\right].
\end{aligned} \tag{5.16}$$

and the (internal) face action has also been defined

$$\begin{aligned}
S_{ab} :=& j_{ab} \ln \frac{\langle m_{ba}|Z_{ba}\rangle^2 \langle Z_{ab}|m_{ab}\rangle^2}{\langle Z_{ba}|Z_{ba}\rangle\langle Z_{ab}|Z_{ab}\rangle} - i\gamma j_{ab} \ln \frac{\langle Z_{ab}|Z_{ab}\rangle}{\langle Z_{ba}|Z_{ba}\rangle} + \\
&+ j_{ab} \ln \frac{\langle m_{ab}|\tilde{Z}_{ab}\rangle^2 \langle \tilde{Z}_{ba}|m_{ba}\rangle^2}{\langle \tilde{Z}_{ab}|\tilde{Z}_{ab}\rangle\langle \tilde{Z}_{ba}|\tilde{Z}_{ba}\rangle} - i\gamma j_{ab} \ln \frac{\langle \tilde{Z}_{ba}|\tilde{Z}_{ba}\rangle}{\langle \tilde{Z}_{ab}|\tilde{Z}_{ab}\rangle},
\end{aligned} \tag{5.17}$$

with

$$Z_{ab} := g_a^\dagger z_{ab} \quad \text{and} \quad Z_{ba} := g_b^\dagger z_{ab}, \tag{5.18}$$

and similarly for the tilded quantities. Note that $z_{ab} \equiv z_{ba}$, but $Z_{ab} \not\equiv Z_{ba}$.

Note the delta functions inserted in the measure over the $SL(2,\mathbb{C})$ group elements. As previously discussed these are needed to get a finite result, by gauge fixing a non-compact symmetry. The choice of gauge fixing the group elements g and $\tilde{g}$ instead of g_a and $\tilde{g}_b$ for some value of a and b is simply dictated by the will of keeping equations as symmetric as possible.

5.4 STRATEGY TO EVALUATE THE MELON GRAPH

To my knowledge there is no way to exactly evaluate the above amplitude. Therefore, one has to deal with estimations of some kind. A possible strategy is the following. Supposing that the amplitude is actually divergent, then the divergence must originate from the sum over spins. [5] In this case, provided the cut-off is large enough, one can always neglect a finite amount of "small" spins without altering the result of the summation significantly. This means that one can focus on the large-spin regime of the summand. The latter, on the other side, can be evaluated via a stationary phase technique. Here, an implicit assumption has been made: the stationary phase approximation can be confidently applied only to functions defined on a compact domain, while the EPRL amplitude requires multiple integrals on $SL(2,\mathbb{C})$ which is non-compact. Therefore special care has to be taken on the behaviour of the $SL(2,\mathbb{C})$ integrand for large group elements, in order to ensure the applicability of the techniques here proposed.

Notice that the external spins are kept fix along the whole calculation (and actually constitute the only scale of the problem apart from the cut-off), and only the internal spins are taken very large. For this reason, the external faces essentially decouple from the analysis of the dominant sector of the graph. This situation perfectly parallels the fact that one usually neglects the external legs in the study of divergences in Feynman diagrams. I will briefly come back on this later.

5.5 SYMMETRIES OF THE INTERNAL ACTION

In the following, it is crucial to have a precise understanding of the symmetries of the relevant "internal"

This action is dubbed "internal" because related to the internal faces of the graph only.

$$S := \sum_{a<b} S_{ab}. \tag{5.19}$$

First of all, at each vertex one can left-rotate all the g_a (resp. $\tilde{g}_a$) by an arbitrary $K \in SL(2,\mathbb{C})$ (resp. $\tilde{K}$), provided one simultaneously transforms the z_{ab} (resp. $\tilde{z}_{ab}$) appropriately:

Note the useful identity for $g \in SL(2,\mathbb{C})$ in its definition representation:

$$\mathcal{J}^{-1} g \mathcal{J} \equiv (g^{-1})^{\dagger},$$

where the dagger stands for Hermitian conjugation, which is the appropriate notion of adjoint for the Hermitian product in the spinor space $\langle w, z\rangle$.

$$\begin{cases} g_a \mapsto K g_a \\ \mathcal{J} z_{ab} \mapsto K \mathcal{J} z_{ab} \end{cases} \quad \text{and} \quad \begin{cases} \tilde{g}_a \mapsto \tilde{K} \tilde{g}_a \\ \mathcal{J} \tilde{z}_{ab} \mapsto \tilde{K} \mathcal{J} \tilde{z}_{ab} \end{cases} . \tag{5.20}$$

Furthermore, one can rotate the spinors on a given edge $\{m_{ab}\}_{b, b\neq a}$ by a $k_a \in SU(2)$, provided one also right-rotates the sets $\{g_a\}$ and $\{\tilde{g}_a\}$ by the inverse of the same k_a:

$$\begin{cases} m_{ab} \mapsto k_a m_{ab} \quad \forall b,\ b \neq a \\ (g_a, \tilde{g}_a) \mapsto (g_a k_a^{-1}, \tilde{g}_a k_a^{-1}) \end{cases} . \tag{5.21}$$

Since these symmetries are symmetries of the total *internal* action and of the "internal path integral" integration measure $\mathcal{D}g\mathcal{D}m\mathcal{D}z$, I will often refer to them as *gauge* symmetries. However, this term is inaccurate. Indeed, the amplitude of the external faces of the melon graph is invariant only under the transformations at the edges (equation 5.21), while it is not invariant under the transformations at the vertices (equation 5.20). This is essentially because of the gauge fixing performed on the external (half) edges (see end of section 5.3). Therefore, the evaluation of $w_M(j_{ab})$ at the critical "points" of the action is rather an average on the *internal*

5. This fact follows from a theorem shown in [91], see also the discussion around equation 4.36.

"gauge" orbits of the solutions to the stationary point equations.

If this may sound odd, there is an equivalent way of performing this calculation that does not need this "averaging". It consists in gauge fixing to the identity a couple of internal holonomies (e.g. g_1 and $\tilde{g}_4$). In this way, there will be no invariance at the vertices for the internal action, and consequently no averaging; nevertheless, the integration over the two $SL(2,\mathbb{C})$ group elements associated to the external edges would play the same role as the averaging in the previous setting. Also, as it will result clear from the forthcoming discussion, the degree of divergence of the graph would stay unaltered: there would be two less integrations on $SL(2,\mathbb{C})$ elements, but also two less $SL(2,\mathbb{C})$ symmetries to take into account. Eventually, the two settings are equivalent. As already stated, the reason why I did not choose the latter option, is to keep an higher degree of symmetry in the critical point equations.

5.6 CRITICAL POINTS AND GEOMETRIC INTERPRETATION

Formally, the large-spin regime I presented in section 5.4 is obtained by rescaling all the *internal* spins $\{j_{ab}\}$ by a common factor $\lambda \to \infty$. In this way one is led to consider the critical points of the internal action S:

$$S := \sum_{a<b} S_{ab}. \tag{5.22}$$

The critical point equations consist of two conditions: maximization of the real part of the action[6] and its stationarity with respect to variations of the variables appearing in it.

Maximization of the real part of the total action (which can be easily shown to be always negative or null via Cauchy-Schwartz inequalities) leads to

$$\Re(S) = 0 \quad \text{iff} \quad \begin{cases} m_{ab} = e^{-i\varphi^a_b}\frac{Z_{ab}}{||Z_{ab}||} \\ m_{ab} = e^{-i\tilde{\varphi}^a_b}\frac{\tilde{Z}_{ab}}{||\tilde{Z}_{ab}||} \end{cases}, \tag{5.23a}$$

for some phases $\varphi^a_b, \tilde{\varphi}^a_b \in [0, 2\pi]$. Using the definition of Z_{ab} (equation 5.18) and the definition of $\mathcal{J}$, this implies:

$$\begin{cases} g_b \mathcal{J} m_{ba} = \frac{||Z_{ab}||}{||Z_{ba}||} e^{i\varphi_{ab}} g_a \mathcal{J} m_{ab} \\ \tilde{g}_b \mathcal{J} m_{ba} = \frac{||\tilde{Z}_{ab}||}{||\tilde{Z}_{ba}||} e^{i\tilde{\varphi}_{ab}} \tilde{g}_a \mathcal{J} m_{ab} \end{cases}, \tag{5.23b}$$

where $\varphi_{ab} := \varphi^a_b - \varphi^b_a = -\varphi_{ba}$, and similarly for their tilded counterparts.

Stationarity of the action with respect to variations of the variables z_{ab}, $\tilde{z}_{ab}$, g_a and $\tilde{g}_a$ gives

$$\delta_{z_{ab}} S = 0 = \delta_{\tilde{z}_{ab}} S \quad \text{iff} \quad \begin{cases} g_b m_{ba} = \frac{||Z_{ba}||}{||Z_{ab}||} e^{-i\varphi_{ab}} g_a m_{ab} \\ \tilde{g}_b m_{ba} = \frac{||\tilde{Z}_{ba}||}{||\tilde{Z}_{ab}||} e^{-i\tilde{\varphi}_{ab}} \tilde{g}_a m_{ab} \end{cases}, \tag{5.23c}$$

$$\delta_{g_a} S = 0 = \delta_{\tilde{g}_a} S \quad \text{iff} \quad \sum_{b, b\neq a} \epsilon_{ab} j_{ab} \vec{m}_{ab} = \vec{0} = \sum_{b, b\neq a} \tilde{\epsilon}_{ab} j_{ab} \vec{m}_{ab}\,. \tag{5.23d}$$

Here, as in equation 3.7, $\vec{m}_{ab} := \langle m_{ab}|\vec{\sigma}|m_{ab}\rangle \in S_2$ and $\epsilon_{ab} = -\epsilon_{ba} = -\tilde{\epsilon}_{ab} \in \{\pm 1\}$. The particular choice of face orientations I implicitly chose in writing (5.15)

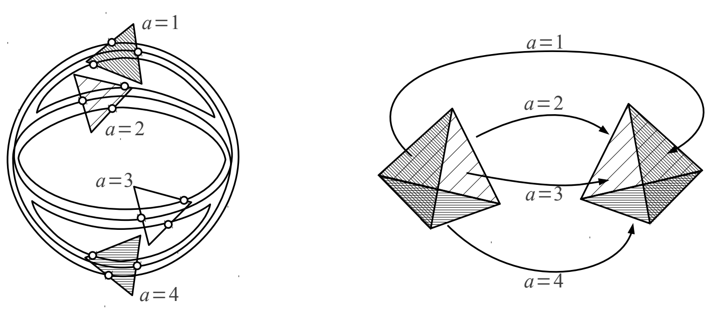
6. In our conventions there is no minus sign in front of the action within the exponential inside the path integral (equation 5.15).

gives (*up to a global sign*) $\epsilon_{ab} = +1$ for $a < b$. Finally, the equations issued by the variations of the m_{ab}'s do not lead to any further condition.[7]

The critical point equations (equations 5.23) are slightly laborious to work out. For this reason, and because of the fact that it is straightforward to apply to this case the techniques used in [40, 128], I am not detailing the steps which lead to the them. The main thing one has to take into account is the fact that in the present case, only the *internal-face* spins, and not all of them, need to be rescaled. Concretely, the action to be extremized is that of equation 5.22, which only involves the data along the internal faces of the melon graph. In this sense the graph external faces drop out of the "closure" equations (equation 5.23d), and no "parallel transport" equations analogous to equation 5.23b and equation 5.23c has to be considered along such faces. At a practical level, it is as if we were treating the large-spin limit of a closed melon graph with four strands only. This is graphically represented in figure 5.2 by the use of different styles for the internal- and external-face strands.

The critical point equations 5.23 can be explicitly and constructively solved. This construction can be found in all its details in the original work [189]. Here, I will just explain its physical meaning and how it can be "read" out of the critical point equations intuitively. In this way, I will be able to argue for the final result, in my opinion quite convincingly, without the need of solving almost any equations.

Let me start from the observation that having reduced the problem to the internal faces is going to lead us to deal with three-dimensional, rather then four dimensional geometries. This fact, which is clearly reflected in the solution of the critical point equations, can be intuited by simply looking at the structure of the graph built out of the internal faces only. This is depicted in figure 5.3a. This graph is dual to two tetrahedra glued together in such a way to form a three-sphere. This subgraph has the same structure as the boundary graph. However, the two must not be confused. Indeed, in this case the spinfoam faces are dual to the sides (one-simplices), and not the faces (two-simplices), of the tetrahedra; similarly, the spinfoam edges are dual to the triangular faces (two-simplices) of the tetrahedra, and not full tetrahedra (three-simplices), which are now dual the spinfoam vertices themselves.



(a) The subgraph of the melon graph M composed by only its internal face

(b) The three-sphere dual to the subgraph of M composed by only its internal faces.

Figure 5.3

This shift in dimensionality can be seen simply as an artefact of the fact of having neglected the external faces. Nonetheless it admits a (heuristic) geometrical inter-

7. Indeed, it is easy to show that the condition implied by the variations with respect to the m_{ab}'s is equivalent to the vanishing of the real part of the action.

pretation in itself. Roughly speaking, the fact of considering very large spins for the internal faces, but not for the external ones, gives to each vertex of the spinfoam a very "spiky" shape: it is as if one vertex of the four-simplex (the one at the centre of the four-ball dual to the melon graph) was pulled very far-away from the basis of the tetrahedron (dual to one of the external half edges). The three-dimensional tetrahedron dual to half of the internal-face subgraph can then be interpreted as the normal section of this spike. See figure 5.4. I will come back on this further on.

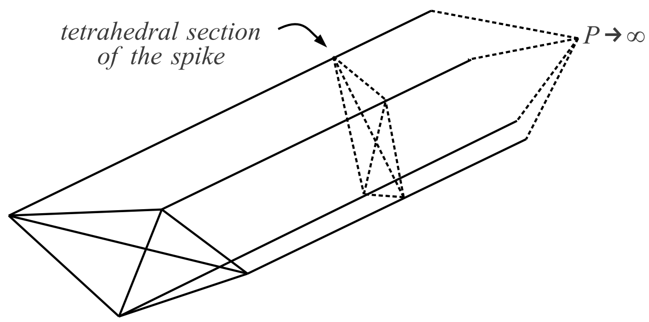


Figure 5.4: A graphical representation of the "spike" interpretation in four space dimensions.

5.6.1 *Geometric Interpretation of the Critical Point Equations at One Vertex*

It is now possible to turn the attention to the critical point equations. The easiest to understand is the closure relation of equation 2.8, which is a sort of "left-over" of Minkowski's closure relation (section 3.1). I say a leftover because in this case there are only three vectors present and they actually define a triangle rather than a polyhedron. Therefore this equation tells that at the critical point the S_2 variables $\{m_{ab}\}_{b,b\neq a}$ must be such that they define vectors of length $\{j_{ab}\}_{b,b\neq a}$ which close into triangles.

The parallel transport equations (equations 5.23b and 5.23c), on the other hand, contain the meaning of the group variables $\{g_a, \tilde{g}_a\}$. Indeed, they tell us how the side vectors m_{ab} and m_{ba}, which are associated to the same side of the tetrahedron, must be identified with one another. In other words, the group elements $\{g_a\}$ (or $\{\tilde{g}_a\}$) tell how the tetrahedron face frame must be parallel transported to the "centre" of the tetrahedron ν (or $\tilde{\nu}$) before being compared and identified among them. The further normalization and phase factors $\|Z_{ba}\|/\|Z_{ab}\| \exp(-i\varphi_{ab})$ measure the mismatch of two face frames once parallel transported in the same common frame: they are found to basically correspond to the dihedral angles between the tetrahedron faces. By the way, this discussion gives also a clean geometrical interpretation to the symmetries of the action (equations 5.20 and 5.21): each face frames can be rigidly rotated, and the whole tetrahedron globally rotated without altering the geometrical content of the action.

Finally, equation 5.23a can be read as a simple fixing of the auxiliary variables $\{z_{ab}, \tilde{z}_{ab}\}$ in terms of the unit vectors $\{m_{ab}\}$, with no further geometrical content.

A question may have occurred to the reader: where does the need for two sets of parallel transport equations, for $\{m_{ab}\}$ and $\{\mathcal{J}m_{ab}\}$ respectively, come from? The two sets are both needed basically because $(m, \mathcal{J}m)$ constitutes a complete basis of the complex vector space $\mathbb{C}^2$. Nonetheless, the two sets of equations contain some redundancy (the normalization and phase factors of the second can be deduced from those of the first using the fact that $\det(g_a) = 1$). Moreover, they would contain exactly the same information, if the group elements $\{g_a, \tilde{g}_a\}$ were in $SU(2)$ instead than in $SL(2, \mathbb{C})$.

It is interesting to notice that the solution to the previous equations is essentially unique (up to the obvious symmetries and a finite number of relevant ambiguities) in terms of the six spins $\{j_{ab}\}$. This descends from the fact that the geometrical structure these equations determine, a tetrahedron, is rigid and completely defined by its side-lengths. On the one hand, this means that the solutions of the equations at the two spinfoam vertices are basically the same, up to a finite set of ambiguities, which can in turn be shown to be related to orientation choices. [8] On the other hand, this allows to fully classify the possible solutions of these equations in terms of the values of the six spins.

To do this, it is useful to start with the following geometrical remark, whose easy demonstration can be found in the appendix of [189]. Any set of six edge lengths $\{j_{ab}\}$ (strictly) satisfying triangular inequalities for any subset $\{j_{ab}\}_{b, b\neq a}$ defines up to global rotations and boosts, and up to space inversion, a (non-degenerate) tetrahdron in $\mathbb{R}^{1,3}$. A tetrahedron in $\mathbb{R}^{1,3}$ is here understood as a set of four points. It is non degenerate if the four vectors identifying these points span a three-dimensional subspace of $\mathbb{R}^{1,3}$. The tetrahedron can be either Euclidean, if the metric restricted to the subspace it defines is positive defined, or Lorentzian, otherwise. An Euclidean tetrahedron can be equivalently defined by requiring that its edge lengths satisfy not only the triangular inequalities at evey face, but also the condition of positivity of the Cayley-Menger determinant. [9] With this notions at hand, it is now possible to classify the critical point equations at *each* vertex of the melon graph into four cases, depending on the particular values of the spins $\{j_{ab}\}$:

◇ NO SOLUTION If the $\{j_{ab}\}_{b, b\neq a}$ do not satisfy the triangular inequalities at each edge a, then there is no solution to the stationary point equations (in particular to the closure equations). The corresponding graph amplitude results exponentially suppressed with the spin scale λ.

◇ EUCLIDEAN SECTOR If the triangular inequalities are strictly satisfied at every edge, and the Caley-Menger determinant of the spins $\{j_{ab}\}$ is positive, then there are (at each vertex) exactly four solutions to the stationary point equations, which correspond to two different geometries characterised by a sign choice. These two geometries can be interpreted as the two different, parity-related [10] Eculidean tetrahedra one can build out of the six side lengths $\{j_{ab}\}$. The two parities are char-

8. This is a crucial point, and will be further discussed later.

9. This is the determinant of a certain matrix constructed out of the squared side lengths of a general n-simplex in Euclidean space, and it is proportional to the *square* of its n-dimensional volume. In particular, given $n(n+1)/2$ numbers, they can be interpreted in a given order as the lengths of the sides of an n-simplex if and only if all the Cayley-Menger determinants associated to the n-simplex and to all its subsimplices of any dimension are positive (see e.g. [163]). The first proof of this theorem is due to Karl Menger [157]. The determinant is named after him and after the great mathematician Arthur Cayley, who introduced it in his first paper back in 1841 [64].

10. This can be seen as the change in sign of the $\hat{z}$-axis, corresponding to $\eta \mapsto -\eta$, defined in equation 5.27.

acterised by a sign $\nu_v = \pm 1$.

◇ LORENTZIAN SECTOR Similarly, if the triangular inequalities are strictly satisfied at every edge, but the Caley-Menger determinant of the spins $\{j_{ab}\}$ is negative, then there are (at each vertex) exactly four solutions to the stationary point equations, which still correspond to two different geometries characterised by a sign choice. However, these two geometries can now be interpreted as the two different, time-reversal-related "Lorentzian tetrahedra" one can build out of the six (Minkowskian) side lengths $\{j_{ab}\}$. Remark that $j_{ab}^2 > 0$, hence the reconstructed tetrahedron has space-like sides and faces. Once again the two possible geometries are characterised by a sign $\nu_v = \pm 1$.

◇ DEGENERATE SECTOR If the triangular inequalities are satisfied at each edge by the $\{j_{ab}\}_{b,b\neq a}$, but they are saturated at least at one edge, then the graph is said to belong to the (geometrically) *degenerate sector*. It shall as well be called degenerate, the sector spanned by the $\{j_{ab}\}$ strictly satisfying the triangular inequalities, whose Caley-Menger determinant is null (zero volume tetrahedra). The study of this sector is left for future work, however see the last part of section 5.8.1 for a partial and tentative treatment of it.

Thus, define a label $\sigma(j)$ function of the six spins $\{j_{ab}\}$ only, such that:

$$\sigma(j) = \begin{cases} \mathrm{NG} & \text{if } \{j_{ab}\} \text{ can } \textit{not} \text{ be identified with the sides of} \\ & \quad \text{a tetrahedron} \\ \mathrm{E} & \text{if } \{j_{ab}\} \text{ are the sides of a non-degenerate} \\ & \quad \text{Euclidean tetrahedron} \\ \mathrm{L} & \text{if } \{j_{ab}\} \text{ are the sides of a non-degenerate} \\ & \quad \text{Lorentzian tetrahedron} \\ \mathrm{D} & \text{if } \{j_{ab}\} \text{ are the sides of a degenerate tetrahedron} \end{cases} . \tag{5.24}$$

The factors ν_v and $\nu_{\tilde{v}}$ are not the only orientation choices appearing in the problem. Another one is that related to the choice of face orientation in the writing of the amplitude. Actually, the EPRL amplitude (for any spinfoam) is invariant under reversal of an internal face orientation. However, this is not the case for the *integrand* of equation 5.15, and neither for the closure equation. In particular, in the latter, the sign coefficients ϵ_{ab} and $\tilde{\epsilon}_{ab}$ depend on such choice at each face. Following the choice I implicitly made when writing equation 5.15, one has

$$\epsilon_{ab} = -\tilde{\epsilon}_{ab} = \begin{cases} \nu & \text{if } a < b \\ -\nu & \text{if } a > b \end{cases} , \tag{5.25}$$

where $\nu \in \{\pm 1\}$ is an arbitrary global choice of sign.

All these subtleties about orientations and sign ambiguities become relevant when combining the critical points at the two vertices to estimate the value of the partial amplitudes $w_M(j_{ab})$. Also, notice the analogy existing between them and the orientation invariance leading to the cosine in the asymptotics of the Ponzano-Regge vertex amplitude.

5.6.2 *Internal Action at Critical Points*

In order to evaluate the phase of the partial amplitude $w_M(j_{ab})$ at its critical points, one needs to relate the algebraic quantities appearing in the spinfoam action $S = \sum S_{ab}$, to the geometrical quantities characterizing the tetrahedral geometry. To do this, let me start by inserting the formal solution of the critical point equation $\mathfrak{R}(S) = 0$ into the spinfoam action:

$$S_{ab}|_{\text{crit}} = 2j_{ab}i\left(-\varphi_{ab} - \gamma \ln\frac{||Z_{ab}||}{||Z_{ba}||}\right) + 2j_{ab}i\left(\tilde{\varphi}_{ab} - \gamma \ln\frac{||\tilde{Z}_{ba}||}{||\tilde{Z}_{ab}||}\right). \tag{5.26}$$

The solution of the critical point equations can be summarized in the values taken by the parameters $\eta_{ab}, \tilde{\eta}_{ab} \in \mathbb{C}$ defined as

$$\eta_{ab} \equiv -\eta_{ba} := i(\varphi_{ab} - \phi_{ab} + \phi_{ba}) + \ln\frac{||Z_{ab}||}{||Z_{ba}||}, \tag{5.27a}$$

$$\tilde{\eta}_{ab} \equiv -\tilde{\eta}_{ba} := i(\tilde{\varphi}_{ab} - \phi_{ab} + \phi_{ba}) + \ln\frac{||\tilde{Z}_{ab}||}{||\tilde{Z}_{ba}||}, \tag{5.27b}$$

In these formulas, the phases ϕ_{ab}, ϕ_{ba} are arbitrary phases actually coming from the definitions of the $\{m_{ab}\}$. This is why they are the *same* in both η_{ab} and $\tilde{\eta}_{ab}$.

In a specific (edge) gauge, the parameters η_{ab} enter the expressions of the critical values of the group elements $g_a, \tilde{g}_a$ via the equations

$$G_{ba} := g_b^{-1} g_a \;\Big|_{\text{crit.}} = e^{-i\theta_{ba}\sigma_y} e^{\eta_{ab}\sigma_z} e^{i\theta_{ab}}. \tag{5.28}$$

See [189] for the details about the gauge, and the construction of the real functions $\theta_{ab} = \theta_{ab}(j)$, which turn out to be the face angles of the tetrahedron. An analogous formula holds for the tilded quantities. Notice that $\{G_{ba}, \tilde{G}_{ba}\}$ are invariant quantities under the $SL(2,\mathbb{C})$ vertex gauge transformations (equation 5.20), and are gauge-fixed quantities with respect to the $SU(2)$ edge gauge transformations (equation 5.21).

The solutions of the critical point equations can then be shown to be

$$2\eta_{ab} \equiv \nu_v \epsilon_{ab} \Theta_{ab}(j) \mod 2i\pi \tag{5.29a}$$

$$2\tilde{\eta}_{ab} \equiv \nu_{\tilde{v}} \tilde{\epsilon}_{ab} \Theta_{ab}(j) \mod 2i\pi, \tag{5.29b}$$

where

$$\Theta_{ab}(j) := \begin{cases} i\,\Theta^E_{ab}(j) & \text{Euclidean} \\ \Theta^L_{ab}(j) & \text{Lorentzian} \end{cases}. \tag{5.30}$$

In the last equation, I introduced the functions Θ^E_{ab} and Θ^L_{ab} of the six spins $\{j_{ab}\}$, here indicated for brevity by the collective name j. These functions are defined as follows. If $\{j_{ab}\}$ defines an Euclidean tetrahedron, then $\Theta^E_{ab}(j) \in [0,\pi]$ represents the unique (absolute value of the) dihedral angle associated to its side ab. While, if $\{j_{ab}\}$ defines a Lorentzian tetrahedron, then $\Theta^L_{ab}(j) \in \mathbb{R}^+ + i\{0,\pi\}$ represents the unique (absolute value of the) dihedral rapidity angle (boost parameter) associated to its side ab. In this case, the possible presence of an imaginary part equal to π is needed to take into account the case in which the two faces sharing the side ab have opposite time orientations.

In terms of the $\{G_{ab}, \tilde{G}_{ab}\}$, this means that they are elements of the $SU(2)$ subgroup of $SL(2,\mathbb{C})$ at the Euclidean critical points, and "genuine" elements of $SL(2,\mathbb{C})$ in the Lorentzian case:

$$G_{ab} \in \begin{cases} SU(2) & \text{Euclidean} \\ SL(2,\mathbb{C}) & \text{Lorentzian} \end{cases} \tag{5.31}$$

This property is reflected onto the $\{g_a, \tilde{g}_a\}$ provided an appropriate vertex gauge is choosen.

Notice that the stationary phase equations lead more naturally to the solution for $2\eta_{ab}$, rather than that for η_{ab} itself. This reflects the fact that the vector geometry described by such solutions is "doubly covered" by the spinorial one encoded in the fundamental variables $\{m_{ab}, g_a, \tilde{g}_a\}$. As explained in detail in [189], and briefly discussed later, it is indeed found that the stationary phase technique here employed is not able to distinguish between face holonomies which differ by a sign (in their definition representation). However, this is not really surprising, since this distinction is somewhat subtle and already in the Ponzano-Regge model it can be seen only through the sum over spins. To see this, it is enough to look at the following equality:

$$\delta(\pm h) = \sum_j d_j \mathrm{Tr}_j(\pm h) = \sum_j d_j e^{2ji\pi} \mathrm{Tr}_j(h) = \sum_j (-1)^{2j} d_j \mathrm{Tr}_j(h)\,. \tag{5.32}$$

Thus, without entering into these details any further, I will simply note the solutions for the η_{ab} by

$$\eta_{ab} \equiv \frac{1}{2}\nu_v \epsilon_{ab} \Theta_{ab}(j) + [i\pi]_{ab} \quad \mathrm{mod}\ 2i\pi\,, \tag{5.33}$$

and similar equations hold for the tilded quantities. Here, the square brackets just symbolize the fact that there could be the need (or the possibility) of adding an extra $i\pi$ to the displayed value of η_{ab}. Note, however, that this additional terms are not independent from one another, hence the face label around the square brackets. In the end, this signs can be shown to exactly mirror the $\mathbb{Z}_2$ ambiguity just discussed.

At this point it is straightforward to write the value of the action at the critical points. In the Euclidean sector, one distinguishes three cases according to the choices of ν_v and $\nu_{\tilde{v}}$:

$$S_{ab}|^{E}_{\text{crit.}} = \begin{cases} 0 + \sum_{a<b} 2j_{ab}[i\pi]^0_{ab} & \text{if } \nu_v = -\nu_{\tilde{v}} = \pm 1 \\ \pm \sum_{a<b} 2j_{ab}\left(i\Theta^E_{ab}(j) + [i\pi]^0_{ab}\right) & \text{if } \nu_v = \nu_{\tilde{v}} = \pm 1 \end{cases} . \tag{5.34}$$

In the Lorentzian sector, one simply finds the obvious transposition of this result.

Notice that in the case where the orientations at the two vertices are opposite, a zero total action is obtained, and no oscillatory feature is left within the path integral (apart for a possible sign oscillation due to the $\{[i\pi]_{ab}\}$). Conversely, if the two orientations agree, one basically recovers the Regge action (possibly generalized to the Lorentzian case) associated to the three-sphere triangulated by the two tetrahedra arising from the geometrical configuration detailed at the beginning of this chapter. [11] In this case the total phase appearing in the path integral is strongly dependent from the specific values of the spins $\{j_{ab}\}$.

11. For clarity, let me specify that the three-sphere I am referring to is the "virtual" one, obtained via the identification of the faces of the two "virtual" tetrahedra arising from the normal section of the spike in figure 5.4.

In particular, notice that such phase depends (exactly) linearly from the scale of the spins. This fact allows me to argue that the summation over the spins in the sector where the two parities agree is suppressed roughly by one power of the cut-off if compared to the sector where they do not agree. This can be most easily seen by performing the summation in the (six-dimensional) spin space in "spherical coordinates". Within a shell of given "radius" the summand is *a priori* slowly varying in the two sectors, and nothing clear can be stated; however, when moving in the radial direction, in the sector where the parities disagree, the summand acquires an oscillating factor, which suppresses the value of the sum by roughly a power of the cut-off. [12]

For this reason, I argue that the dominant contributions to the *sum over spins* are those associated with opposite choices of parity at the two vertices. These can come either from the Euclidean or the Lorentzian sector. [13]

A completely analogous conclusion was reached in another work I did together with Marios Christodoulou, Miklos Långvik, Christian Röken and Carlo Rovelli [66], on a toy model realization of the Ponzano-Regge amplitude for the 1-4 Pachner move. [14] In that work we showed that the presence of the two parities in the asymptotics of the Ponzano-Regge amplitude is crucial to obtain its well-known divergence structure, and that a modified model where this sum over orientations is absent would be less divergent. We also argued that this is intimately related to the properties of BF-theories rather than to those of Einstein-Hilbert and Regge gravity. I will come back on this in the discussion of Chapter 7.

5.7 AMPLITUDE SCALING (IN THE NON-DEGENERATE SECTOR)

In the present setting, a necessary ingredient for estimating the degree of divergence of the melon graph is the estimation of the scaling of each partial amplitude with the spin scale. A careful study of the action at the critical point provides what is needed.

Let me start from the well-known result on the stationary phase approximation of an integral:

$$\int_D dx\, f(x)\, e^{\lambda S(x)} = \sum_{\{x_c\}} \frac{\lambda^{-\frac{1}{2}\mathrm{rank}\, S''(x_c)}}{\sqrt{|\det[\tilde{S}''(x_c)/2\pi]|}} f(x_c) e^{\lambda S(x_c)} \left[1 + O\left(\frac{1}{\lambda}\right)\right], \tag{5.35}$$

where D is a suitable domain of integration (generally required to be compact), x_c are the critical points of the action S (that is $\Re S(x_c)$ is maximal and $S'(x_c) = 0$), S'' is the Hessian of the action S, and $\tilde{S}''$ is the restriction of S'' onto its maximal invertible subspace $(\ker S'')^\perp$. Moreover, $\Re\tilde{S}''(x_c)$ is also required to be positive defined.

To be more precise, the previous formula applies when the reduction of the rank of the Hessian is due to the presence of (compact) symmetries, and not to some

12. To be more precise this is true provided the rest of the integrand is slowly varying with respect to the the oscillation scale. I will assume that this is the case. As I showed in [189], this hypothesis leads essentially to the correct result at least in the context of SU(2) BF-theory on the same graph. See also the analogy with [66] cited later in this section.

13. Similar contributions are expected from the degenerate sector, too. See section 5.8.1.

14. The 1-4 Pachner move graph represents the first natural correction to the vertex amplitude, as much as the melon graph corresponds to the first natural correction to the gluing.

"accidental" vanishing, as for $S(x) = -x^4$. In the latter case, one expects the previous result to constitute only an estimate, by defect, of the divergence degree of the integral.

From this formula it is clear that what is needed is the evaluation of the rank of the Hessian of the action at the critical points discussed in the previous section. The explicit form of the Hessian at the critical points can be found in the work of Muxin Han and Mingyi Zhang [128]. In spite of this, I will not attempt to use their formulas, since the Hessian matrix is quite large and intricate. Rather, I will use the geometrical intuition developed so far, as well as the analysis of the "gauge" symmetries of the action performed already in section 5.5, to evaluate $\operatorname{rank} S''$. [15]

The idea is to subtract from the total dimension of the space one is integrating over in equation 5.15, the dimension of the gauge orbits. It is important to notice that this calculation sets only an upper bound to the rank of S'', which in turn sets a lower bound on the total degree of divergence of this sector. We shall ignore this issue, assuming that the bound is saturated at the non-degenerate fixed points of the action I am considering here. [16] However it will be crucial when dealing with the geometrically degenerate sector, where also further symmetries satisfied by the critical points, but not by the full action, must be taken into account; see section 5.8.1.

Therefore,

$$\begin{aligned}\operatorname{rank} S'' &= \Big(8 \times \dim\, SL(2,\mathbb{C}) + 12 \times \dim\, S_2 + 12 \times \dim\, \mathbb{CP}^1\Big) \\ &\quad - \Big(4 \times \dim\, SU(2) - 2 \times \dim\, SL(2,\mathbb{C})\Big) \\ &= 72 \end{aligned} \tag{5.36}$$

where were taken into account the following integrations variables: 8 group elements $\{g_a, \tilde{g}_a\}$ (i.e. 4 per each inner half-edge), 12 $\{m_{ab}\}$, and as many $\{z_{ab}, \tilde{z}_{ab}\}$. [17] For what concerns the gauge symmetries, both vertex (2 $SL(2,\mathbb{C})$ invariances) and edge symmetries (4 $SU(2)$ invariances) have been considered. [18]

I am now in the position of giving a formula for the partial amplitudes $w_M(j_{ab})$ (see equation 5.15) evaluated in the stationary phase approximation, i.e. for large spins $j_{ab} \sim \lambda \gg 1$, and in the approximation where only the parity agreeing critical points are kept (and the spins do not induce a degenerate geometry):

$$\begin{aligned} w_M(j_{ab}) \sim \sum_{\sigma \in \{E,L\}} & \delta_{\sigma,\sigma(j)} \\ & \int_{SL(2,\mathbb{C})} dK d\tilde{K} \left[\prod_a \langle j_a, n_a | Y_\gamma{}^\dagger K g_a^\sigma Y_\gamma Y_\gamma{}^\dagger (\tilde{g}_a^\sigma)^{-1} \tilde{K}^{-1} Y_\gamma | j_a, \tilde{n}_a\rangle\right] \times \\ & \qquad \times \left[\prod_{a<b} \mu(j_{ab}) d^4_{j_{ab}}\right] \lambda^{-\frac{1}{2}\operatorname{rank} S''|^{\sigma}_{\text{crit.}}} + \\ & + \delta_{NG,\sigma(j)}\, O\left(\lambda^{-\infty}\right), \end{aligned} \tag{5.37}$$

where the index σ labels the Euclidean (E) and Lorentzian (L) sectors, and $\{g_a^\sigma, \tilde{g}_a^\sigma\}$ are the solutions to the critical point equations in the two sectors, and $\sigma(j)$ is a

15. I will ignore the study of the positive-definiteness of $\Re \tilde{S}''$.
16. As shown in [189], this assumption leads to the correct result when applied to $SU(2)$ BF-theory on the same graph.
17. Recall that $m_{ab} \neq m_{ba}$, but $z_{ab} = z_{ba}$ and $\tilde{z}_{ab} = \tilde{z}_{ba}$.
18. Notice that, had I gauged fixed two internal $SL(2,\mathbb{C})$ group elements, say $g_4 = \mathbb{I} = \tilde{g}_2$, I would have had to consider two less group integration, and I would have lost the vertex gauge symmetry. Therefore this counting would not have changed, as it should be.

label depending of the six spins $\{j_{ab}\}$ defined in equation 5.24. Notice, that the degenerate sector ($\sigma(j) = D$) is being neglected, and that the non-geometric sector, corresponding to the absence of solution to the critical point equations ($\sigma(j) = NG$) is infinitely suppressed. [19]

For what concerns the integration over the $SL(2,\mathbb{C})$ group elements K and $\tilde{K}$, it is possible to think of it either as a mean of restoring the gauge symmetry broken by the choice of one particular solution $\{g_a^\sigma, \tilde{g}_a^\sigma\}$, or as an integral over the critical manifold parametrized by the vertex gauge symmetry. [20]

In the previous formula the fourth power of $d_{j_{ab}}$ comes from the integration measures over the variables $\{m_{ab}\}$ and $\{z_{ab}, \tilde{z}_{ab}\}$. It is is interesting to notice how this factor cancels the contributions of those variables to the rank of the Hessian. This is a good sign, since both these variables are auxiliary and are not conceptually needed to describe the model itself: the $\{m_{ab}\}$ were introduced via the insertion of some resolutions of the identity, while the $\{z_{ab}, \tilde{z}_{ab}\}$ were needed to explicitly write the scalar products in the unitary irreducible representations of $SL(2,\mathbb{C})$ and are therefore more bound to the specific representation chosen rather than to the physics of the problem.

Thus, putting into evidence the scale λ:

$$
\begin{aligned}
&w_M(j_{ab} \sim \lambda) \sim \\
&\sim \sum_{\sigma \in \{E,L\}} \delta_{\sigma,\sigma(j)}\, \lambda^{6\bar{\mu}-12} \int_{SL(2,\mathbb{C})} dK d\tilde{K} \\
&\qquad\qquad \left[\prod_a \langle j_a, n_a | Y_\gamma{}^\dagger K g_a^\sigma Y_\gamma Y_\gamma{}^\dagger (\tilde{g}_a^\sigma)^{-1} \tilde{K}^{-1} Y_\gamma | j_a, \tilde{n}_a \rangle \right] + \\
&\quad + \delta_{NG,\sigma(j)}\, O\left(\lambda^{-\infty}\right)
\end{aligned}
\tag{5.38}
$$

where the face-weight scaling $\bar{\mu}$ has been introduced via

$$
\mu(\lambda \gg 1) \sim \lambda^{\bar{\mu}} . \tag{5.39}
$$

In particular, the $SU(2)$ face-weight $\mu(j) = (2j+1)$ advocated in the Chapter 4 gives $\bar{\mu} = 1$, while the $SL(2,\mathbb{C})$ face-weight $\mu(j) = (\gamma^2+1)j^2$ gives $\bar{\mu} = 2$.

5.8 RENORMALIZED GLUING

Equation 5.38 can be further simplified within the Euclidean sector, i.e. provided $\sigma(j) = E$. To see this one needs to use the following facts. First, $g_a^E, \tilde{g}_a^E \in SU(2)$ in a suitable (vertex) gauge (equation 5.31). Second, the previous fact implies that the $\{g_a^E, \tilde{g}_a^E\}$ commute with the Y_γ map. And finally, in the disagreeing-parity sector considered here $\nu_v = -\nu_{\tilde{v}}$, which implies [21] via equation 5.33 $\eta_{ab} = \tilde{\eta}_{ab} + [i\pi]_{ab}$

19. For the degenerate sector, see the last part of section 5.8.1. The meaning of "infinitely suppressed" means that

$$
\lim_{\lambda\to\infty} \lambda^N O\left(\lambda^{-\infty}\right) = 0 \quad \forall N \in \mathbb{N} .
$$

20. Or as the integral over the external $g, \tilde{g}$ had they not been gauge fixed at the beginning.
21. Recall that $\epsilon_{ab} = -\tilde{\epsilon}_{ab}$, see equation 5.25.

and therefore [22] $g_a = [\pm]_a \tilde{g}_a$ (in the appropriate vertex gauge). Therefore, using these facts, one has immediately

$$w_M(j_{ab})\big|_{\sigma(j)=E} \sim \lambda^{6\bar{\mu}-12} \int_{SL(2,\mathbb{C})} dK d\tilde{K} \prod_a \langle j_a, n_a | Y_\gamma{}^\dagger K Y_\gamma Y_\gamma{}^\dagger \tilde{K}^{-1} Y_\gamma | j_a, \tilde{n}_a \rangle \,. \tag{5.40}$$

The interesting thing about this formula is that at the leading order the dependence on the spins $\{j_{ab}\}$ is found to be confined to the global scaling factor and does not enter the amplitude of the external faces any more.

This observation allows an easy estimation of the cut-off amplitude of the melon graph when the summation over all the possible spins is restricted to the Euclidean sector: [23]

$$\begin{aligned} Z^M_{\text{EPRL},\Lambda}(j_a, n_a, \tilde{n}_a)\big|_{\text{Eucl.}} &:= \\ &:= \sum_{\{j_{ab} < \Lambda\}} \delta_{E,\sigma(\{j_{ab}\})}\, w_M(j_a, n_a, \tilde{n}_a; j_{ab}) \\ &\sim \Lambda^{6(\bar{\mu}-1)} \int_{SL(2,\mathbb{C})} dK d\tilde{K} \prod_a \langle j_a, n_a | Y_\gamma{}^\dagger K Y_\gamma Y_\gamma{}^\dagger \tilde{K}^{-1} Y_\gamma | j_a, \tilde{n}_a \rangle \,, \end{aligned} \tag{5.41}$$

where the fact was used that one is summing over six independent spins.

What about the sectors other than the Euclidean one?

The non-geometric sector is clearly suppressed. The degenerate one - as already said - is for the moment left aside. Finally, the Lorentzian sector should *a priori* scale in the same way as the Euclidean one. However, the external face amplitudes do not decouple from the summation, and it is therefore simply not possible to give a formula similar to equation 5.41 for its amplitude. Nonetheless, this same fact allows to conjecture that in the end, the dependence of the external faces on the internal spins has the net result upon summation of slightly suppressing the amplitude in this sector.

Even if such a conjecture is not fully justified, it is anyway appealing to assume that - when divergent - the total amplitude is dominated by the Euclidean sector, which presents the simplest dependence on the boundary states one could have expected. This constitutes at least a useful working hypothesis for advancing further considerations on the renormalization of the melon graph. For this reason, while keeping in mind all the *caveats* I mentioned, I will *assume* for the rest of this thesis the following result:

$$Z^M_{\text{EPRL},\Lambda} \approx Z^M_{\text{EPRL},\Lambda}\big|_{\text{Eucl.}} \,. \tag{5.42}$$

As emphasized in the first paragraph of this chapter, one of the most interesting ways of think of the melon graph, is in terms of a "1-bubble" renormalization of the gluing function between tetrahedra.

Within this interpretation a few questions are very natural: when is the melon graph amplitude divergent at all? Is the bare gluing modified by the 1-bubble corrections, and how? Is it still a projector, and which consequences would a negative

22. For a discussion of the signs see [189]. Anyway, the meaning of the square brackets is similar to that in equation 5.33.

23. One can show that the signs $[\pm]_a$ appearing in the relation between the critical values of the g_a and $\tilde{g}_a$ simplifies when calculating the total amplitude.

answer have on the model? Are the known semiclassical results for the EPRL model "endangered" by the the renormalized gluing? Finally, one can also ask the following more "spinfoam-minded" quesiton: can the divergence of the melon graph be given a geometrical interpretation? I will dedicate the rest of this chapter to answering them.

Though, before delving into this discussion, I want to briefly highlight a result of the present analysis, which is valid for both the Euclidean and Lorentzian sectors. Assuming once again that the parity agreeing critical points are sub-dominant, one finds that $g_a^\sigma = [\pm]_a \tilde{g}_a^\sigma$. This is as if the amplitude were dominated by BF (i.e. flat) solutions, exactly up to the sign discussed around equation 5.32.

5.8.1 *Melon-Graph Divergence Degree*

From equation 5.41 it is immediate to read out when the melon graph is expected to be divergent. This is when

$$\bar{\mu} \geqslant 1\,. \tag{5.43}$$

This equation puts to the forefront the quite obvious fact that the divergence degree of a spinfoam crucially depends on the amplitudes assigned to its faces. A similar conclusion would apply for the edge weights. However, they do not appear in the previous formula. This is because they had been set to 1, or equivalently, one can say they had been reabsorbed into the vertex amplitude.

The $SL(2,\mathbb{C})$-BF face weight is such that $\bar{\mu} = 2$ and leads to a divergence degree of Λ^6. On the other hand, the face weight which assures composition invariant of the spinfoam amplitudes is the $SU(2)$-BF one, which is such that $\bar{\mu} = 1$. In this case, the melon graph diverges at most logarithmically in the cut-off:

$$\left| Z^{M}_{\mathrm{EPRL},\Lambda}(j_a \sim \bar{j}) \right|_{SU(2)} \sim \log\left(\frac{\Lambda}{\bar{j}}\right)\,, \tag{5.44}$$

where $\bar{j}$ is the scale of the *external*-face spins $\{j_a\}$. Indeed, in the case of a logarithmic divergence a scale must be introduced for "dimensional reasons", and the only available scale in the problem is that fixed by the external faces. Mathematically, one has to expect this scale to be the relevant one (unless it is of order 1), since it is starting from this scale that the stationary point equations for the internal and external faces decouple. Note that this is analogous to what happens when evaluating logarithmically divergent Feynman diagrams in quantum field theory.

It is important to compare this result with others which appeared previously in the literature. In fact, even if no calculations were performed in the context of the EPRL model for Lorentzian quantum gravity, two other papers had already estimated the divergence degree of the melon graph in the context of the EPRL model for Euclidean quantum gravity (which is in turn equivalent for $0 < -\gamma < 1$ to the model proposed by Laurent Freidel and Kirill Krasnov in [103]). These papers used completely different techniques with respect to the one proposed here. In [181], Claudio Perini, Simone Speziale, and Carlo Rovelli had estimated the divergence degree of the melon graph (and also of the vertex correction via the 1-5 Pachner move) via the scaling of the $\{9j\}$ and $\{15j\}$ symbols through which the Euclidean model can be expressed (but not the Lorentzian one). In [144], Thomas Krajewski, Jacques Magnen, Vincent Rivasseau, Adrian Tanasa, and Patrizia Vitale obtained a consistent result by linearising the model around the BF-solution, i.e. around trivial holonomies ($g_a = \mathbb{I}$) and extended the analysis of Perini *et al.* to the degenerate sector. A few comments are in order.

First of all, these results are in agreement with mine, in the sense that they display the same divergence degree as in the Lorentzian theory. Even if these may look a bit surprising at first sight, the fact that the symmetries of the two models, as well as the dimensions of the respective symmetry groups ($SL(2,\mathbb{C})$ and $SO(4)$) are the same, makes the claim much more reasonable. In particular, the fact that $SL(2,\mathbb{C})$ is non-compact plays no role at all in my analysis.

Another useful comparison is that with the topological $SU(2)$ BF-theory on the same graph. Within this theory, the melon graph is found to diverge as Λ^9. Therefore, the EPRL model gives rise to much less diverging amplitudes.[24] This is in perfect accord with the results obtained when renormalizing group field theories, where it is by now clear that the renormalizability of the models is improved by adding more structure (such as gauge invariance) to them. It is also interesting to note that one can double-check the techniques used here to evaluate the melon graph amplitude within $SU(2)$ BF-theory. In fact, it is possible in this case to use both the saddle point scheme as well as simpler and more direct regularization schemes (see e.g. [43, 189]). The result is that the two are in agreement, somehow reinforcing the results obtained for the EPRL model. One caveat has nonetheless to be taken into account and is discussed at the end of this section.

For what concerns the degenerate sector, the estimation by Krajewski *et al.* points toward its domination over the non-degenerate one, by one power in the cut-off. Then the question arises on how to deal with this sector within the Lorentzian model, and possibly within the geometric construction presented in this chapter. This can be done, but the problem is that the final result is not a well-defined expression, therefore the procedure is not conclusive. More specifically, the problem is the following: consider the case in which the spins $\{j_{ab}\}$ can be identified with the sides of a maximally degenerate tetrahedron (i.e. a tetrahedron whose vertices are all aligned). Then, on the top of the usual edge and vertex symmetries, one finds extra boost and rotation symmetries at every half-edge. Intuitively they corresponds respectively to rotating *around* and boosting *along* the direction spanned by the degenerate triangle, now squashed onto a line. (The meaning of boost symmetry is not completely transparent, though). Therefore, one should integrate over six extra boosts (two of them can and must be gauged fixed) and eight rotations (there is no necessity of gauge fixing these ones). Neglecting the integrations over the angle variables which *a priori* pose no difficulty, since they take place over a compact space, let me focus on the structure of the integrations over the boost variables. For this I will not take care of possible signs and phase subtleties. Sketchily:

$$\left| Z^{M}_{\text{EPRL},\Lambda}(j_a) \right|_{\text{deg.}} \sim \Lambda^{\Delta} \int_{SL(2,\mathbb{C})} dK d\tilde{K} \int_{-\infty}^{\infty} \mathcal{D}\eta_a \mathcal{D}\tilde{\eta}_a \; \delta(\eta_1)\delta(\tilde{\eta}_1) \left[\prod_a \langle j_a, n_a | Y_\gamma{}^{\dagger} K e^{\eta_a \sigma_z} Y_\gamma Y_\gamma{}^{\dagger} e^{-\tilde{\eta}_a \sigma_z} \tilde{K}^{-1} | j_a, \tilde{n}_a \rangle \times \right. \\ \left. \times\, e^{\sum_{a<b} 2 j_{ab} i\gamma[-(\eta_a - \tilde{\eta}_a) + (\eta_b - \tilde{\eta}_b)]} \right] \tag{5.45}$$

24. Since the two theories share the same kinematical Hilbert space, the most reasonable way to compare them is to keep the same face weight ($\mu(j) = 2j+1$) and change only the vertex amplitude. Note in fact that the face-weight choice imposed by the gluing condition discussed in section 4.2 involves only the kinematical Hilbert space of the boundary states.

where Δ is a putative divergence degree to be calculated later via the rank of the Hessian of the action, $\{\eta_a, \tilde{\eta}_a\}$ are the boost parameter discussed above, the deltas are the gauge fixing conditions, and the measure $\mathcal{D}\eta$ is defined by (see [25] [205])

$$\mathcal{D}\eta := \frac{1}{2\pi}(\sinh 2\eta)^2 d\eta\,. \tag{5.46}$$

This measure for large values of the boost η scales as $\sim \exp|4\eta|$. Usually, this factor is compensated by the presence of enough representation matrices which typically scale as

$$D^{(\rho,k)}(g_\eta) \sim e^{-2|\eta|+i\rho|\eta|}\,, \tag{5.47}$$

where g_η is an $SL(2,\mathbb{C})$ element with boost component η, i.e. which can be put in the form $g_\eta = h_1 e^{\eta\sigma_z} h_2$, for some $h_i \in SU(2)$. Therefore, one sees that on the one hand in the expression for the amplitude within the degenerate sector I gave the integrand diverges exponentially [26] in each of the $\{\eta_a, \tilde{\eta}_a\}$, and that on the other hand there are multiple (and competing) oscillatory factors which render the integral *a priori* completely ill-defined.

Thus, from this analysis I can only conclude that the method devised for estimating the divergence degree and the renormalized gluing associated to the non-degenerate sector, breaks down in the case of the degenerate one.

Nonetheless, one can compute the "putative" degree of divergence Δ, defined in analogy to the non-degenerate case, as [27]

$$\Delta := -\frac{1}{2}\mathrm{rank}S''|_{\mathrm{deg.}} + (6\bar{\mu} - 12) + 6 - 3\,, \tag{5.48}$$

where the last contribution accounts for the 3 constraints over the internal spins required to restrict the sum over their (maximally) degenerate sector. To estimate $\mathrm{rank}S''|_{\mathrm{deg.}}$ one has to take into account the symmetries typical of the degenerate configuration, which further reduce this rank. In particular, as discussed there are 8 new $U(1)_\mathbb{C}$ symmetries. However, not all of them are "new": indeed, two full $U(1)_\mathbb{C}$ symmetries can be reabsorbed into the vertex gauge symmetry (say at the half edges 1 and $\tilde{2}$), while extra 4 *real* $U(1)$ symmetries can be reabsorbed via the edge gauge symmetry (say at half edges $\tilde{1}$ and 2, 3, 4). Therefore one is left with

$$\begin{aligned} \mathrm{rank}S''|_{\mathrm{deg.}} - \mathrm{rank}S''|_{\mathrm{non\ deg.}} &= \\ &= -8 \times \dim U(1)_\mathbb{C} + 2 \times \dim U(1)_\mathbb{C} + 4 \times \dim U(1) \\ &= -8\,. \end{aligned} \tag{5.49}$$

Hence,

$$\Delta = 6(\bar{\mu} - 1) + 1\,, \tag{5.50}$$

which exceeds by one the divergence degree found for the non-degenerate sector. This result is also in agreement with the scaling found by Krajewski *et al.* in [144]. However, the non-compactness of $SL(2,\mathbb{C})$ and the presence of oscillatory factors, prevent any precise statement.

25. Note that $2\eta_{\mathrm{here}} = \eta_{\mathrm{there}}$.
26. This is not the case for the non-degenerate sector, since the group elements $K, \tilde{K}$ appear in four copies - and not just a single one - in equation 5.41.
27. Recall the discussion after equation 5.35: such a "putative" degree of divergence - even admitting that it is meaningful - is rather an estimate by defect of the true one.

Notice that in the previous counting there is an interplay between the number of constraints on the spins associated to the fact that degenerate configurations are not generic, and the augmented number of symmetries associated to these configurations. The first tend to reduce the divergence degree associated with the degenerate configurations, while the second tend to increase it by reducing the rank of the Hessian of the action.

In the context of the evaluation of the melon graph within the $SU(2)$ BF-theory, a similar interplay exists when dealing with the degenerate sector. However, the final result is the opposite to the one just discussed. In fact, in the context of $SU(2)$ BF theory the symmetries one gains by looking at the maximally degenerate configurations belong to the *real* $U(1)$ group and therefore span a lower dimensional space. The net result can be shown to be a suppression of the degenerate sector with respect to the non-degenerate one [28] [189]. This observation highlights the fact that having different symmetry groups at the vertices and at the edges, with the one at the vertices being larger, could be the source of the dominance of degenerate configurations.

Anyway, at the light of the difficulties discussed above, one cannot draw any definite conclusion about the degenerate sector other than the fact that it deserves a more thorough study.

5.8.2 *The One-bubble Renormalized Gluing is Not a Projector*

The interpretation of the melon graph as a radiative correction to the gluing of tetrahedra puts to the forefront the question whether the corrected gluing is still a projector or not. Before going into the details of the calculation, it is useful to spend a few words commenting what it is meant by saying that the melon graph can be interpreted as a radiative corrections to the gluing and why it is important for a gluing to be a projector. Since these issues are probably most clearly addressed within the group field theoretical framework, I need to briefly introduce this formalism.

In Chapter 3, I presented spinfoams in a way which had its main focus on the properties of spinfoam faces. However, another formulation is possible, where spinfoam vertices and edges play the prominent role. Indeed, when formulated this way, spinfoam amplitudes formally resemble a Feynman diagram evaluation. This statement can be made precise, by introducing a quantum field theory whose formal Feynman diagram expansions reproduces spinfoam amplitudes.

The first example of a group field theory was devised by Dimitrij Boulatov in 1992 [57], in the context of three-dimensional quantum gravity,as a generalization of matrix models, which were already knonw to provide a quantization two-dimensional quantum gravity [74]. In 2000 it was realized by a group of authors (Roberto De Pietri, Laurent Freidel, Kirill Krasnov, and Carlo Rovelli) [73] that the Barrett-Crane spinfoam amplitudes could also be generated by a group field theory. [29] Soon afterwards, Michael Reisenberger and Carlo Rovelli [188] understood that the existence

Barrett and Crane's model is the first spinfoam model of four-dimensional quantum gravity [39]

28. The relevant calculation is easily done: $\Delta^{\text{deg.}}_{SU(2)} = \Delta^{\text{non deg.}}_{SU(2)} + \frac{1}{2}\text{rank}S''|_{\text{non deg.}} - \frac{1}{2}\text{rank}S''_{SU(2)}|_{\text{deg.}} - 3$, where I used that the relevant number of constraints is the same as in the EPRL model (and equal to 3, see [189]). Then,

$$\text{rank}S''|_{\text{deg.}} - \text{rank}S''|_{\text{non deg.}} = -8 \times \dim U(1) + 2 \times \dim U(1) + 4 \times \dim U(1) = 2,$$

yielding $\Delta^{\text{deg}}_{SU(2)} = \Delta^{\text{non deg.}}_{SU(2)} - 2$, and hence the suppression.

29. The wording "group field theory" was actually coined in this work.

of a group field theory formulation for spinfoam models is a generic feature. Since then the efforts of understanding quantum gravity in terms of group field theories have kept increasing. I refer to [98, 171] for reviews.

Generally a group field theory associated to a $(D+1)$-dimensional spinfoam model is defined by the action:

$$\begin{aligned} S_{\mathrm{GFT}}[\phi] = & \frac{1}{2}\int_{G^{2D}} dg_a d\tilde{g}_a\ \bar{\phi}(g_a)\mathcal{K}(g_a\tilde{g}_a^{-1})\phi(\tilde{g}_a) + \\ & + \frac{\alpha}{(D+1)}\int_{G^{D(D+1)}} dg_{ab}\ \mathcal{V}(g_{ab}g_{ba}^{-1})\phi(g_{1b})\ldots\phi(g_{D+1\,b}) + \text{c.c.} \end{aligned} \tag{5.51}$$

where $\phi : G^D \to \mathbb{C}$ is a complex field, and dg is the Haar measure over the group G, which turns out to be the gauge group of the boundary spin networks. Note that in this action only one type of interaction vertex is assumed, however under renormalization higher order operators are generated and must be included in the Lagrangian density. [30]

The kinetical and interaction are required to satisfy the following invariance properties:

I will often commit a slight abuse of notation, using the same symbol $\mathcal{K}$ for both the integral kernel $\mathcal{K}(g_a, \tilde{g}_a)$ and the function $\mathcal{K}(G_a)$ evaluated at $G_a = g_a\tilde{g}_a^{-1}$. Similarly for $\mathcal{V}$

$$\mathcal{K}(kG_a\tilde{k}) = \mathcal{K}(G_a) \quad \text{and} \quad \mathcal{V}(k_aG_{ab}k_b^{-1}) = \mathcal{V}(G_{ab}), \tag{5.52}$$

where $G_a := g_a\tilde{g}_a^{-1}$, $G_{ab} := g_{ab}g_{ba}^{-1}$ and the previous equality holds for any $k, k_a \in G$. These invariance properties will then be reflected in the spinfoam amplitudes into the well known edge gauge symmetries, discussed multiple times in this chapter. Indeed, the field $\phi(g_a)$ can be thought as the quantum field associated to a *spin network* node, and the gauge invariance of the kernels projects this field to its gauge invariant component:

The vertex gauge symmetries are, on the other side, implemented by the specific form of the interaction kernel $\mathcal{V}$

$$\phi(kg_a) = \phi(g_a), \tag{5.53}$$

which in turn creates and annihilates "closed" D-simplices in the terminology of section 3.1. Note that the action is already invariant under

$$\phi(g_i) \mapsto \phi(g_i k). \tag{5.54}$$

Then, spinfoam amplitudes result from an expansion in powers of [31] α of the gauge invariant n-point functions of ϕ, which represent boundary spin network states with n-nodes. The amplitude is then found by contracting together field propagators and interaction kernels. Notice that if the kernel $\mathcal{K}$ contains a projector, i.e. it has some zero eigenvalues, then properly speaking it has no inverse to be used as a propagator. One should then restrict the space of fields to be in the subspace stabilized by $\mathcal{K}$ and work out the amplitudes. This basically means that only those "components" of the field are propagated, which are in the image of the projector. In turn this means that the relevant boundary spin network states of the theory are also built out of fields in the image of the same projector. In particular, the gauge invariance of the kernel $\mathcal{K}$ expressed in equation 5.52 induces the closure constraint

30. This statement is only qualitative at this level, since a lot of work has to be done to make sense of the renormalization flow in this context. See e.g. [61].

31. Actually, the physical meaning of the coupling constant α is still obscure. A possibility is that it gets dynamically tuned via renormalization flow to a critical value at which the number of simplices relevant for the calculation of the amplitude diverges. This mechanism is inspired to what happens in the matrix model large-N limit.

on the spin network states [32] (equations 2.6 and 2.8).

A particular case of the previous discussion is that in which $\mathcal{K}$ *is* a projector. In this case one can effectively use $\mathcal{K}$ itself as a propagator when writing the spinfoam amplitude, since the only thing which matters is to trivially propagate only those degrees of freedom which are in its image. This case is prototypical of the topological models (like the Boulatov model reproducing Ponzano-Regge amplitudes). On the other side, in order to trigger the ronormalization flow, one needs a propagator with a non-trivial spectrum, that is a propagator containing some notion of scale. At this purpose, in a very nice piece of work [43], Joseph Ben Geloun and Valentin Bonzom showed that in the Bulatov model the kinetical kernel $\mathcal{K}$ requires upon renormalization the addition of a group Laplacian operator. Indeed, by calculating all the divergent terms arising from the (three-dimensional) melon graph amplitude, they could show that if at the leading order in the cut-off one obtains just a mass renormalization, at the first (and only) sub-leading order one obtains a wave function renormalization, which in turn needs a Laplacian as a counter term in the Lagrangian. A complete renormalization of the coloured version of the Bulatov model was performed by Sylvain Carrozza during his Ph.D. thesis [61] and it is also the subject of a publication of his together with Daniele Oriti and Vincent Rivasseau [62]. There seem also to be strong hints [33] toward the fact that the model is asymptotically free [34] in the "ultraviolet" (in the sense of the group manifold, i.e. at high spin values), and there is the hope that it develops an infinite mass in the "infrared" (always in the sense of the group manifold, i.e. at small spins) effectively recovering a projector kinematical kernel which is a projector as in the bare model.

So, what about the 1-bubble renormalization of the EPRL propagator? First, let me define the EPRL model in the above terms. This is readily done: [35]

$$\mathcal{K}_{\text{EPRL}}(H_a) = \int_{SU(2)} dk \prod_{a=1}^{4} \delta(H_a k) \tag{5.55a}$$

$$\mathcal{V}_{\text{EPRL}}(H_{ab}) = \int_{SU(2)} dk_a dX_{ab} \int_{SL(2,\mathbb{C})} d'g_a \prod_{a\neq b} K_\gamma\left(g_a g_b^{-1}, X_{ab}\right) \delta(k_a H_{ab} k_b^{-1} X_{ab}) \tag{5.55b}$$

where $H_a = h_a \tilde{h}_a^{-1}$ and $H_{ab} = h_{ab} h_{ba}^{-1}$, with $h_a, h_{ab} \in SU(2)$. In these expressions $k_a \in SU(2)$ play exactly the same role as in equations 5.20 and the prime over the $SL(2,\mathbb{C})$ measure in the vertex symbolizes the gauge fixing $d'g_a = \prod_{a=1}^{4} dg_a \delta(g_1)$.

Note that this is an $SU(2)$ group field theory, and $SL(2,\mathbb{C})$ appears only in the interaction kernel. As a consequence, the corresponding spinfoam amplitudes are

32. One should not confuse the field $\phi(g_a)$ with the spin network wave function $\psi_\Gamma(h_\ell)$. Indeed, the latter is built out of a gauge invariant contraction of n copies of the field, where n is the number of nodes contained in Γ: it is somehow a gauge invariant "n-particle" state of the group field theory. See [172].

33. I thank Sylvain Carrozza for sharing his work in progress with me.

34. Asymptotic freeness seems to be a generic feature of tensor group field theories, even if the one studied by Carrozza *et al.* is for the moment the only studied example which implements gauge invariance.

35. The notation for the interaction kernel is redundant, but this writing is supposed to make easier the identification with the spinfoam amplitude:

$$Z^{\Delta}_{\text{EPRL}} = \sum_{j_f} \int_{SL(2,\mathbb{C})} \prod_{(ev)}{}' dg_{ev} \prod_f d_{j_f} \mathrm{Tr}_{j_f} \left(\overleftarrow{\prod_{v\subset f}} Y_\gamma{}^\dagger g_{e'v} g_{ve} Y_\gamma \right),$$

obtained from equation 4.38 by integration over the coherent states via equation 3.8.

characterized by the $SU(2)$ face weight $\mu(j) = 2j+1$.

In this model, the bare gluing between tetrahedra is given by $\mathcal{K}_{\text{EPRL}}$ which implements the identification between the tetrahedra (the delta), up to gauge (the integral), i.e. up to rotations. It is therefore by definition equivalent to the projector onto $SU(2)$ invariant states with kernel $\mathbb{P}(h_a, \tilde{h}_a)$. The leading order term in the renormalization of the group field theory propagator can then be worked out of equation 5.41, yielding:[36]

$$Z^{M}_{\text{EPRL},\Lambda}(h_a, \tilde{h}_a) \sim \\ \sim \log\left(\frac{\Lambda}{\bar{j}}\right) \frac{|\alpha|^2}{(D+1)^2} \int_{SL(2,\mathbb{C})} dK d\tilde{K} \int_{SU(2)} dk_a d\tilde{k}_a \\ \prod_a K_\gamma(K, k_a) K_\gamma(\tilde{K}, \tilde{k}_a) \delta\left(h_a k_a \tilde{k}_a^{-1} \tilde{h}_a^{-1}\right), \tag{5.56}$$

where I have also introduced the powers of the coupling constant suitable in the group field theoretical context. This expression is in turn proportional to the square of the operator

$$\mathbb{T}_\gamma(h_a, \tilde{h}_a) := \int_{SL(2,\mathbb{C})} dK \int_{SL(2,\mathbb{C})} dk_a \prod_a K_\gamma(K, k_a) \delta(h_a k_a \tilde{h}_a^{-1}). \tag{5.57}$$

Note that, *morally* this is nothing else but

$$\text{“}\mathbb{T}_\gamma = K_\gamma \triangleright \mathbb{P}\text{”}. \tag{5.58}$$

Anyway, differently from $\mathbb{P}$, $\mathbb{T}_\gamma$ is *not* a projector:

$$\mathbb{T}_\gamma{}^2 \neq \mathbb{T}_\gamma. \tag{5.59}$$

Note also that $\mathbb{T}_\gamma$ is actually equivalent to the kernel of a two-valent EPRL vertex, and as such it is gauge invariant: $\mathbb{T}_\gamma = \mathbb{P}\mathbb{T}_\gamma = \mathbb{T}_\gamma\mathbb{P}$.

The renormalization procedure for the EPRL group field theory would now need to add in the original Lagrangian the renormalized gluing i.e. a term proportional to $\mathbb{T}_\gamma{}^2$ as a new operator. However, since $\mathbb{T}_\gamma$ is not a projector, one can immediately conclude that the model is *not* renormalizable even when corrected in the way just described. What is meant by this statement, is that by reiterating the procedure one is forced to add further new terms to the original Lagrangian, which hence does not contain all the relevant operators yet.[37] Whether this procedure stabilizes at some point is still an open question, even when considering the “melonic” sector only.

Despite the fact that I framed the discussion in the group field theoretical language, it should not be taken as the only possible language in which renormalization can be made sense of. The debate is still very open on this issue [99]. Therefore the previous result should be better considered in a wider perspective. I will come back on this topic in Chapter 7.

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36. To pass from the coherent state notation, to this holonomy notation one proceeds as follows. First one notices that any $H \in SU(2)$, in the fundamental representation, can be written (non-uniquely) as $H = |\tilde{m}\rangle\langle m| + |\mathcal{J}\tilde{m}\rangle\langle \mathcal{J}m|$ for some $m, \tilde{m} \in S_2$. Therefore, by summing $Z(j, m, m') \sim \langle j, m| \cdots |j, m'\rangle$ and $Z(j, \mathcal{J}m, \mathcal{J}\tilde{m}) \sim \langle j, \mathcal{J}m| \cdots |j, \mathcal{J}\tilde{m}\rangle$, one obtains $Z(j, H) \sim \text{Tr}_j(\cdots H)$. Finally, the weighted sum over the representations $\{j\}$ gives $Z(H) \sim \sum d_j Z(j, H) \sim \delta(\cdots H)$. Setting $H = h\tilde{h}^{-1}$ and identifying $Z(h, \tilde{h})$ with $Z(h\tilde{h}^{-1})$ one obtains the sought result. As long as $Z(h, \tilde{h})$ is a function of $h\tilde{h}^{-1}$ only, the previous procedure is reversible.

37. I thank Joseph Ben Geloun for clarifying to me the common parlance in this context.

ADDENDUM In principle, it is also possible to write an $SL(2,\mathbb{C})$ group field theory for the EPRL model, where the vertices, instead of the propagators, are trivial, by which I mean composed by a bunch of deltas functions on the group and group averaging. However, one is forced to relax the property of the kinematical kernel of depending only on the combination $g_i\,\tilde{g}_i^{-1}$ of its arguments, and also to "unfreeze" the Barbero-Immirzi parameter. Let me briefly explain why this is the case. One is forced to abandon the simple form of the kinematic kernel because the invariance property of equation 5.54 does not hold when $k \in SL(2,\mathbb{C})$, but only for $k \in SU(2)$ (see e.g. equation 5.21). The reason why one needs to "unfreeze" the Barbero-Immirzi parameter is that when integrating over the group arguments to evaluate the Feynman amplitude one uses Schur's orthornormality relation to identify the representations appearing along a single spinfoam face. However, if the $SL(2,\mathbb{C})$ representation labels $R = (\rho, k) \in \mathbb{R} \times \frac{1}{2}\mathbb{N}$ are always related by $\rho = -\gamma k$ for a fixed value of γ as in the EPRL model, from the delta functions identifying the continuous representation labels one obtains nothing but meaningless infinities:

$$\delta(R-R') \equiv \delta(\rho-\rho')\delta_{k,k'} = \delta\left[\gamma(k-k')\right]\delta_{k,k'} = \delta(0)\delta_{k,k'}\,. \tag{5.60}$$

This would not be the case if the Barbero-Immirzi parameter is left *a priori* fluctuate from edge to edge. The same type of freedom on the Barbero-Immirzi parameter has been matter of study in a work by Maïté Dupuis and Etera Livine [82].[38] Passing by, I would like to mention that it has also been proposed to promote the Barbero-Immirzi parameter to a full-fledged matter (scalar or pseudo-scalar) field. At this purpose, see e.g. the works by Alexander Torres-Gomeza and Kirill Krasnov, those by Victor Taveras, Nicolás Yunes, Simone Mercuri and Gianluca Calcagni, as well as those by Francesco Cianfrani and Giovanni Montani (e.g. [217, 211, 160, 159, 59, 67]). However, in spite of very attractive features this formalism may have, it makes the loop quantum gravity quantization scheme much more involved and less transparent. For this reason, I will not consider this possibility any further.

5.8.3 *Semiclassical Limit*

As I discussed in section 4.3, one of the major indications of the viability of the EPRL model is the fact that it reproduces the Regge action in the large spin limit, i.e. in a semiclassical regime.[39] Therefore, a natural question is whether the one-bubble renormalized gluing spoils this semiclassical property of the EPRL model or not. To investigate this question, one needs to study the semiclassical regime of the melon graph amplitude $Z^{M}_{\mathrm{EPRL},\Lambda}(j_a, n_a, \tilde{n}_a)$, i.e. the limit of this expression when the $\{j_a\}$ are uniformly large (but still much smaller than the cut-off).

The result can be expressed in terms of the asymptotics of

$$\mathbb{T}_\gamma(j_a, n_a, \tilde{n}_a) = \int_{SL(2,\mathbb{C})} dK \prod_a \langle j_a, n_a | Y_\gamma{}^{\dagger} K Y_\gamma | j_a, \tilde{n}_a\rangle\,, \tag{5.61}$$

for $j_a \gg 1$. This was studied by Jacek Puchta [184]. His result, maybe a bit surprisingly, is that

$$\mathbb{T}_\gamma \xrightarrow{j_a \gg 1} \frac{1}{16\pi^2}\left[\frac{6\pi}{(1+\gamma^2)\sum_a j_a}\right]^{\frac{3}{2}} \mathbb{P} + O\left(\bar{j}^{-\frac{5}{2}}\right)\,. \tag{5.62}$$

38. There, this freedom was imagined to be useful as a mean of changing locally the length (area) scale, e.g. in a process of inhomogeneous coarse graining of the spin network. To my knowledge this idea has not been developed further.

39. See the cited section for the subtleties related to the semiclassical limit.

This is quite reassuring, since it means that the semiclassical properties are not spoiled by the use of the corrected gluing and the viability of the model is not endangered by the first "melonic" radiative correction to the propagator. Whether the gluing in a fully renormalizable version (even within the "melonic" sector only) of the EPRL model spoils or not this result, is though still an open question.

However, the previous remarks should not be taken too blindly: in the next chapter I will discuss an example in which a *semiclassical* observable *is* actually affected by the one-bubble corrected gluing, already at the leading order.

Finally, notice that equation 5.62 shows that the kinetic part of the EPRL group field theory acquires a spin-dependent term, once corrected at one-bubble. This is reminiscent, though quantitatively quite different, to what happens in the Boulatov model, where a Laplacian must be added to the Lagrangian in order for the theory to (have a chance of) being renormalizable.

6

RADIATIVE CORRECTIONS AND CORRELATIONS ON A SPINFOAM
- THE EXAMPLE OF THE GRAVITON PROPAGATOR

This chapter is extensively based on my paper [190].

The goal of this chapter is to understand some consequences of the result presented in the previous chapter. In order to do so, I show how the insertion of a melonic radiative correction modifies the spinfoam calculation of the metric-correlation functions across a region of spacetime, ie. of the graviton two-point function. This is done on the basis of previous work, reviewed in the next section, which is the result of the efforts of many people: Emanuele Alesci, Eugenio Bianchi, Luisa Doplicher, Etera Livine, Elena Magliaro, Davide Mamone, Federico Mattei, Leonardo Modesto, Claudio Perini, Carlo Rovelli, Simone Speziale, Massimo Testa, Ding You, and surely many others.

The final result is somehow surprising: even if the calculation is done in the large-spin regime in which the insertion of the melon graph should reduce to a projector, one still finds some residual effects of its presence already at leading order. Unluckily, the calculation is very involved and I was not able to explicitly extract and analyse these new terms.

From the graviton-propagator point of view, these corrections are found to influence only those terms which were already considered as spurious and never calculated. I will also briefly comment on this fact.

6.1 SPINFOAM GRAVITON PROPAGATOR

6.1.1 *General-Boundary picture*

The task of recovering n-point functions from a background independent theory is non trivial. To capture the basic difficulty it is enough to observe that in a formal expression like

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \int \mathcal{D}\phi\,\phi(x_1)\cdots\phi(x_n)\exp iS[\phi]\,, \tag{6.1}$$

where both the measure and the action on the right hand side are taken to be diffeomorphism invariant, the left hand side cannot be anything but a constant in the $\{x_i\}$. A few years ago, Carlo Rovelli, Leonardo Modesto, Eugenio Bianchi, and Simone Speziale [193, 165, 49] showed how to tackle the problem via the general-boundary formalism devised by Robert Oeckl and his collaborators [167, 168, 169]. In a nutshell, the idea is that an expression like equation 6.1 can be made sense of by performing the path integral in a (possibly) confined spacetime region $\mathcal{M}$ with the values of all the dynamical fields kept fixed on its boundary $\partial\mathcal{M}$:

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle_{\mathcal{M},\varphi} = \int_{\phi|_{\partial\mathcal{M}}=\varphi} \mathcal{D}\phi\,\phi(x_1)\cdots\phi(x_n)\exp iS_M[\phi]. \tag{6.2}$$

This prescription, which reminds of the original Feynman's path integral for non-relativistic quantum mechanics, is decisive for the following reason: among the fields which are kept fixed on $\partial\mathcal{M}$ there is also the gravitational field, *with respect to which* it is now meaningful to talk about positions. [1] Ultimately, the previous expression is diffeomorphisms *invariant* only in the bulk, and the boundary can be thought as "frozen" by some measurement. This still allows the n-point functions to be diffeomorphism *covariant*. The general-boundary formalism is a generalization (and *a priori* not a modification) of the usual path integral quantization [169]: the original quantization prescription is recovered whence $\mathcal{M}$ is taken to be the four dimensional space between two equal time hypersurfaces in Minkowski spacetime.

6.1.2 *Spinfoam Realization*

As anticipated, these ideas can be adapted to the spinfoam quantization program. The main obstacle arises from the need of letting the continuous (field) picture talk to the discrete (spinfoam) one. Two, naturally related, form of discreteness enter the game: the discreteness of the boundary state, in the form of an $SU(2)$ spin network state, and that of the bulk, in the form of a spinfoam with given boundary. The choice of a particular (superposition of) spin networks as a boundary state is dictated by the boundary geometry and by the variety of scales one wants to describe: [2] if only large scale modes of the boundary geometry and fields are of interest, then relatively small spin network will be sufficient, and their colorings (and superposition coefficients) will be such that to reproduce the boundary intrinsic and extrinsic geometry [193]. The expansion in the number of spinfoam (bulk) vertices follows a similar logic and can be seen either as a refinement of the bulk discretization lattice capturing more and more degrees of freedom, or as a development in the perturbative group field theory expansion expansion. I briefly comment about the two prescriptions in the last section of this chapter.

In [49] it is discussed how this approach yields the expression for the spinfoam lowest order graviton two point function:

$$G^{abcd}_{\mathbf{q}}(\mathbf{x},\mathbf{y}) = \frac{\sum_s Z[s]\hat{h}^{ab}(\mathbf{x})\hat{h}^{cd}(\mathbf{y})\Psi_{\mathbf{q}}[s]}{\sum_s Z[s]\Psi_{\mathbf{q}}[s]}. \tag{6.3}$$

Here, $\mathbf{q}$ represents the classical intrinsic and extrinsic boundary geometry, and $\Psi_{\mathbf{q}}$ is the quantum minimal-dispersion coherent state describing it. $\Psi_{\mathbf{q}}[s] := \langle s|\Psi_{\mathbf{q}}\rangle$ is its projection on the spin-network basis element $|s\rangle = |\Gamma, j_\ell, v_n\rangle$, where Γ is an abstract spin-network graph, and $\{j_\ell\}$ ($\{v_n\}$) are spins (intertwiners) labelling its links (nodes). See equation 3.13. Then, $\hat{h}^{ab}(\mathbf{x})$ is the spinfoam operator corresponding to the linearized gravitational field operator; it acts at the spin-network node corresponding to the point $\mathbf{x}$ on the 3d metric manifold $(\partial\mathcal{M}, \mathbf{q})$. Finally, $Z[s]$ is the (dynamical) amplitude associated to the spin network s. It can be calculated order by order in a vertex expansion. In this paper I focus on the one- and on a three-vertex spinfoam expansion calculated via the EPRL spinfoam model.

A useful interpretation of the spinfoam expansion is the following. Clearly, a little number of nodes and vertices corresponds to a coarse discretization, however this should not be interpreted too "rigidly" as a rough cutting up of spacetime into large chunks. Rather, it should be thought as a mean to truncate the number of degrees of freedom involved in the physical process under investigation, as well as their possible interactions. In particular, the spinfoam selects those "modes" which

1. For an extensive discussion of this point, see Carlo Rovelli's book [194].
2. Remark that the spin-network superpositions here considered involve only different colorings of the *same* graph. Considering more general situations would bring into the problem new conceptual difficulties.

fit within the chosen triangulation; e.g. back to the case of a triangulation with just a few vertices, only a few of the "longest wave-length modes" will be taken in consideration. The viability of a given truncation is naturally dictated by the physical question one is asking as well as by the complexity of the state one is considering. All of this is very nicely discussed at the kinematic (spin network) level by Seramika Ariwahjoedi, Carlo Rovelli, and collaborators in their recent work [11]. Nonetheless, the consistency of the approximation under refinement has also to be checked, especially in the presence of large radiative corrections. The latter question is one of the motivations of the work presented in this chapter.

6.1.3 *Lorentzian EPRL Boundary State Construction*

As shown in detail in [49], within the EPRL model, the first non trivial order of equation 6.3 is given by a one-vertex spinfoam. Name this spinfoam σ_4, and its boundary $\partial\sigma_4 = \Gamma_5$. They are represented in figure 6.1). For definiteness, call them the "pentagon spinfoam" and the "pentagon spin network", respectively. Geometrically, the pentagon spinfoam σ_4 is dual to a four-ball triangulated by a single-four simplex; in particular, its boundary is a three-sphere triangulated by five tetrahedra, which are dual to the nodes of the pentagon spin network Γ_5, and which are glued to one another via their faces, in turn dual to the ten links of Γ_5. The numbers $a \in \{1, \dots, 5\}$ label the nodes of the spin network. In figure 6.1 solid lines represent its links (labelled by $\{ab, a < b\}$), while dashed lines represent the spinfoam edges (labelled by $\{a\}$), and the black dot the only spinfoam vertex.

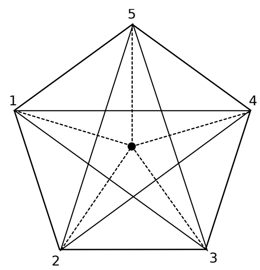


Figure 6.1: The pentagon spinfoam σ_4. It is dual to a four-ball triangulated by a single four-simplex. Its boundary graph Γ_5 (in solid lines) is dual to a three-sphere triangulated by five tetrahedra.

Since we are interested in the graviton propagator on flat spacetime, the boundary state (which is the only place where one can store information about the physics of the process) must be chosen in such a way it triangulates a three-dimensional (closed) hypersurface contained in four-dimensional Minkowski spacetime. Also, to fit the usual Lorentzian EPRL boundary states, this triangulated hypersurface must be taken space-like, i.e. with time-like normals.

In [49, 47, 44], the construction of the boundary state $\Psi_{\mathbf{q}}$ is carried out with great care. I am not going in any detail through the full construction, which involves many subtleties. I just briefly review its main features in order to fix notations.

At each node of a consider a Livine-Speziale coherent intertwiner (equation 3.9) $\Phi_a := \|j_{ba}, \vec{n}_{ba}\rangle$. E.g., the components of Φ_5 are:

$$\Phi_5^{m_1,\dots,m_4}(j_{15},\vec{n}_{15};\dots;j_{45},\vec{n}_{45}) := \int_{SU(2)} dh \prod_{b=1}^{4} \langle j_{b5}, m_b | D^{j_{b5}}(h) | j_{b5}, \vec{n}_{b5}\rangle \,. \tag{6.4}$$

Note that Φ_a is labelled by four spins and four unit three-vectors. These are taken to satisfy a non-degenerate closure condition $\sum_{a=1}^{4} j_a \vec{n}_a = \vec{0}$ by hypothesis, since Ψ_q is supposed to describe a semiclassical Minkowskian boundary state. Taking all the normals to the tetrahedra outward pointing, there will be both future and past pointing tetrahedra. Supposing the tetrahedron $a = 5$ is always future pointing, define

$$\Upsilon_a := \begin{cases} \Phi_a & \text{if tetrahedron } a \text{ is future pointing} \\ e^{-i\sum_{b>a}\Pi_{ab} j_{ab}}\Phi_a & \text{if tetrahedron } a \text{ is past pointing} \end{cases}, \tag{6.5}$$

where [3]

$$\Pi_{ab} := \begin{cases} 0 & \text{if wedge } (ab) \text{ is thick} \\ \pi & \text{if wedge } (ab) \text{ is thin} \end{cases}. \tag{6.6}$$

Naming $\iota^{(v,j_\ell)}_{m_1,\dots,m_4}$ the standard recoupling basis for intertwiners, define the coefficients

$$\Upsilon^v_a(\{\vec{n}_{ab}\}) := \iota^{(v,j_\ell)}_{m_1,\dots,m_4} \Upsilon_a^{m_1,\dots,m_4}. \tag{6.7}$$

Hence, define the coherent Lorentzian spin network

$$|\Gamma_5, j_{ab}, \Upsilon_a(\{\vec{n}_{ab}\})\rangle := \sum_{v_1,\dots,v_5} \left(\prod_{a=1}^{5} \iota_a^{v_a} \right) |\Gamma_5, j_{ab}, v_a\rangle. \tag{6.8}$$

This state is peaked on a given intrinsic geometry described by its labels, which have to be carefully chosen in such a way to guarantee its"geometricity". For the state to be peaked also on a given extrinsic geometry, one has to take a superposition of such coherent states [4] [193, 49]. Schematically:

$$|\Psi_o\rangle = \sum_{j_{ab}} \psi_{j^o}(j) |j, \Upsilon_a(\vec{n})\rangle, \tag{6.9}$$

with coefficients $\psi_{j^o}(j)$ given by a Gaussian distribution times a complex phase: [5]

$$\psi_{j^o}(j) = \exp\left(-i \sum_{ab} \gamma \Theta_o^{ab} (j_{ab} - j^o_{ab}) - \sum_{ab,cd} \gamma \alpha^{(ab),(cd)} \frac{j_{ab} - j^o_{ab}}{\sqrt{j^o_{ab}}} \frac{j_{cd} - j^o_{cd}}{\sqrt{j^o_{cd}}} \right). \tag{6.10}$$

Here, $\Theta_o^{ab} = \Theta_o^{ab}(\{j^o\})$ is the simplicial extrinsic curvature, i.e. the dihedral angle, associated to the triangle (ab) shared by the tetrahedra a and b within the flat four simplex defined by the triangle areas j^o_{ab}; the 10×10 matrix α is supposed to be complex with positive-defined real part. The Barbero-Immirzi parameter γ multiplying the matrix α has been introduced for later convenience.

3. The "thin" and "thick" wedge terminology is standard in this context. The wedge composed by two space-like tetrahedra is said to be "thin" if their normals are both future or past pointing, and "thick" otherwise.
4. This is why these states are sometimes called *semi*-coherent.
5. This construction parallels that of standard quantum-mechanical coherent one-particle states, with wave function:

$$\psi(x) \propto \exp\left(-\frac{(x - x_o)^2}{2\sigma_x^2} + i p^o x \right).$$

6.1.4 *EPRL Graviton Propagator*

The only missing piece of the spinfoam graviton-propagator construction is the metric operator. For simplicity, and following [49], I work with the (densitized-)inverse-metric operator, which can be easily written in terms of the LQG flux operators through the triangles shared by the boundary tetrahedra. The flux operator at one boundary point, acts at one specific node n along the link between the nodes n and a. For each of the three tangential directions i, write this operator as $(E_n^a)^i$; therefore the inverse-metric operator reads:

$$g_n^{ab} = \delta_{ij}(E_n^a)^i(E_n^b)^j. \tag{6.11}$$

When acting on a $SU(2)$ state $|j_{cd}, \vec{n}_{cd}\rangle$ associated to a coherent Livine-Speziale intertwiner (equation 6.4), the flux operators acts as

$$(E_n^a)^i|j_{cd}, \vec{n}_{cd}\rangle = 8\pi G\hbar\gamma(\delta_c^a\delta_d^n + \delta_c^n\delta_d^a)J^i|j_{cd}, \vec{n}_{cd}\rangle, \tag{6.12}$$

where G is the Newton constant and J^i is the i-th component of the angular momentum operator in representation j_{cd}.

It is then a matter of calculations to show [44] that the connected two point function of the metric operator at this order of perturbation theory in the EPRL model can be recast into the path-integral form:

$$\begin{aligned} G_{nm}^{abcd} =& \frac{\sum_j \psi_j \int \mathcal{D}^4 g \mathcal{D}^{10} z \, q_n^{ab} q_m^{cd} e^S}{\sum_j \psi_j \int \mathcal{D}^4 g \mathcal{D}^{10} z \, e^S} - \\ & - \frac{\sum_j \psi_j \int \mathcal{D}^4 g \mathcal{D}^{10} z \, q_n^{ab} e^S}{\sum_j \psi_j \int \mathcal{D}^4 g \mathcal{D}^{10} z \, e^S} \frac{\sum_j \psi_j \int \mathcal{D}^4 g \mathcal{D}^{10} z \, q_m^{cd} e^S}{\sum_j \psi_j \int \mathcal{D}^4 g \mathcal{D}^{10} z \, e^S}, \end{aligned} \tag{6.13}$$

where ψ_j are the coefficients of equation 6.10, and

$$\int \mathcal{D}^4 g \mathcal{D}^{10} z := \left[\prod_a \int_{SL(2,\mathbb{C})} dg_a \, \delta(g_5)\right] \left[\prod_{a<b} \int_{\mathbb{CP}^1} \left(\frac{d_{j_{ab}}}{\pi}\right) \omega_2^{g_a, g_b}\right] \tag{6.14}$$

(see section 5.3 for further details on the notations). Also, with a slight abuse of notation, define the unit norm spinor $n_{ab} \equiv |n_{ab}\rangle \equiv |\frac{1}{2}, \vec{n}_{ab}\rangle \in \mathbb{C}^2$. Then, the insertion q_n^{ab} can be written using the $\mathbb{C}^2$ Hermitian scalar product $\langle\cdot,\cdot\rangle$:

$$q_n^{ab} = \delta_{ij}(A_n^a)^i(A_n^b)^j, \quad (A_n^a)^i = \gamma j_{an}\frac{\langle\sigma^i Z_{an}, n_{an}\rangle}{\langle Z_{an} n_{an}\rangle}, \tag{6.15}$$

σ^i being the Pauli matrices, and $Z_{ab} := g_a^\dagger z_{ab}$ as in equation 5.18. Remark that, despite sharing the same physical meaning, g_n^{ab} (respectively $(E_n^a)^i$) and q_n^{ab} (respectively $(A_n^a)^i$) are different objects: the former is an operator acting on a spin-network state, while the latter is a complex-valued function of the variables entering the path integral. Finally the action S is nothing but the usual vertex action augmented by the thin/thick wedge phases of equation 6.5:[6]

$$S(g,z) = \sum_{a<b} S_{ab}(g_a, g_b, z_{ab}) \tag{6.16}$$

$$S_{ab} = j_{ab}\log\frac{\langle n_{ab}|Z_{ab}\rangle^2\langle Z_{ba}|n_{ba}\rangle^2}{\langle Z_{ab}|Z_{ab}\rangle\langle Z_{ab}|Z_{ab}\rangle} + i\gamma j_{ab}\log\frac{\langle Z_{ab}|Z_{ab}\rangle}{\langle Z_{ab}|Z_{ab}\rangle} - i\Pi_{ab}j_{ab}. \tag{6.17}$$

6. See also [40] for details. However, notice that the amplitude used here is slightly different from the one I used in [190], and also the one used by Eugenio Bianchi and You Ding in [44]. This change is dictated by the will of keeping a uniform notation throughout the thesis. Anyway, none of the results is affected by this change (however, some intermediate steps are).

Now, in order to extract some intelligible physics out of equation 6.13, one can simplify the formulas by taking the limit of large distances. Remark that this is exactly the physical regime of interest in order to make contact with the graviton propagator on a semiclassical background far away from the Planck scale. Formally, this limit is simply achieved by uniformly rescaling all the spins appearing in equation 6.13 by a common factor $\lambda \to \infty$:

$$j_{ab} \mapsto \lambda j_{ab}, \quad j^{o}_{ab} \mapsto \lambda j^{o}_{ab}. \tag{6.18}$$

However, before proceeding with the analysis of equation 6.13 in the large distance regime, it is convenient to manipulate it one last time. Following [44] once more, observe that a total effective action can be introduced, which still scales linearly in λ:

$$S_{\text{tot}} = S + \log \psi_j \mapsto \lambda S_{\text{tot}}. \tag{6.19}$$

Furthermore, in the large-spin limit, the discreteness of the spins themselves becomes less and less relevant and the sum over the spins can be substituted with an integral (at least close to the region where the integrand is peaked):

$$\sum_j \mapsto 2^{10} \int d^{10} j. \tag{6.20}$$

In this way, equation 6.13 can be formally written as [7]

$$\lambda^{-4} G^{abcd}_{nm} = \frac{\int \mathcal{D}x \, q^{ab}_n q^{cd}_m e^{\lambda S_{\text{tot}}}}{\int \mathcal{D}x \, e^{\lambda S_{\text{tot}}}} - \frac{\int \mathcal{D}x \, q^{ab}_n e^{\lambda S_{\text{tot}}}}{\int \mathcal{D}x \, e^{\lambda S_{\text{tot}}}} \frac{\int \mathcal{D}x \, q^{cd}_m e^{\lambda S_{\text{tot}}}}{\int \mathcal{D}x \, e^{\lambda S_{\text{tot}}}}, \tag{6.21}$$

where x summarizes all the 24 variables (j, g, z).

Applying the stationary phase approximation to this expression taking $\lambda \to \infty$, one finds [8] [47] that at leading order in negative powers of λ:

$$\lambda^{-4} G^{abcd}_{nm} = \lambda^{-1} (H^{-1})^{ij} (q^{ab}_n)'_i (q^{cd}_m)'_j \Big|_{x_o} + O(\lambda^{-2}), \tag{6.22}$$

where both the Hessian of the total action $H_{ij} = \partial^2 S_{\text{tot}} / \partial x^i \partial x^j$ and the partial derivatives of the insertions $q'_i = \partial q / \partial x^i$ are understood to be evaluated at the critical point x_o. In turn, x_o is defined as the solution of $\Re(S_{\text{tot}})|_{x_o} = \sup \Re(S_{\text{tot}}) = 0$ and $\partial S_{\text{tot}} / \partial x^i |_{x_o} = 0$.

Remark that at this order of approximation the details of the measure $\mathcal{D}x$ do not play any role. This fact will be relevant in the next section.

To work out the structure of the Hessian of the total action, it is useful to take advantage of a clever parametrization of variables around the stationary point x_o and of the fact that the action S is linear in the spins. However, before entering into such details, let me recall what the stationary point looks like. The condition $\Re(S_{\text{tot}})|_{x_o} = \sup \Re(S_{\text{tot}}) = 0$ implies that both $\Re(S)|_{x_o}$ and $\Re(\log \psi_j)|_{x_o}$ vanish. [9] In particular, the latter condition implies

$$j_{ab} = j^{o}_{ab}. \tag{6.23}$$

7. The λ^{-4} factor in front of G comes from the scaling $q^{ab}_n \mapsto \lambda^2 q^{ab}_n$.
8. The idea is first found in [149].
9. This is the case because of the non-positivity of $\Re(S)$ (by Cauchy-Schwarz inequality) and of the positivity of the real part of the matrix α.

Since in building the superposition of spin-network states which is Ψ_o, care was taken of choosing the vectors $\vec{n}_{ab}$ as functions of the spins in such a way they always[10] describe a geometrical four simplex, so do the $\vec{n}^o_{ab} = \vec{n}_{ab}(\{j^o\})$. As first shown for the EPRL model by John Barrett and collaborators in [40], the stationary point equations for the action S select, then, those unique holonomies $g^o_a(\{j^o\})$ which parallel transport the vectors $\vec{n}^o_{ab}$ (and the spinors n^o_{ab}) at the centre of the four-simplex and onto one another (following the four-simplex combinatorics, of course), as well as those $\mathbb{CP}^1$ variables $z^o_{ab}(\{j^o\})$ which are essentially[11] equal to $(g^o_a)^{\dagger-1} n^o_{ab}$ (with $a < b$). Also, it turns out that the action S evaluated at the stationary point x_o is (numerically) equal to the Regge action of the four-simplex:[12]

$$S|_{x_o} = iS_R(\{j^o\}) = i\sum_{ab} \gamma\, j^o_{ab}\, \Theta^{ab}_o(\{j^o\})\,. \tag{6.24}$$

Notice that the phase choice of equation 6.10 is the only one which would have provided a solution to the equation $\partial S/\partial j|_{x_o} = 0$, a fact whcih can be easily shown by using the linearity of S in the spins. Also, notice that this same phase choice selects only one of the two possible orientations for the vertex action at the critical points.

Back to the particular parametrization needed to simplify the form of the Hessian, the uniqueness of the solution I just described for a given value of the spins $\{j_{ab}\}$ suggests to parametrize the variables g and z in a neighbourhood of the stationary point in the following way:

$$x = (j_{ab}, g_a, z_{ab}) = (j_{ab}, h_a g^o_a(j_{ab}), z^o_{ab}(j_{ab}) + \delta z_{ab}) \mapsto x = (j_{ab}, \beta^a_i, \delta z_{ab}), \tag{6.25}$$

where $SL(2,\mathbb{C}) \ni h_a \approx \mathbb{I} + \beta^a_i \sigma^i$, and $\beta^a_i \in \mathbb{C}$. In words, for any set of values of the spins $\{j_{ab}\}$ close to the critical one, the data x at a point $\beta^a_i = 0$, $\delta z_{ab} = 0$ are always taken to describe a geometrical four simplex. This parametrization which "follows the geometricity" makes the form of the Hessian quite simple. In particular it sets $\partial^2 S/\partial j \partial\beta = \partial^2 S/\partial j \partial(\delta z) = 0$. Indeed, the previous parametrization assures that one gets simply zero when varying with respect to the spins the first variations of the action with respect to the β (i.e. the g) and δz. The reason is that these first variations are nothing else than the critical point equations for S, which in turn encode exactly the geometricity conditions in terms of closures and parallel transports (see e.g.[128] for more details on how to to obtain these formulas). Hence:

$$H = \left(\begin{array}{c|cc} Q_{10\times10} & 0_{10\times24} & 0_{20\times20} \\ \hline 0_{24\times20} & H^{\beta,\beta}_{24\times24} & H^{\beta,\delta z}_{24\times20} \\ 0_{0\times20} & H^{\delta z,\beta}_{20\times24} & H^{\delta z,\delta z}_{20\times20} \end{array}\right) = \left(\begin{array}{c|cc} Q & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$X_{44\times44}$}} \\ 0 & & \end{array}\right), \tag{6.26}$$

where

$$Q_{(ab)(cd)} = \left.\frac{\partial^2 S_{tot}}{\partial j_{ab}\partial j_{cd}}\right|_{x_o} = -\frac{\gamma\alpha_{(ab)(cd)}}{\sqrt{j^o_{ab} j^o_{cd}}} + i\frac{\partial^2 S_R}{\partial j_{ab}\partial j_{cd}}. \tag{6.27}$$

Remark that also the last equality of the previous equation stems crucially from the choice of parametrization.

This concludes the leading order calculation of the graviton propagator in the EPRL model, since the quantities $(q^{ab}_n)'$ and H can be calculated,[13] and the Hessian

10. In a neighbourhood of $\{j^o_{ab}\}$, at least.
11. Actually, up to a $U(1)$ phase and a normalization factor. See the references for the details.
12. To obtain this formula without any additional phase, the introduction of the thin- and thick-wedge phases Π_{ab} (equation 6.5) played an essential role. Cf. equation 52 of [40].
13. At least in principle: the previous parametrization is highly non trivial for practical purposes, and turns this otherwise easy task into a difficult problem.

can be (at least ideally) inverted. At this point, observe that it would have been probably more satisfactory if H contained *only* the second derivatives of the (area) Regge action appearing in Q. In fact, in such a case the result

$$G^{abcd}_{nm} \sim \sum_{(ef)(gh)} Q^{-1}_{(ef)(gh)} \frac{\partial q^{ab}_n}{\partial j_{ef}} \frac{\partial q^{cd}_m}{\partial j_{gh}} \tag{6.28}$$

would match with the two-point function computed in perturbative Regge calculus with the appropriate boundary state (at this purpose, see in particular [49, 48]). In the next subsection I review an idea on how, and in which sense, one can recover this result.

6.1.5 *The Bojowald-Bianchi-Magliaro-Perini Limit*

Since the first work on the graviton propagator within the EPRL model [47], it has been realized that in order to recover the quantum Regge calculus result for the graviton propagator one had to consider the limit where $\gamma \to 0$ while keeping γj_{ab} constant and large. This particular limit first appeared in a paper about the semiclassical limit of loop quantum cosmology by Martin Bojowald in 2001 [51], and has been recently revived by Claudio Perini [180], and in particular by Muxin Han [123, 125, 124], as a tool to go around the so-called flatness problem in the spin-foam semilcassical limit [52, 131, 132, 180] by selecting the right parameter-space region. However, in the context of the present calculation, there is a simpler way to understand this limit, which we will call from now on the Bojowald-Bianchi-Magliaro-Perini (BBMP) limit.

First, introduce the physical area associated to a (large) spin j (equation 2.16): $a_j \approx a_{Pl}\, \gamma j$, where $a_{Pl} := 8\pi G\hbar$ is Planck area, and the spacing in the area spectrum (always in the large spin regime) is $\Delta a \approx \frac{1}{2}\gamma a_{Pl}$. Then, rewrite all the quantities involving the spins in terms of these physical areas, in particular those entering the action S. At this point, it is straightforward to realize that the large distance, i.e. large area regime, considered so far is formally equivalent to the semiclassical [14] $\hbar \to 0$ limit. However, these two regimes should not be confused with the "full" classical regime: the inverse Hessian H^{-1} appearing in the leading order expression of the graviton propagator G^{abcd}_{nm} is proportional to $\hbar$, and this should not be sent to zero, while higher order terms are. Now, with this premise, it is clear that the $\gamma \to 0$, $\gamma j \sim \mathrm{const}$ limit is a limit where quantum effects are not neglected in general, but where the spacing of the area spectrum is nonetheless sent to zero. Therefore, it is in this limit, that on can hope to recover known results from standard perturbative general relativity.

Another more technical, but even more direct way to address this same limit, is the following. Given any general spinfoam amplitude, once expressed in terms of the physical areas, the limit $\gamma \to 0$ can be taken by performing a stationary phase analysis on those terms of the action which happen to be proportional to γ^{-1}. A careful inspection shows that the stationary phase conditions one obtains in this way are exactly equivalent to the geometricity conditions obtained in the large spin limit of the full action, in turn equivalent to the semiclassical limit [15] $\hbar \to 0$. The consequence of this fact is that the zero Immirzi parameter limit projects the spin-foam amplitude to a (very specific) form of quantum (area-)Regge calculus on the

14. And to the weak gravity $G \to 0$ limit, too. Indeed, the only physical scale which matters in pure quantum gravity is the Planck area a_{Pl}, which is in turn the product of $\hbar$ and G.
15. Indeed, remark that in equation 6.17 γ and j do not always appear together.

given triangulation, by killing all the non-geometric fluctuations which are generically present in the full spinfoam amplitude.

Now that the meaning of the BBMP limit has been clarified, it is possible to check whether it actually satisfies expectations. In order to apply this limit to the spinfoam graviton problem, one must first investigate the dependence of the Hessian H from both the Barbero-Immirzi parameter γ and the spin scale j, defined by setting $j_{ab} = j\epsilon_{ab}$, with $\epsilon_{ab} \sim O(1)$. A careful analysis, and many calculations, show the following result (see e.g. the work by Eugenio Bianchi and You Ding [44]):

$$Q = \frac{\gamma j}{j^2} Q_\epsilon \quad \text{and} \quad X = j\left[X_\epsilon + O(\gamma)\right], \tag{6.29}$$

where Q_ϵ, X_ϵ depend only on ϵ (and neither on γ nor j). Therefore, in the BBMP limit, the inverse of H is:

$$H^{-1} \overset{\text{BBMP}}{=} \left(\begin{array}{c|c} j^2(\gamma j)^{-1} Q_\epsilon^{-1} & 0 \\ \hline 0 & j^{-1}(X_\epsilon^{-1} + O(\gamma)) \end{array}\right). \tag{6.30}$$

Consider now the derivatives of the isertions $(q_n^{ab})'$. A moment of reflection shows that they scale as (see [44] for more details)

$$\frac{\partial q}{\partial j} = \gamma(\gamma j)(q')_j^\epsilon \sim O\left(\gamma(\gamma j)\ \right), \tag{6.31a}$$

$$\frac{\partial q}{\partial \beta} = (\gamma j)^2 (q')_\beta^\epsilon \sim O\left((\gamma j)^2\right), \tag{6.31b}$$

$$\frac{\partial q}{\partial \delta z} = (\gamma j)^2 (q')_z^\epsilon \sim O\left((\gamma j)^2\right). \tag{6.31c}$$

Hence, schematically

$$\begin{aligned} G &\overset{\text{BBMP}}{=} (\gamma j)^3 Q_\epsilon^{-1} (q')_j^\epsilon (q')_j^\epsilon + O\left(j^{-1}(\gamma j)^4\right) + O\left(\gamma j^{-1}(\gamma j)^4\right) \\ &\overset{\text{BBMP}}{=} (\gamma j)^3 \left[Q_\epsilon^{-1} (q')_j^\epsilon (q')_j^\epsilon + O(\gamma)\right]. \end{aligned} \tag{6.32}$$

So, the sought semiclassical result (equation 6.28) was shown to arise from the BBMP limit. Remark, that the correlations expressed in equation 6.32 scale with the inverse squared distance between the two nodes, i.e. as $\sim (\gamma j)^{-1}$. To realize this one has to recall the fact that each of the correlated objects (the q's) scales by itself as $(\gamma j)^2$ (see [47]).

Maybe, a faster way to get directly to this formula, would have been doing the change of variables $\gamma j_{ab} \mapsto k_{ab}$, in order to take the limit $\gamma \to 0$ directly at the level of equation 6.13. As observed by Elena Magliaro and Claudio Perini in [155], a leading order saddle point appoximation of this limit automatically projects the EPRL spinfoam calculations to a specific quantum Regge calculus, where the only variable are the k_{ab}. Then, the limit $k_{ab} \to \infty$ would be simply the semiclassical limit of the quantum Regge calculus and would directly lead to the previous result. The obvious drawback of this procedure is the fact that the possibility is lost of investigating the corrections in the Barbero-Immirzi parameter.

6.2 RADIATIVE CORRECTIONS TO THE PENTAGON SPINFOAM

After having reviewed the calculation of the first-order spinfoam graviton propagator, it is time to study to the main subject of this chapter, i.e. the corrections to the previous calculation engendered by the presence of a "melonic" bubble on one of the edges of σ_4 (see figure 6.2). Name this new spinfoam σ_M.

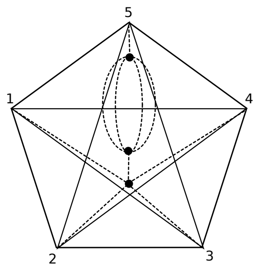


Figure 6.2: The melon-bubble corrected pentagon spinfoam σ_M.

Contrary toσ_4, σ_M contains two types of faces: external and internal. The external faces have spin fixed by the boundary state. The internal faces have by definition unconstrained spins and, in this case, all of them are two-edge-long faces confined within the melonic insertion. These unconstrained spins must be summed over when calculating the spinfoam amplitude. The (approximative) result of this procedure was the subject of the previous chapter of this thesis. Modulo the *caveats* discussed there, this is given by the following formula, where, for simplicity, the $SU(2)$ face weight for the EPRL model has been picked:

$$Z^{\sigma_M}_{\mathrm{EPRL},\Lambda} \sim \log\left(\frac{\Lambda}{\bar{j}}\right) Z^{\sigma_4'}_{\mathrm{EPRL}} + \text{finite terms.} \tag{6.33}$$

Here, σ_4' is given in figure 6.3, Λ is a cut-off on the spins (to be taken $\Lambda \gg \bar{j}$), and $\bar{j}$ is the scale of the boundary spins ($\bar{j} \sim j^o_{ab} \gg 1$). The exact form of the coefficient in front of the logarithm and of $\bar{j}$ is not known.

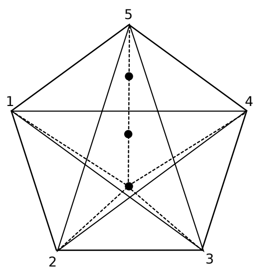


Figure 6.3: The spinfoam σ_4'.

It is now enough to go through the whole procedure described in the previous section, just substituting the amplitude $Z^{\sigma_4}_{\mathrm{EPRL},\Lambda}$ with $\log\left(\Lambda/\bar{j}\right) \times Z^{\sigma_4'}_{\mathrm{EPRL}}$. In particular, the boundary state on $\partial\sigma_M = \Gamma_5 = \partial\sigma_4$ is left unchanged.

The first important remark is that the (divergent) normalization $\log\left(\Lambda/\bar{j}\right)$, when staying within same approximation as in the previous section, just modifies the "path-integral" measure $\mathcal{D}x$ by a spin-dependent factor. As observed in the paragraph following equation 6.22, the details of this measure do not influence the leading order result I am going to investigate. [16] Nevertheless, the presence of two more

16. However, they *do* influence sub-leading orders.

vertices in $Z^{\sigma'_4}_{\text{EPRL}}$ has some consequences even at this order. Specifically, more variables are needed to write the spinfoam amplitude and the action gets consequently modified:

$$S'(g,z) = \sum_{a<b} S'_{ab}(g_a, g_b, z_{ab}) \tag{6.34}$$

$$\begin{aligned}
a,b \neq 5, \quad S'_{ab} &= j_{ab} \log \frac{\langle n_{ab}|Z_{ab}\rangle^2 \langle Z_{ba}|n_{ba}\rangle^2}{\langle Z_{ab}|Z_{ab}\rangle\langle Z_{ab}|Z_{ab}\rangle} \\
&\quad + i\gamma j_{ab} \log \frac{\langle Z_{ab}|Z_{ab}\rangle}{\langle Z_{ab}|Z_{ab}\rangle} - i\Pi_{ab} j_{ab} \\
b = 5, \quad S'_{a5} &= j_{a5} \log \frac{\langle n_{a5}|Z_{a5}\rangle^2 \langle z_{5a}|m_a\rangle^2}{\langle g_a^\dagger z_{a5}|Z_{a5}\rangle} + i\gamma j_{a5} \log \frac{1}{\langle Z_{a5}|Z_{a5}\rangle} \\
&\quad + j_{a5} \log \frac{\langle m_a|w_a\rangle^2 \langle W_a|m'_a\rangle^2}{\langle W_a|W_a\rangle} + i\gamma j_{a5} \log \langle W_a|G_a\rangle \\
&\quad + j_{a5} \log \frac{\langle m'_a|W'_a\rangle^2 \langle w'_a|n_{5a}\rangle^2}{\langle W'_a|W'_a\rangle} + i\gamma j_{a5} \log \frac{1}{\langle W'_a|W'_a\rangle} \\
&\quad - i\Pi_{a5} j_{a5}\,,
\end{aligned} \tag{6.35}$$

where the gauge fixing was already taken into account (and "concentrated" along the row of edges stemming from node 5). Remark that it was necessary to introduce eight more (normalized) spinors $\{m_a, m'_a\}$, eight more $\mathbb{CP}^1$ variables $\{w_a, w'_a\}$ (taken in the unit-norm section), and two [17] more $SL(2,\mathbb{C})$ variables $\{G, G'\}$ (see figure 6.4). Also, I defined $W_a := G^\dagger w_a$ and $W'_a := (G')^\dagger w'_a$. Notice that both Z_{5a} and z_{5a} (as well as both W_a and w_a, and both W'_a and w'_a) appear because of the gauge fixing to the identity of some of the $SL(2,\mathbb{C})$ variables.

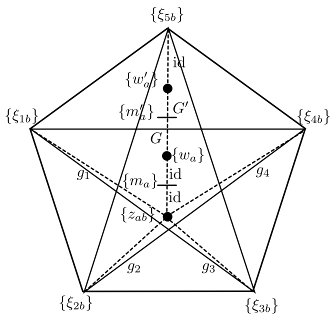


Figure 6.4: The spinfoam σ'_4, where the variables living on it were highlighted.

Finally, in order to calculate the correlation of the metric at the node 5 with the metric at some other node a, care should be taken of this adaptation of the insertion:

$$q_5^{ab} = \delta_{ij}(A_5^a)^i (A_5^b)^j, \quad (A_5^a)^i = \gamma j_{a5} \frac{\langle \sigma^i w'_a, n_{5a}\rangle}{\langle w'_a, n_{5a}\rangle}, \tag{6.36}$$

(while all other insertions stay unchanged).

Now, we have all the elements to carry out the saddle point analysis of the corrected graviton propagator, via the formula

$$G^{abcd}_{nm} = \langle\langle q_n^{ab} q_m^{cd}\rangle\rangle - \langle\langle q_n^{ab}\rangle\rangle\langle\langle q_m^{cd}\rangle\rangle, \tag{6.37}$$

17. Two other $SL(2,\mathbb{C})$ elements have been tacitly gauge-fixed to the identity. See figure 6.4.

where

$$\langle\langle Q \rangle\rangle = \frac{\sum_j \psi_j \int \mathcal{D}^4 g \mathcal{D}^2 G \mathcal{D}^{10} z \mathcal{D}^8 w \mathcal{D}^8 m \, Q \, e^{S'}}{\sum_j \psi_j \int \mathcal{D}^4 g \mathcal{D}^2 G \mathcal{D}^{10} z \mathcal{D}^8 w \mathcal{D}^8 m \, e^{S'}}. \tag{6.38}$$

6.3 SEMICLASSICS OF THE CORRECTED GRAVITON-PROPAGATOR

The analysis of the stationary point of the new action is only slightly modified by the presence of the two new vertices. Indeed, as shown explicitly in the appendix of [190], the group elements G and G' at the stationary point have to be in the $SU(2)$ subgroup of $SL(2,\mathbb{C})$, making them almost irrelevant, since their effect can always be reabsorbed in the redefinition of other variables. This conclusion could be reached more directly from the asymptotics of the "melonic gluing" discussed at the end of the last chapter and first worked out by Jacek Puchta [184]. In fact, this result can be roughly be restated as saying that, in the large-spin limit, one can ignore any insertion of additional vertices along a single edge. This is what happens to the two additional vertices of figure 6.4 with respect to figure 6.1. Therefore, the stationary point geometry is always that of the four simplex induced by the boundary state.

Then, what is more compelling to study is the way *correlations* between insertions behave, and whether or not differences arise with respect to the spinfoam σ_4. At a practical level, what one has to study is therefore the Hessian H' of the action S'.

The first thing to notice is that H' can be made block diagonal, therefore isolating the spin-spin correlations, by choosing a parametrization of the space around a stationary point which follows the "geometricity" conditions. This follows exactly the discussion of the previous section (in particular, see the paragraph of equation 6.25). For what concerns the other variables, name the "old" ones existing also in σ_4 (i.e. $\{g_a, z_{ab},\}$) x', and the "new" ones (i.e. $\{w_a, w'_a, m_a, m'_a, G, G'\}$) y. So, the Hessian H' can be schematically written as

$$H' = \left(\begin{array}{c|cc} Q' & 0 & 0 \\ \hline 0 & X' & Z \\ 0 & Z^T & Y \end{array}\right), \tag{6.39}$$

where

$$Q' = \left.\frac{\partial^2 S'_{tot}}{\partial j \partial j}\right|_{x'_o, y_o}, \quad X' = \left.\frac{\partial^2 S'_{tot}}{\partial x' \partial x'}\right|_{x'_o, y_o},$$
$$Z = \left.\frac{\partial^2 S'_{tot}}{\partial x' \partial y}\right|_{x'_o, y_o}, \quad Y = \left.\frac{\partial^2 S'_{tot}}{\partial y \partial y}\right|_{x'_o, y_o}. \tag{6.40}$$

A moment of reflection reveals that[18] $Q' = Q$ and $X' = X$ (see equation 6.26). On the other hand, the matrix Y, can be worked out only by looking at the actions S'_{a5} for the four faces $\{(a5)\}$ and is *a priori* quite complicated. Of main interest, the matrix Z has quite a simple form since it contains many null elements. Before starting to analyse this matrix, notice that

$$\left.\frac{\partial q_a}{\partial x'}\right|_{x'_o, y_o} \neq 0 \text{ and } \left.\frac{\partial q_5}{\partial x'}\right|_{x'_o, y_o} = 0, \tag{6.41a}$$

18. After having carefully parametrized the space as discussed above.

while

$$\left.\frac{\partial q_a}{\partial y}\right|_{x_o',y_o} = 0 \text{ and } \left.\frac{\partial q_5}{\partial y}\right|_{x_o',y_o} \neq 0, \tag{6.41b}$$

with $a \in \{1, \dots, 4\}$.

Looking at S'_{a5} in equation 6.35, it is immediate to see that the only non-null terms in Z are

$$Z_{z_{5a},m_a} = \left.\frac{\partial^2 S'_{tot}}{\partial z_{5a} \partial m_a}\right|_{x_o',y_o}. \tag{6.42}$$

Unluckily, these terms are enough to waste any simple attempt to reduce (even part of) the inverse of this Hessian $(H')^{-1}$ to H^{-1}. Indeed, it is easy to realize that the variables m_a (z_{a5}) are - more or less directly - correlated to essentially all the other x' (respectively y). In turn, this means that all the variables $\{x', y\}$ are cross-correlated among them. Therefore, abandoning for the rest of the paper the hope of any finer analysis, define

$$\begin{pmatrix} Y' \end{pmatrix} = \begin{pmatrix} X' & Z \\ Z^T & Y \end{pmatrix}. \tag{6.43}$$

It is simple to see that the scale dependence of Q' and Y' from j_o and γj_o (I have the BBMP limit in mind) is the same as that of Q and X. This fact allows to immediately affirm that at leading order in the BBMP limit, the graviton propagator gets no modifications from the insertion of the melonic bubble within one of its edges. Once more, as discussed at the end of section 6.2, this could have been directly deduced by making first the change of variables $\gamma j_{ab} \to k_{ab}$ followed by the two limits (taken in this order) $\gamma \to 0$ and $k_{ab} \to \infty$.

6.4 SUMMARY AND DISCUSSION

I have reviewed in some detail the calculation of the Lorentzian EPRL spinfoam graviton propagator on the spinfoam σ_4 at leading order in an expansion over the external spin scale. I recalled that the quantum Regge result can be recovered in the Bojowald-Bianchi-Magliaro-Perini (BBMP) limit. I have then gone through the same calculation steps after inserting a "melonic" radiative correction to one of the spinfoam edges.

To deal with the new spinfoam, I first considerably simplified the problem by applying the result on the dominant order of the melon graph obtained in Chapter 5. In this way it was possible to effectively "contract" the bubble to two spinfoam vertices while forgetting its internal structure. The divergent factor this "contraction" comes with is irrelevant at the considered order of approximation. As a side remark, observe that this happens because of the normalization scheme used, i.e. ultimately because of the denominator of equation 6.3.

Then, I noticed that the dominating process (which can be identified with the stationary point of the path integral form of the amplitude) is geometrically equivalent to the original one. Nonetheless, the correlation matrix of graviton fluctuations gets modified by the presence of the bubble. Moreover, being the correlation matrix the *inverse* of the Hessian of the effective action of the spinfoam, this happens in a non-trivial way: *a priori*, *all* the cross correlations are modified, and not only those involving the node "close" to where the bubble was inserted (node 5 in my notation). Finally, I showed that such non-trivial modifications of the spinfoam graviton propagator involve only those terms which disappear in the BBMP limit ($\gamma \to 0$, $\gamma j^o \sim cnst.$), and therefore do not modify the final leading-order Regge result.

To conclude, I stress a couple of points about the consistency of the melonic-bubble radiative corrections with respect to previous results and known physics. First, the consistency is obtained only in the BBMP limit, where spins are large and the Barbero-Immirzi parameter is reciprocally small, and couldn't be obtained otherwise. The reason for this is two-fold: on the one hand the melonic bubble is equivalent to the trivial gluing only in the case of large external spins (which here happen to be the same boundary spins which must be large in order to make contact with known low-energy physics); on the other hand, leading-order corrections are present in the $1/j$ expansion (before taking the $\gamma \to 0$ limit) in both the pentagon and melonic foams, and there is no reason for such corrections to be same in the two cases.

At last, precisely the fact that the leading corrections stemming from the presence of the melonic bubble affect those terms which disappear in the $\gamma \to 0$ limit, is preventing us to advance any physical interpretation of such radiative correction, for the simple reason that it affects terms on their own of difficult interpretation.

Part III

- DISCUSSION -
DIVERGENCES AND GEOMETRY

This is the last part of the thesis, and it is devoted to the consequences the results obtained in the second part may have on the EPRL model. I also connect these results to more general problems such as the issues of recovering diffeomorphism symmetry in spinfoam gravity, the interpretation of the renormalization flow of the closely related group field theories, and the possible consequences of the presence of a cosmological constant.

7 DIVERGENCES AND GEOMETRY

In this chapter, I want to add some further comments and thoughts about the calculations performed in the last two chapters. This will be the occasion to pinpoint their physical meaning on one side, and on the other to try and abstract from them some general features of divergences in spinfoam quantum gravity. In particular, I will discuss the possibility that the divergences have a geometrical origin, their relation to the flatness problem, and to the semiclassical limit.

7.1 DIFFEOMORPHISMS

In section 4.4, I discussed how divergences are related to residual diffeomorphism symmetry in the Ponzano-Regge model. It is then tempting to ask if the same is true in the EPRL model.

If a relation with diffeomorphism symmetry was actually present, one would expect the divergence to be related with the freedom of "moving around in spacetime" the only vertex of the triangulation Δ dual to the melon graph, which is not constrained by the boundary tetrahedra. This is the point "at the centre" of the three-ball dual to the melon graph, that is P in figure 5.4. Since this point would span a four dimensional space, and the spins have the dimension of an area, a diffeomorphism-related divergence is expected to scale with the cut-off as Λ^2.

However, in the previous analysis the two "natural" choices of face-weights, the $SU(2)$ and $SL(2,\mathbb{C})$ ones, lead to divergences which scale as $\log\Lambda$ and Λ^6, respectively. Therefore, there seems to be no way to associate the divergences in these versions of the model with diffeorphisms, at least in the case of the melon graph.

A possibility would be that none of the two face weights I considered is the correct one. Using the requirement that the melon graph divergence scales as Λ^2, one finds that the face weight should scale for large spins as

$$\mu(j \gg 1) \sim j^{\frac{4}{3}} . \tag{7.1}$$

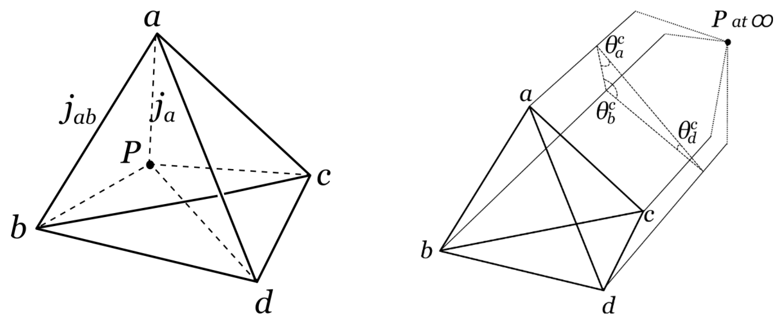
Such a face-weight looks however slightly unlikely, at least from a group field theoretical point of view, where face-weights cannot be fixed by hand, but are related to the representation theory of the group the theory is defined on.

Nevertheless, this is not concluding either, because until one is dealing with a single graph, one can also tweak the edge weights (or vertex amplitude normalization, which is the same) to get the desired result. In this case, fractional powers may arise naturally from expressions of the type of equation 5.62.

7.2 FLATNESS

Talking about geometrical interpretation, I want to go back to another important point: the fact that the sector dominating the amplitude corresponds to two congruent four-simplices of opposite orientations. This sector is selected by its property of having a null action. This is reminiscent of a flatness constraint, but it is not equivalent to it. Moreover, in the absence of such orientation-disagreeing configurations, the amplitude would be less divergent, and with an SU(2) face weight, not divergent at all. [1]

In order to investigate in more depth this issue, I briefly introduce here another work I did together with Marios Christodoulou, Miklos Långvik, Christian Röken, and Carlo Rovelli [66]. In this work, it was argued that a similar mechanism takes place in the context of a toy model for three-dimensional quantum gravity. The considered model is a continuum model of three dimensional quantum gravity "inspired" by the asymptotic regime of the Ponzano-Regge model. In the large spin regime, the one relevant for the divergence calculations, the two models should therefore give extremely similar results. Indeed, in this regime, the off-shell (non-geometrical) fluctuations of the Ponzano-Regge model are highly suppressed and the spacing between the spins becomes essentially "infinitesimal". In that work, it was considered, rather than the melonic bubble, the graph associated to the 1-4 Pachner move vertex correction. This corresponds to the subdivision of a tetrahedron into four other tetrahedra, by connecting all its vertices to a newly introduced extra one.



(a) The 1-4 Pachner move. P is the extra vertex, the only one internal to the triangulation. Generally, P is *not* required to belong to the same $\mathbb{R}^3$ as the initial tetrahedron.

(b) A spiky configuration of the 1-4 Pachner move. In spiky configurations, certain sets of dihedral angles, e.g. those highlighted in the figure, sum up to (almost) π.

Figure 7.1

Note that in three-dimensions the spins correspond to length, and not area, quantum numbers

Like the three-dimensional melon graph, also this one contains a bubble, which is associated to the only internal vertex of the triangulation (figure 7.1a). It is also found to diverge with the third power of the cut-off, exactly as it is expected from the diffeomorphism-symmetry interpretation. However, the configurations contributing to the most diverging sector of the integral are shown to come only from a particular set of orientations of the four tetrahedra (three positively and one negatively oriented), all the others being subdominant. The dominating sector is the only one where orientations are such that, at least in the limit of infinitely "spiky" configurations (figure 7.1b), the *oriented* dihedral angles can cancel each other without the flatness requirement to be satisfied. A slight refinement of the argument [2] presented there, shows also that the usual intuition that the divergence comes from those configurations in which the flatly embedded four tetrahedra span $\mathbb{R}^3$ is wrong: such configurations do not contribute at all to the amplitude, since the

1. I am still neglecting the possibility of a dominance by the degenerate sector.
2. The same refinement shows also that the suspected ill-definiteness of the continuum integral measure in the limit of degenerate tetrahedra is not there.

quantum measure is strictly zero on them. [3]

Summarizing, one can state that the flatness problem consists in the tendency of null-spinfoam-action configurations to dominate the sum over spins appearing in the spinfoam amplitude. This feature is concretely realized in both the examples discussed so far, the 1-4 Pachner move within the Ponzano-Regge model and the melon graph divergence. Actually, this dominance is even more important here because (infinitely) extended regions of the spin-space exist where the spinfoam action is null.

However, it is also clear that such null-action configurations are not quite "flat", in particular they would not be flat within the usual Regge calculus, where the orientations of the building block are coherently fixed from the outset. Indeed, one can interpret this feature present in both the Ponzano-Regge and the EPRL model as coming from the quantization of *tetrad* gravity. At this purpose see the discussion in Chapter 1, and also the paper by Edward Wilson-Ewing and Carlo Rovelli [204]. Furthermore, a modification of the EPRL model has been proposed where the summation over orientation does not arise; this was done by John Engle in [88, 86]. [4]

Anyhow, at the light of the discussion above, it is clear that the issues of (non-infinitesimal [5]) diffeomorphism symmetry and of the flatness constraint must be yet understood completely, especially in relation to the role played by the sum over orientations.

7.2.1 *Ditt-Invariance*

One last point which is worth mentioning is the so-called *Ditt*-invariance. This name was coined by Carlo Rovelli in the paper [195] and comes from the combination of the expression "diff-infariance" indicating invariance under diffeomorphisms, and the name of Bianca Dittrich, who studied its emergence in the context of discretised models of classical and quantum gravity [77, 27, 26, 28, 78, 79, 81]. More specifically, *Ditt*-invariance is the property discretized parametrized systems have of regaining paramtetrization invariance, otherwise broken by the discretization process, in the limit of finer and finer discretization. This has been discussed in detail by Bianca Dittrich and her collaborators Benjamin Bahr and Sebastian Steinhaus in [29], and then reprised by Carlo Rovelli in [195, 196].

The *Ditt*-invariance mechanism in quantum gravity is the conjecture that a similar mechanism allows diffeomorphism symmetry to emerge in the limit of fine triangulations from discretized theories analogous to quantum Regge calculus. More concretely, this happens because approximate invariances of the amplitude of an extremely fine triangulation under (certain) small variations of the geometrical data can give rise collectively to diffeomorphism symmetry at a larger scale. Sometimes,

3. Technically, this is due to the fact that the quantum measure scales as the four-volume of the four-simplex defined by the vertices $(abcdP)$.

4. To be more precise, John Engle's viewpoint on the problem is slightly different: he argues that the arousal in the EPRL model of what we interpreted as two oppositely oriented sectors is actually due to the fact that the linear simplicity constraints do not distinguish *a priori* the two relevant Plebanski sectors, $(\mathrm{I}\pm)$ in equation 1.17 (note that the numbering of the sectors here is different from that in the cited references). The See [85, 87]. Though appealing, this argument does not really explain the presence of the two sectors in the Ponzano-Regge asymptotics, where no simplicity constraint imposition is needed.

5. Infinitesimal diffeomorphisms are much better understood, at least in the context of three-dimensional quantum gravity. See for examples the work by Laurent Freidel and David Louapre [106], as well as the more recent one by Aristide Baratin, Florian Girelli and Daniele Oriti [31]. Note that divergences are, however, *a fortiori* related to non infinitesimal diffeomorphism symmetry.

it is also suggested that in this regime the theory approaches locally a topological BF-like behaviour, which makes the theory "almost" discretization invariant.

Then, the question arises whether in the calculations presented in this thesis there is some trace of such a mechanism. Unluckily, these calculations provide too little evidence for or against *Ditt*-invariance. The main reasons are first that *Ditt*-invariance is a collective phenomenon and therefore more than a couple of "extra" vertices is needed to claim whatsoever, and second that the melon graph has a degenerate geometrical structure, and hence is probably not quite suited for this kind of analysis (a study of the 1-5 Pachner move vertex correction [181] would already be more adequate). However, in spite of these pessimistic remarks, the present calculations can point out some interesting general features of the models considered so far: putting aside for a moment the orientation change issue, what emerges is that the large spin almost-flat regime which dominates the partition function of the internal triangulation vertices [6] may constitute a possible realization of "quasi" triangulation invariance, which is one of the main features of *Ditt*-invariance. However, this comes with the counter-intuitive twist of this happening "correctly" in a deeply quantum regime, but being therefore characterized by large "semiclassical" values of the spins. The relevance and meaning of the large-spin regime is discussed in the next section.

7.3 RADIATIVE CORRECTIONS AND THE SEMICLASSICAL LIMIT

In estimating the melon graph amplitude, extensive use has been made of saddle point techniques as well as of geometry reconstruction theorems which are usually employed to study the semiclassical regime of spinfoam models. At first, I found this point surprising and confusing: why is one led to semiclassical considerations when peering into the most quantum of the processes, that is into radiative corrections?

Spinfoam amplitudes are by construction written as *discrete* sums over spin labels of some (better finite) partial amplitudes. Therefore the only possible source of divergences is in the large spin regime. In turn, large spin regime is known to be related to semiclassical properties of the spinfoam building block. Therefore the study of spinfoam divergences is bound to the study of a semiclassical regime of appropriate spinfoam (sub)structures. This argument is very simple, so why does it sound odd at all? The tension arises from the comparison with our usual intuition of quantum processes, which has developed mostly in the study of quantum field theory. Consider the case of a field theory in a box where suitable boundary conditions are imposed. In this way possible infrared divergences are cured in a way analogous to the fact that spinfoam divergences are "cut-off" by the minimal spin $j = 1/2$. Furthermore, this makes the momenta of the particles discrete, in analogy to the discreteness of the spin sequence. In such a system, Feynman diagrams divergences arise from large momenta, too. [7] However, why these large momentum quantum numbers are not treated using semiclassical methods? What makes spinfoam amplitudes really different from Feynman diagrams in this respect?

The point is that the semiclassical regime used in evaluating spinfoam amplitude divergences has *nothing to do* with the semiclassical regime of the (group) field theory generating those divergences! To make this point clearer and stronger, let me

6. These are the vertices associated to the spinfoam bubbles.
7. I am completely neglecting those divergences arising by the resummation of the series of Feynman diagrams. In a group field theoretical setting one must face with this problem, too.

stress the fact that at this level of the discussion there is no reason to treat the group field theory partition function as a quantum path integral: it can be treated with equivalent results [8] as a "thermal" partition function. The only thing one needs is the formal development of the partition function in powers of the (somewhat mysterious) coupling constant.

Another way of highlighting this fact is by observing that it is the analogue of the *ordinary integral in momentum space* defining a specific Feynman amplitude that is reinterpreted as a regularized version of a *full fledged path integral* for quantum gravity. And it is in this second interpretation that the semiclassical limit is taken.

Now that this point has been clarified, one can turn the attention to a closely related issue. When renormalizing group field theories in a way as close as possible to ordinary quantum field theories, one observes that the renormalization group flows from large spins to small spins. This is natural in this context, since the renormalization group naturally flows "out of" the region where there are infinitely many degrees of freedom. One usually talks about "decimation" of degrees of freedom, and this can only happen in one direction. This direction is what *defines* the meaning of "ultraviolet" and "infrared" in this context. [9] However, the consequent definition of ultraviolet and infrared is clearly the opposite to the one given by the gravitational interpretation of the same amplitudes. Is there something more to this than a vocabulary mismatch?

Once more, I do not have a definite answer to this question, which in my opinion will be answered only when more experience in renormalizing the models will be gained. Nevertheless, looking at the particularly simple examples of the Ponzano-Regge and Boulatov models, it is quite clear that a tension is present. From a lattice perspective, the divergences of the Ponzano-Regge model are interpreted as gauge artefacts and properly gauge fixed. On the other hand, its group field theoretical realization, that is the Boulatov model, turns out to be non-renormalizable and the model must be modified with the introduction of the group-Laplacian in order to make sense of it. Whether the so-modified Boulatov model flows towards an infinite-mass regime giving Ponzano-Regge amplitude in some limit, [10] it seems to me a different question which does not alleviate the tension between the two schemes. [11]

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8. In some applications of the group field theory formalism the distinction is relevant - e.g. when considering quantum condensate of the field [116], or when looking at the group field theory as a second quantized formalism of loop quantum gravity [172]. I thank Daniele Oriti for discussing this fact with me.

9. A common perplexity this argument raises is: what happens if the cut-off on the spins is physical as it is the case for quantum-group based models of quantum gravity? If there is no infinite reservoir of degrees of freedom, in either direction, what selects the ultraviolet and the infrared? These models moreover turn out to be attractive because they seem to implement the presence of a cosmological constant in quantum gravity. [218, 122, 121, 94]. At this purpose, see next section.

10. For the moment there are vague indications that this may be the case. I thank Sylvain Carrozza for explaining this point to me.

11. At this point, let me highlight that another possibility exists to make sense of the Boulatov model as a field theory, without invoking any renormalization procedure. That is defining it via a generalized $1/N$-expansion. To do this, one first cuts-off the spin representations at some value Λ, making the number of propagating modes finite. The group field theory can now be manipulated as a tensor model in the large N regime. For this kind of models, techniques have been recently developed to make sense of the $1/N$-expansion [118]. However, even if the presence of phase transitions is proven, it is still hard to obtain extended smooth geometries out of them [54, 119], in a way reminding the issues with branched polymers encountered in the dynamical triangulations [9]. In the latter context, the problem has been solved to a good extension by the introduction of *causal* dynamical triangulations. For more informations about causal dynamical triangulation, see the first paper by Jan Ambjørn and Renate Loll [10], and the recent review [8].

Furthermore, at least in the case of the melon graph, at least, what one is led to consider is *not* quite a true semiclassical limit: what is going semiclassical is actually a "sub"-geometry, not the full four-dimensional one. Indeed, the solution of the stationary point equations discussed in Chapter 5 allows to reconstruct a *three*-dimensional geometry only. This can be interpreted as the three-dimensional section of a four-dimensional object (in the spike picture) only pictorially, since such an interpretation is not completely justified mathematically. E.g., there is no necessity that the tetrahedron at the basis of the spike, which is given by the boundary state of the graph, is "closed" in the sense of equation 3.2. Nevertheless, for a more complicated spinfoam, dual to a triangulation presenting many internal vertices, i.e. for a spinfoam with a complicated and extended bubble structure, one expects that in the dominant sector of the spinfoam amplitude pieces of the bulk geometry far enough from the boundary decouple and go semiclassical altogether. In this case a full four-dimensional semiclassical geometry would arise from the analysis, and only its semiclassical three-dimensional boundary would actually be the analogue of the semiclassical tetrahedra discussed in the context of the melon graph.

Finally, let me mention one possible way out of this IR-UV tension which is sometimes advocated: it may be that the geometrical interpretation of the quantum spin labels as areas is not viable, not meaningful, *before* a continuum smooth geometry has been extracted from the quantum dynamics. This is the point of view taken for example in group field theory, where smooth space is argued to arise as some sort of "condensate" of group field theory quanta and the classical gravitational dynamics arises only in the form of excitations living on the top of such a condensate [170, 173, 116]. However, it is not clear to me how this can be fully realized, since I do not see for the moment how the right group field theoretical vacuum can be chosen, or engineered, without any reference to the geometrical content of the quanta out of which it is formed. Anyhow, the techniques for dealing with this proposal appeared only recently, and may shed some new light on it in a close future.

7.4 COSMOLOGICAL CONSTANT AND AMPLITUDE FINITENESS

The Ponzano-Regge model can be "twisted" to obtain the Turaev-Viro model [218], via a q-deformation of the group theoretical data defining its amplitude. The main interest from the quantum gravity perspective is probably the fact that in this way the model encompasses the presence of a non-vanishing cosmological constant Λ_{CC} in a very natural way, since it happens to be a function of the parameter q. However, this is by far not the only attraction of the Turaev-Viro model. Indeed, its amplitudes result to be finite and well defined for any triangulation, and constitute a true topological invariant of the (triangulated) three-manifold over which the model is considered (and this constitutes the main interest from the mathematician's perspective).

In particular, the amplitudes are made finite by the appearance of a natural cut-off on the spins, which also depends on q, and is of the order of Λ_{CC}^{-1}. This can be heuristically understood as the fact that in presence of a (positive) cosmological constant a cosmological horizons is present and bounds the maximal radius of the accessible universe.

The ERPL model can be extended to the case of general relativity with cosmological constant in an analogous way. This is the case in both its Euclidean [121, 122, 94] and Lorentzian [94] versions. Such an extension uses the q-deformed Lorentz group, with the q-deformation parameter related to the cosmological constant, and turns out to be finite on any spinfoam. However, the existence of a finite model does not mean that the issue of large radiative corrections can be ignored: it may still happen that some graphs have large amplitudes and therefore drive some renormalization flow, possibly even through phase transitions. Also in this case, the scale which imposes the finiteness of the model is given qualitatively by the inverse cosmological constant, which is of the order of the radius of the universe; therefore, at our - or smaller - length scales, it can be considered as infinite for most practical purposes.

In this thesis, to study the simplest EPRL divergence, I introduced a cut-off Λ to the $SU(2)$ representation labels j. The physical meaning of such a cut-off is that of imposing a maximal value for the area operator, which can be thought as the introduction of a finite maximal size within our universe. A bound to the area operator is typical of the q-deformed version of the EPRL model, in both their Euclidean and Lorentzian versions. Therefore the introduction of such a cut-off can be hoped to be a simple implementation of the main feature of the q-deformed EPRL model within the much more manageable non-deformed version. At the light of this (qualitative) correspondence, the calculation of this paper can be also given a more physical, though possibly naïve, interpretation in which the cut-off Λ *is* a physical quantity and corresponds - at least in order of magnitude - to the inverse [12] cosmological constant Λ_{CC}^{-1} expressed in Planck units: $\Lambda \approx (\Lambda_{CC} a_{Pl})^{-1} \sim 10^{120}$ (in this estimation the possible running of the cosmological constant from a bare value to the observed one is completely neglected).

Probably the most interesting case to analyse is the one characterized by the $SU(2)$ face weight. Indeed, in spite of the largeness of $\Lambda \sim 10^{120}$, its logarithm would not be very large at all:

$$\ln \Lambda \sim \ln(10^{120}) \approx 280, \tag{7.2}$$

which is a very small number when compared to anything one would expect when considering representations of the order of Λ itself. The fact that the melon graph is expected to be one of the most diverging appearing in the model makes this observation even more pressing.

Moreover, notice that the previous estimation seems to be meaningful only for radiative corrections of phenomena taking place at the Planck scale: if one takes into account the fact that there exists a scale at which our analysis in terms of the stationary point approximation starts to be viable, and that this scale is basically set by the boundary spins, then as I have already discussed:

$$Z^{M}_{EPRL,\Lambda} \sim O\left(\ln \frac{\Lambda}{\bar{j}}\right). \tag{7.3}$$

Hence, making a crude identification of scales, for galactic physics one has $\bar{j} \sim (10^5 ly)^2/a_{Pl} \approx 10^{110}$, which gives

$$\ln \frac{\Lambda}{\bar{j}} \sim \ln(10^{10}) \approx 20, \tag{7.4}$$

which changes of just one order of magnitude the previous estimate. [13]

Therefore, this setting leads us to speculate that in a theory with cosmological constant, the supposedly very large divergences associated to melon graphs could

12. I apologize to the reader for the inconvenient notation relating the value of the cut-off Λ, with the *inverse* value of the cosmological constant Λ_{CC}.
13. At our scale $\bar{j} \sim 1m^2/a_{Pl} \approx 10^{70}$, hence $\ln(\Lambda/\bar{j}) \sim \ln(10^{50}) \approx 115$.

be, in the end, not that large at all. Remark, however, that the framework we used for the calculations breaks down at the cosmological scale, since it is strongly based on the existence of a neat scale hierarchy, $\bar{j} \ll \Lambda$.

Considering the fact that the melon graph should be in absolute [14] one of the most diverging graphs appearing in the calculation of the spinfoam amplitudes, this scenario would mean that there is basically no divergent graph, not even in the limit of a really large Λ. It is then compelling to ask how natural would be such a scenario.

In the case of the Turaev-Viro model, having finite amplitudes is crucial and also attractive because the model is topological and hence triangulation invariant. Therefore, one does not have to deal with complicated triangulations or spinfoams and compare them to check if their amplitudes converge in the infinitely-fine triangulation limit. The EPRL model, however, is manifestly not topological (as the results of Chapter 5, by the way, testify) and the need of comparing more and more complicated triangulations is present more than ever. Then, given this premise, the scenario previously described becomes maybe less attractive, at least from a practical viewpoint, since it makes even more difficult to study the effects of refinement on the amplitudes.

Nevertheless, as a concluding general remark which turns away from the EPRL model, it is interesting to notice that triangulation invariant models for *four*-dimensional quantum gravity have been advocated by different authors. E.g. by John Barrett's 1995 work [38], where it is suggested that topological quantum field theory is the correct framework for quantum gravity (even in presence of gravitons and propagating degrees of freedom), and more recently by Bianca Dittrich and collaborators who advocate triangulation invariance to be essential to recover diffeomorphism invariance, since these are basically equivalent [77, 28, 81]. In such a situation, again, having a model which gives finite amplitudes would be mostly desirable, since it woul make it possible to control its amplitudes by more general, and powerful, means.

14. Neglecting too-degenerate and pathological graphs.

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