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Contribution à la modélisation et à la simulation numérique multi-échelle du transport cinétique électronique dans un plasma chaud

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Après avis des Rapporteurs :

M. Guy BONNAUD, Professeur INSTN-CEA, Saclay.

M. Pierre DEGOND, Professeur, Université Paul Sabatier, Toulouse.

Devant la commission d'examen formée de :

M. Guy BONNAUD, Professeur INSTN-CEA, Saclay,

M. Pierre DEGOND, Professeur, Université Paul Sabatier, Toulouse,

M. Alain BACHELOT, Professeur, Université de Bordeaux 1,

M. Christophe BERTHON, Professeur, Université de Nantes,

M. Francis FILBET, Professeur Université Claude Bernard, Lyon 1,

M. Luc MIEUSSENS, Professeur, Université de Bordeaux 1,

M. Stéphane BRULL, Maître de conférence, Université Bordeaux 1,

M. Vladimir TIKHONCHUK, Professeur, Université Bordeaux 1,

M. Emmanuel D'HUMIERES, Maître de conférence, Université de Bordeaux 1,

M. Bruno DUBROCA, Expert Senior CEA/DAM, Professeur, Univ. Bordeaux 1,

Rapporteur.

Rapporteur.

Examineur.

Examineur.

Examineur.

Examineur.

Invité.

Invité.

Co-Directeur de thèse.

Directeur de thèse.

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CHAPTER 1

INTRODUCTION

Since 1973 and the first oil crisis, the issue of exhaustion of oil resources is ubiquitous in the energy domain. In 1973, the oil resources were estimated at 30 years but thanks to the development of oil yields, wind and water energy, the new estimation is about 40 years of oil consumption. Production and energy demand has never stopped growing and humanity has to find an other way to remedy oil exhaustion.

For a long time now nuclear fission has been an alternative. It is a process where a nuclear reaction is initiated by the splitting of the nucleus of a heavy atom. This nuclear reaction also results in the emission of neutrons and a very large release of energy (about 200 MeV). This reaction produces a certain percentage of waste elements which are difficult to get ride of because of their radioactivity. Moreover fuel uranium supplies are limited, even though another of fission fuel plutonium can be obtained from the wastes produced in the uranium fission cycle. Moreover a nuclear accident can release a large variety of radioactive elements in the environment as we have seen with Chernobyl (1986) and more recently Fukushima (2011). This last events lead many countries to improve the use of fission energy and develop renewable energy (wind, geothermal, solar...) but the energy obtained is not sufficient for the world.

An alternative way to obtain sufficient energy could be the nuclear fusion called sometimes thermonuclear fusion which generate less radioactive waste. Thermonuclear fusion is the energy source of the universe, our sun and stars. The nuclear fusion chosen in this work consists in the fusion of two hydrogen isotopes deuterium (${}^2_1D$) and tritium (${}^3_1T$).

There are three main advantages of fusion reactions of deuterium and tritium. The first one concerns the fuel reserves. Deuterium is abundant in the ocean water and it can be extracted at a very low cost. On the contrary, there is no natural tritium. However, tritium is obtained by breeding with the lithium isotope Li^6 which is present in important quantity on earth. The environment impact of fusion is the next advantage. Indeed, no greenhouse emission, no other harmful chemical materials are emitted into atmosphere. Finally, safety of fusion reaction is also an advantage. Only tritium is a radioactive element which is quickly disintegrates (the tritium half-life is about 12 years) contrary to radioactive elements in fission reactions.

The main characteristic of this fusion reaction is its high specific energy generated by the fusion of deuterium (${}^2_1D$) and tritium (${}^3_1T$). This reaction provides (per unit mass) eight times more energy than fission of uranium and between 106 to 107 times more energy than fossil fuels. The reason is in the binding energy, which for nuclear force within a core, is of the order of MeV whereas for the electromagnetic force (within the electronic structure of an atom), it is of the order of eV. Since the twentieth century, humanity searches to create artificial conditions to trigger these fusion reactions and control them in order to obtain energy usable for all human activities in terms of electricity and heat flux. The generator scheme is currently planned to produce continuously energy via the exo-energetic nuclear reaction



The amount of energy released depends on the number of reactions caused in the volume of plasma confinement and its temperature. The gain G of the reaction becomes positive when

the energy released by fusion reactions becomes greater than the one invested in the reaction. This condition can be expressed in terms of density n , temperature T of the plasma and its confinement time τ through the Lawson criteria ([13]). For a Deuterium-Tritium plasma of 10 KeV, this criteria can be rewritten as

$$n \tau > 10^{20} \text{ m}^{-3} \text{ s}.$$

The helium nuclei in (1.1) (also called α particles) maintain the environment's temperature at some $10^7 K$, the usable energy is carried by created neutrons. The conditions for fusion reactions lead to the management of a very warm material environment and so very ionized, which is called plasma. The plasma is the fourth state of matter, it is a fully ionized fuel constituted of charged moving particles interacting by electromagnetic forces. Such a plasma can not come into contact with a solid without being volatilized.

If the plasma can be stably maintained far away from the walls, by magnetic fields, a continuous production is possible: this is the process of magnetic confinement fusion called MCF ([9, 10]). The well-known MCF device is the tokamaks which confine a gas mixture of hydrogen isotopes using a magnetic field produced by coils and an induced current flowing in the plasma. The longitudinal confinement is achieved by choosing a toroidal geometry for the magnetic fusion reactor.

If nothing is done to confine the plasma, continuous energy generation can not be realized but the production of energy can be obtained by repeating the process at a rate of a few Hz. This is the way of the inertial confinement fusion called ICF ([8, 12, 14]). Fusion by inertial confinement, in which fits this thesis, was introduced 50 years ago after the development of lasers. ICF consists of compressing and heating by means of laser beams a target (1 mm of diameter) containing a solid or liquid DT fuel. Then the heating by lasers compresses the target and the aim of ICF is to achieve compression around 1000 times liquid density with minimal heating of the target until the final phase when the compressed fuel is heated to thermonuclear reaction temperature. This approach is called the direct drive (see figure 1.1 (a)). However to avoid the development of hydrodynamic instabilities, another way has been studied: the indirect drive (see figure 1.1 (b)). The target capsule is surrounded by a hohlraum. Light enters in the hohlraum and produces X-rays which in turn provide target heating and compression. It is a more robust but less efficient approach.

To understand processes in ICF, in particular in direct drive approach, a mathematical modeling is necessary. Indeed, the major issue for direct drive approach is the description of the transport of particles, interactions between particles through electromagnetic fields and of course collisions between particles. These phenomenons are important to understand physics of plasmas inside the pellet. The aim of this thesis is to propose different numerical approaches sufficiently robust and relatively low cost in computational time to illustrate the electron transport. Classically kinetic theory is used to describe the evolution of different species in a collisional plasma with accuracy. The Fokker-Planck-Landau equation describes electron transport, accel-

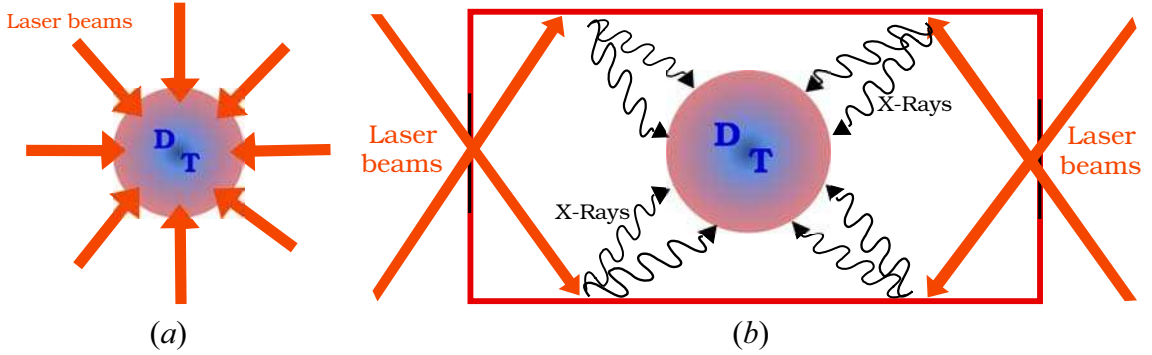


Figure 1.1: Schemes for irradiation of a target by intense laser light by direct drive approach (a) and indirect drive approach (b).

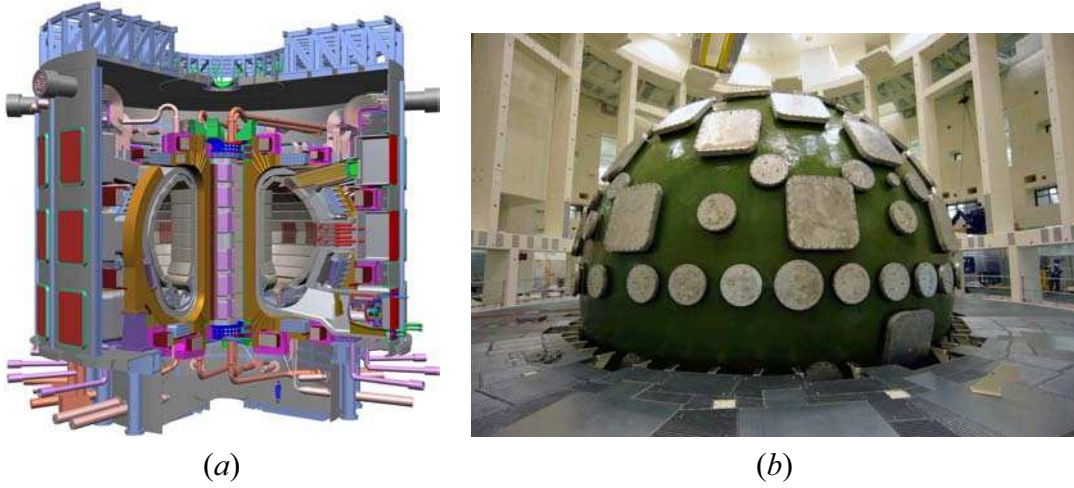


Figure 1.2: “International Thermonuclear Experiment Reactor” project (ITER - Cadarache) for the process of magnetic confinement fusion (a) and the laser installation “Laser MegaJoule” (LMJ - Bordeaux) for Inertial Confinement Fusion (b) .

eration of particles and consider binary collisions between particles whereas Maxwell equations are used to describe the evolution of electromagnetic fields. The solutions of the kinetic equations are non-negative distribution functions $f_\alpha(t, x, v)$ specifying the density of each specie α of particles with velocity v at time t and position x . For the sake of simplicity, we assume in this thesis that the plasma consists of electrons and one ion specie which is considered as immobile in the plasma. Here, all study on the plasma are with small collisional time or plasma period, that is why we can consider ions as immobile because of their important mass compared to electrons. In order to approximate the solution of such problems, many computational methods have been developed up to now. A variety of models are used to describe electrons transport: there exists some hydrodynamic models ([3, 4, 5]) and some kinetic models ([1, 2, 6, 7, 21]). In order to take into account all collective effects, we first focus on a kinetic description and

we improve a code to solve Fokker-Planck-Landau equation coupled with Maxwell system. This code called *KETS* issued from the work of N. Crouseilles and F. Filbet ([6]) has been then developed by R. Duclous ([7]). *KETS* is an acronym for *Kinetic Electron Transport Simulation*. *KETS* is based on a high order deterministic direct numerical method to solve the Vlasov-Maxwell system coupled with Fokker-Planck-Landau collision operators. The phase space considered is a planar geometry of two dimensions in space and three dimensions in velocity. This scheme enjoys many fundamental properties such as conservation of flux particles, positivity of the distribution function... In the present work, *KETS* code is applied for some crucial processes in direct drive approach. Indeed the electron collisional transport between the laser absorption zone and the ablation front is studied. Moreover the formation of a hot spot obtained during first thermonuclear reactions is then simulated.

However the computational time considered for simulations with *KETS* code is too much expensive to be useful in practical computations for Inertial Confinement Fusion. To overcome this drawback, we chose in this work another approach by considering an intermediate model between kinetic model and hydrodynamic one. This new reduced model called M_N model involves an angular closure in the phase space and retains only the energy of particles as the kinetic variable instead of the velocity variable. This model presents the same accuracy than a kinetic model and reproduces our understanding of the electron transport for ICF. Moreover the use of moment model presents two other advantages. First the model simplify the kinetic description and then it is appropriate for massively parallel computation with minimal data transfer. It is important to note that the idea of using this model for kinetic equation is not new. The originality of this work comes from the collision operator. Indeed to preserve physical conditions and to satisfy fundamental properties, a new approximation for the collision operator is proposed here. We consider first the M_1 model for this new description. The closure on the angle variable is crucial to guarantee physical properties. There already exists a moment model called P_N model ([15, 16]) but the positivity of the underlying distribution function is not ensured. Here we propose instead a model based on a minimum entropy principle. The problem of minimization under moments constraints have been firstly introduced by Levermore ([17]) and then clarified by Junk and Schneider ([11, 20]). The present model enjoys many interesting properties for the continuous way and also semi-discretized way. Some simulations are presented here to illustrate these properties.

Numerical simulations shown in this thesis has been realized with parallelized versions of the code *KETS* and M_1 . Indeed, to reduce the computational time, each program has been massively parallelized. Both code (*KETS* and M_1 code) has been parallelized in *mpi* by using the library *PETSC*. Most of results presented in this paper has been launched on the two national supercomputing french centers called *CINES* and *CCRT*. Most of *KETS* simulations has been performed with 42 processors to improve the time computation, on the *CEA – CCRT – platine* facility. This facility is a cluster of NovaScale 3045 servers with a power of 48 Tflops, including 932 nodes, each one having 4 Intel Itanium dual core processors, at 1.6 Ghz. Each processor has a memory of 24 Go. The NovaScale servers are interconnected by a Voltaire Network, with technology Infiniband DDR. Concerning the *CINES – Jade* facility, it is a cluster

SGI Altix ICE8200 with a power of 267 Tflop/s. This cluster includes 23040 nodes, each one having 2 processors Intel Quad-Core E5472 or X5560.

This thesis is a combination of plasma physics, numerical analysis and applied mathematics. In order to highlight the interaction between physics, mathematics and computing, we choose to divide this work in five chapters. The content is then organized as follows.

In the second chapter, some basis of classical kinetic models are recalled for plasma physics. The collisionless kinetic theory is first presented with Vlasov equation. Then the collisional kinetic theory is described with the non-relativistic Maxwell-Fokker-Planck system. Moreover fundamental properties of the collision operators are detailed such as conservation laws, entropy dissipation, etc.

The third chapter is devoted to the improvement of a deterministic direct numerical method introduced in [7] to solve the non-relativistic $2D_x \times 3D_v$ Vlasov-Maxwell system coupled with Fokker-Planck-Landau type operators. Its construction is improved to obtain a conservative scheme for the phase space. Finite volume method is used in order to obtain the approximation of the solution of the Maxwell-Fokker-Planck system. Some positive high order numerical MUSCL schemes are presented in the first section to obtain a good compromise between accuracy and computation time. Then in a second part, some boundary conditions are proposed to obtain a conservative scheme.

The efficiency of each high order scheme is compared in chapter four. Several test cases will be considered to check properties of the numerical scheme. Then some applications which describe crucial processes in ICF are presented. On one hand, the evolution of the hot spot is studied when first thermonuclear reactions are obtained in the center of the D-T pellet in a weakly collisional regime. On the other hand, we focus on the simulation of the kinetic electron collisional transport in ICF between the laser absorption zone and the ablation front.

In the fifth chapter, the new approach is introduced to reduce the huge computation time obtained with kinetic models. This chapter is devoted to the development and the validation of the reduced M_1 model. M_1 consists in reducing computational time by considering minimum entropy for the angular distribution of particle. M_1 model is based on two equations depending only of the tow first moments. The two equations obtained characterize the mass conservation and momentum conservation. The principle of the angular moment closure is first explained on a simple kinetic BGK model. Then the moment model is generalized for the Maxwell-Fokker-Planck-Landau kinetic equation and different properties are shown such as H-Theorem. The derivation of an entropic semi-discrete scheme for the space homogeneous moments system is then presented with entropic property. Finally some test cases are shown to illustrate that the full discrete scheme satisfies the properties of the semi-discrete scheme.

The sixth chapter is the most theoretical chapter because no simulations are presented inside. The kinetic continuous equation in spherical domain is first described and a new model is chosen for collisions in order to preserve collision properties. The derivation of a semi-discrete scheme is then presented. Next, the N-moments model is derived from the continuous kinetic

equation and the model is shown to preserve fundamental properties such as conservation laws. The semi-discrete scheme associated is finally proved to be an entropic scheme for the moments system.

Finally we finish in the last chapter by conclusions and perspectives to this work.

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Tables (1, 1) gather parameters, symbols and constants used through this thesis. The units system is the international units system. The main symbols and parameters used in this work are given by

Each constant used here are expressed as:

Quantity	Denomination	Units (S.I.)
n_e, n_i	electron and ion density	m^{-3}
T_e, T_i	electron and ion temperature	K
Z	atomic number	-
$\mathbf{E}$	electric field	$V.m^{-1}$
$\mathbf{B}$	magnetic field	T
ρ	charge density	$C.m^3$
$\mathbf{j}$	current density	$A.m^{-2}$
v_{th}	plasma (electron) thermal velocity	$m.s^{-1}$
ω_{pe}	electronic plasma pulse	s^{-1}
λ_{De}	Debye length	m
λ_{laser}	laser wavelength	m
I	laser intensity	$W.m^{-2}$
n_c	critical density	m^{-3}

Figure 1.3: *Glossary of symbols and parameters.*

Constant	Denomination	Value
e	positive electron charge	$e = 1,60219 \times 10^{-19} C$
$q_e = -e$	signed electron charge	$q_e = -1,60219 \times 10^{-19} C$
m_e	electron mass	$m_e = 9,1094 \times 10^{-31} kg$
c	speed of light	$c = 299792458 m.s^{-1}$
k_B	Boltzmann constant	$k_B = 1,3807 \times 10^{-23} J.K^{-1}$
ϵ_0	vacuum permittivity	$\epsilon_0 = 8,8542 \times 10^{-12} F.m^{-1}$
μ_0	vacuum permeability	$\mu_0 = 4\pi \times 10^{-7} H.m^{-1}$

Figure 1.4: *Glossary of constants.*

CHAPTER 2

BASIC CONCEPT ON TRANSPORT OF ELECTRONS AND KINETIC MODEL IN THE PLASMA PHYSICS

2.1 Introduction

To describe the state of a plasma, there exists two ways of modeling, each description point out different physical aspects. The first one is hydrodynamical model which consists in determining macroscopic quantities of the plasma such as density, temperature, pressure...

Despite their computational efficiency, this approach is not sufficiently accurate to describe the majority of observed phenomena. Indeed the hydrodynamical method is inadequate for some phenomena like electromagnetic study... For more accuracy, the plasma can be studied in smaller scale. For example the molecular status of the plasma can be considered, this treatment is called kinetic theory ([11, 12]). The way widely used consists to study the movement of each particle which form the plasma. However this way can be too much expensive and very difficult if the density of the considered plasma is huge.

Nevertheless, such a detailed description is not necessarily required to understand the behavior of the plasma. To describe plasma, a statistical method is sufficient. The statistical description of particles is used by means of their non-negative distribution function $f_\alpha(t, x, v)$ specifying the density of each specie α of particles with velocity v at time t and position x . This chapter concerns modeling of this theory. For the following chapter, the condition of plasma neutrality is assumed for the equilibrium of the plasma in the absence of external forces. Moreover we assume in this thesis that the plasma consists only of electrons and one ion specie which is considered as immobile in the plasma. Here, all study on the plasma are with small collisional time or plasma period, that is why we can consider ion as immobile because of their important mass compared to electrons.

In this chapter, we recall some basis of classical kinetic models for plasma physics. First, basis of the collisionless kinetic model is presented. Then collisional kinetic theory is described with the non relativistic Maxwell-Fokker-Planck system and some properties of the collision operator are detailed such as conservation laws, entropy dissipation, etc.

2.2 Description of the collisionless kinetic model: from Liouville equation to Vlasov equation

Let us consider a system of N punctual particles of the same specie. The state of each particle at time t is determined by a vector position $\mathbf{x}_i$ and a vector momentum $\mathbf{p}_i$ with $i \in [0; N]$. The state of the system at time t is given by the vectors

$$\mathbf{x}_1, \mathbf{x}_2 \dots \mathbf{x}_N ,$$

$$\mathbf{p}_1, \mathbf{p}_2 \dots \mathbf{p}_N .$$

This set of vectors defines a point in a 6 dimensions space called phase space. To describe the plasma constituted with these N particles, we introduce the density

$$f_N(\mathbf{x}_1, \mathbf{x}_2 \dots \mathbf{x}_N, \mathbf{p}_1, \mathbf{p}_2 \dots \mathbf{p}_N) ,$$

where

$$dN = f_N d^3x_1 \dots d^3x_N d^3p_1 \dots d^3p_N ,$$

is the probability that the phase of the system is at time t inside a specific element of $6N$ dimensions. In fact, dN corresponds to the probability that particle 1 is inside the volume d^3x_1 around $\mathbf{x}_1$ with a momenta inside the element d^3p_1 around $\mathbf{p}_1$ when at the same time particle 2 to be inside the volume d^3x_2 around $\mathbf{x}_2$ with a momenta inside the element d^3p_2 around $\mathbf{p}_2$ and the same thing for the remaining $N - 2$ particles. Liouville theorem leads to conclude that f_N is conserved in time. By using partial derivatives, we obtain the Liouville equation

$$\frac{df_N}{dt} = \frac{\partial f_N}{\partial t} + \sum_{i=1}^N \mathbf{v}_i \frac{\partial f_N}{\partial \mathbf{x}_i} + \sum_{i=1}^N \mathbf{W}_i \frac{\partial f_N}{\partial \mathbf{p}_i} = 0 . \quad (2.1)$$

Here $\mathbf{v}_i = \frac{d\mathbf{x}_i}{dt}$ represents the velocity of the particle i and $\mathbf{W}_i = \frac{d\mathbf{p}_i}{dt}$ is the force applied on this particle.

Since $\mathbf{p}_i = m\mathbf{v}_i$, by considering that the velocity of particles is less than the light speed, Liouville equation (2.1) can be rewritten into

$$\frac{\partial f_N}{\partial t} + \sum_{i=1}^N \mathbf{v}_i \cdot \frac{\partial f_N}{\partial \mathbf{x}_i} + \sum_{i=1}^N \frac{\mathbf{W}_i}{m} \cdot \frac{\partial f_N}{\partial \mathbf{v}_i} = 0 . \quad (2.2)$$

Nevertheless, the density function f_N presents too many data on the system of N particles and it is very difficult to handle. To describe with less accuracy the system, let us introduce a simple distribution function f_1 such as the number of particle dN_1 inside the element d^3p_1 is given by

$$dN_1 = f_1 d^3x_1 d^3p_1 ,$$

with $f_1 = N \int f_N d^3x_2 \dots d^3x_N d^3p_2 \dots d^3p_N$. In the same way, let us introduce the function f_2 which represents the probability that the particle 1 is inside the volume d^3x_1 with a velocity vector which extremity is inside the volume d^3p_1 while the particle 2 is inside the volume d^3x_2 with a velocity vector which extremity is inside the volume d^3p_2 . The number of particles which verify such conditions is given by $dN_2 = f_2 d^3x_1 d^3x_2 d^3p_1 d^3p_2$ with $f_2 = N(N-1) \int f_N d^3x_3 \dots d^3x_N d^3p_3 \dots d^3p_N$. By multiplying Liouville equation (2.2) by N and then integrate on variable $d^3x_2 \dots d^3x_N d^3p_2 \dots d^3p_N$, we obtain the equation of f_1 evolution

$$\frac{\partial f_1}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f_1}{\partial \mathbf{x}_1} + \frac{\mathbf{W}_1}{m} \cdot \frac{\partial f_1}{\partial \mathbf{v}_1} = \mathbb{B}(f_2) , \quad (2.3)$$

with

$$\mathbb{B}(f_2) = - \int \frac{\mathbf{W}_{12}}{m} \cdot \frac{df_2}{d\mathbf{v}_1} d\mathbf{x}_2 d\mathbf{v}_2 , \quad (2.4)$$

where $\mathbf{W}_{12}$ represents the force acting on the particle 1 due to the particle 2. The second term of the left hand side of the equation (2.4) expresses the transport of particles while the third term

specifies the action of external forces.

In the same way, by using one integration less, we obtain an equation on the evolution of f_2 depending on f_3 and so forth until the equation on the evolution of f_N . This method called regressive method (see [7, 8] for more details) leads to a system of N equations referred as the Born-Bogolioubov-Green-Kirkwood-Yvon system (BBGKY). This system is closed by choosing an assumption on the distribution function of the highest degree.

The closure method for equation (2.3) leads to give an approximated expression of $\mathbb{B}(f_2)$ depending only on f_1 . The chosen closure determines the equation obtained. The three main approximations lead to Boltzmann equation with a second member, Boltzmann equation without a second member ($\mathbb{B}(f_2) = 0$) and Vlasov equation. We will focus on the last one. Function f_2 can also be rewritten such as

$$f_2 = f_1 f_1 + C_2 ,$$

where C_2 denotes correlation function between two particles. When the density of particles is such that we can not neglect interaction, the simplest assumption consists to neglect correlation between particles (i.e. long-range interaction are just taken into account). Thus, in $\mathbb{B}(f_2)$, we can set

$$f_2 = f_1 f_1 .$$

Therefore, equation (2.3) can be simplified into

$$\frac{\partial f_1}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f_1}{\partial \mathbf{x}_1} + \frac{\mathbf{W}'_1}{m} \cdot \frac{\partial f_1}{\partial \mathbf{v}_1} = 0 .$$

This is the Vlasov equation which is fundamental in plasma theory. $\mathbf{W}'_1$ represents applied fields (external fields and macroscopic fields generated by the plasma) and is given by

$$\mathbf{W}'_1 = \frac{-e}{m} (\mathbf{E} + \mathbf{v}_1 \times \mathbf{B}) ,$$

where $\mathbf{E}$ represents the electric field, $\mathbf{B}$ is the magnetic field and e is the electron charge.

Vlasov equation consists in considering only collective interaction which determine the trajectory of particle. Collectives action are defined as the action of the charge mean field on each charge particle, this mean field is generated by other particle.

Neglecting the correlations between particles leads to consider the phenomenon whose scope is greater than the correlation length λ_{De} called Debye length. λ_{De} writes in the case of an electronic population as

$$\lambda_{De} = \frac{v_{th}}{\omega_{pe}} = \sqrt{\frac{\epsilon_0 k_B T_e}{n_e e^2}} ,$$

where T_e is the electronic temperature, n_e is the electronic density of the plasma, k_B is the Boltzmann constant, $v_{th} = \sqrt{k_B T_e m_e}$ is the thermal velocity and $\omega_{pe} = \sqrt{\frac{n_e e^2}{\epsilon_0 m_e}}$ is the plasma period

frequency. This Debye length expresses the fact that the charge of an electron of the plasma is screened by the other charge particles, λ_{De} is the screened length.

2.3 Collisional kinetic theory: the Maxwell-Fokker-Planck system

The second fundamental equation to describe collisional plasma is the Fokker-Planck equation. It is derived from the Boltzmann equation where multiple collisions between charged particles with small deviations are taken into account ([3]). This can be justified by the long-range Coulomb force. To take more into account the correlation effects, the particles collide only if the impact parameter p (the distance between two particles which will come into collision) is less than λ_{De} . Thus a particle come into collision with all particles included in a Debye sphere. Classically in kinetic theory, a Fokker-Planck-Landau equation coupled with Maxwell equations is used to describe the evolution of different species in a collisional plasma (see [5, 6, 2, 9] for instance). The Fokker-Planck-Landau equation is the most accomplished model to describe collisions. In this section, we present this system. Then we briefly review some backgrounds about the collision operator properties and the property of the system to conserve energy.

2.3.1 Maxwell-Fokker-Planck equations

In collisional plasma kinetic theory, the time evolution of the electrons distribution function $f(t, x, v)$, where $x \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$, is governed by the Fokker-Planck-Landau equation

$$\partial_t f + \mathbf{v} \cdot \nabla_x f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \cdot \nabla_v f = C(f, f), \quad (2.5)$$

coupled with the Maxwell equations describing the time evolution of the electric field $\mathbf{E}(t, \mathbf{x})$ and magnetic field $\mathbf{B}(t, \mathbf{x})$. This Maxwell equations are expressed as

$$\left\{ \begin{array}{l} \partial_t \mathbf{E} - c^2 \nabla_x \wedge \mathbf{B} = -\frac{\mathbf{j}}{\varepsilon_0}, \\ \partial_t \mathbf{B} + \nabla_x \wedge \mathbf{E} = 0, \\ \nabla_x \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \\ \nabla_x \cdot \mathbf{B} = 0, \end{array} \right. \quad (2.6)$$

where $q = -e$ is a charge of an electron, m its mass, ε_0 represents the permittivity of vacuum, c the speed of the light.

Macroscopic quantities ρ and j represent the plasma charge density and the plasma current density and are defined by

$$\rho = q(n_e - n_0) = q \left(\int_{\mathbb{R}^3} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v} - Zn_i \right) \text{ and } \mathbf{j}(t, \mathbf{x}) = q \int_{\mathbb{R}^3} f(t, \mathbf{x}, \mathbf{v}) \mathbf{v} d\mathbf{v} .$$

Here $C(f, f)$ is the collision operator introduced in [2]. It is called Fokker-Planck-Landau operator and defined by

$$C(f, f) = C_{ee}(f, f) + C_{ei}(f) .$$

The operator C_{ee} stands for the electron-electron collision operator

$$C_{ee}(f, f) = \alpha_{ee} \operatorname{div}_{\mathbf{v}} \left(\int_{\mathbb{R}^3} \Phi(\mathbf{u}) [f(\mathbf{v}') \nabla_{\mathbf{v}} f(\mathbf{v}) - f(\mathbf{v}) \nabla_{\mathbf{v}'} f(\mathbf{v}')] d\mathbf{v}' \right) , \quad (2.7)$$

where $\mathbf{u} = \mathbf{v} - \mathbf{v}'$ is the relative velocity of electrons and $\Phi(u)$ is an operator acting on the relative velocity u

$$\Phi(\mathbf{u}) = \frac{1}{|\mathbf{u}|^3} (|\mathbf{u}|^2 Id - \mathbf{u} \otimes \mathbf{u}) , \quad (2.8)$$

where Id is the unit tensor. The conservative form of (2.7) is called Landau form. There exists an other equivalent form of this operator called Rosenbluth form. It is derived by developing (2.7) as a non-linear combination of a diffusive operator and a friction operator.

Remark 1. *The operator Φ is called the Landau tensor. It corresponds to the projection matrix on the orthogonal plane of $\mathbf{u}$. The tensor Φ is a symmetric positive semi definite matrix.*

The coefficient α_{ee} writes

$$\alpha_{ee} = \frac{e^4 \ln \Lambda}{8\pi\epsilon_0^2 m_e^2} , \quad (2.9)$$

where $\ln \Lambda$ is the Coulomb logarithm, which is supposed to be constant over the domain, $-e$ represents the charge of an electron, m_e its mass, Z is the ion charge and ϵ_0 is the permittivity of vacuum.

The electron-ion collision operator $C_{ei}(f)$ is defined by

$$C_{ei}(f) = \alpha_{ei} \operatorname{div}_{\mathbf{v}} [\Phi(\mathbf{u}) \nabla_{\mathbf{v}} f(\mathbf{v})] , \quad (2.10)$$

where the coefficient α_{ei} is given by

$$\alpha_{ei} = \frac{Zn_0 e^4 \ln \Lambda}{8\pi\epsilon_0^2 m_e^2} = Zn_0 \alpha_{ee} . \quad (2.11)$$

Remark 2. *The third equation of Maxwell equations (2.6) called Poisson equation is redundant. Indeed the first equation of (2.6) called Ampere equation and Poisson equation are equivalent if Poisson equation is verified at initial time.*

2.3.2 Properties of the collision operator

Next we present conservation laws and dissipation entropy property of the collision operator C_{ee} and C_{ei} (see [1, 4] for more details). The electron-electron collision operator satisfies mass, momentum and energy conservations property

$$\int_{\mathbb{R}^3} C_{ee}(f, f) \left(\frac{1}{|\mathbf{v}|^2} \right) d\mathbf{v} = 0, \quad t \geq 0, \quad (2.12)$$

while the electron-ion collision operator satisfies only the conservation of mass and energy

$$\int_{\mathbb{R}^3} C_{ei}(f) \left(\frac{1}{|\mathbf{v}|^2} \right) d\mathbf{v} = 0, \quad t \geq 0. \quad (2.13)$$

They both satisfy the entropy dissipation, i.e.

$$\int_{\mathbb{R}^3} C_{ei}(f) \log f \, d\mathbf{v} \leq 0 \text{ and } \int_{\mathbb{R}^3} C_{ee}(f, f) \log f \, d\mathbf{v} \leq 0.$$

Therefore the Boltzmann entropy $H = \int_{\mathbb{R}^3} (f \log f - f) \, d\mathbf{v}$ is a Lyapounov function, i.e.

$$\frac{dH}{dt}(t) \leq 0.$$

Property 3. *The equilibrium state of the electron-electron collision operator C_{ee} is given by the Maxwellian distribution function*

$$\mathcal{M}_{u_e} = n_e \left(\frac{m_e}{2\pi k_B T_e} \right)^{\frac{3}{2}} \exp \left(-\frac{m_e(\mathbf{v} - \mathbf{u}_e)^2}{2k_B T_e} \right), \quad (2.14)$$

where k_B is the Boltzmann constant, n_e is the electronic density, T_e is the temperature and u_e is the mean velocity defined as

$$\begin{cases} n_e = \int_{\mathbb{R}^3} f(t, \mathbf{x}, \mathbf{v}) \, d\mathbf{v}, \\ \mathbf{u}_e = \frac{1}{n_e} \int_{\mathbb{R}^3} f(t, \mathbf{x}, \mathbf{v}) \mathbf{v} \, d\mathbf{v}, \\ T_e = \frac{m_e}{3n_e} \int_{\mathbb{R}^3} f(t, \mathbf{x}, \mathbf{v}) \|\mathbf{v} - \mathbf{u}_e\|^2 \, d\mathbf{v}. \end{cases}$$

The equilibrium state of the electron-ion collision operator C_{ei} is given by the isotropic functions

$$f = f(|\mathbf{v}|). \quad (2.15)$$

Proof. We aim to prove here that isotropic distribution functions are equilibrium state of the electron-ion collision operator, i.e.

$$C_{ei}(f) = 0 . \quad (2.16)$$

After integrating (2.16), we get

$$\int_{\mathbb{R}^3} C_{ei}(f) d\mathbf{v} = 0 . \quad (2.17)$$

From the definition of the operator C_{ei} , the right-hand side of (2.17) writes

$$\int_{\mathbb{R}^3} \alpha_{ei} \nabla_{\mathbf{v}} [\Phi(\mathbf{v}) \nabla_{\mathbf{v}} f] d\mathbf{v} = 0 ,$$

which can be simplified into

$$\int_{\mathbb{R}^3} \alpha_{ei} \Phi(\mathbf{v}) |\nabla_{\mathbf{v}} f| \cdot \nabla_{\mathbf{v}} f d\mathbf{v} = 0 .$$

Therefore we get

$$\Phi(\mathbf{v}) \nabla_{\mathbf{v}} f \cdot \nabla_{\mathbf{v}} f = 0 \quad \forall \mathbf{v} .$$

Since Φ is an orthogonal projector on $\Omega = \frac{1}{|\mathbf{v}|} \mathbf{v}$, we have $\Phi(\mathbf{v}) \nabla_{\mathbf{v}} f$ proportional to $\mathbf{v}$. So we obtain

$$\mathbf{v} \cdot \nabla_{\mathbf{v}} f = 0 , \quad \forall \mathbf{v} \in \mathbb{R}^3 .$$

So $\mathbf{v} \perp \nabla_{\mathbf{v}} f$.

In Spherical coordinate, i.e. $\mathbf{v} = (v_r, v_\theta, v_\varphi)$, the divergence $\nabla_{\mathbf{v}} f$ can be written as

$$\nabla_{\mathbf{v}} f = \frac{\partial f}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \vec{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi} \vec{e}_\varphi .$$

Since $(\vec{e}_\theta, \vec{e}_\varphi) \perp \mathbf{v}$, we obtain the following equalities

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial \varphi} = 0 ,$$

and (2.15) follows. □

2.3.3 Conservation of total energy

The choice of the Maxwell-Fokker-Planck system to describe a plasma presents an other interesting property. Indeed under some conditions on total energy flux, the system conserves the total energy given by

$$E_{\text{tot}} = \varepsilon_0 (\mathbf{E}^2 + (c\mathbf{B})^2) + \int_{\mathbf{v}} m \mathbf{v}^2 f d\mathbf{v} . \quad (2.18)$$

Proposition 1. *The Maxwell-Fokker-Planck system (2.5-2.6) conserves the total energy under the following conditions on the frontier γ of Ω*

$$\varepsilon_0 c^2 (\mathbf{E} \wedge \mathbf{B}) \cdot \mathbf{n} + \int_{\mathbf{v}} m \mathbf{v}^2 (\mathbf{v} \cdot \mathbf{n}) f d\mathbf{v} = 0 . \quad (2.19)$$

Remark 4. *The case widely used to obtain conservation of energy, consists to consider periodic boundary. You can also consider that $\varepsilon_0 c^2 (\mathbf{E} \wedge \mathbf{B}) \cdot \mathbf{n} = 0$, i.e. if the regime is periodic or the magnetic field is null on the boundary, and at the same time $\int_{\mathbf{v}} m \mathbf{v}^2 (\mathbf{v} \cdot \mathbf{n}) f d\mathbf{v} = 0$, i.e. if the regime is periodic or the domain is infinite or limit conditions are specular ($f(-\mathbf{v} \cdot \mathbf{n}) = f(\mathbf{v} \cdot \mathbf{n})$).*

Proof. Consider here the two first equations of Maxwell equations (2.6), i.e. Ampere equation and Faraday equation

$$\text{Ampere : } \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} - \text{rot}_{\mathbf{x}} \vec{\mathbf{B}} = -\frac{\mathbf{j}}{c^2 \varepsilon_0} , \quad (2.20)$$

$$\text{Faraday : } \frac{\partial \mathbf{B}}{\partial t} + \text{rot}_{\mathbf{x}} \mathbf{E} = 0 . \quad (2.21)$$

Now, multiply respectively equation (2.20) and equation (2.21) by $\mathbf{E}$ and $\mathbf{B}$ and sum the two obtained equations. So we get

$$\frac{1}{c^2} \frac{1}{2} \frac{\partial \mathbf{E}^2}{\partial t} + \frac{1}{2} \frac{\partial \mathbf{B}^2}{\partial t} - \mathbf{E} \cdot \text{rot}_{\mathbf{x}} \vec{\mathbf{B}} + \mathbf{B} \cdot \text{rot}_{\mathbf{x}} \mathbf{E} = \frac{-1}{c^2 \varepsilon_0} \mathbf{j} \cdot \mathbf{E} . \quad (2.22)$$

Equation (2.22) can be simplified into

$$\frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\mathbf{E}^2}{c^2} + \mathbf{B}^2 \right) + \text{div}_{\mathbf{x}} (\mathbf{E} \wedge \mathbf{B}) = -\frac{1}{c^2 \varepsilon_0} \mathbf{j} \cdot \mathbf{E} . \quad (2.23)$$

After integrating equation (2.23) over Ω , we obtain

$$\frac{1}{2} \frac{\partial}{\partial t} \int_{\Omega} \left(\frac{\mathbf{E}^2}{c^2} + \mathbf{B}^2 \right) dx + \int_{\partial\Omega} (\mathbf{E} \wedge \mathbf{B}) \cdot \mathbf{n} d\sigma = -\frac{1}{c^2 \varepsilon_0} \int_{\Omega} \mathbf{j} \cdot \mathbf{E} dx . \quad (2.24)$$

Now, by multiplying the Fokker-Planck-Landau equation defined in system (2.5-2.6) by $m \frac{\mathbf{v}^2}{2}$ and integrating it over the domain of velocity $\mathbf{v}$, we get

$$\frac{\partial}{\partial t} \int_{\mathbf{v}} m \frac{\mathbf{v}^2}{2} f d\mathbf{v} + \text{div}_{\mathbf{x}} \left(\int_{\mathbf{v}} m \frac{\mathbf{v}^2}{2} f \mathbf{v} d\mathbf{v} \right) + q \int_{\mathbf{v}} \text{div}_{\mathbf{v}} ((\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) f) \frac{\mathbf{v}^2}{2} d\mathbf{v} = \int_{\mathbf{v}} C(f, f) \frac{\mathbf{v}^2}{2} d\mathbf{v} . \quad (2.25)$$

The third term of the right-hand side of equation (2.25) can be rewritten as

$$\text{div}_{\mathbf{v}} ((\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) f) \frac{\mathbf{v}^2}{2} = \text{div}_{\mathbf{v}} \left((\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) f \frac{\mathbf{v}^2}{2} \right) - f (\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \cdot \nabla_{\mathbf{v}} \left(\frac{\mathbf{v}^2}{2} \right) .$$

Since $\nabla_{\mathbf{v}} \left(\frac{\mathbf{v}^2}{2} \right) = \mathbf{v}$, we obtain

$$\operatorname{div}_{\mathbf{v}} \left((\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) f \right) \frac{\mathbf{v}^2}{2} = \operatorname{div}_{\mathbf{v}} \left((\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) f \frac{\mathbf{v}^2}{2} \right) - \mathbf{E} \cdot \mathbf{v} .$$

Thus, we get

$$\int_{\mathbf{v}} \operatorname{div}_{\mathbf{v}} \left((\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) f \right) \frac{\mathbf{v}^2}{2} d\mathbf{v} = - \int_{\mathbf{v}} f \mathbf{v} d\mathbf{v} \cdot \mathbf{E} = -\frac{1}{q} j \cdot \mathbf{E} .$$

Therefore, since collision operator $C(f, f)$ preserves energy (2.12, 2.13), equation (2.25) can be simplified into

$$\frac{\partial}{\partial t} \left(\int_{\mathbf{v}} f m \frac{\mathbf{v}^2}{2} d\mathbf{v} \right) + \operatorname{div}_{\mathbf{x}} \left(\int_{\mathbf{v}} f m \frac{\mathbf{v}^2}{2} \mathbf{v} d\mathbf{v} \right) - j \cdot \mathbf{E} = 0 . \quad (2.26)$$

where $\varepsilon = \int_{\mathbf{v}} f m \frac{\mathbf{v}^2}{2} d\mathbf{v}$ is the plasma energy and $\mathbb{F} = \int_{\mathbf{v}} f m \frac{\mathbf{v}^2}{2} \mathbf{v} d\mathbf{v}$ is the plasma energy flux. Finally the energetic balance equation is obtained by summing equation (2.26) and equation (2.24) multiplied by ε_0 . The obtained equation can be written as

$$\frac{1}{2} \frac{\partial}{\partial t} \left(\varepsilon_0 (\mathbf{E}^2 + (c\mathbf{B})^2) + \int_{\mathbf{v}} m \mathbf{v}^2 f d\mathbf{v} \right) + \operatorname{div}_{\mathbf{x}} \left(\varepsilon_0 c^2 (\mathbf{E} \wedge \mathbf{B}) + \int_{\mathbf{v}} m \mathbf{v}^2 \mathbf{v} f d\mathbf{v} \right) = 0 . \quad (2.27)$$

By integrating over the space domain, equation (2.27) can be rewritten as

$$\frac{1}{2} \frac{\partial}{\partial t} \int_{\Omega} \left(\varepsilon_0 (\mathbf{E}^2 + (c\mathbf{B})^2) + \int_{\mathbf{v}} m \mathbf{v}^2 f d\mathbf{v} \right) d\mathbf{x} + \int_{\partial\Omega} \left(\varepsilon_0 c^2 (\mathbf{E} \wedge \mathbf{B}) \cdot \mathbf{n} + \int_{\mathbf{v}} m \mathbf{v}^2 (\mathbf{v} \cdot \mathbf{n}) f d\mathbf{v} \right) d\sigma = 0 ,$$

and the proposition 1 follows. $\square$

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CHAPTER 3

HIGH ORDER KINETIC SCHEME FOR MAXWELL-FOKKER-PLANCK SYSTEM

3.1 Introduction

In the previous chapter, we have presented an overview of some kinetic models. In this chapter, we will focus on the numerical schemes that we used.

Numerous works have considered constructing discretizations of the Maxwell-Fokker-Planck system that obey the physical properties of conservation and entropy described previously. The first scheme in this direction was established in [7]. Unfortunately it turned out to be very expensive in terms of computing time. Later various fast algorithms have been constructed to reduce this cost (see [3, 4]). For example in [11], a time implicit scheme is used to solve the homogeneous Fokker-Planck-Landau equation in both the isotropic and 3-dimensional geometries. To solve the involved linear system, kinetic methods are proposed to conserve fundamental properties. In this chapter, we propose to use the deterministic direct numerical method introduced in [8] to solve the non-relativistic Vlasov-Maxwell system coupled with Fokker-Planck-Landau type operators. The space considered is a planar geometry of two dimensions in space and three dimensions in velocity. The kinetic code *KETS* developed by R. Duclous is based on a “splitting method” to split the Vlasov part and collisional part. A finite difference method is used to describe the discrete collision operator as shown in [8]. A multigrid formulation of the Fokker-Planck-Landau operator is employed to reduce the cost of computation of such discrete operator. We proposed here to improve this kinetic code to obtain a conservative scheme for the phase space. A finite volume method is used in order to obtain the approximation of the solution of the system (2.5-2.6). Some positive high order numerical MUSCL schemes are presented in the first section to solve the transport part. Their efficiency will be shown in the next chapter. Then in a second part, a discretization of Maxwell equations is explained by using a finite difference method. Finally, some boundary conditions are proposed to obtain a conservative scheme.

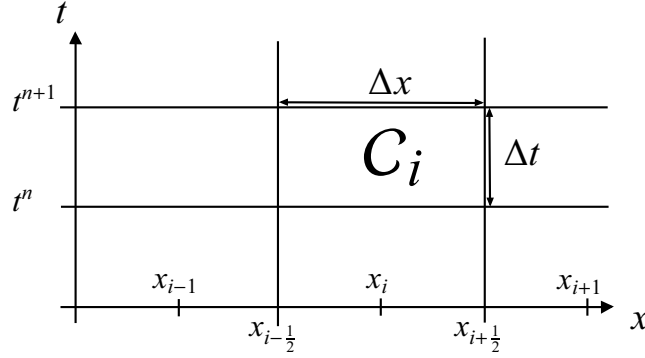
3.2 The high order numerical scheme

First, for the sake of simplicity, we focus on discretization in one dimensional space of the kinetic transport equation in order to describe different high order schemes. Let us consider the following equation for $t \geq 0$ and $x \in (0, L)$,

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = 0 \quad (3.1)$$

where $v > 0$ represents the velocity.

Let us define the mesh $\mathcal{M}$, for the space variable x , decomposed into a family of rectangles $\mathcal{M}_i =]x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}[$, $\forall i \in [0, n_x]$, where $x_{i+\frac{1}{2}} = (i + \frac{1}{2})\Delta x$ and $n_x \in \mathbb{N}$ the number of points which discretize the space domain $(0, L)$. Δx represents the uniform step of discretization. Let Δt be the time step and $t^n = n \Delta t$.

Figure 3.1: Discretization of the mesh $\mathcal{M}_h$

Integrating equation (3.1) on a control volume $C_i = [x_{i-1/2}, x_{i+1/2}] \times (t^n, t^{n+1})$ (see figure 3.1), it yields

$$f_i^{n+1} = f_i^n - \frac{\Delta t}{\Delta x} [\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n], \quad (3.2)$$

where $f_i^n = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} f(t^n, x) dx$ is the mean value of f on the control volume C_i at time t^n .

Here, the flux at the interface $x_{i+1/2}$ and at time t^n denoted by $v \int_{t^n}^{t^{n+1}} f(t, x_{i+1/2}) dt$ is approximated by the high order flux $\mathcal{F}_{i+1/2}^n$.

The main step to get an accurate solution is to reconstruct the distribution function in the space direction. To this aim, the distribution function $f(t^n, x)$ is approximated by $f_h(x)$ with different kind of reconstructions. This reconstruction will determine the order of the scheme. Higher the order and more the scheme is accurate but less stable. This section is devoted to the presentation of different high order numerical schemes. For the sake of simplicity, we denote the approximation $f_h(x_i)$ for positive velocity by f_i^+ and f_i^- for negative velocity. Therefore, the approximated flux $\mathcal{F}_{i+1/2}^n$ at the interface $x_{i+1/2}$ from (3.2) writes

$$\mathcal{F}_{i+1/2}^n = v^+ f_i^+ + v^- f_{i+1}^-. \quad (3.3)$$

3.2.1 Second order transport scheme

Some methods of reconstruction are presented here, they allow to obtain a second order scheme which preserves the positivity of the scheme using slope correctors.

Generalized *minmod* limiter: Van Leer's parameter

Starting with the given cell average f_i^n , the first step consists in *reconstructing* a non-oscillatory piecewise polynomial of the form

$$f_i^\pm = f_i^n \pm \frac{1}{2} \Delta x p_i . \quad (3.4)$$

This method is conservative, i.e

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(t^n, x) = f_i^n .$$

Indeed, the slope p_i is the same for a positive or a negative velocity thus $f_i = \frac{1}{2}(f_i^+ + f_i^-)$ and the conservative property follows. To ensure a non-oscillatory nature of the reconstruction (3.4), one needs to use a nonlinear limitation in the computation of this slope. Here we have used “Van Leer's one parameter family of the *minmod* limiters” (see [10, 14])

$$p_i = \minmod\left(\frac{f_{i+1} - f_{i-1}}{2\Delta x}, \theta \frac{f_{i+1} - f_i}{\Delta x}, \theta \frac{f_i - f_{i-1}}{\Delta x}\right) , \quad (3.5)$$

where $\theta \in [1; 2]$.

Recall that for two variables $(x, y) \in \mathbb{R}^2$, the usual *minmod* limiter is defined as

$$\minmod(x, y) = \max(0, \min(x, y)) + \min(0, \max(x, y)).$$

For three variables $(x, y, z) \in \mathbb{R}^3$, it is given by

$$\minmod(x, y, z) = \max(0, \min(\min(x, y), z)) + \min(0, \max(\max(x, y), z)) .$$

Remark 5. Another choice of limiter which gives second order accuracy is the so called *superbee* limiter introduced by Roe ([13]) and defined as

$$p_i = \maxmod\left(\minmod\left(\frac{f_{i+1} - f_i}{\Delta x}, 2\left(\frac{f_i - f_{i-1}}{\Delta x}\right)\right), \minmod\left(2\left(\frac{f_{i+1} - f_i}{\Delta x}\right), \frac{f_i - f_{i-1}}{\Delta x}\right)\right) . \quad (3.6)$$

For two variables $(x, y) \in \mathbb{R}^2$, the *maxmod* limiter is defined as

$$\maxmod(x, y) = \min(0, \max(x, y)) + \max(0, \min(x, y)).$$

Proposition 2. If $\theta = 1$, the Van Leer limiter (3.5) is equivalent to the usual *minmod* corrector and if $\theta = 2$, it is equivalent to the *superbee* limiter.

Proof: If $\frac{f_{i+1} - f_i}{\Delta x} \cdot \frac{f_i - f_{i-1}}{\Delta x} < 0$, we get

$$\minmod\left(\frac{f_{i+1} - f_i}{\Delta x}, \frac{f_i - f_{i-1}}{\Delta x}\right) = \text{superbee}\left(\frac{f_{i+1} - f_i}{\Delta x}, \frac{f_i - f_{i-1}}{\Delta x}\right) = \minmod\left(\frac{f_{i+1} - f_{i-1}}{2\Delta x}, \theta \frac{f_{i+1} - f_i}{\Delta x}, \theta \frac{f_i - f_{i-1}}{\Delta x}\right) = 0. \quad (3.7)$$

If the slopes $\frac{f_{i+1} - f_i}{\Delta x}$ and $\frac{f_i - f_{i-1}}{\Delta x}$ have the same sign, so $\frac{f_{i+1} - f_{i-1}}{2\Delta x}$ is with the same sign. Therefore if the slopes are positive, the usual *minmod* limiter can be minimized such as

$$\minmod\left(\frac{f_{i+1} - f_i}{\Delta x}, \frac{f_i - f_{i-1}}{\Delta x}\right) \leq \frac{1}{2} \left(\frac{f_{i+1} - f_i}{\Delta x} + \frac{f_i - f_{i-1}}{\Delta x} \right) = \frac{f_{i+1} - f_{i-1}}{2\Delta x}.$$

Moreover if the slopes are negative, we obtain

$$\minmod\left(\frac{f_{i+1} - f_i}{\Delta x}, \frac{f_i - f_{i-1}}{\Delta x}\right) \geq \frac{1}{2} \left(\frac{f_{i+1} - f_i}{\Delta x} + \frac{f_i - f_{i-1}}{\Delta x} \right) = \frac{f_{i+1} - f_{i-1}}{2\Delta x},$$

and the equivalence with the usual *minmod* limiter follows. In the same way, we can show the equivalence between the *superbee* limiter and the *Van Leer* corrector. $\square$

Remark 6. Notice that the larger is θ in (3.5) corresponds to less dissipative but still non-oscillatory limiters. For example in the case $\theta = 1$, (3.5) is equivalent to the usual *minmod* limiter which is more diffuse. Whereas for case $\theta = 2$, we get the *superbee* limiter which is less diffuse but less stable than the *minmod* limiter. Therefore each limiter presents some disadvantages and some advantages. So the limiter is chosen depending on the test case.

Remark 7. It is also called the *monotonized central-difference limiter* (MC limiter). This limiter appears to be a good default choice for a wide class of problems as shown in [12].

Proposition 3. If we use the “*Van Leer’s one parameter family of the minmod limiters*”, the approximate distribution function given by

$$f_i^+ = f_i + \frac{1}{2} \Delta x p_i^n, \quad (3.8)$$

is defined as $f_i^+ \in [f_i^n; f_{i+1}^n]$. Particularly, by using the usual *minmod* limiter ($\theta = 1$), the approximate distribution function

$$f_i^+ = f_i^n + \frac{1}{2} \Delta x p_i^n, \quad (3.9)$$

is given by $f_i^+ \in [f_i^n; \frac{f_i^n + f_{i+1}^n}{2}]$.

Proof: First we prove the property of $f_i^+ \in [f_i^n, f_{i+1}^n]$ with $\theta \in [1, 2]$. If the slope $\frac{f_{i+1}^n - f_i^n}{\Delta x}$ and $\frac{f_i^n - f_{i-1}^n}{\Delta x}$ are not with the same sign, the property on f_i^+ is trivial because $f_i^+ = f_i^n$. If the two terms are with the same sign and if $f_{i+1}^n > f_i^n$ for example, we get

$$0 \leq f_i^+ - f_i^n \leq \frac{1}{2}\theta(f_{i+1}^n - f_i^n) \leq f_{i+1}^n - f_i^n ,$$

because $\theta \in [1, 2]$. Therefore, we obtain

$$f_i^n \leq f_i^+ \leq f_i^n + (f_{i+1}^n - f_i^n) = f_{i+1}^n .$$

If $f_{i+1}^n < f_i^n$, we can prove in the same way that $f_i^+ \in [f_i^n, f_{i+1}^n]$.

Secondly we show that, with the usual *minmod* limiter, $f_i^+ \in [f_i^n, \frac{f_i^n + f_{i+1}^n}{2}]$. If $\frac{f_{i+1}^n - f_i^n}{\Delta x}$ and $\frac{f_i^n - f_{i-1}^n}{\Delta x}$ are with the same sign and if $f_{i+1}^n > f_i^n$ for example, we get

$$0 \leq f_i^+ - f_i^n \leq \frac{1}{2}(f_{i+1}^n - f_i^n) .$$

Therefore, we obtain

$$f_i^n \leq f_i^+ \leq f_i^n + \frac{1}{2}(f_{i+1}^n - f_i^n) = \frac{f_i^n + f_{i+1}^n}{2} .$$

□

Remark 8. *The minmod slope limiter preserves the order of the function (increasing or decreasing) unlike the superbee corrector with which reconstruction intersect.*

Let $TV(f) = \sum_{i=1}^{n_x-1} |f^n(x_{i+1}) - f^n(x_i)|$ denote the total variation of a piecewise constant function f given by $f(x) = f \in \mathcal{P}_0$, $x_i = [x_i, x_{i+1}]$.

Proposition 4. *The scheme (3.2) with the use of usual minmod slope limiter has a diminishing total variation (TVD),*

$$TV(f^{n+1}) \leq TV(f^n) ,$$

provided the fixed mesh ratio $\lambda = v \frac{\Delta t}{\Delta x}$ satisfies

$$\lambda \leq \frac{2}{3} .$$

Moreover the scheme (3.2, 3.5) is TVD under the condition $\lambda \leq \frac{1}{2}$.

Proof: After integrating equation (3.1) on a control volume $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, it yields

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} + v \frac{f_i^+ - f_{i-1}^+}{\Delta x} = 0 .$$

Therefore we obtain

$$f_i^{n+1} = f_i^n - \lambda(f_i^+ - f_{i-1}^+), \quad (3.10)$$

with $\lambda = v \frac{\Delta}{\Delta x}$. We first consider the usual *minmod* limiter. Expression (3.10) can be rewritten as

$$f_i^{n+1} = f_i^n - \lambda \left[f_i^n + \frac{1}{2} \minmod(f_{i+1}^n - f_i^n, f_i^n - f_{i-1}^n) - f_{i-1}^n - \frac{1}{2} \minmod(f_i^n - f_{i-1}^n, f_{i-1}^n - f_{i-2}^n) \right]. \quad (3.11)$$

For the sake of clarity, we set $\Delta f_{i-\frac{1}{2}}^n = f_i^n - f_{i-1}^n$ and $r_{i+\frac{1}{2}}^n = \frac{\Delta f_{i-\frac{1}{2}}^n}{\Delta f_{i+\frac{1}{2}}^n}$. Equation (3.11) writes

$$f_i^{n+1} = f_i^n - \lambda \left(1 + \frac{\lambda}{2} \minmod\left(1, \frac{1}{r_{i+\frac{1}{2}}^n}\right) - \frac{\lambda}{2} \minmod(1, r_{i-\frac{1}{2}}^n) \right) \Delta f_{i-\frac{1}{2}}^n. \quad (3.12)$$

The incremental form of (3.12) gives

$$f_i^{n+1} = f_i^n + C_{i+\frac{1}{2}} \Delta f_{i+\frac{1}{2}}^n - D_{i-\frac{1}{2}} \Delta f_{i-\frac{1}{2}}^n,$$

with $C_{i+\frac{1}{2}} = 0$ and $D_{i-\frac{1}{2}} = \lambda \left(1 + \frac{\lambda}{2} \minmod\left(1, \frac{1}{r_{i+\frac{1}{2}}^n}\right) - \frac{\lambda}{2} \minmod(1, r_{i-\frac{1}{2}}^n) \right)$. The scheme is TVD under two conditions, i.e if $C_{i+\frac{1}{2}}, D_{i-\frac{1}{2}} \geq 0$ and if $C_{i+\frac{1}{2}} + D_{i-\frac{1}{2}} \leq 1$. Since $\minmod(1, r_{i+\frac{1}{2}}^n) \geq 0$, we obtain $D_{i-\frac{1}{2}} \geq 1 - \frac{\lambda}{2} \minmod(1, r_{i-\frac{1}{2}}^n) \geq \frac{1}{2}$ and the first condition is ensured. Besides we get $\minmod(1, r_{i+\frac{1}{2}}^n) \leq 1$ and so

$$C_{i+\frac{1}{2}} + D_{i-\frac{1}{2}} \leq \lambda \left(1 + \frac{1}{2} \right) \leq \frac{3}{2} \lambda.$$

Therefore, the scheme is TVD if $\frac{3}{2} \lambda \leq 1$, i.e. if $\lambda \leq \frac{2}{3}$.

Now by considering the “Van Leer’s one parameter family of the *minmod* limiters”, we obtain

$$C_{i+\frac{1}{2}} + D_{i-\frac{1}{2}} \leq 2\lambda,$$

because $0 \leq \minmod\left(2, 2r_{i+\frac{1}{2}}^n, \frac{1+r_{i-\frac{1}{2}}^n}{2}\right) \leq 2$. Thus, with this limiter, the scheme is TVD if $\lambda \leq \frac{1}{2}$.

□

Primitive technique

The high order approximation $f_h(x)$ is obtained here from a reconstruction using primitives ([6]). The main difference with the previous usual method comes from the fact that the slope corrector p_i depends on the sign of the velocity. By using the primitive technique, a second

order accurate approximation of the distribution function on the interval $[x_{i-\frac{1}{2}}; x_{i+\frac{1}{2}}]$ is obtained and can be expressed as

$$f_i^\pm(x) = f_i^n \pm \frac{1}{2}p_i^\pm \quad (3.13)$$

with

$$p_i^+ = \begin{cases} 0 & \text{if } (f_{i+1}^n - f_i^n)(f_i^n - f_{i-1}^n) < 0, \\ \min(\Delta f_{i+\frac{1}{2}}, 2(f_i^n - f_{\max})) & \text{if } (f_{i+1}^n - f_i^n) \leq 0, \\ \min(\Delta f_{i+\frac{1}{2}}, 2(f_i^n - f_{\min})) & \text{else,} \end{cases} \quad (3.14)$$

and

$$p_i^- = \begin{cases} 0 & \text{if } (f_{i+1}^n - f_i^n)(f_i^n - f_{i-1}^n) < 0, \\ \min(\Delta f_{i-\frac{1}{2}}, 2(f_{\min} - f_i^n)) & \text{if } (f_i^n - f_{i-1}^n) \leq 0, \\ \min(\Delta f_{i-\frac{1}{2}}, 2(f_{\max} - f_i^n)) & \text{else.} \end{cases} \quad (3.15)$$

The approximation of the distribution function defined by (3.13) with the slope limiters (3.14) and (3.15) satisfies the conservation of the average and the positivity of $f_h(x)$ ([6]).

3.2.2 Third order transport scheme

A general family of TVD discretization with 5-point stencil is the Monotone Upstream centered Scheme for Conservation Laws (MUSCL). To improve the accuracy, we turn to a third order MUSCL TVD scheme proposed in [1]. MUSCL schemes use a κ -parameter family of interpolation formulas with a limiter function $\Psi(R) : \mathbb{R} \rightarrow \mathbb{R}$ such as

$$\begin{cases} f_i^+ = f_i^n + \frac{1+\kappa}{4}\Psi(R_i)\Delta f_{i-\frac{1}{2}}^n + \frac{1-\kappa}{4}\Psi(\frac{1}{R_j})\Delta f_{i+\frac{1}{2}}^n, \\ f_i^- = f_i^n - \frac{1+\kappa}{4}\Psi(\frac{1}{R_j})\Delta f_{i+\frac{1}{2}}^n - \frac{1-\kappa}{4}\Psi(R_i)\Delta f_{i-\frac{1}{2}}^n. \end{cases} \quad (3.16)$$

Here R_i is the ratio of successive solution increments $R_i = \frac{\Delta f_{i+\frac{1}{2}}^n}{\Delta f_{i-\frac{1}{2}}^n}$. In this section, the third order scheme is defined by choosing $\kappa = \frac{1}{3}$. By using the “usual” limiter, we obtain

$$\begin{cases} \Psi(R_i)\Delta f_{i-\frac{1}{2}}^n = \minmod\left(\theta, \frac{\Delta f_{i+\frac{1}{2}}^n}{\Delta f_{i-\frac{1}{2}}^n}\right)\Delta f_{i-\frac{1}{2}}^n = \minmod\left(\theta\Delta f_{i-\frac{1}{2}}^n, \Delta f_{i+\frac{1}{2}}^n\right), \\ \Psi(\frac{1}{R_j})\Delta f_{i+\frac{1}{2}}^n = \minmod\left(\theta, \frac{\Delta f_{i-\frac{1}{2}}^n}{\Delta f_{i+\frac{1}{2}}^n}\right)\Delta f_{i+\frac{1}{2}}^n = \minmod\left(\theta\Delta f_{i+\frac{1}{2}}^n, \Delta f_{i-\frac{1}{2}}^n\right). \end{cases} \quad (3.17)$$

In order to use a less diffusive scheme, the *superbee* limiter ($\theta = 2$) can be chosen but the scheme is less stable. In the same way, another limiter can be chosen, similar in construction

with limiters (3.14) and (3.15), defined as

$$\Psi(R_i)\Delta f_{i-\frac{1}{2}} = p_{i,1} = \begin{cases} 0 & \text{if } \Delta f_{i-\frac{1}{2}}^n \cdot \Delta f_{i+\frac{1}{2}}^n < 0, \\ \min(\theta(f_i^n - f_{\min}), \Delta f_{i+\frac{1}{2}}^n) & \text{if } \Delta f_{i+\frac{1}{2}}^n \geq 0, \\ \max(\theta(f_i^n - f_{\max}), \Delta f_{i+\frac{1}{2}}^n) & \text{if } \Delta f_{i+\frac{1}{2}}^n \leq 0, \end{cases} \quad (3.18)$$

and

$$\Psi\left(\frac{1}{R_i}\right)\Delta f_{i+\frac{1}{2}} = p_{i,2} = \begin{cases} 0 & \text{if } \Delta f_{i-\frac{1}{2}}^n \cdot \Delta f_{i+\frac{1}{2}}^n < 0, \\ \min(\Delta f_{i-\frac{1}{2}}^n, \theta(f_{\max} - f_i^n)) & \text{if } \Delta f_{i+\frac{1}{2}}^n \geq 0, \\ \max(\Delta f_{i-\frac{1}{2}}^n, \theta(f_{\min} - f_i^n)) & \text{if } \Delta f_{i+\frac{1}{2}}^n \leq 0. \end{cases} \quad (3.19)$$

Proposition 5. *By using limiters (3.18) and (3.19), the reconstructed distribution function obtained for a positive velocity*

$$f_i^+ = f_i^n + \frac{1+\kappa}{4}p_{i,1} + \frac{1-\kappa}{4}p_{i,2},$$

belongs to $[0; f_{\max}]$. In the same way, for a negative velocity, the approximation

$$f_i^- = f_i^n - \frac{1+\kappa}{4}p_{i,2} - \frac{1-\kappa}{4}p_{i,1},$$

belongs to $[0; f_{\max}]$.

Proof: Consider the case $\Delta f_{i+\frac{1}{2}}^n \geq 0$ (the case with $\Delta f_{i+\frac{1}{2}}^n \leq 0$ is similar). Recall that

$$\forall x, y \in \mathbb{R}, \minmod(x, y) \geq 0.$$

Hence

$$f_i^+ \geq 0.$$

Moreover f_i^+ writes

$$f_i^+ = f_i^n + \frac{1+\kappa}{4}\theta(f_{\max} - f_i^n) + \frac{1-\kappa}{4}(f_{i+1}^n - f_i^n). \quad (3.20)$$

Expression (3.20) can be simplified into

$$f_i^+ = \frac{1-\kappa}{4}(f_i^n + f_{i+1}^n) + \frac{1+\kappa}{4}((2-\theta)f_i^n + \theta f_{\max}). \quad (3.21)$$

Since $f_i^n \leq 0$ and $f_{i+1}^n \leq 0$, we obtain $f_i^+ \leq f_{\max}$.

For the reconstruction f_i^- , it comes that $f_i^- \leq f_i^n \leq f_{\max}$. Moreover f_i^- can be rewritten such as

$$f_i^- = f_i^n - \frac{1+\kappa}{4}\theta f_i^n - \frac{1-\kappa}{4}(f_i^n - f_{i-1}^n). \quad (3.22)$$

Expression (3.22) can be simplified into

$$f_i^- = f_i^n \left(1 - \frac{1+\kappa}{4}\theta - \frac{1-\kappa}{4} \right) + \frac{1-\kappa}{4} f_{i-1}^n . \quad (3.23)$$

Since $\theta \in [1; 2]$, we get

$$f_i^- \geq \frac{1-\kappa}{4} (f_i^n + f_{i-1}^n) .$$

Therefore we obtain $f_i^- \geq 0$. □

Proposition 6. *The scheme (3.2, 3.3, 3.16) is L^∞ stable by using limiters (3.18) and (3.19) provided $\theta \leq \frac{5}{2}$ and the CFL condition satisfy*

$$\lambda \leq \frac{6}{5} ,$$

where $\lambda = \frac{\Delta t}{\Delta x}$.

Proof: Let us consider the case where the velocity v is positive and $\Delta f_{i+\frac{1}{2}}^n \geq 0$ (other cases can be demonstrated in the same way). By denoting λ such as $\lambda = \frac{\Delta t}{\Delta x}$, we get

$$f_i^{n+1} = f_i^n - \lambda(f_i^+ - f_{i-1}^+) . \quad (3.24)$$

By using reconstructions (3.16), equation (3.24) can be rewritten into

$$\begin{aligned} f_i^{n+1} = f_i^n - \lambda & \left(f_i^n + \frac{1+\kappa}{4} \min(\Delta f_{i-\frac{1}{2}}^n, \theta(f_{\max} - f_i^n)) + \frac{1-\kappa}{4} \min(\Delta f_{i+\frac{1}{2}}^n, \theta f_i^n) \right) \\ & + \lambda \left(f_{i-1}^n + \frac{1+\kappa}{4} \min(\Delta f_{i-\frac{3}{2}}^n, \theta(f_{\max} - f_{i-1}^n)) + \frac{1-\kappa}{4} \min(\Delta f_{i-\frac{1}{2}}^n, \theta f_{i-1}^n) \right) . \end{aligned}$$

Then, it comes

$$f_i^{n+1} \leq f_i^n - \lambda \Delta f_{i-\frac{1}{2}}^n + \lambda \frac{1+\kappa}{4} \theta (f_{\max} - f_{i-1}^n) + \lambda \frac{1-\kappa}{4} \Delta f_{i-\frac{1}{2}}^n . \quad (3.25)$$

By definition of $\Delta f_{i-\frac{1}{2}}^n$, the right-hand side of (3.25) writes under the form

$$f_i^{n+1} \leq \alpha f_i^n + \beta f_{i-1}^n + \gamma f_{\max} ,$$

with

$$\alpha = 1 - \lambda \frac{3+\kappa}{4}, \quad \beta = \lambda \left(\frac{3+\kappa}{4} - \frac{1+\kappa}{4} \theta \right) \text{ and } \gamma = \lambda \frac{1+\kappa}{4} \theta .$$

Notice that $\alpha + \beta + \gamma = 1$ and $\gamma \geq 0$. The first term α is positive provided the numerical coefficient λ satisfy

$$\lambda \leq \frac{4}{3 + \kappa}.$$

Moreover $\beta \geq 0$ under the condition

$$\theta \leq \frac{3 + \kappa}{1 + \kappa}.$$

Therefore the third order scheme ($\kappa = \frac{1}{3}$) is L^∞ stable with the CFL condition

$$\lambda \leq \frac{6}{5},$$

and for $\theta \leq \frac{5}{2}$. □

Proposition 7. *The scheme (3.2, 3.3, 3.16) is a positive scheme for the limiters (3.18) and (3.19)*

Proof. As the previous proof, let us consider the case where the velocity v is positive. From equations (3.2, 3.3, 3.16), we get

$$\begin{aligned} f_i^{n+1} &= f_i^n + \lambda \left(f_i^n + \frac{1 + \kappa}{4} \min(\Delta f_{i-\frac{1}{2}}, \theta(f_{\max} - f_i^n)) + \frac{1 - \kappa}{4} \min(\Delta f_{i+\frac{1}{2}}, \theta f_i^n) \right) \\ &\quad - \lambda \left(f_{i-1}^n + \frac{1 + \kappa}{4} \min(\Delta f_{i-\frac{3}{2}}, \theta(f_{\max} - f_{i-1}^n)) + \frac{1 - \kappa}{4} \min(\Delta f_{i-\frac{1}{2}}, \theta f_{i-1}^n) \right). \end{aligned}$$

Then we obtain

$$f_i^{n+1} \geq f_i^n - \lambda \Delta f_{i-\frac{1}{2}}^n - \lambda \frac{1 + \kappa}{4} \Delta f_{i-\frac{1}{2}}^n - \lambda \frac{1 - \kappa}{4} f_i^n. \quad (3.26)$$

The inequality (3.26) can be simplified into

$$f_i^{n+1} \geq \alpha f_i^n + \beta f_{i-1}^n,$$

where

$$\alpha = 1 - \frac{3}{2}\lambda \text{ and } \beta = \lambda \frac{5 + \kappa}{4}.$$

Notice that $\beta \geq 0$ and $\alpha \geq 0$ under the CFL condition

$$\lambda \leq \frac{2}{3},$$

and the proposition follows. □

Remark 9. *By using $\kappa = -1$ in reconstructions (3.16), we obtain the classical order 2 scheme with limiter given by*

$$\begin{cases} p_i^+ = \minmod(f_{i+1}^n - f_i^n, \theta(f_i^n - f_{i-1}^n)), \\ p_i^- = \minmod(f_i^n - f_{i-1}^n, \theta(f_{i+1}^n - f_i^n)), \end{cases}$$

with $\theta \in [1; 2]$.

3.2.3 Fourth order transport scheme

A fourth-order compact MUSCL TVD scheme (see [15]) is proposed in this subsection. However this fourth order scheme do not ensured the positivity of the distribution function (because no more conservative). This scheme has been extended in [2] by proposing an additional limitation to preserve positivity of the distribution function f in the frame of VFRoe schemes for the shallow water equations.

The present scheme with this additional limitation is explained simply by using the one dimensional scalar conservation equation (3.1). In the same way as previously, the technique is based on a polynomial reconstruction f_i^+ and f_i^- depending of $x_{i\pm\frac{1}{2}}$ piecewise constant approximation. This reconstruction does not satisfy positivity of the distribution function because of the non conservative form of the scheme. However in [2], a modification of the scheme is proposed to preserve the positivity property. Indeed by assuming that the reconstructed state f_i^+ and f_i^- belongs to $\Omega = [0, \infty[$, an intermediate state f_i^* is introduced and given by

$$f_i^* = \frac{1}{3}(f_i^* + f_i^+ + f_i^-) , . \quad (3.27)$$

According to a CFL condition divided by three ([2]), this fourth order scheme preserves the non-negativity of the distribution function f_i^* if f_i^* belongs to Ω . The fourth-order MUSCL TVD reconstruction proposed in [15] is defined as

$$\begin{cases} f_i^+ = f_i^* + \Delta f_i^+ , \\ f_i^- = f_i^* + \Delta f_i^- , \end{cases} \quad (3.28)$$

Following [15], reconstructions reads

$$(\Delta f)_i^- = -\frac{1}{6} \left(2\Delta^* \bar{f}_{i-1/2} + \Delta^* \tilde{f}_{i+1/2} \right), \quad (\Delta f)_i^+ = \frac{1}{6} \left(\Delta^* \bar{f}_{i-1/2} + 2\Delta^* \tilde{f}_{i+1/2} \right), \quad (3.29)$$

where

$$\Delta^* \bar{f}_{i-1/2} = \minmod(\Delta^* f_{i-1/2}, 4\Delta^* f_{i+1/2}) , \quad (3.30)$$

$$\Delta^* \tilde{f}_{i+1/2} = \minmod(\Delta^* f_{i+1/2}, 4\Delta^* f_{i-1/2}) ,$$

and

$$\Delta^* f_{i+1/2} = \Delta f_{i+1/2} - \frac{1}{6} \Delta^3 \bar{f}_{i+1/2} , \quad (3.31)$$

$$\Delta^3 \bar{f}_{i+1/2} = \Delta \bar{f}_{i-1/2} - 2\Delta \bar{f}_{i+1/2} + \Delta \bar{f}_{i+3/2} .$$

Here $\Delta \bar{f}_{i-1/2}$, $\Delta \bar{f}_{i+1/2}$ and $\Delta \bar{f}_{i+3/2}$ are given by

$$\begin{aligned} \Delta \bar{f}_{i-1/2} &= \minmod(\Delta f_{i-1/2}, 2\Delta f_{i+1/2}, 2\Delta f_{i+3/2}) , \\ \Delta \bar{f}_{i+1/2} &= \minmod(\Delta f_{i+1/2}, 2\Delta f_{i+3/2}, 2\Delta f_{i-1/2}) , \\ \Delta \bar{f}_{i+3/2} &= \minmod(\Delta f_{i+3/2}, 2\Delta f_{i-1/2}, 2\Delta f_{i+1/2}) . \end{aligned} \quad (3.32)$$

Concerning the limitation involving the intermediate state, we have to ensure the three following restrictions

$$\begin{cases} f_i^n + \theta \Delta f_i^+ \geq 0, \\ f_i^n - \theta \Delta f_i^- \geq 0, \\ f_i^* = f_i^n - \theta \Delta f_i^+ - \theta \Delta f_i^-. \end{cases} \quad (3.33)$$

Therefore to ensure these restrictions, a process of over-limitation is defined as

$$\begin{cases} f_i^+ = f_i^n + (\Delta f)_i^{\text{lim},-}, \\ f_i^- = f_i^n + (\Delta f)_i^{\text{lim},+}, \end{cases} \quad (3.34)$$

The modified limitation proposed in [2] to illustrate this new restrictions is defined as

$$\begin{aligned} (\Delta f)_i^{\text{lim},-} &= \theta \max((\Delta f)_i^-, -f_i^n), \\ (\Delta f)_i^{\text{lim},+} &= \theta \max((\Delta f)_i^+, -f_i^n), \end{aligned} \quad (3.35)$$

where

$$\theta = \begin{cases} 1, & \text{if } \max((\Delta f)_i^-, -f_i^n) + \max((\Delta f)_i^+, -f_i^n) \leq 0, \\ \min\left(1, \frac{f_i^n}{\max((\Delta f)_i^-, -f_i^n) + \max((\Delta f)_i^+, -f_i^n)}\right) & \text{otherwise.} \end{cases} \quad (3.36)$$

3.2.4 Verification of the scheme's order

In this part, we do not prove analytically the order of schemes although it has been verified with a strong and popular software called Maple.

In order to verify this, numerical results will be shown to illustrate the order of the different methods. The order in space has been analyzed for each schemes on the transport equation. To maintain the TVD property of each schemes, a suitable Runge-Kutta time discretization for each scheme is used. These total variation diminishing Runge-Kutta schemes are described in [9].

A general Runge-Kutta method of order m for (3.1) is written in the form

$$\begin{aligned} f^{(i)} &= \sum_{k=0}^{i-1} \left(\alpha_{ik} f^{(k)} + \Delta t \beta_{ik} L(f^{(k)}) \right), \quad i = 1, \dots, m, \\ f^{(0)} &= f^n, \quad f^{(m)} = f^{n+1}, \end{aligned} \quad (3.37)$$

where $L(f) = v \frac{\partial f}{\partial x}$.

In [9], the authors propose suitable coefficients α_{ik} and β_{ik} to obtain optimal m^{th} order TVD

Runge-Kutta method (optimal in the sens of CFL condition and cost of computation). For example, under the CFL condition $c = 1$, the optimal second order TVD Runge-Kutta method (3.37) is given by

$$\alpha_{10} = 1, \quad \beta_{10} = 1,$$

$$\alpha_{20} = \frac{1}{2}, \quad \beta_{20} = 0,$$

$$\alpha_{21} = \frac{1}{2}, \quad \beta_{21} = \frac{1}{2}.$$

Now we consider a numerical application to illustrate the different order of schemes. For this test case, the initial distribution function is chosen to be simple and periodic in order to obtain more easily the analytical solution. More precisely, f is defined by

$$f(t = 0, x) = \sin(2\pi x).$$

Moreover we consider transport equation (3.1) with a transport coefficient $v = 1$. The space domain is discretized with different number of points from 10 to 100 points and the length of the space domain is denoted by $x_{\max} = 6$. We choose a time step satisfying the CFL condition. The numerical results are computed at $t = 40$.

Figure 3.2 shows the ratio of the L^1 norm of the error between analytical and numerical solution depending on the space step Δx . The slope coefficient for the second and the third order scheme correspond to the order of the scheme. Indeed we obtain 1.9986 for the second order scheme and 2.8154 for the third order scheme.

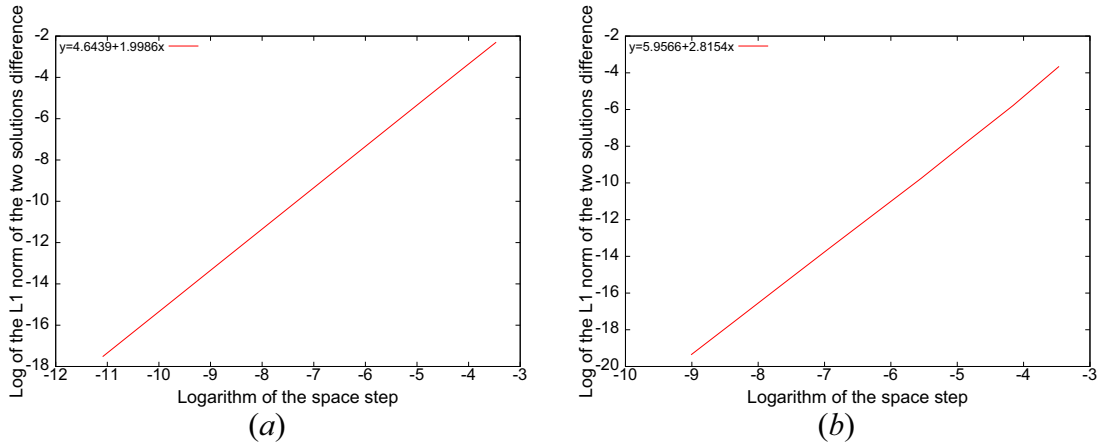


Figure 3.2: Evolution of the logarithm of the L^1 norm of the difference between the analytical and the numerical solution, and the logarithm of the space step for the second order scheme (a) and the third order scheme (b).

Concerning the fourth order scheme, results are different of what we expect. As you can see on figure 3.3, the slope coefficient is equal to 1.9835 which does not match with the fourth order

scheme. This paradox difference is explained by the use of the additional limitation. Indeed if we analyze the order of the scheme on the part of the solution where the limiter does not affect the calculation (regular zone), a slope coefficient of 4.0016 is obtained as shown on figure 3.3. This difference of order may influence the accuracy of the fourth order scheme. So the third order scheme is maybe a better scheme in fact, this accuracy will be check in the next chapter.

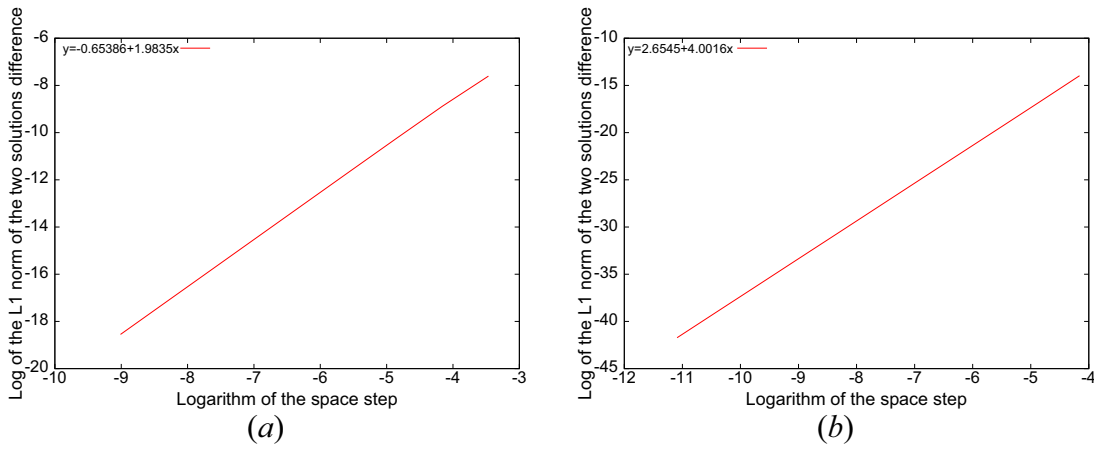


Figure 3.3: Evolution of the logarithm of the L^1 norm of the difference between the analytical and the numerical solution, and the logarithm of the space step for the fourth order scheme where limiters take action (a) and where limiters do not take action (b).

3.3 Numerical scheme for Maxwell equations

Now we consider the phase space $(x, \mathbf{v}) \in \mathbb{R} \times \mathbb{R}^3$. Let $\mathbf{E}$ be the electric field $\mathbf{E}$ denoted by $\mathbf{E} = {}^t(E_x, E_y, 0)$ and $\mathbf{B}$ the magnetic field $\mathbf{B}$ given by $\mathbf{B} = {}^t(0, 0, B_z)$.

3.3.1 Maxwell equations

The non relativistic FKP-Maxwell system (2.5-2.6) can be rewritten as

$$\left\{ \begin{array}{l} \frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + \frac{q}{m} (E_x + B_z v_y) \frac{\partial f}{\partial v_x} + \frac{q}{m} (E_y - B_z v_x) \frac{\partial f}{\partial v_z} = C_{ee}(f) + C_{ei}(f) , \\ \frac{\partial E_x}{\partial t} = \frac{-j_x}{\epsilon_0} , \\ \frac{\partial E_y}{\partial t} = -c^2 \frac{\partial B_z}{\partial x} - \frac{j_y}{\epsilon_0} , \\ \frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x} . \end{array} \right. \quad (3.38)$$

In order to have no time constraints about plasma frequency, electric and magnetic field ($\mathbf{E}, \mathbf{B}$) are expressed in implicit format while sources ($\rho, \mathbf{j}$) will be explicit. For the discretization of the Maxwell equations, taking the centered formulation for the temporal derivatives gives

$$\left(\frac{\partial E_{x,i}}{\partial t} \right)^{n+\frac{1}{2}} = \frac{E_{x,i}^{n+1} - E_{x,i}^n}{\Delta t} . \quad (3.39)$$

Concerning the space discretization, a finite difference method to second order is simply used and writes

$$(\partial_x E_{x,i})^n = \frac{E_{x,i+1}^n - E_{x,i-1}^n}{2\Delta x} . \quad (3.40)$$

By reducing the time of equation (3.40) to $n + \frac{1}{2}$, we get

$$(\partial_x E_{x,i})^{n+\frac{1}{2}} = \frac{1}{2} \left(\frac{E_{x,i+1}^{n+1} - E_{x,i-1}^{n+1}}{2\Delta x} + \frac{E_{x,i+1}^n - E_{x,i-1}^n}{2\Delta x} \right) = \frac{1}{2} \left(\frac{E_{x,i+1}^{n+\frac{1}{2}} + E_{x,i-1}^{n+\frac{1}{2}}}{\Delta x} \right) , \quad (3.41)$$

where

$$\mathbf{E}_i^{n+\frac{1}{2}} = \frac{1}{2} (\mathbf{E}_i^{n+1} + \mathbf{E}_i^n) . \quad (3.42)$$

The choice of the space discretization is very important. Indeed this is one of few scheme that conserve energy. Next for the magnetic field $\mathbf{B}$ we proceed as previously. The electric field $\mathbf{E}$

and the magnetic field $\mathbf{B}$ are solution of the following Maxwell discretized system

$$\left\{ \begin{array}{l} \frac{E_{x,i}^{n+1} - E_{x,i}^n}{\Delta t} = -\frac{J_{x,i}^n}{\epsilon_0}, \\ \frac{E_{y,i}^{n+1} - E_{y,i}^n}{\Delta t} + c^2 \frac{B_{z,i+1}^{n+1/2} - B_{z,i-1}^{n+1/2}}{2\Delta x} = -\frac{J_{y,i}^n}{\epsilon_0}, \\ \frac{B_{z,i}^{n+1} - B_{z,i}^n}{\Delta t} + \frac{E_{y,i+1}^{n+1/2} - E_{y,i-1}^{n+1/2}}{2\Delta x} = 0, \end{array} \right. \quad (3.43)$$

where $\mathbf{E}^{n+1/2}$ and $\mathbf{B}^{n+1/2}$ are given by

$$\mathbf{E}^{n+1/2} = \frac{1}{2}(\mathbf{E}^{n+1} + \mathbf{E}^n) \quad \text{and} \quad \mathbf{B}^{n+1/2} = \frac{1}{2}(\mathbf{B}^{n+1} + \mathbf{B}^n). \quad (3.44)$$

The approximation for the current in (3.43) J_x^n and J_y^n has been chosen such as

$$J_{x,i}^n = \sum_{j \in \mathbb{Z}^3} \Delta v^3 v_{x,j} f_{i,j}^n \quad \text{and} \quad J_{y,i}^n = \sum_{j \in \mathbb{Z}^3} \Delta v^3 v_{y,j} f_{i,j}^n, \quad (3.45)$$

where $\Delta v = \Delta v_x = \Delta v_y$. The system (3.43) can be simplified into

$$\left\{ \begin{array}{l} E_{x,i}^{n+1} = E_{x,i}^n - \Delta t \frac{J_{x,i}^n}{\epsilon_0}, \\ E_{y,i}^{n+1} + \frac{c^2 \Delta t}{4\Delta x} (B_{z,i+1}^{n+1} - B_{z,i-1}^{n+1}) = E_{y,i}^n - \frac{\Delta t}{\epsilon_0} J_{y,i}^n - \frac{c^2 \Delta t}{4\Delta x} (B_{z,i+1}^n - B_{z,i-1}^n), \\ B_{z,i}^{n+1} + \frac{\Delta t}{4\Delta x} (E_{y,i+1}^{n+1} - E_{y,i-1}^{n+1}) = B_{z,i}^n - \frac{\Delta t}{4\Delta x} (E_{y,i+1}^n - E_{y,i-1}^n). \end{array} \right. \quad (3.46)$$

3.3.2 Poisson equation

In the case of non-magnetic field, the only equations from (2.6) to consider are Ampere equation and Poisson equation. Therefore the Maxwell system (2.6) becomes in on space dimension

$$\left\{ \begin{array}{l} \frac{\partial E_x}{\partial t} = -\frac{j_x}{\epsilon_0}, \\ \frac{\partial E_y}{\partial t} = -\frac{j_y}{\epsilon_0}, \\ \frac{\partial E_x}{\partial x} = -\frac{\rho}{\epsilon_0}. \end{array} \right. \quad (3.47)$$

By using the same discretization than previously (3.43), the discretized system of (3.47) writes

$$\left\{ \begin{array}{l} \frac{E_{x,i}^{n+1} - E_{x,i}^n}{\Delta t} = -\frac{J_{x,i}^n}{\epsilon_0}, \\ \frac{E_{y,i}^{n+1} - E_{y,i}^n}{\Delta t} = -\frac{J_{y,i}^n}{\epsilon_0}, \\ \frac{E_{x,i}^{n+\frac{1}{2}} - E_{x,i-1}^{n+\frac{1}{2}}}{\Delta x} = \rho_i^{n+1}. \end{array} \right. \quad (3.48)$$

We choose here to discretize the charge density ρ at time $n+1$ like a predictor-corrector method. Indeed electric field has to check Poisson Equation after each computation because the Poisson equation is not ensured.

3.4 Boundary conditions and CFL condition

In this section, different phenomena of electron-wall interaction considered in our model are described. Then our numerical approach is presented and finally, the CFL condition is detailed.

3.4.1 Boundary conditions on the distribution function in space x

A brief description is given here to present different processes considered in our model when an electron reaches the boundary of our domain.

Specular reflection

The first one is the specular reflection (see figure 3.4 (a)). The particle is reflected on the wall like a light ray on a plane mirror. Classically the electron is reflected from the wall in a random direction which can be explained by the uncertainty about the state of the wall. The number of particles hitting the wall is equal to the number of particle bouncing with an inverse velocity. At the boundary $x_{\min}$, for positive velocity $v_x \geq 0$, it can be expressed as

$$f(t, x_{\min}, v_x, v_y, v_z) = f(t, x_{\min}, -v_x, v_y, v_z). \quad (3.49)$$

Diffusive reflection

Now we will consider the diffuse reflection as a boundary condition (see figure 3.4 (b)). A diffuse reflection is the result of interaction between electron and the electric field in the wall.

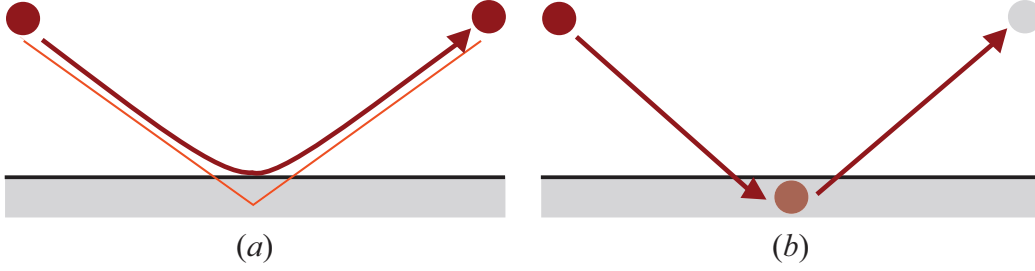


Figure 3.4: Representation of different conditions during interaction particle-wall: specular reflection (a) and diffusive reflection (b)

The electron is first trapped in the wall then its temperature changes to be the same as the one in the wall and finally the particle is ejected from the wall in an isotropic way with the wall temperature. The number of particles which go in the wall is the same as the number of particles which go out of the wall.

The distribution function at the boundary $x = x_{\min}$ for positive velocity is given by a Maxwellian

$$f(t, x_b, v_x, v_y, v_z) = \alpha K e^{\frac{-v^2}{2k_B T}} + (1 - \alpha) f(t, x_b, -v_x, v_y, v_z) \quad (3.50)$$

with $\mathbf{v}$ the mean velocity output, $\alpha \in [0; 1]$ the accommodation parameter. We will take $\alpha = 1$ in our studies. T represents the temperature and the mean velocity issued from the wall tends to 0.

In order to conserve the charge, the current flow must be null, i.e. we obtain

$$j_x = q \int_{\mathbb{R}^3} v_x f(t, x_{\min}, v_x, v_y, v_z) d\mathbf{v} = 0. \quad (3.51)$$

By splitting velocity v_x into positive and negative terms, equation (3.51) can be simplified into

$$q \int_{(v_y, v_z) \in \mathbb{R}^2} \left[\int_{v_x \geq 0} v_x f(t, x, v_x, v_y, v_z) dv_x + \int_{v_x \leq 0} v_x f(t, x, v_x, v_y, v_z) dv_x \right] dv_y dv_z = 0.$$

Therefore K is solution of equation

$$q \int_{(v_y, v_z) \in \mathbb{R}^2} \left[K \int_{v_x \geq 0} v_x e^{\frac{-v^2}{2k_B T}} dv_x + \int_{v_x \leq 0} v_x f(t, x, v_x, v_y, v_z) dv_x \right] dv_y dv_z = 0.$$

Thus K is given as the ratio between the half inflow and the half outflow such as

$$K = \frac{- \int_{v_x \leq 0} v_x f(t, x, v_x, v_y, v_z) dv_x}{\int_{v_x \geq 0} v_x e^{\frac{-v^2}{2k_B T}} dv_x} = \frac{\int_{|v_x| \leq 0} v_x f(t, x, v_x, v_y, v_z) dv_x}{\int_{v_x \geq 0} v_x e^{\frac{-v^2}{2k_B T}} dv_x}. \quad (3.52)$$

3.4.2 Boundary conditions on the distribution function in velocity v

For the sake of simplicity, we consider in this subsection the discretization of the transport equation (3.1) with the first order scheme

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} + \frac{\mathcal{F}_{j+\frac{1}{2}}^n - \mathcal{F}_{j-\frac{1}{2}}^n}{\Delta v} = 0 . \quad (3.53)$$

The transport coefficient is given by $v = 1$. Since $f \geq 0$, we get $\mathcal{F}_{j+\frac{1}{2}}^n = f_j^n$ and equation (3.53) can be rewritten as

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} + \frac{f_j^n - f_{j-1}^n}{\Delta v} = 0 .$$

To obtain a conservative scheme, we get

$$\sum_0^{n_{vx}} f_j^{n+1} - \sum_0^{n_{vx}} f_j^n = -\frac{\Delta t}{\Delta v} \sum_0^{n_{vx}} \mathcal{F}_{j+\frac{1}{2}}^n - \mathcal{F}_{j-\frac{1}{2}}^n = 0 . \quad (3.54)$$

By exchanging the indexes j in the sum of flux $\mathcal{F}$, equation (3.54) can be simplified into

$$-\left(\mathcal{F}_{n_{vx}+\frac{1}{2}}^n - \mathcal{F}_{-\frac{1}{2}}^n\right) \frac{\Delta t}{\Delta v} = -\left(f_{n_{vx}}^n - f_{-1}^n\right) \frac{\Delta t}{\Delta v}$$

Therefore, in order to have a conservative scheme, we can impose $f_{-1}^n = f_{n_{vx}}^n = 0$.

3.4.3 Boundary conditions on the electromagnetic field in non periodic cases

The boundary condition concerning the electromagnetic field for the space direction satisfies Newman conditions on the electric field and Dirichlet conditions on the magnetic field. On the boundary, electromagnetic field is given by

$$\left\{ \begin{array}{l} \frac{\partial E_y}{\partial x}(x_{\min}, t) = \frac{\partial E_y}{\partial x}(x_{\max}, t) = 0 , \\ B_z(x_{\min}, t) = B_R(t) , \\ B_z(x_{\max}, t) = B_L(t) . \end{array} \right. \quad (3.55)$$

The linear system (3.46) with conditions (3.55) can be rewritten in conservative form as

$$AX = B . \quad (3.56)$$

The matrix A , whom the first and last lines are defined by the condition (3.55), is defined as

$$A = \begin{pmatrix} \mathbf{1} & \mathbf{a} & 0 & \mathbf{a} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathbf{b} & \mathbf{1} & \mathbf{b} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathbf{a} & \mathbf{1} & 0 & 0 & \mathbf{a} & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathbf{b} & 0 & 0 & \mathbf{1} & \mathbf{b} & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{a} & \mathbf{1} & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathbf{b} & 0 & 0 & \mathbf{1} & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & \mathbf{1} & 0 & 0 & \mathbf{a} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \mathbf{1} & \mathbf{b} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & -\mathbf{a} & \mathbf{1} & 0 & 0 & \mathbf{a} \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & -\mathbf{b} & 0 & 0 & \mathbf{1} & \mathbf{b} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -\mathbf{a} & \mathbf{1} & -\mathbf{a} \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -\mathbf{b} & 0 & \mathbf{b} & \mathbf{1} \end{pmatrix}.$$

Here coefficients a and b write

$$a = \frac{c^2 \Delta t}{4\Delta x} \quad \text{and} \quad b = \frac{\Delta t}{4\Delta x}.$$

Vectors $\mathbf{X}$ and $\mathbf{B}$ from (3.56) are given by

$$X = \begin{pmatrix} E_{y,0}^{n+1} \\ B_{z,0}^{n+1} \\ E_{y,1}^{n+1} \\ B_{z,1}^{n+1} \\ \vdots \\ E_{y,n_x-2}^{n+1} \\ B_{z,n_x-2}^{n+1} \\ E_{y,n_x-1}^{n+1} \\ B_{z,n_x-1}^{n+1} \end{pmatrix}, \quad B = \begin{pmatrix} E_{y,0}^n \\ B_{z,0}^n \\ E_{y,1}^n \\ B_{z,1}^n \\ \vdots \\ E_{y,n_x-2}^n \\ B_{z,n_x-2}^n \\ E_{y,n_x-1}^n \\ B_{z,n_x-1}^n \end{pmatrix} - \frac{1}{4\Delta x} \begin{pmatrix} c^2(B_{z,1}^n - B_{z,-1}^n) \\ E_{y,1}^n - E_{y,-1}^n \\ c^2(B_{z,2}^n - B_{z,0}^n) \\ E_{y,2}^n - E_{y,0}^n \\ \vdots \\ c^2(B_{z,n_x-1}^n - B_{z,n_x-3}^n) \\ E_{y,n_x-1}^n - E_{y,n_x-3}^n \\ c^2(B_{z,n_x}^n - B_{z,n_x-2}^n) \\ E_{y,n_x}^n - E_{y,n_x-2}^n \end{pmatrix} - \frac{1}{\epsilon_0} \begin{pmatrix} J_{y,0}^n \\ 0 \\ J_{y,1}^n \\ 0 \\ \vdots \\ J_{y,n_x-2}^n \\ 0 \\ J_{y,n_x-1}^n \\ 0 \end{pmatrix} - \frac{c^2}{2\Delta x} \begin{pmatrix} B_{x=x_{\min}} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ -B_{x=x_{\max}} \end{pmatrix}$$

3.4.4 CFL condition for the numerical scheme

CFL condition and electron-ion collision operator

Here we focus on the CFL condition of the approximation of the electron-ion collision operator. Since the space variable is a parameter, only the space homogeneous equation is considered

$$\frac{\partial f}{\partial t} = C_{e,i}(f, f). \quad (3.57)$$

Recall that the Fokker-Planck-Landau $C_{e,i}(f, f)$ stands for the electron-ion collision operator and is given by

$$C_{e,i}(f) = \alpha_{e,i} \nabla_{\mathbf{v}} \cdot [\Phi(\mathbf{v}) \nabla_{\mathbf{v}} f] ,$$

where $\alpha_{e,i}$ is defined in (2.11). Recall that $\Phi(u)$ is an operator acting on the relative velocity $\mathbf{u}$

$$\Phi(\mathbf{u}) = \frac{\|\mathbf{u}\|^2 \|Id - \mathbf{u} \otimes \mathbf{u}\|}{\|\mathbf{u}\|^3} .$$

For the sake of simplicity, we simplify this expression as $C_{e,i}(f) = \nabla_{\mathbf{v}} \cdot (\mathbf{M} \nabla_{\mathbf{v}} f)$ where $\mathbf{M} = \Phi(\mathbf{v}) \alpha_{e,i}$.

For $n \geq 0$, $(f_{i,j}^{n+1})_{i,j}$ is assumed to approximate the solution on the control volume $C_{i,j}$. It satisfies

$$\frac{f_{i,j}^{n+1} - f_{i,j}^n}{\Delta t} = \frac{1}{\Delta \mathbf{v}} \left(\mathbf{M}_{i+\frac{1}{2}} \frac{f_{i,j+1}^n - f_{i,j}^n}{\Delta \mathbf{v}} + \mathbf{M}_{i-\frac{1}{2}} \frac{f_{i,j}^n - f_{i,j-1}^n}{\Delta \mathbf{v}} \right) , \quad (3.58)$$

with $\mathbf{M}_{i+\frac{1}{2}} = \Phi(\frac{\mathbf{v}_i + \mathbf{v}_{i+1}}{2}) \alpha_{e,i}$, $\Delta \mathbf{v} = \mathbf{v}_{j+\frac{1}{2}} - \mathbf{v}_{j-\frac{1}{2}}$ is the velocity space step and Δt is the time step. So equation (3.58) can be rewritten as

$$f_{i,j}^{n+1} = f_{i,j}^n \left(1 - \frac{\Delta t}{\Delta \mathbf{v}^2} (\mathbf{M}_{i+\frac{1}{2}} + \mathbf{M}_{i-\frac{1}{2}}) \right) + f_{i-1}^n \frac{\Delta t}{\Delta \mathbf{v}^2} \mathbf{M}_{i-\frac{1}{2}} + f_{i+1}^n \frac{\Delta t}{\Delta \mathbf{v}^2} \mathbf{M}_{i+\frac{1}{2}} .$$

Since $\mathbf{M}$ is a positive matrix, the positivity of the scheme is ensured under the CFL condition

$$\max(\mathbf{M}_{i+\frac{1}{2}} + \mathbf{M}_{i-\frac{1}{2}}) \frac{\Delta t}{\Delta \mathbf{v}^2} \leq 1 . \quad (3.59)$$

The CFL condition (3.59) can be simplified into

$$\max(\mathbf{M}) \frac{2\Delta t}{\Delta \mathbf{v}^2} \leq 1 . \quad (3.60)$$

Moreover eigenvalues of Φ are defined as

$$\text{SP}(\Phi) = \{0, \frac{1}{|\mathbf{v}|}\} .$$

Therefore we obtain

$$\max(\mathbf{M}) = \max(\Phi(\mathbf{v}) \alpha_{e,i}) = \frac{\alpha_{e,i}}{\mathbf{v}_{\min}} = \frac{\alpha_{e,i}}{\Delta \mathbf{v}/2} . \quad (3.61)$$

From (3.60) and (3.61), the CFL condition for one dimensional velocity model for equation (3.57) writes

$$\alpha_{e,i} \frac{4\Delta t}{(\Delta \mathbf{v})^3} \leq 1 .$$

In 3D velocity space, the CFL condition is given by

$$\alpha_{e,i} \left(\frac{4\Delta t}{(\Delta \mathbf{v}_x)^3} + \frac{4\Delta t}{(\Delta \mathbf{v}_y)^3} + \frac{4\Delta t}{(\Delta \mathbf{v}_z)^3} \right) \leq 1 . \quad (3.62)$$

Transport part and CFL condition

Now consider the Vlasov equation in 1D space given by

$$\frac{\partial f}{\partial t} + \mathbf{v} \nabla_x f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \nabla_v f = 0 . \quad (3.63)$$

The numerical scheme associated can be written as

$$\frac{f_{i,j}^{n+1} - f_{i,j}^n}{\Delta t} + \mathbf{v}^+ \frac{f_{i,j}^n - f_{i-1,j}^n}{\Delta x} + \mathbf{v}^- \frac{f_{i+1,j}^n - f_{i,j}^n}{\Delta x} + \frac{q}{m} (\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \frac{f_{i,j+1}^n - f_{i,j-1}^n}{\Delta v} = 0 .$$

Concerning the transport of particles depending on x , an upwind scheme is used to ensure the positivity of the distribution function f . Therefore the well known CFL condition writes

$$\frac{\max |\mathbf{v}| \Delta t}{\Delta x} \leq 1 . \quad (3.64)$$

For the acceleration of the particles, i.e. the transport of particles depending on v , the positivity of the scheme is preserved under the CFL condition

$$\frac{q}{m} \frac{(\max(\mathbf{E}) + \mathbf{v}_{\max} \wedge \max(\mathbf{B})) \Delta t}{\Delta v} \leq 1 . \quad (3.65)$$

Fokker-Planck-Landau equation

Now the CFL condition of the Fokker-Planck-Landau equation in 3D velocity space can be expressed. From (3.62, 3.64, 3.65), the CFL condition is given by the sum of the CFL conditions of the different part of Fokker-Planck-Landau equation

$$\begin{aligned} & \frac{\max |\mathbf{v}| \Delta t}{\Delta x} + \frac{q}{m} (\max(\mathbf{E}) + \mathbf{v}_{\max} \times \max(\mathbf{B})) \Delta t \left(\frac{1}{\Delta \mathbf{v}_x} + \frac{1}{\Delta \mathbf{v}_y} + \frac{1}{\Delta \mathbf{v}_z} \right) \\ & + \alpha_{e,i} \left(\frac{4\Delta t}{(\Delta \mathbf{v}_x)^3} + \frac{4\Delta t}{(\Delta \mathbf{v}_y)^3} + \frac{4\Delta t}{(\Delta \mathbf{v}_z)^3} \right) \leq 1 . \end{aligned}$$

Remark 10. *Classical condition for the use of Poisson equation is to consider time step smaller than plasma period. Here Maxwell equations have been expressed in implicit form. That is why we do not find a such condition in this work.*

3.5 Conclusion

In this chapter we have proposed a numerical scheme based on a finite volume method to solve the Fokker-Planck-Landau equation. The first numerical scheme presented of order two presents many interesting properties such as TVD property. Then we have developed a third order scheme to improve accuracy which preserves the positivity of the scheme and the L^∞ stability. Finally a fourth order scheme is proposed in the same way and also preserves the positivity. In next chapter, all these high order schemes will be used to study their good physical behavior and to determine the best choice between them. To reproduce the behavior of particles at the border, some boundary conditions have been presented for these high order scheme.

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CHAPTER 4

VALIDATION AND NUMERICAL RESULTS FOR THE KINETIC MODEL: APPLICATION TO ELECTRON ENERGY TRANSPORT

4.1 Introduction

There exists already many codes which solve the Vlasov-Fokker-Planck equation. For example, M. Tzoufras *et al.* have developed in [14] a relativistic $2D_x \times 3D_v$ Vlasov-Fokker-Planck code called *OSHUN*. This code describes the electron transport by incorporating a spherical harmonic expansion of the electron distribution function. Nevertheless this decomposition while limited to a finite number, does not ensure in the non-negativity of the distribution function whereas in the *KETS* code, positivity of the electron distribution function is ensured. In this chapter, we develop the kinetic code *KETS* with the improvements described in the previous chapter such as new boundary conditions and new high order schemes (order 3). For this improvements, we keep parallelism aspect of the code. Moreover the efficiency of each high order schemes described previously will be compared here through representative configurations of ICF.

Several test cases will be considered to check the properties of the numerical scheme. First the interaction of two counter streaming electrons beams is studied to validate the good behavior of the schemes described in the previous chapter. Then we are interested in laser-plasma interaction, a significant part in ICF. Finally applications in ICF related to the local and non-local regimes of electron energy transport are proposed to validate our solver in different regimes. This test case will underline the robustness in different regimes. The difference between both regimes is given by the Knudsen number K_n which quantifies the thermodynamic non-equilibrium of the system, it is given by

$$K_n = \frac{\lambda_{ei}}{\lambda_T},$$

where λ_{ei} defines the electron-ion collision mean free path and λ_T is the electron temperature gradient length. The local regime is defined by a small Knudsen parameter $K_n \ll 1$ which is characteristic for a highly collisional regime. The local regime refers to the hydrodynamic limit. The non-local regime is defined with a high Knudsen number $K_n \gg 1$ which is characteristic for a weakly collisional regime. The limit achieved here is the kinetic limit.

To better understand the physics of thermal conduction, the relaxation of a single hot spot is studied to illustrate results in non-local regime. Then the evolution of a plasma with a high temperature gradient is studied to represent the highly collisional regime. For each test case, different scaling are used and described in appendix at the end of the chapter.

4.2 Two-stream instability

To study the good behavior of previous numerical schemes implemented in the *KETS* code and explore each high order scheme, two collisionless instabilities are examined in regime that involve strong electromagnetic fields, highly anisotropic distributions and trapped particles.

The physical context of this case is to study the interaction of electron oppositely propagating

beams with a perturbation parallel to the beam velocities. This example, called the Two-Stream Instability, is very important in the ICF physics because it reproduces the behavior of the plasma in interaction with the laser beam. Indeed, the impact of the laser beam on the plasma involves generation of a beam of fast electrons and a return current from that ensures the quasi neutrality of the plasma. This leads to a very unstable configuration favorable to the excitation of plasma waves. This is a relevant test in the study of the ICF physics.

We consider here the phase space with one dimension in space and one dimension in velocity space without collision. Vlasov-Maxwell system is used to describe the time evolution of the distribution function. We choose a scaling with the plasma frequency detailed in Annex 4.7. The plasma density is normalized by the initial density n_0 and the energy by the initial temperature T_0 . The Vlasov-Maxwell system writes in these units

$$\begin{cases} \frac{\partial f}{\partial t} + \nabla_{\mathbf{x}} \cdot (\mathbf{v}f) - \nabla_{\mathbf{v}} \cdot (\mathbf{E}f) = 0, \\ \frac{\partial \mathbf{E}}{\partial t} = n \mathbf{u}, \\ \nabla_{\mathbf{x}} \cdot \mathbf{E} = (1 - n), \end{cases} \quad (4.1)$$

where the density and the current are expressed such as

$$n(t, \mathbf{x}) = \int_{\mathbb{R}^3} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v} \quad \text{and} \quad \mathbf{u}(t, \mathbf{x}) = \frac{1}{n} \int_{\mathbb{R}^3} \mathbf{v} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}.$$

The initial conditions consist of two opposite streams of charged particles (electrons) with a neutralizing background of immobile ions as sketched on figure 4.1.

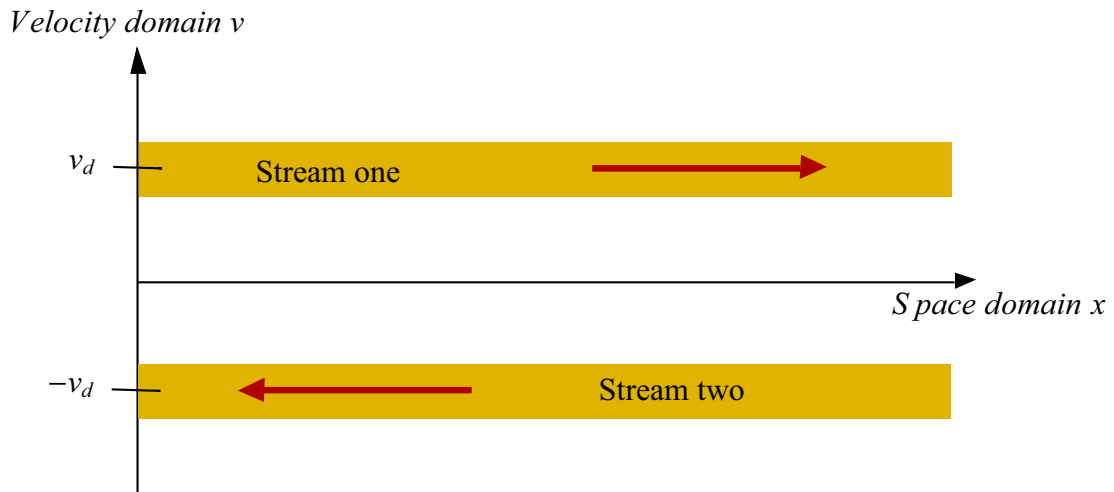


Figure 4.1: The two opposite streams in phase space at the start of the problem, $t = 0$.

A small amplitude perturbation $A \ll 1$ with the wave number k is considered as an instability seed. Thus the initial distribution function and electric field are given by

$$\begin{cases} f(t=0, x, v) = \frac{1}{2} [(1 + A \cos(kx))\mathcal{M}_{v_d}(v) + (1 - A \cos(kx))\mathcal{M}_{-v_d}(v)], \\ E(t=0, x) = 0, \end{cases} \quad (4.2)$$

where

$$\mathcal{M}_{v_d}(v) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \exp\left(-\frac{(v - v_d)^2}{2}\right).$$

is the Maxwellian distribution function centered around v_d . The temperature of the plasma is chosen as $T_e = 1$ and the density $n_e = 1$. Boundary conditions on each side are chosen to be periodic. At $t = 0$, both streams have mean velocities $\pm v_d = \pm 4$ and are strongly perturbed in position by choosing a perturbation amplitude $A = 0.1$. We discretized the periodic space domain which length is $x_{\max} = 25$ with 64 points, the velocity domain $[-v_{\max}; v_{\max}]$ is also discretized with 64 points and the truncation of the velocity domain is $v_{\max} = 12$. In order to respect the CFL condition of the scheme, we choose a time step $\Delta t = 1/200$.

For all high order schemes, we first observe the same behavior for the two counter-streaming beams as shown on figure 4.2. The two beams are initially well separated in the phase space (see figure 4.2 (a)). Then as the instability develops they change direction and mix together as illustrated by figure 4.2 (d) to create a phase space vortex form which persists. This phenomenon is called an electrostatic instability. In fact, both streams move through each other. Then a density perturbation on one stream is reinforced due to the bunching of particles in the other stream which slows the stream and some particles of the beam stop and then reverse. The same behavior is observed with the other stream. Finally the opposing streams system has formed a vortex in the phase space as illustrated in figure 4.2 (d).

The main differences between the high order schemes can be shown with the following diagnostic. On the one hand, the conservation of the total dimensionless is studied. On the other hand, the logarithm of the dimensionless total electrostatic energy (which represents the plasma wave structure) is analysed.

Figure 4.3 shows the impact of the use of different second order limiters on results. In figure 4.3 (a), the conservation of total energy is better with the slope limiter (3.14)/(3.15) than with the corrector (3.5) where $\theta = 1$ and $\theta = 2$. The total energy variation is about 1.5% with the slope limiter (3.14)/(3.15) while it is more than 5% with the corrector (3.5). Therefore, the second order scheme with the slope limiter (3.14)/(3.15) improves the results compared with other correctors. Moreover, the value of the coefficient θ is important in the proprieties of the scheme. If θ is great, the energy dissipation is less important than for a small value of θ .

In figure 4.3 (b), the evolution of the electrostatic energy is studied. Initially the instability grows exponentially which corresponds to a linear regime. Then the beams has reached their

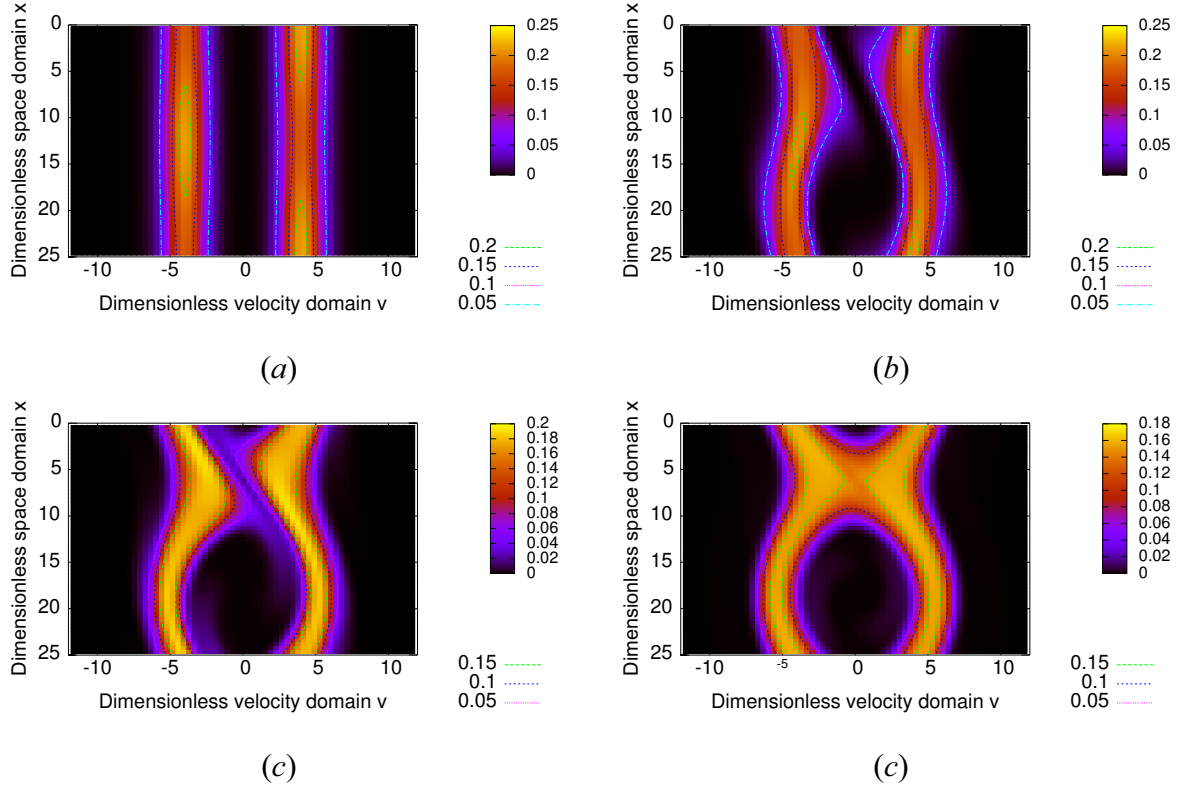


Figure 4.2: Representation of the beams in the phase space at initial time (a), at 6 plasma periods (b), at 10 plasma periods (c) and after 30 plasma periods (d) with 64×64 grid points .

maximum kinetic energy. The subsequent oscillations and particle bouncing die out as the particles do not all have the same frequency of oscillation in the wave potential. The semi-analytical solution for the linear regime is computed in [5]. Whatever the slope limiter chosen for the second order scheme, the linear regime is well reproduced. The use of corrector (3.5) with $\theta = 1$ creates less oscillations in the non-linear regime compared to other correctors.

Therefore, for the second order scheme, the use of slope limiters (3.14)/(3.15) and (3.5) with $\theta = 2$ present better results than corrector (3.5) with $\theta = 1$.

Figure 4.4 represents also the evolution of total energy and electrostatic energy but with the use of the third order scheme (3.16) and the corrector (3.18)/(3.19). The amplitude of the electrostatic energy first increases (linear regime) and then, it oscillates periodically around a constant as shown with the use of the second order scheme. The use of corrector (3.18)/(3.19) with $\theta = 2$ increases the total energy with only a rate of 2,2% instead of other correctors for which difference is more than 5%. Moreover the slope limiter (3.18)/(3.19) with $\theta = 2$ generates more periodic oscillations for the electrostatic energy. Therefore, the use of the corrector (3.18)/(3.19) is the best choice for the third order scheme. Moreover the use of the third order scheme with

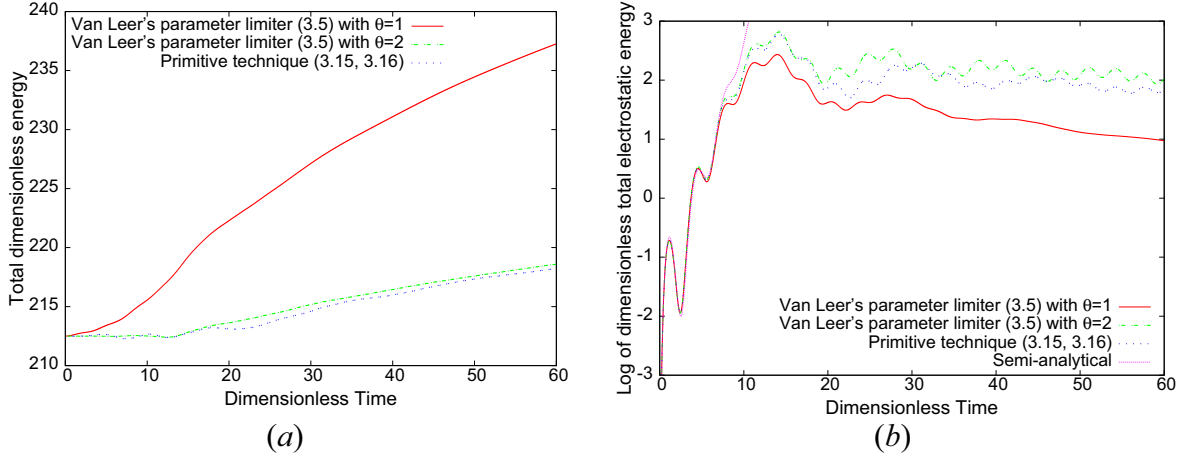


Figure 4.3: Evolution of the total energy (a) and the electrostatic energy (b) of the second order with the slope limiter (3.14)/(3.15) and slope limiter (3.5) where $b = 1$ and $b = 2$ with 64×64 grid points .

the slope limiter (3.18)/(3.19) seems to be more accurate than the second order scheme. The accuracy is demonstrated in figure 4.5.

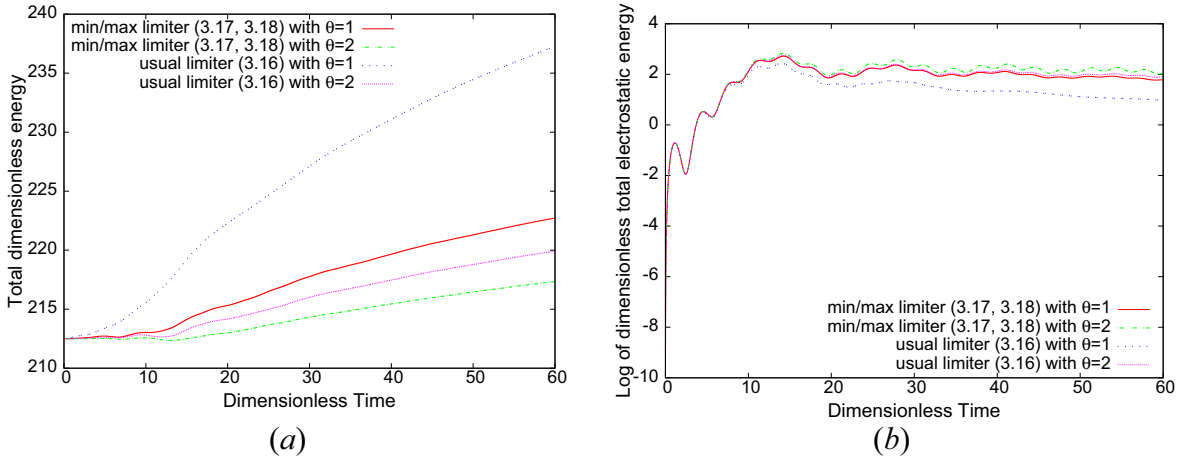


Figure 4.4: Evolution of the total energy (a) and the electrostatic energy (b) of the third order case with the slope limiter (3.18)/(3.19) where $b = 1$ and $b = 2$ and with the limiter (3.17) where $b = 1$ and $b = 2$ with 64×64 grid points .

Figure 4.5 (a) shows the variation of the total energy increases for all high order schemes but the phenomenon seems to be attenuated with the fourth order scheme for which fluctuation is only about 1%. With a longer simulation such as 40 more periods, the total energy for the fourth order scheme is stabilized around a variation of 0.1%. Comparing the electrostatic energy in

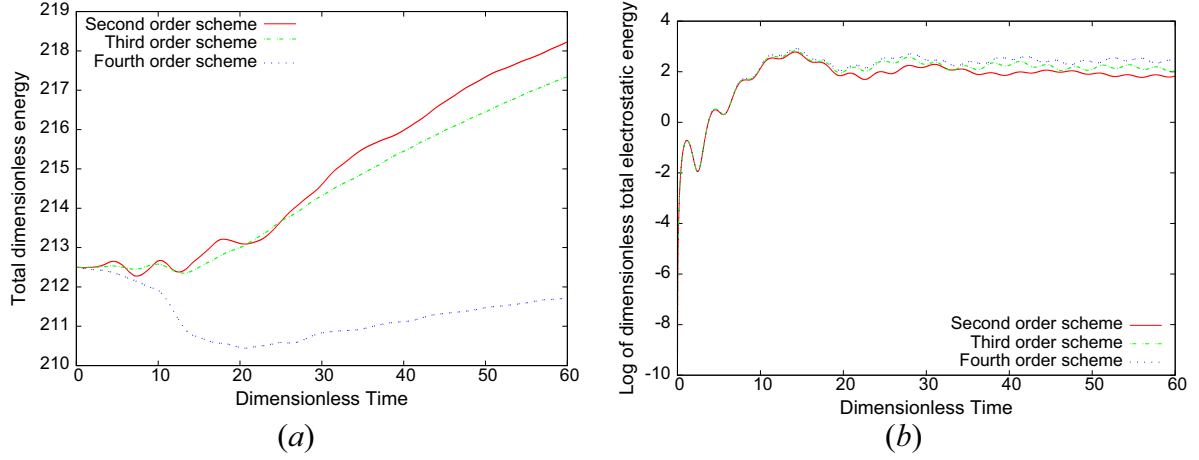


Figure 4.5: Evolution of the total energy (a) and the electrostatic energy (b) of the high order case with the different slope limiters with 64×64 grid points.

figure 4.5 (b), each high order schemes present the same linear regime and some oscillations around a constant value in the non-linear regime. Therefore, the fourth order scheme is the most accurate scheme for the Vlasov equation.

Figure 4.6 shows the effect of the use of more discretization points on our results. For the third order, the use of discretization points from 32×32 to 256×256 does not affect the results. However, the use of at least 128×128 points for the second and the fourth order scheme seems to be suitable to have some periodic oscillations around a constant value in the non-linear regime. Table 4.1 illustrates the computationnal time needed for the different high order schemes. The

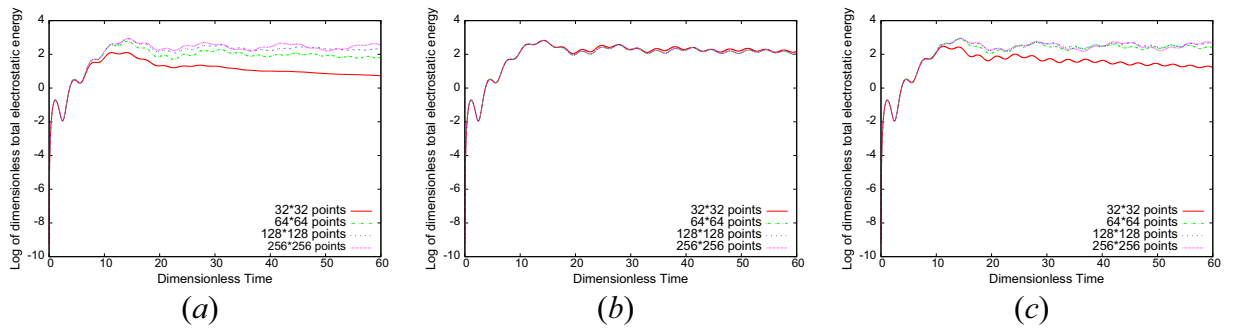


Figure 4.6: Evolution of the electrostatic energy of the second order (limiter (3.14)/(3.15)) (a), the third order (limiter (3.18)/(3.19)) (b) and the fourth order (c) for 32^2 , 64^2 , 128^2 and 256^2 grid points.

higher is the scheme's order, the longer is the simulation, which is understandable because the higher is the scheme's order, the more operation is required. Nevertheless, the computation time

with the third order scheme and the second order scheme does not increase too much with a large number of discretization points. While difference in computational time when using a large number of discretization points is more critical with the fourth order scheme.

	2 nd order	3 rd order	4 th order
32×32	2.51	2.71	8.1
64×64	8.13	8.31	28.96
128×128	28.33	30.93	120.2

Table 4.1: *CPU times (in seconds) made by the program to solve the Vlasov equation for each scheme order (two processors Intel(R) Core(TM)2 Duo CPU T960 @ 2.80 GHz were used).*

To conclude, the third order scheme is more interesting for application in physics. Indeed, this scheme is a good compromise between calculation time and accuracy of the numerical solution. To complete this analyze, a comparison by using an other approach call M_1 model is proposed in section 5.9.2.

4.2.1 Comparison with the Particle-In-Cell code

To demonstrate the good behavior and the accuracy of the fourth order scheme in the linear regime but also in the non-linear regime, we compare our numerical results with results obtained with the *PIC* code which is one of the most widely used kinetic code in plasma. The *PIC* method introduced in [13] is described in appendix 4.10. The essential feature of *PIC* codes is that the plasma is described as a large number of discrete particles moving in their self-consistent electromagnetic fields. Here the *PIC* code used for comparison, called *PICLS* has been developped by Y. Sentoku [13] and use a fourth order scheme.

For this numerical comparison, we chose a one dimensional space direction and one dimensional velocity direction. The initial distribution function f is the Maxwellian (4.2) with a perturbation amplitude about $A = 0, 1$. The space domain $L = [0; 25]$ is discretized with 64 points and the velocity domain $[-v_{\max}; v_{\max}]$ with 64 points where $v_{\max} = 12$. We chose a time step satisfying the CFL condition. The scaling used here is described in Annex 4.7. The numerical results are computed at time $t = 60$.

Figure 4.7 confirms the good behavior of the fourth order scheme. For the linear regime, *PIC* results are identically to those obtained with the fourth order scheme. In the non-linear regime, both results present the same behavior, the eletrostatic energy obtained by both codes are very close. Let us remark that the computationnal time need for *PICLS* is of the same order than the computationnal time need for *KETS*, i.e. less than 5 minutes.

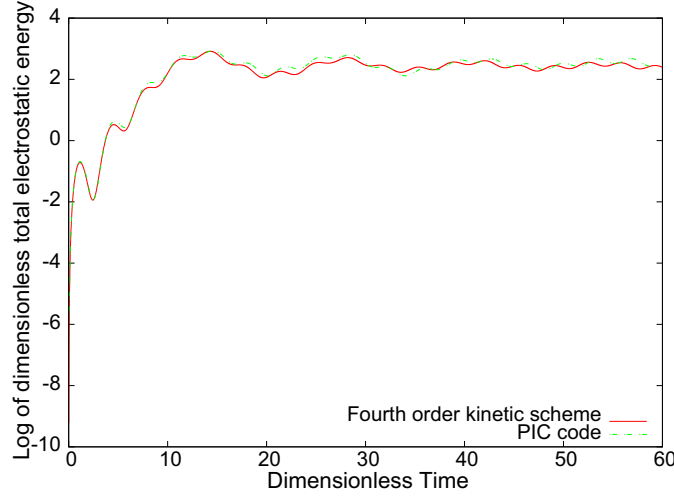


Figure 4.7: Evolution of the total energy with respect to time by using the fourth order kinetic scheme with 64×64 discretization points and the PICLS code.

4.3 Laser-plasma interaction

Although the physics of laser-plasma interaction is of a fundamental interest (see [8, 15]), it is the wide range of significant applications of laser produced in plasmas which makes so worthwhile. Notable among these applications is ICF. Indeed the deposited energy to heat the target of D-T via direct drive approach leads to a large variety of phenomena and mainly the energy is brought to plasma with a laser. In this section, the interaction of a laser with a plasma is studied in collisionless regime. The laser wave penetrates into a plasma only if the electron density does not exceed the density described as "critical". This critical density is given by

$$n_c = \frac{\epsilon_0 m_e}{e^2} \cdot \frac{4\pi^2 c^2}{\lambda_{\text{laser}}^2}, \quad (4.3)$$

where $\lambda_{\text{laser}} = \frac{2\pi c}{\omega_{\text{laser}}}$ is the laser wavelength and ω_{laser} is the pulse of the laser wave. In laser plasma physics, the critical density is given by

$$n_c (\text{cm}^{-3}) = \frac{1.1 \times 10^{21}}{\lambda_{\text{laser}}^2 (\mu\text{m}^2)}. \quad (4.4)$$

The propagation of an electromagnetic wave in a hot plasma follows the Maxwell equations and does not depend on the collisions between particles. The plasma is considered as neutral, that is to say $\rho = 0$. Furthermore, only electrons contribute to the current because of the inertia of the ions which are set more difficultly in motion by the wave. So we can assume that the velocity of ions is null and thus consider our kinetic model. For this test case, the laser wavelength scaling

chosen is explained in Annex 4.8. In these units, the Vlasov-Maxwell system becomes

$$\left\{ \begin{array}{l} \frac{\partial f}{\partial t} + \nabla_{\mathbf{x}} \cdot (\mathbf{v}f) - \nabla_{\mathbf{v}} \cdot ((\mathbf{E} + \mathbf{v} \times \mathbf{B})f) = 0, \\ \frac{\partial \mathbf{E}}{\partial t} - \nabla_{\mathbf{x}} \times \mathbf{B} = (2\pi)^2 n \mathbf{u}, \\ \frac{\partial \mathbf{B}}{\partial t} + \nabla_{\mathbf{x}} \times \mathbf{E} = 0, \\ \nabla_{\mathbf{x}} \cdot \mathbf{E} = \left(\frac{\omega_{pe}}{\omega_{laser}} \right)^2 (1 - n), \\ \nabla_{\mathbf{x}} \cdot \mathbf{B} = 0, \end{array} \right. \quad (4.5)$$

where the density and the current are expressed such as

$$n(t, \mathbf{x}) = \int_{\mathbb{R}^3} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v} \quad \text{and} \quad \mathbf{u}(t, \mathbf{x}) = \frac{1}{n} \int_{\mathbb{R}^3} \mathbf{v} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}.$$

The test case consists in analysing the interaction of a laser wave with a plasma which density n_e varies between $0.5 n_c$ and $1.5 n_c$. We consider a box split in two parts, the first part with vacuum and the second one with a plasma as shown in figure 4.8. The physical processes of this study are described in three parts. First, the propagation of the laser in vacuum is examined, then the expansion of plasma in vacuum and finally the laser-plasma interaction as shown on figure 4.8.

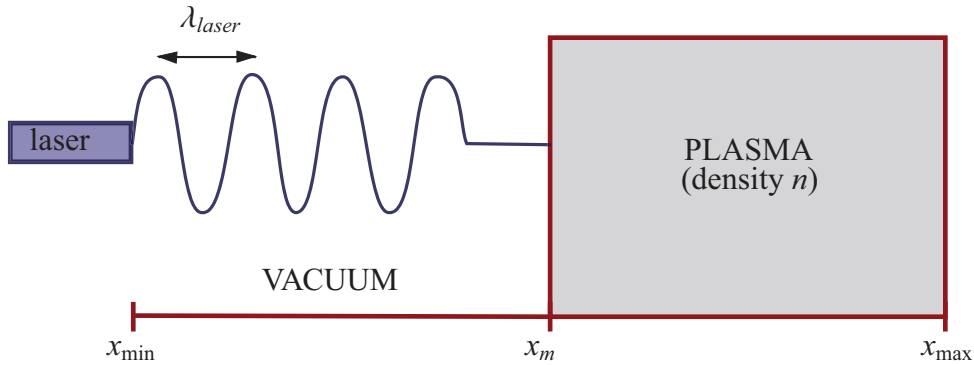


Figure 4.8: *Laser-plasma interaction.*

4.3.1 Laser propagation in vacuum

Propagation of a laser wave is an important process which impacts on laser-plasma interaction. This is why we turn now to study the propagation of a laser wave in vacuum before studying

the interaction with a plasma.

In numerical modeling of the shock ignition process, it is very important to have a wave which goes out of the simulation domain without being reflected. Definition of non reflecting boundary conditions is our goal. In this subsection, the appropriate boundary conditions for the use of the centered numerical interpolation for electromagnetic fields are presented. Then some test cases on the propagation of a laser wave illustrate the good behaviour of the scheme.

Characteristic method and numerical scheme

In the scaling detailed in Annex 4.8, electric and magnetic fields ($\mathbf{E}$, $\mathbf{B}$) are given by

$$\begin{cases} \frac{\partial \mathbf{E}}{\partial t} - \nabla_{\mathbf{x}} \times \mathbf{B} = -(2\pi)^2 \mathbf{j}, \\ \frac{\partial \mathbf{B}}{\partial t} + \nabla_{\mathbf{x}} \times \mathbf{E} = 0, \end{cases} \quad (4.6)$$

The phase space considered here consists in one dimension space domain and three dimensions velocity domain. The electromagnetic field has three components: E_x , E_y and B_z . In this configuration, system (4.6) writes

$$\begin{cases} \partial_t E_x = -(2\pi)^2 j_x, \\ \frac{\partial}{\partial t}(E_y + B_z) + \frac{\partial}{\partial x}(E_y + B_z) = -(2\pi)^2 j_y, \\ \frac{\partial}{\partial t}(E_y - B_z) + \frac{\partial}{\partial x}(E_y - B_z) = -(2\pi)^2 j_y. \end{cases} \quad (4.7)$$

By setting $Ep = (E_y + B_z)$ and $Em = (E_y - B_z)$, system (4.7) can be simplify into

$$\begin{cases} \partial_t E_x = -(2\pi)^2 j_x, \\ \frac{\partial Ep}{\partial t} + \frac{\partial Ep}{\partial x} = -(2\pi)^2 j_y, \\ \frac{\partial Em}{\partial t} + \frac{\partial Em}{\partial x} = -(2\pi)^2 j_y. \end{cases} \quad (4.8)$$

In the same way as (3.39), a centered formulation is chosen to discretize the temporal derivative which gives

$$(\partial_t Ep_i)^{n+\frac{1}{2}} = \frac{Ep_i^{n+1} - Ep_i^n}{\Delta t}.$$

By using a space discretization with a finite difference method to first order, space derivatives can be expressed as

$$(\partial_x Ep_i)^n = \frac{Ep_{i+1}^n - Ep_{i-1}^n}{2\Delta x}. \quad (4.9)$$

By reducing (4.9) at time $n + \frac{1}{2}$, we obtain

$$(\partial_x E p_i)^{n+\frac{1}{2}} = \frac{1}{2} \left(\frac{E p_{i+1}^{n+1} - E p_{i-1}^{n+1}}{2\Delta x} + \frac{E p_{i+1}^n - E p_{i-1}^n}{2\Delta x} \right) = \frac{1}{2} \left(\frac{E p_{i+1}^{n+\frac{1}{2}} + E p_{i-1}^{n+\frac{1}{2}}}{\Delta x} \right), \quad (4.10)$$

where

$$E p^{n+\frac{1}{2}} = \frac{1}{2} (E p^{n+1} + E p^n).$$

The discretization of the system (4.8) by impliciting fields as in (4.10) gives

$$\begin{cases} \frac{E_{x,i}^{n+1} - E_{x,i}^n}{\Delta t} = -(2\pi)^2 j_{x,i}^n, \\ \frac{E p_i^{n+1} - E p_i^n}{\Delta t} + \frac{E p_{i+1}^{n+\frac{1}{2}} - E p_{i-1}^{n+\frac{1}{2}}}{2\Delta x} = -(2\pi)^2 j_{y,i}^n, \\ \frac{E m_i^{n+1} - E m_i^n}{\Delta t} + \frac{E m_{i+1}^{n+\frac{1}{2}} - E m_{i-1}^{n+\frac{1}{2}}}{2\Delta x} = -(2\pi)^2 j_{y,i}^n, \end{cases} \quad (4.11)$$

with $E m_i^{n+\frac{1}{2}} = \frac{1}{2} (E m_i^{n+1} + E m_i^n)$ and $i \in [0; n_x]$. System (4.11) can be simplified into

$$\begin{cases} \frac{E_{x,i}^{n+1} - E_{x,i}^n}{\Delta t} = -(2\pi)^2 j_{x,i}^n, \\ E p_i^{n+1} + \frac{\Delta t}{2\Delta x} \left(\frac{E p_{i+1}^{n+1} - E p_{i-1}^{n+1}}{2} \right) = E p_i^n - \frac{\Delta t}{2\Delta x} \left(\frac{E p_{i+1}^n - E p_{i-1}^n}{2} \right) - \Delta t (2\pi)^2 j_{y,i}^n, \\ E m_i^{n+1} - \frac{\Delta t}{2\Delta x} \left(\frac{E m_{i+1}^{n+1} - E m_{i-1}^{n+1}}{2} \right) = E m_i^n + \frac{\Delta t}{2\Delta x} \left(\frac{E m_{i+1}^n - E m_{i-1}^n}{2} \right) - \Delta t (2\pi)^2 j_{y,i}^n. \end{cases} \quad (4.12)$$

Boundary conditions

To introduce a laser wave from the left boundary, characteristics into the box are given by

$$\begin{cases} E p_{-1}^{n+1} = 2E_0 \sin(\omega_{laser} t), \\ E m_{-1}^{n+1} = E m B_0^{n+1}. \end{cases}$$

with an electric and magnetic amplitude given by $E_0 = B_0$. In order to have no reflected wave at the boundary, characteristics out to the box are expressed as

$$\begin{cases} E p_{n_x}^{n+1} = E p_{n_x-1}^{n+1}, \\ E m_{n_x}^{n+1} = 0. \end{cases}$$

A first order upwind method will be taken into account to describe boundary. For the left boundary, system (4.12) writes

$$\begin{cases} Ep_0^{n+1} \left(1 + \frac{\Delta t}{2\Delta x} \right) = Ep_0^n \left(1 - \frac{\Delta t}{2\Delta x} \right) + \frac{\Delta t}{\Delta x} Ep_{-1} - \Delta t (2\pi)^2 j_{y,0}^n, \\ Em_0^{n+1} - \frac{\Delta t}{\Delta x} \left(\frac{Em_1^{n+1} - Em_0^{n+1}}{2} \right) = Em_0^n + \frac{\Delta t}{\Delta x} \left(\frac{Em_1^n - Em_0^n}{2} \right) - \Delta t (2\pi)^2 j_{y,0}^n, \end{cases}$$

and on the right boundary, we get

$$\begin{cases} Ep_{n_x-1}^{n+1} + \frac{\Delta t}{\Delta x} \left(\frac{Ep_{n_x-1}^{n+1} - Ep_{n_x-2}^{n+1}}{2} \right) = Ep_{n_x-1}^n + \frac{\Delta t}{\Delta x} \left(\frac{Ep_{n_x-1}^n - Ep_{n_x-2}^n}{2} \right) - \Delta t (2\pi)^2 j_{y,n_x-1}^n, \\ Em_{n_x-1}^{n+1} \left(1 + \frac{\Delta t}{2\Delta x} \right) = Em_{n_x-1}^n \left(1 - \frac{\Delta t}{2\Delta x} \right) + \frac{\Delta t}{\Delta x} Em_{n_x} - \Delta t (2\pi)^2 j_{y,n_x-1}^n. \end{cases}$$

Propagation of a laser wave in vacuum

To describe the propagation of the laser wave in vacuum, the initial distribution function is zero in the vacuum.

The space domain is discretized with 800 points and its length is given by $L = 10$. The numerical results are computed at time $t = 20$ with the time dimensionned by the laser period. The objective of this test is to demonstrate that the previous scheme propagates laser light with the velocity $v = 1$. Figure (4.9) shows propagation of the laser wave in the vacuum with the numerical scheme of fourth order. The laser wave penetrates the vacuum by the left boundary and propagates through the vacuum. The length of 10 wavelengths has been propagated in the box which dimension is 10 wavelengths. This propagation with velocity $v_x = 1$ translates the good accuracy of the numerical scheme. This propagation is very close to the exact solution. Indeed, the L_2 relative error between numerical and analytical solution is about 6.10^{-3} .

4.3.2 Poisson equation: equilibrium vacuum plasma

This test case is devoted to study the charge separation effects in the collisionless plasma expansion into vacuum with no external magnetic field. Our particular interest is to study the electric field at the plasma-vacuum front which is governed by the equilibrium between the electrostatic potential Φ and the electron density n_e . The amplitude of electric field on the boundary between vacuum and plasma is defined in [9] and given by

$$E_{x,\max} = \frac{T_e}{e \lambda_{De}}, \quad (4.13)$$

where λ_{De} is the electron Debye length. We consider here again $1D_x \times 1D_v$ problem as the velocity components in the plane (y, z) do not contribute to the equilibrium of plasma. Some

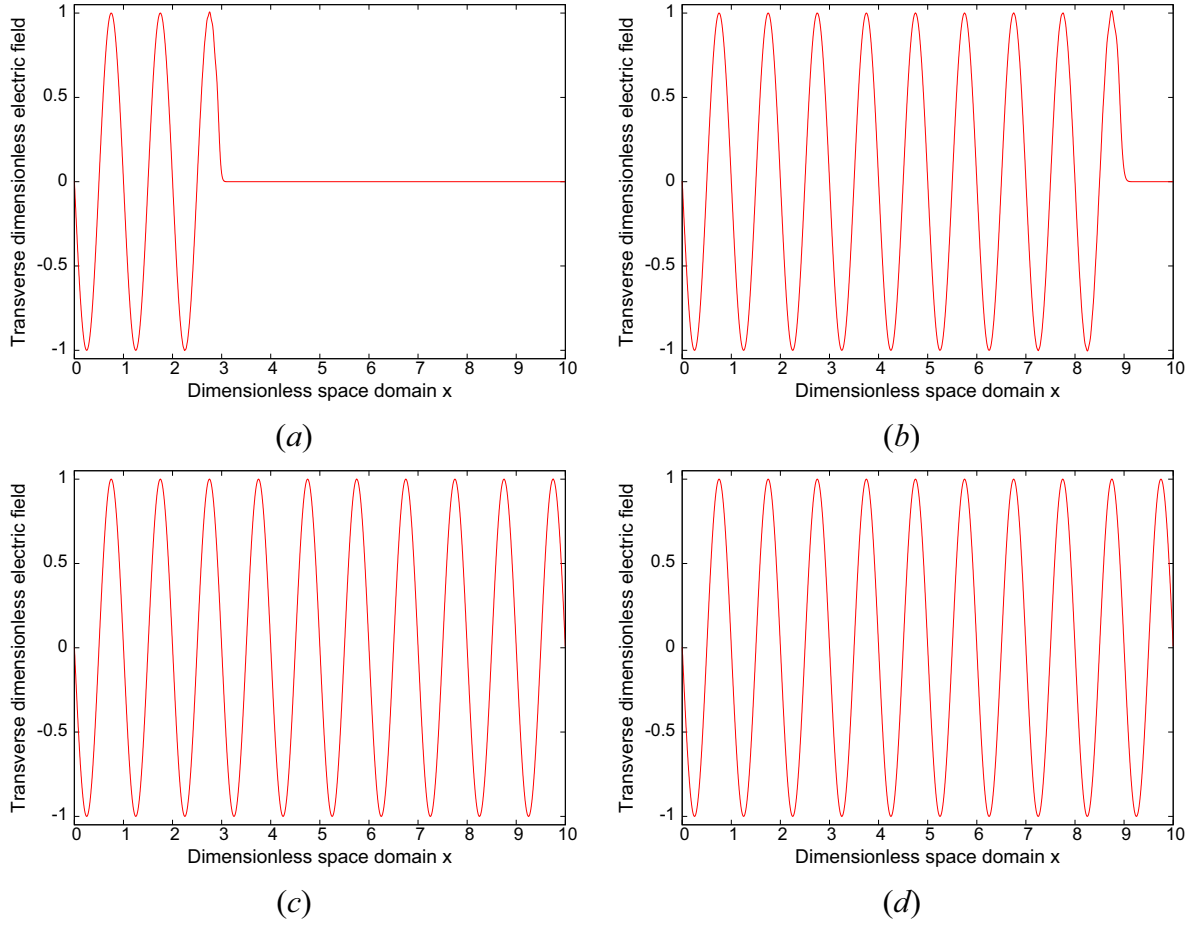


Figure 4.9: Evolution of the transverse electric field E_y by using the fourth order scheme at time $t = 3$ (a), $t = 9$ (b), $t = 15$ (c) and $t = 20$ (d) for 800×32 grid points

	T = 1 KeV	T = 5 KeV	T = 10 KeV
$n_e = 0, 1 n_c$	$-1, 40.10^{-2}$	$-3, 13.10^{-2}$	$-4, 40.10^{-2}$
$n_e = n_c$	$-4, 42.10^{-2}$	$-9, 89.10^{-2}$	$-1, 40.10^{-1}$
$n_e = 10 n_c$	$-1, 40.10^{-1}$	$-3, 13.10^{-1}$	$-4, 42.10^{-1}$

Table 4.2: Values of the electric field $E_{x,\max}$ on the boundary between plasma and vacuum for different electronic density and temperatures.

theoretical values are referenced in table 4.2 for different values of electronic density n_e . The space domain $L = [0; 30]$ is discretized with 5000 points to capture the formation of the electric field at the plasma border. The velocity domain $[-0.5; 0.5]$ normalized by the light velocity along v_x is discretized with 64 points. We choose a time step satisfying the CFL condition. The

scaling used here gives the time step of the order of the laser period (cf Annex 4.8 for more details on the scaling). The numerical results are computed at time $t = 200$.

At time $t = 0$, a plasma is assumed to occupy the two third space i.e. $x > 9$ with a density going from $n_e = 0.1 n_c$ down to $n_e = 10 n_c$ with n_c the critical density. The temperature of the plasma is chosen from $T_e = 1 \text{ KeV}$ to $T_e = 10 \text{ KeV}$. Electric field is supposed to be zero at initial time. Figure 4.3.2 shows the electric field E_x as a function of space at time $t = 150$. The plasma front stands at $x = 9$ where the electric field peaks. Note that the electric field at the plasma front is very close to the theoretical values described in the table 4.2. Indeed, the minimum value of the electric field E_x is greater than theoretical value from only 10%.

4.3.3 Laser-plasma interaction

Now after checking that the laser light is propagating with a good velocity and the vacuum-plasma interface is well described, the laser-plasma interaction can be studied. Let us recall the physical backgrounds of laser light propagation in plasma. Let k_0 be the wave number given by the dispersion relation

$$k_0^2 - \frac{\omega_{\text{laser}}^2}{c^2} = -\frac{\mu_0 n_e e^2}{m_e} = -\frac{\omega_{pe}^2}{c^2}, \quad (4.14)$$

with electron plasma frequency ω_{pe} is expressed as $\omega_{pe} = \sqrt{\frac{n_e e^2}{m_e \epsilon_0}}$.

If ω_{laser} is greater than ω_{pe} , k_0 is real and the wave can propagate in the plasma. However if ω_{laser} is less than ω_{pe} then k_0 is imaginary and the wave in the plasma becomes evanescent and penetrates only a small depth (skin effect).

For this test case, we choose a box divided in two parts. The first one is vacuum and the second one is a plasma where the density varies between $n_e = 0.5 n_c$ and $n_e = 1.5 n_c$. The plasma front is localized at the middle of the box. The space domain $L = [0; 30]$ is discretized with 2000 points to capture the formation of the electric field at the boundary. The velocity domain $[-0.5; 0.5]$ is discretized with 64 points. We choose a time step satisfying the CFL condition. The scaling used here gives the time steps of the order of the laser period (cf Annex 4.8 for more details on the scaling). The numerical results are computed at time $t = 200$.

Inside the box, electric and magnetic fields are initially zero. To simulate the laser propagating from the left side of the box, we chose the same specific boundary conditions as for the propagation of laser in vacuum (subsection 4.3.1) i.e.

$$\begin{cases} E_x(x = -1) = 0, \\ E_y(x = -1) = E_0 \sin(\omega_{\text{laser}} t), \\ B_z(x = -1) = E_y(x = -1), \\ \omega_{\text{laser}} = 1. \end{cases}$$

The amplitude of the electric field E_0 and the magnetic field B_0 is chosen such as

$$E_0 = B_0 = 0.1.$$

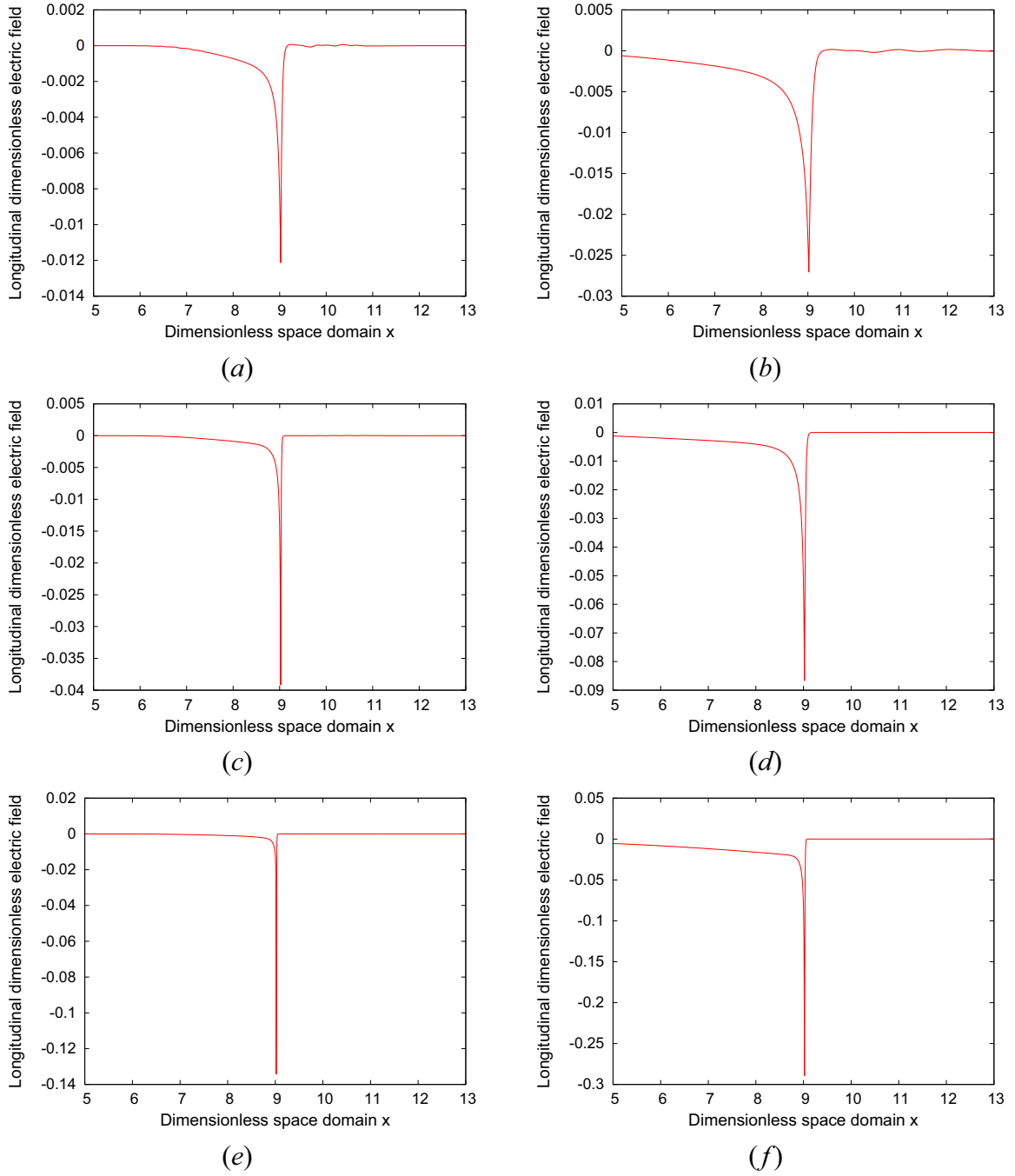


Figure 4.10: Evolution of the Electric field E_x depending on the space domain x with density and temperature such as $n_e = 0.1 n_c$ and $T_e = 1 \text{ KeV}$ (a), $n_e = 0.1 n_c$ and $T_e = 5 \text{ KeV}$ (b), $n_e = n_c$ and $T_e = 1 \text{ KeV}$ (c), $n_e = n_c$ and $T_e = 5 \text{ KeV}$ (d), $n_e = 10 n_c$ and $T_e = 1 \text{ KeV}$ (e), $n_e = 10 n_c$ and $T_e = 5 \text{ KeV}$ (f) for 5000×32 grid points.

These dimensionless values corresponds to a laser with an intensity about $I = 10^{16} \text{ W/cm}^2$ ($I = \frac{1}{2} c \epsilon_0 E_0^2$) and laser wavelength defined by $\lambda_{\text{laser}} = 0,35.10^{-6} \text{ m}$.

Figure 4.11 represents the laser propagation from the vacuum to the plasma with a density $n_e = 0,9 n_c$. The numerical results are shown at the time from $t = 20$ to $t = 200$. At time $t = 20$, the laser wave did not interact with the plasma yet. The propagation is just effective in vacuum and the laser wavelength corresponds to the laser wavelength choosen. At time $t = 35$, the laser begins to interact with the plasma. The laser amplitude inside the plasma increases because refractive index $N = \sqrt{1 - \frac{n_e}{n_c}}$ is less than 1.

Indeed as according to (4.14) the laser wave number k in plasma decreases and is given by

$$k = k_0 * N, \quad (4.15)$$

which can be rewritten as

$$\lambda_{\text{laser}} = N * \lambda_{\text{laser/plasma}}, \quad (4.16)$$

where $\lambda_{\text{laser/plasma}}$ stands for the laser wavelength in the plasma. Moreover the conservation of the Poyntig vector $kE^2 = c^{te}$ implies that the laser amplitude increases and it can be expressed as

$$E = \frac{E_0}{\sqrt{N}}. \quad (4.17)$$

This is a linear approach and it is valid in a strongly under-dense plasma.

If the density plasma is close to the critical density, this approach is not sufficiently accurate as it does not take into account some effects such as plasma heating weakly. That is why, higher is the density and worst is this approach. For the density $n_e = 0,9 n_c$, the laser wavelength in the plasma is given by $\frac{\lambda_{\text{laser/plasma}}}{\lambda_{\text{laser}}} = 2,33$ which is different from the theoretical value by 26,3%.

Besides we can note a partial reflexion from the plasma boundary. This reflexion comes from the creation of the eletromagnetic fields at the plasma boundary.

Figure 4.12 shows the propagation of the laser wave at time $t = 100$ for different values of the plasma density from $n_e = 0,1 n_c$ to $n_e = 2 n_c$. For plasma with a low density, the interaction with the laser is not significant compared to over-dense plasma. Indeed, for $n_e = n_c$, the laser wavelength penetrates only the plasma on small distance and becomes evanescent. In the case with a density $n_e = 2 n_c$ the laser does not penetrate the plasma. This behavior for the two last cases can be explained by the fact that ω_{laser} is less than ω_{pe} .

Values of the ratio $\frac{\lambda_{\text{laser/plasma}}}{\lambda_{\text{laser}}}$ for different densities are detailed in table 4.3. This value is compared with the approximation (4.16) in the second line of this table. For a strongly under dense plasma, the results are very close to the approximation (4.16) whereas for plasma density close to the critical density, results are different. The phenomenon can be explained by the fact that the previous approximation is exact for plasma which density is very low. Indeed, the approach do not consider effects associated to high density such as heating effect. Moreover the coupling between the plasma and the laser increases for density close to critical density. That is why results are different for high density.

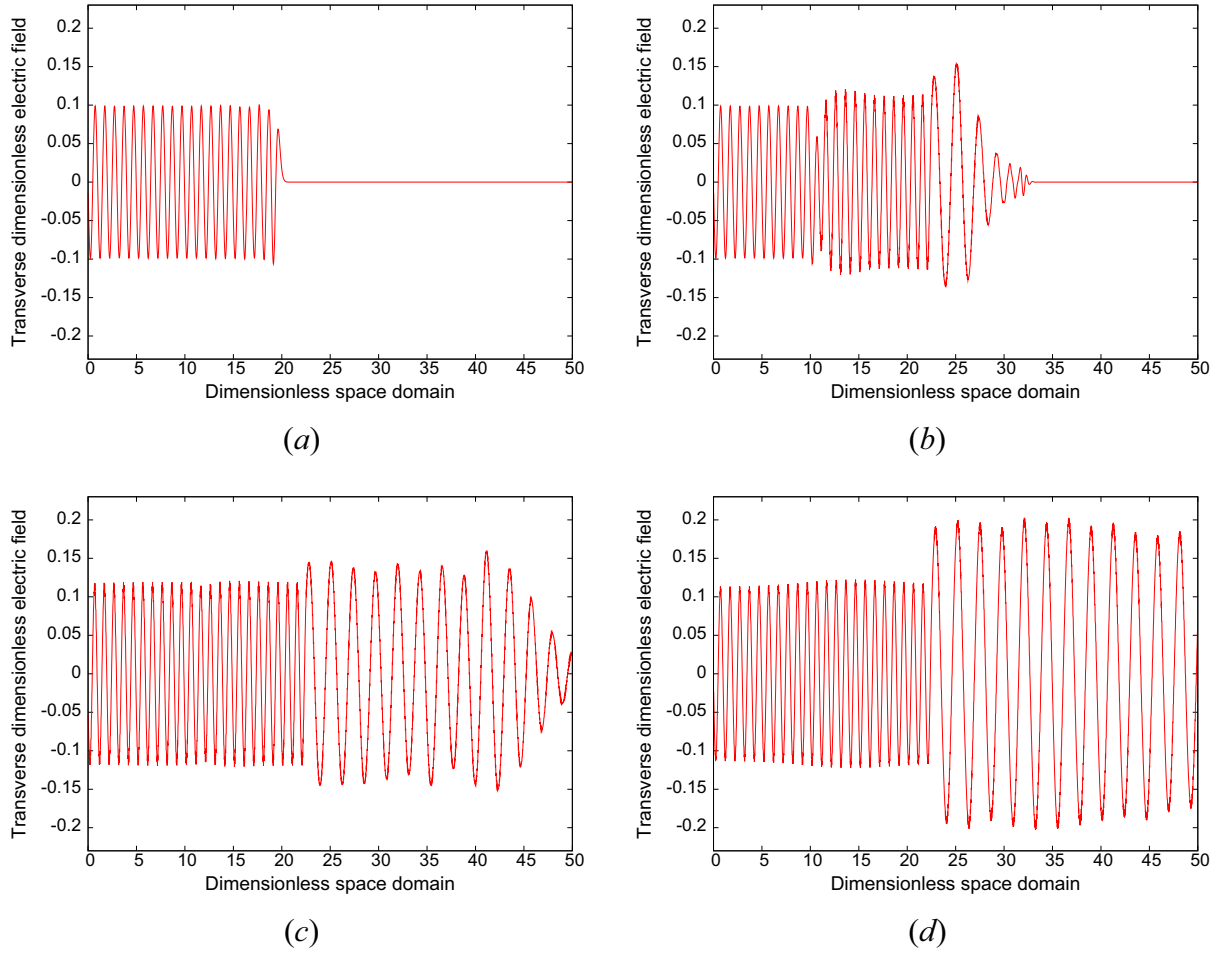


Figure 4.11: *Evolution of the transverse electric field E_y in the space domain x at time $t = 20$ (a), $t = 35$ (b), $t = 80$ (c) and $t = 200$ (d) for 2000×64^3 grid points and a density plasma of $n = 0.9 n_c$.*

	$0,05 n_c$	$0,1 n_c$	$0,2 n_c$	$0,3 n_c$	$0,4 n_c$	$0,5 n_c$	$0,9 n_c$
$\frac{\lambda_{\text{laser/plasma}}}{\lambda_{\text{laser}}}$	1	1,01	1,03	1,08	1,11	1,15	2,33
Relative error	2,5%	3,8%	7,7%	9,2%	14,7%	18,4%	26,3%

Table 4.3: *Value of the ratio between the laser wavelength in the plasma and the laser wavelength in vacuum and value of the relative error with the approximation (4.16) for different values of plasma density n_e .*

4.4 Application for the electron energy transport: the Batishchev test

In order to check the implementation of collisional terms in the electron kinetic equation, we consider the non-local transport theory, i.e. a regime where the characteristic scale of the prob-

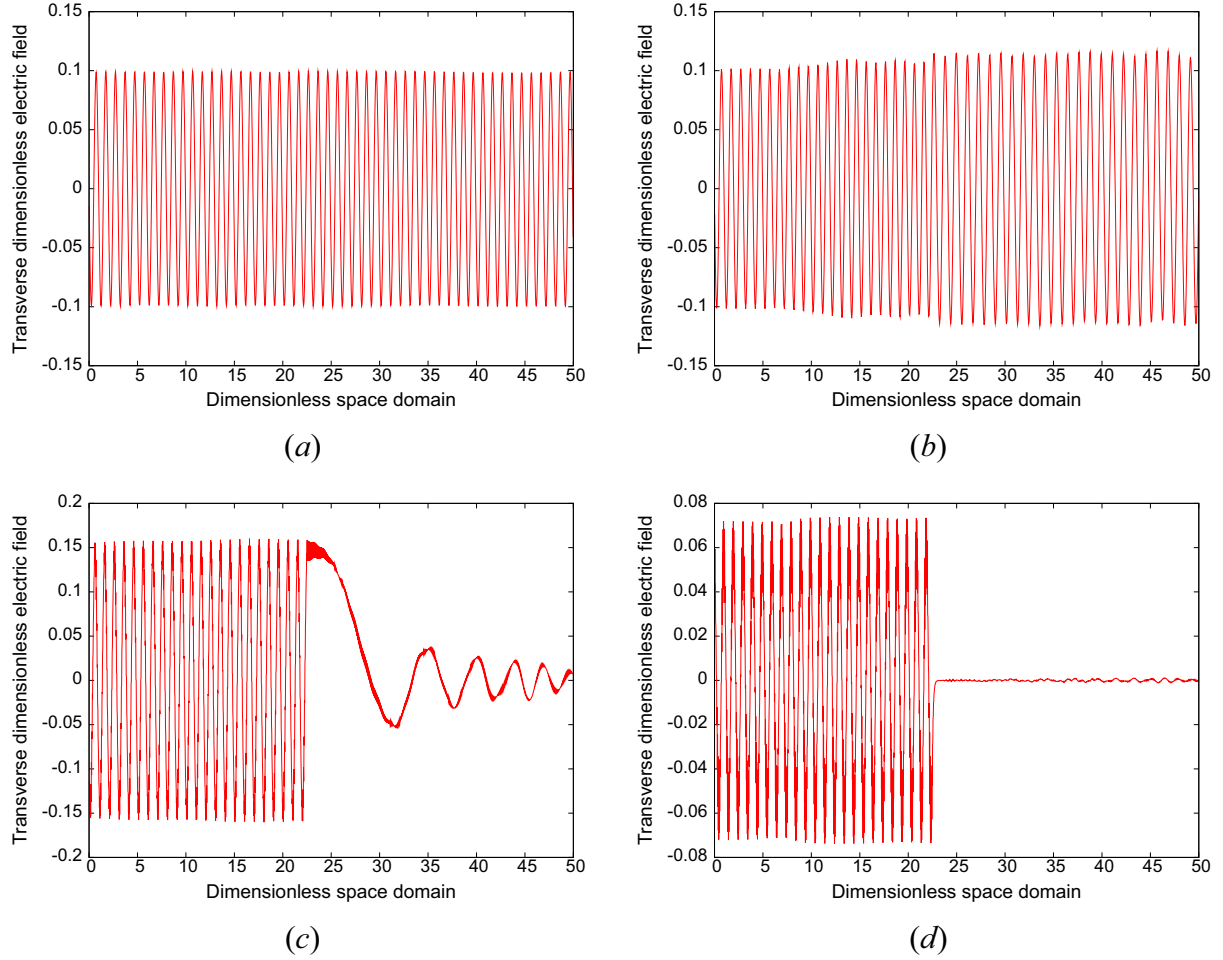


Figure 4.12: Evolution of the transverse electric field E_y in the space domain x at time $t = 200$ for $n_e = 0, 1 n_c$ (a), $n_e = 0, 5 n_c$ (b), $n_e = n_c$ (c) and $n_e = 2 n_c$ (d) for 2000×64^3 grid points.

lem is comparable with the electron mean free path. The relaxation of an initially localized temperature enhancement is studied here. This problem has been studied by Batishchev in [1, 4] for the planar geometry and by Senecha *et al.* [11] for cylindrical geometry. This test problem corresponds to the localized heating of a plasma over a short time period and the subsequent relaxation of the temperature. Steep temperature gradients due to the localized heating by nonuniform laser beams give rise to non-local heat transport. We consider here the collisional regime defined by the Maxwell-Fokker-Planck-Landau system. We choose the scaling defined in Annex 4.9 to use a time step of the order of the electron-ion collision time τ_{ei} . Thus in these

units, the Maxwell-Fokker-Planck-Landau system is given by

$$\begin{cases} \frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} - \nabla_v \cdot (\mathbf{E}f) = \frac{1}{Z} C_{e,e}(f, f) + C_{e,i}(f), \\ \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\alpha^2} n \mathbf{u}, \\ \frac{\partial E_x}{\partial x} = \frac{1}{\alpha^2} (1 - n), \end{cases} \quad (4.18)$$

where $\alpha = \frac{v_{e,i}}{\omega_{pe}}$ and $\beta = \frac{v_{th}}{c}$. The collision terms $C_{e,e}(f, f)$ and $C_{e,i}(f)$ are given in (4.37).

Initially in this test case, electrons are represented by a local Maxwellian distribution function with a Gaussian temperature profile (see figure 4.13) given by

$$T_e(x, t = 0) = T_0 + T_1 \exp\left(-\frac{x^2}{D^2}\right), \quad (4.19)$$

with $T_0 = 1 \text{ KeV}$, $T_1 = 3 \text{ KeV}$ and $D = 14, 25 \lambda_{ei}$. We choose here the boundary condition corresponding to specular reflection. The spatial scale length L (the hot spot size) is varied over

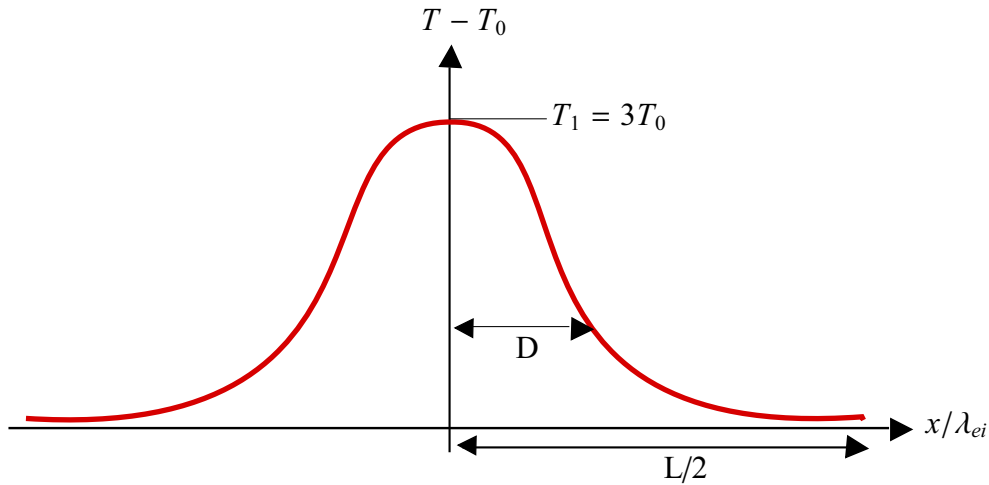


Figure 4.13: *Spatial distribution of the electron temperature*

a range from $L = 80 \lambda_{ei}$ down to $L = -80 \lambda_{ei}$. This space domain is discretized with 160 points to obtain one cell for one mean free path. The velocity domain $[v_{i,min}; v_{i,max}] = [-8; 8]$ along each direction i of $\mathbf{v}$ is discretized with 32 points. We choose a time step satisfying the CFL condition. The numerical results are computed at time $t = 10$.

40 processors were used for this simulation (CINES facility). The domain decomposition on the space domain allows each processor to deal only with 4 points in space. The typical run

time is about 8 hours for $t = 10$ collision times, with this set of parameters.

Figure 4.14 (a) shows the electron temperature evolution. After few iterations, at time $t = 2 \tau_{ei}$, the temperature profile begins to relax to a colder temperature. Until time $t = 10 \tau_{ei}$, the electron temperature of the plasma relaxes to a colder profile. The numerical scheme traduces the good behavior of the relaxation of the hot spot. Moreover hot spot relaxation is characterized by small density perturbations. On figure 4.14 (b), the variation of the density is plotted. The density indeed fluctuates weakly. On figure 4.15 (a), the dimensionless transverse electric field

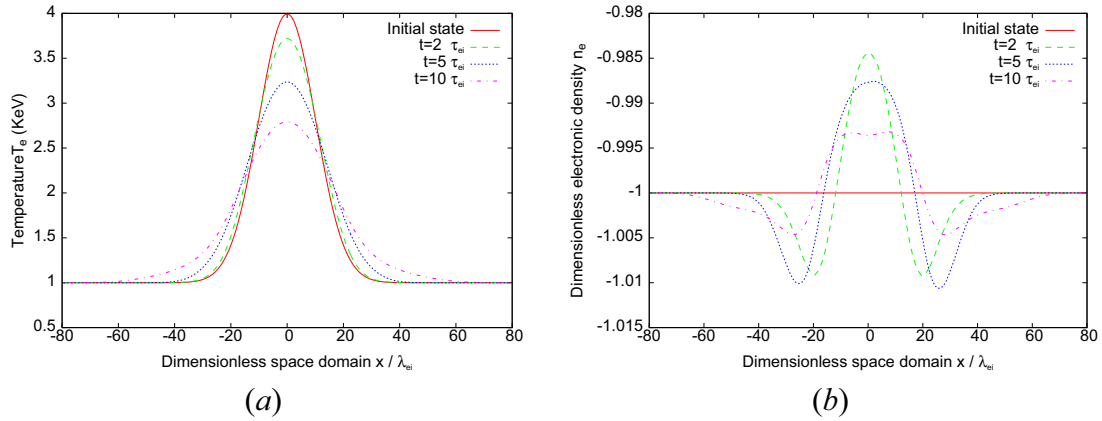


Figure 4.14: Evolution of electronic temperature T_e (KeV) (a) and dimensionless electronic density n_e (b) over a time from $t = 0$ down to $t = 10 \tau_{ei}$ for 160×32^3 grid points.

E_x is represented as a function of time. From system (4.18), the electric field is given by

$$-eE - \frac{1}{n_e} \nabla P_e = 0, \quad (4.20)$$

with P_e the electron pressure. With a constant density n_e , equation (4.20) can be simplified and the electric field writes

$$E = -\frac{1}{e} \nabla T_e. \quad (4.21)$$

Figure 4.15 (a) indeed shows that electric field extremas are localized at the positions where temperature gradients are present. This behavior is consistent with equation (4.21). This configuration leads to the the creation of a heat flux. The flux density of heat carried by particles is given by the vector

$$\mathbf{q} = \frac{m_e n_e}{2} \int_{\mathbb{R}^3} (\mathbf{v} - \mathbf{u})^2 \mathbf{v} f d\mathbf{v}, \quad (4.22)$$

where $\mathbf{u}$ represents the relative velocity of electrons. The evolution of the heat flux is represented on figure 4.15 (b) and shows the same properties as the electric field. These behavior of electric field, temperature or heat flux are consistent with results presented in [1]. Indeed results have the same behaviour than in [1]. Results are not exactly the same because initial configuration

is slightly different than Batishchev's configuration used in [1]. Our configuration has been chosen to be used for shock ignition application, one of the new approach in ICF. This test case

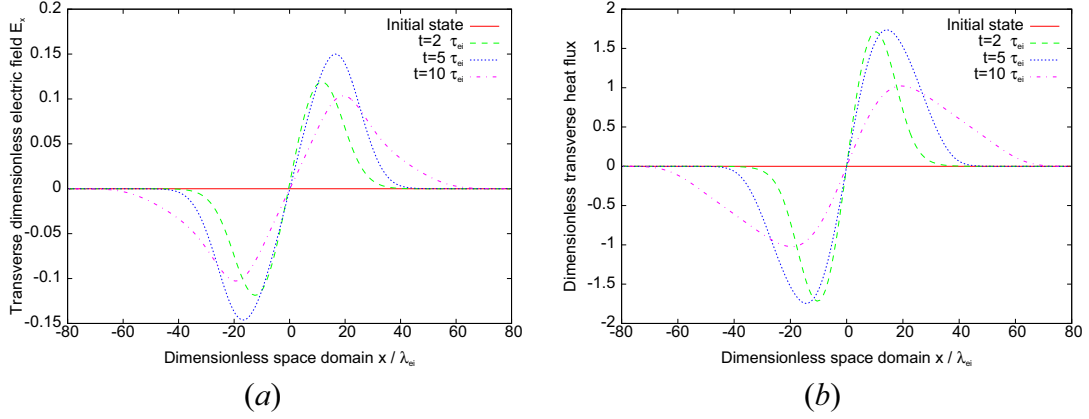


Figure 4.15: *Evolution of the dimensionless transverse electric field E_x (a) and dimensionless heat flux Q (b) over a time from $t = 0$ down to $t = 10 \tau_{ei}$ for 160×32^3 grid points.*

puts forward the suitable use of the code *KETS* in the non-local regime. In the next subsection, the use of the numerical schemes described previously is extended to the local regime. This simulation is subject to some comparisons with hydrodynamical models based on [10].

4.5 Validation for the electron transport in the collisional local regime

In order to validate the numerical modeling electron transport in the collisional local regime, a simple gradient temperature configuration is proposed here. The test case represents the moment during which a laser deposits its energy on the hot temperature side and the absorbed energy is transported to the cold temperature side as shown on figure 4.16. This configuration leads to the the creation of heat flux which contribute to the preheating of the region down to the temperature gradient. An electric current and electric field are generated by the movement of particle. We propose here to compare the heat flux $\mathbf{Q}_{FP}$ and electric field $\mathbf{E}_{FP}$ computed from the numerical solution of the Fokker-Planck-Landau dimensionless equation and Maxwell system, with those obtained analytically. Dimensionless equations are given by the scaling defined

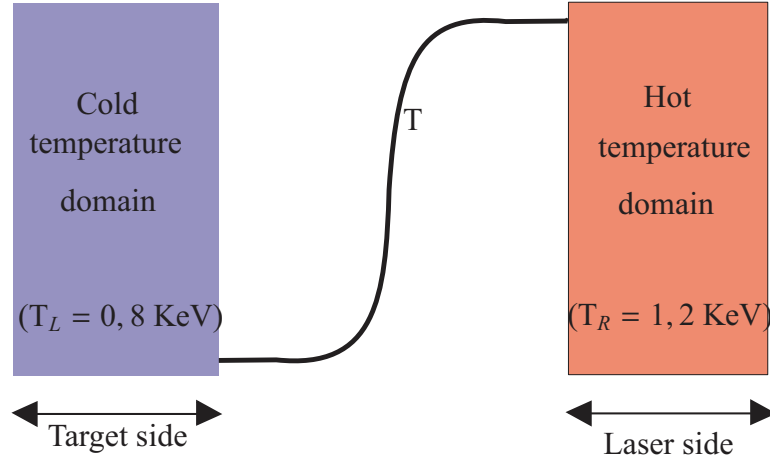


Figure 4.16: Representation of the temperature gradient in local regime.

in Annex 4.9. In these units, Maxwell-Fokker-Planck-Landau system is given by

$$\begin{cases} \frac{\partial f}{\partial t} + \nabla_{\mathbf{x}} \cdot (\mathbf{v}f) - \nabla_{\mathbf{v}} \cdot ((\mathbf{E} + \mathbf{v} \times \mathbf{B})f) = \frac{1}{Z} C_{e,e}(f, f) + C_{e,i}(f), \\ \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\alpha^2} n \mathbf{u}, \\ \nabla_{\mathbf{x}} \cdot \mathbf{E} = \frac{1}{\alpha^2} (1 - n), \end{cases} \quad (4.23)$$

where $\alpha = v_{e,i}/\omega_{pe}$ and $\beta = v_{th}/c$. The collision terms $C_{e,e}(f, f)$ and $C_{e,i}(f)$ are given in (4.37). To describe the behavior of particles at the boundary, the diffusive reflection was assumed. An analytical solution for the electron heat flux were computed from a kinetic model by Braginskii and the Ridgers theory ([3]).

First, we define the average over velocity for a microscopic quantity $A(\mathbf{v})$ as

$$\langle A \rangle = \frac{1}{n_e} \int_{\mathbb{R}^3} A f d\mathbf{v}, \quad (4.24)$$

where $n_e(t, \mathbf{x}) = \int_{\mathbb{R}^3} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}$ is the density of electrons.

The basic macroscopic quantities used in the Braginskii theory are given by

$$\begin{cases} \mathbf{j} = q_e n_e \langle \mathbf{v} \rangle \text{ (electric current),} \\ \mathbf{q} = \frac{1}{2} m_e n_e \langle (\mathbf{v} \cdot \mathbf{v}) \mathbf{v} \rangle \text{ (electron heat flux),} \\ \mathbf{R} = \int_{\mathbb{R}^3} m_e \mathbf{v} C_{e,i}(f) d\mathbf{v} \text{ (electron-ion friction force),} \end{cases} \quad (4.25)$$

and

$$\left\{ \begin{array}{l} p = n_e T_e = \frac{1}{3} m_e n_e \langle (\mathbf{v} - \langle \mathbf{v} \rangle) \cdot (\mathbf{v} - \langle \mathbf{v} \rangle) \rangle \text{ (electron pressure),} \\ \mathbf{\Pi} = \frac{1}{3} m_e n_e \langle (\mathbf{v} - \langle \mathbf{v} \rangle) \otimes (\mathbf{v} - \langle \mathbf{v} \rangle) \rangle - p \mathbf{I} \text{ (stress tensor),} \\ \mathbf{q}_{\text{loc}} = \frac{1}{2} m_e n_e \langle [(\mathbf{v} - \langle \mathbf{v} \rangle) \cdot (\mathbf{v} - \langle \mathbf{v} \rangle)] (\mathbf{v} - \langle \mathbf{v} \rangle) \rangle \text{ (diffusion heat flux).} \end{array} \right. \quad (4.26)$$

There, $\mathbf{j}$ is the electric current, $\mathbf{q}$ the total heat flow, $\mathbf{R}$ the friction force accounting for the transfer of momentum from ions to electrons in collisions, T_e is the temperature, p is the scalar intrinsic pressure, $\mathbf{\Pi}$ is the stress tensor, $\mathbf{q}_{\text{loc}}$ is the intrinsic heat flow and $\mathbf{I}$ the unit diagonal tensor.

Quantities p , $\mathbf{\Pi}$ and $\mathbf{q}_{\text{loc}}$ are defined in the local reference frame of the electrons, whereas $\mathbf{j}$, $\mathbf{q}$ and $\mathbf{R}$ are defined relative to the ion center of mass frame. Ions are supposed to be at rest.

$\mathbf{q}_{\text{loc}}$ can be expressed in function of $\mathbf{j}$ and $\mathbf{\Pi}$ and is given by

$$\mathbf{q}_{\text{loc}} = \mathbf{q} + \mathbf{j} \cdot \left(\frac{5}{2} p \mathbf{I} + \mathbf{\Pi} \right) / (n_e e) + \frac{m_e \langle \mathbf{v} \rangle^2}{2} \frac{\mathbf{j}}{e}. \quad (4.27)$$

The term $\mathbf{q}$ represents the diffusive flux, the second term is the product of the velocity and the enthalpy and the last term is the product of the velocity and the kinetic energy.

Braginskii transport theory

In [6], the anisotropic pressure tensor $\mathbf{\Pi}$ and the electron viscosity $m_e \langle \mathbf{v} \rangle^2$ are neglected. In this section, we shall not neglect these parameters and verify that they are indeed negligible.

The Braginskii theory expresses the electric field $\mathbf{E}$ in terms of $\nabla_{\mathbf{x}} p$, $\nabla_{\mathbf{x}} T_e$ and $\mathbf{j}$, and $\mathbf{q}$ the heat flux, in terms of $\nabla_{\mathbf{x}} T_e$ and $\mathbf{j}$. Transport relations adopted by Braginskii (the classical Ohm's law and heat flow equation) are given by

$$\left\{ \begin{array}{l} q = -\frac{5}{2} T_e \frac{\mathbf{j}}{e} - \kappa_{ep} \cdot \nabla_{\mathbf{x}} T_e - T_e \beta_{ep} \cdot \frac{\mathbf{j}}{e} - \frac{5}{2} \frac{\mathbf{j} \cdot \mathbf{\Pi}}{n_e e} - m_e \langle \mathbf{v} \rangle^2 \frac{\mathbf{j}}{e}, \\ \mathbf{E} = \frac{1}{en_e} \left(-\nabla_{\mathbf{x}} p + \mathbf{j} \times \mathbf{B} + \frac{\alpha_{ep} \cdot \mathbf{j}}{n_e e} - \beta_{ep} \cdot \nabla_{\mathbf{x}} T_e \right). \end{array} \right. \quad (4.28)$$

In papers [6, 5], terms in $\mathbf{\Pi}$ and $m_e \langle \mathbf{v} \rangle^2$ are neglected. We choose here to keep these terms contrary to R. Ducloux's work in order to evaluate their impact. The total heat flow $\mathbf{q}$ and the electric field $\mathbf{E}$ are expressed, in the formalism of Braginskii, in a three-fold geometry, defined

by the temperature gradient $\nabla_x T_e$ and the magnetic field B_0 . The current $\mathbf{j}$, electric field $\mathbf{E}$ and the anisotropic pressure tensor $\mathbf{\Pi}$ are expressed according equations (4.25, 4.26, 4.28). The stress tensor $\mathbf{\Pi}$ is symmetric, with a zero valued trace. The specific dependence of the system to that direction is accounted by the electrical resistivity α_{ep} , the thermoelectric and thermal conductivity tensors, β_{ep} and κ_{ep} respectively, that are calculated from the kinetic theory.

The strong collisional local limit is considered in this section. The tensors are decomposed in $\parallel$, $\perp$, and $\wedge$ components relative to the magnetic field unit vector and the electric current. Directions denoted with $\parallel$ and $\perp$ are respectively parallel and perpendicular to the magnetic field. Consequently, the parallel and perpendicular components of a vector $\mathbf{u}$ are respectively $u_{\parallel} = \mathbf{b}(\mathbf{u} \cdot \mathbf{b})$ and $u_{\perp} = \mathbf{b} \times (\mathbf{b} \times \mathbf{u})$, where $\mathbf{b}$ is the unit vector in the direction of the magnetic field. The direction defined by the third direction in a direct orthogonal frame is denoted by $\wedge$. In the system (4.28), transport coefficients φ are described with reference to the magnetic field and the driving force $\mathbf{u}$ behind the transport, i.e.

$$\varphi \cdot \mathbf{u} = \varphi_{\parallel} \mathbf{b}(\mathbf{b} \cdot \mathbf{u}) + \varphi_{\perp} \mathbf{b} \times (\mathbf{u} \times \mathbf{b}) \pm \varphi_{\wedge} \mathbf{b} \times \mathbf{u} . \quad (4.29)$$

The heat flux and the electric field can then be rewritten in terms of dimensionless quantities, at a reference point denoted by the subscript R, for the particular 1D geometry of our temperature gradient configuration. In that case, the scaling using a collision frequency is used as previously and shown in Annex 4.9.

We obtain

$$\left\{ \begin{array}{l} q_x = -\frac{5}{2} T_e j_x - \frac{3}{7} \sqrt{\frac{\pi}{2}} \kappa_{ep,\perp}^c \nabla_x T_e^{\frac{7}{2}} - T_e (\beta_{ep,\perp}^c j_x - \beta_{ep,\wedge}^c j_y) - \frac{5}{2} n_e^{-1} \Pi_{1,1} j_x \\ \quad - \frac{5}{2} n_e^{-1} \Pi_{1,2} j_y - \frac{1}{2} j_x < v >^2, \\ q_y = -\frac{5}{2} T_e j_y - \frac{3}{7} \sqrt{\frac{\pi}{2}} \kappa_{ep,\wedge}^c \nabla_x T_e^{\frac{7}{2}} - T_e (\beta_{ep,\perp}^c j_y - \beta_{ep,\wedge}^c j_x) - \frac{5}{2} n_e^{-1} \Pi_{1,2} j_x \\ \quad + \frac{5}{2} n_e^{-1} \Pi_{1,1} j_y - \frac{1}{2} j_y < v >^2, \\ E_x = n_e^{-1} j_y B_z - n_e^{-1} \nabla_x p - \beta_{ep,\perp}^c \nabla_x T_e + \frac{2}{3} \sqrt{\frac{2}{\pi}} T_e^{-\frac{3}{2}} (\alpha_{ep,\perp}^c j_x + \alpha_{ep,\wedge}^c j_y) , \\ E_y = -n_e^{-1} j_x B_z - \beta_{ep,\wedge}^c \nabla_x T_e + \frac{2}{3} \sqrt{\frac{2}{\pi}} T_e^{-\frac{3}{2}} (\alpha_{ep,\perp}^c j_y - \alpha_{ep,\wedge}^c j_x) . \end{array} \right. \quad (4.30)$$

where α_{ep}^c , β_{ep}^c and κ_{ep}^c are the dimensionless transport coefficients which are functions of the ion charge Z . χ denotes here the Hall parameters given by

$$\chi = \omega_e \tau_{ei} ,$$

where ω_c is the electron gyrofrequency and τ_{ei} is the electron-ion collision time. It defined the electrons trajectories.

Dimensionless transport coefficient are defined as

$$\left\{ \begin{array}{l} \text{dimensioned quantities} \rightarrow \text{dimensionless quantities} \\ \alpha_{ep} \left(\frac{\tau_{ei}}{m_e n_e} \right) \Big|_R \rightarrow \alpha_{ep} = \frac{n_e m_e}{\tau_{ei}} \alpha_{ep}^c, \\ \beta_{ep} \rightarrow \beta_{ep} = \beta_{ep}^c, \\ \kappa_{ep} \left(\frac{m_e}{n_e \tau_{ei} T_e} \right) \Big|_R \rightarrow \kappa_{ep} = \frac{n_e T_e \tau_{ei}}{m_e} \kappa_{ep}^c. \end{array} \right. \quad (4.31)$$

$$\text{where } \tau_{ei} = \frac{12\pi^2 \epsilon_0^2 \sqrt{m_e} T_e^{\frac{3}{2}}}{\sqrt{2\pi} n_i Z^2 e^4 \ln \Lambda} = \frac{3}{2} \sqrt{\frac{\pi}{2}} \frac{1}{v_{ei}}.$$

In the system (4.30), all collision frequencies are chosen with an homogeneous, constant Coulomb logarithm. This does not alter the accuracy of the analytical solution since the logarithmic approximation has already been performed.

4.5.1 Electron transport in the local regime

Here, a configuration with small Knudsen parameters $Kn \ll 1$ is considered to obtain local regime. This test will underline the accuracy of the collision operator in highly collisional regime. The electron transport will be studied in a plasma with a cold temperature T_L at the left end and a hot temperature T_R at the right end. Let us denote x_L and x_R such as the leftmost and rightmost point of the domain. The initial gradient temperature profile T_e is given by a third order polynomial in $x - x_m$ where x_m stands for the middle of the gradient temperature given by $x_m = \frac{x_L + x_R}{2}$. The coefficient of this polynomial function such that it verify on boundaries

$$\left\{ \begin{array}{l} T_e(x_L) = T_L, \\ T_e(x_R) = T_R, \\ \frac{\partial T_e}{\partial x}(x_L) = \frac{\partial T_e}{\partial x}(x_R) = 0. \end{array} \right. \quad (4.32)$$

Moreover, on the middle point x_m , T_e is defined by

$$\left\{ \begin{array}{l} T_e(x_m) = \frac{T_R + T_L}{2}, \\ \frac{\partial T_e}{\partial x}(x_m) = \frac{T_R - T_L}{(x_R - x_L)/\lambda}. \end{array} \right. \quad (4.33)$$

Here, λ is a parameter that determine the initial stiffness of the temperature gradient.

The simulation is performed with the following parameters. The size of the dimensionless domain $L = x_R - x_L = 5400 \lambda_{ei}$ is discretized with 126 points, the size of the velocity domain $[v_{i,min}; v_{i,max}] = [-12, 12]$ for each direction i discretized with 32 points, the ion charge $Z = 1$. The initial value of the magnetic field is $B_z(t = 0) = 1$ and the initial electric field is zero over the domain, i.e. $E_x(t = 0) = E_y(t = 0) = 0$. In this simulation, B_z is constant. Therefore in equations (4.30), the Hall parameter χ is expressed in terms of the dimensionless quantities B_z and T_e as

$$\chi = \frac{3 \sqrt{\pi} B_z T_e^{\frac{3}{2}}}{2 \sqrt{2} Z}.$$

The initial distribution function is a Maxwellian function depending on the local temperature, with a density that is constant over the domain. The initial temperature profile is chosen such as $T_L = 0.8$, $T_R = 1.2$ and $\lambda = 10$ for the stiffness of the gradient. This set of parameters corresponds to the local regime close to the diffusion transport (the Knudsen number is about 1/500). The dimensionless time step size is $\Delta t = 1/1500$. The scaling used is defined in Annex 4.9.

42 processors were used for each simulation (CINES facility). The domain decomposition on the space domain allows each processor to deal only with 3 points in space. The typical run time is 24 hours for $t = 40$ collision times, with this set of parameters.

By using the second order scheme, results are represented in figures 4.17. Maximum electric field $\mathbf{E}_{FP}$ and maximum heat flux $\mathbf{Q}_{FP}$ computed from KETS code are compared with analytical solution given in [3, 6]. We can see that longitudinal macroscopic quantities obtained with the second order scheme is nearly the same as the analytical longitudinal quantities in the order of less than 5% accuracy. Transverse quantities agree with the theory with about 20% accuracy in the asymptotic limit. The oscillations for electric field observed in figure 4.17 are the consequences of the plasma waves excited by the initial condition. These oscillations can be reduced by using a large time steps in the limit of the CFL condition as shown in [5]. We can also note that with the use of the second order scheme, a slight heating effect appears. This means that the second order scheme is a little diffusive.

Besides in [5], Duclous *et al.* propose the same test case without negligible terms, i.e. terms in Π in the definition of heat flux (4.28). By comparing their results with those obtained here, we can remark that negligible terms can be effectively considered as very small. The same behavior of electric field and heat flux is observed by using the third and the fourth order scheme described in chapter 3. Indeed, as you can see on figures 4.18 and 4.19, longitudinal quantities agree with the theory with about 7% and 5% accuracy by using respectively the third and the fourth order scheme. Concerning transverse quantities, it agree with the theory with about 27% and 24% accuracy for the third and the fourth order scheme. Results disagree with

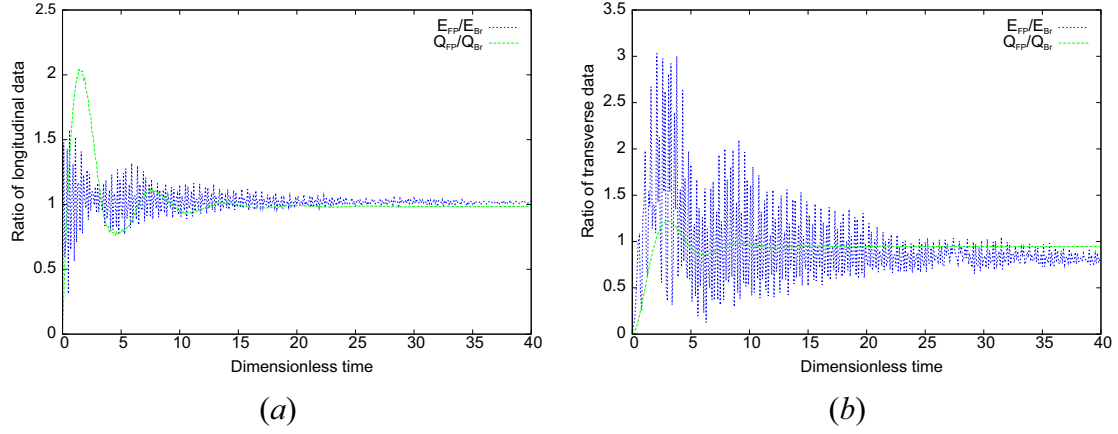


Figure 4.17: Longitudinal (a) and transverse (b) heat flux $\frac{\max_{x_1}(Q_{FP})}{\max_{x_1}(Q_{BR})}$ and electric field $\frac{\max_{x_1}(E_{FP})}{\max_{x_1}(E_{BR})}$ against the dimensionless time for the second order numerical scheme.

those obtained with the two-streams instability test. Indeed, there is no logical in the accuracy of results, mainly with the third order. But we can see that the fourth order gives a better solution than the one obtained with the second order scheme.

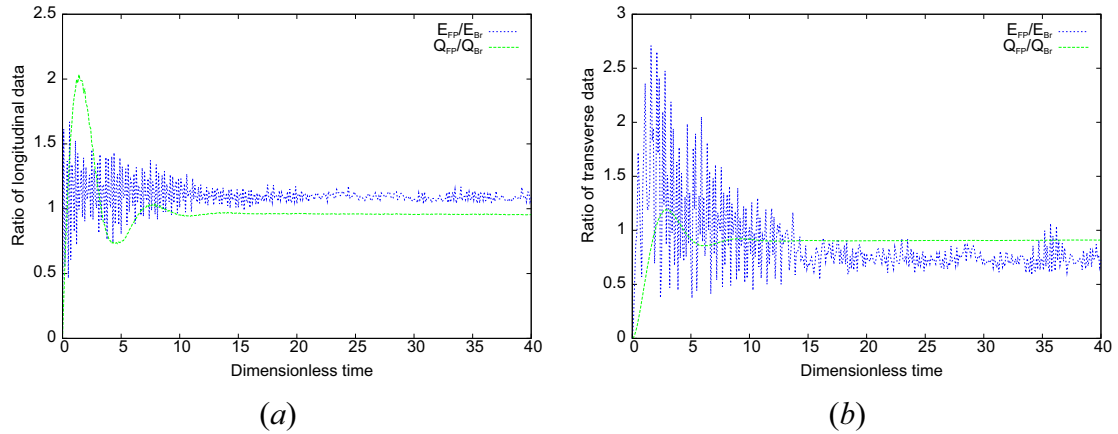


Figure 4.18: Longitudinal (a) and transverse (b) heat flux $\frac{\max_{x_1}(Q_{FP})}{\max_{x_1}(Q_{BR})}$ and electric field $\frac{\max_{x_1}(E_{FP})}{\max_{x_1}(E_{BR})}$ against the dimensionless time for the third order numerical scheme.

We can see that results do not match exactly with the Epperlein solution. It comes from that fields obtained on the boundaries are not very close to the Epperlein field as you can see on the figure 4.20. We can explain this difference by the fact that the Poisson equation is not verified globally on the domain but locally. The use of finite volume scheme means that we do not analyze quantities on the boundaries of cells but at their center, therefore fields obtained on

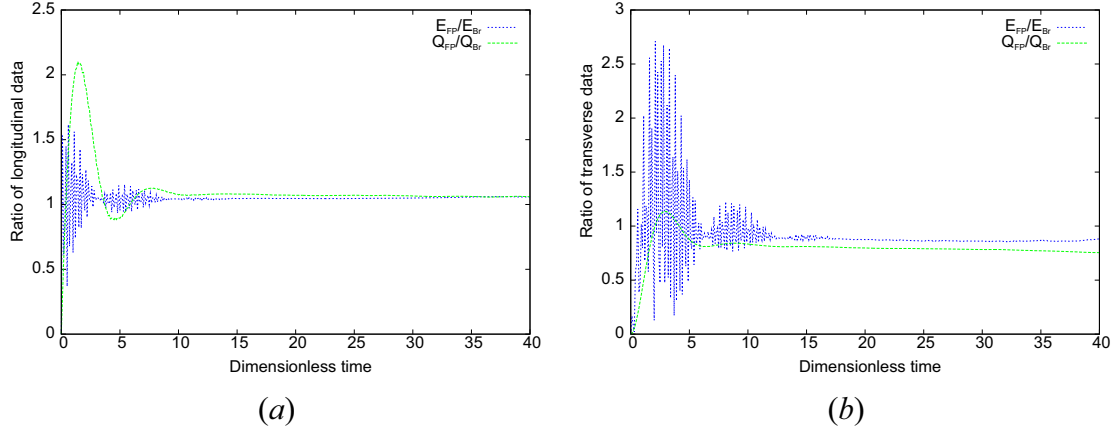


Figure 4.19: Longitudinal (a) and transverse (b) heat flux $\frac{\max_{x_1}(Q_{FP})}{\max_{x_1}(Q_{BR})}$ and electric field $\frac{\max_{x_1}(E_{FP})}{\max_{x_1}(E_{BR})}$ against the dimensionless time for the fourth order numerical scheme.

boundaries are in fact fields obtained with an interpolation of fields calculated on the center of the first boundary. To compensate this error and improve the scheme, a predictor-corrector method described in the article [12] can be used.

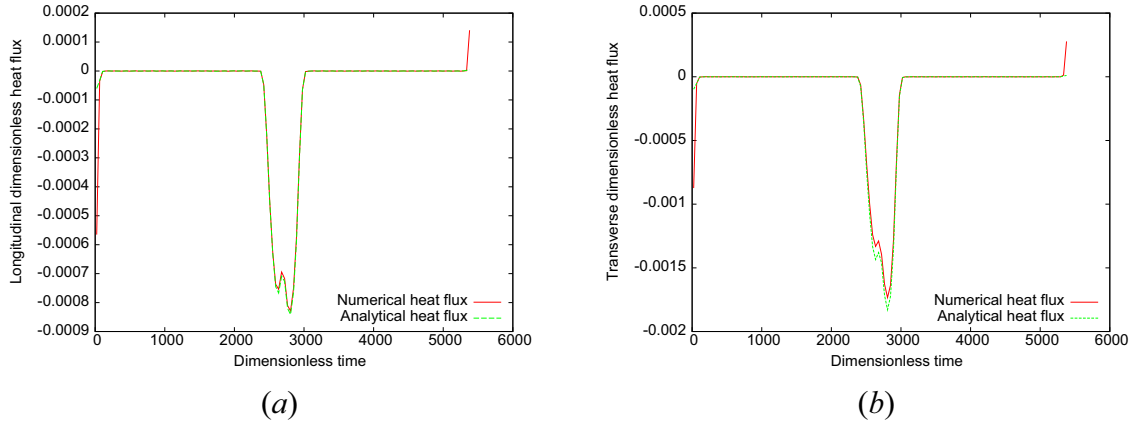


Figure 4.20: Longitudinal (a) and transverse (b) heat flux are shown against the dimensionless time for the fourth order numerical scheme.

4.6 Conclusion

The objectives of this chapter have been achieved. Each numerical scheme has been validated through the different test cases. Indeed, results obtained for the four configurations of ICF

are consistent with our knowledge of physics and reproduce a good behavior. Concerning the efficiency, the third order scheme is more interesting to be used. Indeed, this scheme is a good compromise between time executed by the simulation and accuracy of the numerical solution. The fourth order scheme remains the most accurate. Moreover, this scheme is well adapted for both collisionless situations and the collisional regime. Whatever the regime considered, results obtained are very close to physical results. Nevertheless, the computational time need for each simulation is an important drawback even with the use of massive parallelisation. For example, the typical run time for the collisional local test case is 24 hours for 40 collision times by using the fourth order scheme. Although computer speed and memory are continuously improving, the ability to use kinetic codes is still difficult. To reduce the computational time, a new model is proposed in the next chapter.

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4.7 Appendix A: Scaling with the plasma frequency

We adopt a system of normalized units in order to obtain a dimensionless form of the Vlasov-Maxwell-Fokker-Planck equations. The plasma frequency ω_{pe} , the Debye length λ_{De} , the thermal velocity of electrons v_{th} , and the cyclotron frequency ω_{ce} are defined as follows

$$\omega_{pe} = \sqrt{\frac{n_e e^2}{\epsilon_0 m_e}}, \quad \lambda_{De} = \sqrt{\frac{\epsilon_0 \kappa_B T_0}{n_0 e^2}}, \quad v_{th} = \sqrt{\frac{\kappa_B T_0}{m_e}}, \quad \omega_{ce} = \frac{eB}{m_e}. \quad (4.34)$$

These parameters enable us to define dimensionless parameters marked with a tilde. The time is normalized to the inverse electron plasma frequency ω_{pe}^{-1} , distance to Debye length λ_{De} and velocity to the thermal velocity v_{th} . Electric fields, magnetic field and distribution function are respectively normalized such as

$$\tilde{\mathbf{E}} = \frac{e \mathbf{E}}{m_e v_{th} \omega_{pe}}, \quad \tilde{\mathbf{B}} = \frac{e \mathbf{B}}{m_e \omega_{pe}} = \frac{\omega_{ce}}{\omega_{pe}}, \quad \tilde{f} = f \frac{v_{th}^3}{n_0}. \quad (4.35)$$

In these units, the Fokker-Planck-Landau-Maxwell system becomes

$$\left\{ \begin{array}{l} \frac{\partial f}{\partial t} + \nabla_{\mathbf{x}} \cdot (\mathbf{v} f) - \nabla_{\mathbf{v}} \cdot ((\mathbf{E} + \mathbf{v} \times \mathbf{B}) f) = \frac{\nu}{Z} C_{e,e}(f, f) + \nu C_{e,i}(f), \\ \frac{\partial \mathbf{E}}{\partial t} - \frac{1}{\beta^2} \nabla_{\mathbf{x}} \times \mathbf{B} = n \mathbf{u}, \\ \frac{\partial \mathbf{B}}{\partial t} + \nabla_{\mathbf{x}} \times \mathbf{E} = 0, \\ \nabla_{\mathbf{x}} \cdot \mathbf{E} = (1 - n), \quad \nabla_{\mathbf{x}} \cdot \mathbf{B} = 0, \end{array} \right. \quad (4.36)$$

where $\beta = v_{th}/c$, ν is the ratio between electron-ion collision frequency and plasma frequency

$$\nu = \frac{Z n_0 e^4 \ln \Lambda}{8 \pi \epsilon_0^2 m_e^2 v_{th}^3 \omega_{pe}} = \frac{Z \ln \Lambda}{8 \pi n_0 \lambda_{De}^3} = \frac{\nu_{e,i}}{\omega_{pe}} \quad \text{with} \quad \nu_{e,i} = \frac{Z n_0 e^4 \ln \Lambda}{8 \pi \epsilon_0^2 m_e^2 v_{th}^3}.$$

The zero and first order moments of the distribution function are

$$\left\{ \begin{array}{l} n(t, \mathbf{x}) = \int_{\mathbb{R}^3} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}, \\ \mathbf{u}(t, \mathbf{x}) = \frac{1}{n(t, \mathbf{x})} \int_{\mathbb{R}^3} f(t, \mathbf{x}, \mathbf{v}) \mathbf{v} d\mathbf{v}. \end{array} \right.$$

Moreover, in (2.22) the dimensionless collision operators are considered

$$\left\{ \begin{array}{l} C_{e,e}(f, f) = \nabla_{\mathbf{v}} \cdot \left(\int_{\mathbb{R}^3} \Phi(\mathbf{v} - \mathbf{v}') [f(\mathbf{v}') \nabla_{\mathbf{v}} f(\mathbf{v}) - f(\mathbf{v}) \nabla_{\mathbf{v}'} f(\mathbf{v}')] d\mathbf{v}' \right), \\ C_{e,i}(f) = \nabla_{\mathbf{v}} \cdot [\Phi(\mathbf{v}) \nabla_{\mathbf{v}} f(\mathbf{v})], \end{array} \right. \quad (4.37)$$

with Φ given by (2.8).

4.8 Appendix B: Scaling with the laser wavelength

We adopt a system of normalized units in order to obtain a dimensionless form of the Vlasov-Maxwell equations. The laser period T_{laser} , the laser wavelength λ_{laser} , the light speed c , and the laser pulsation ω_{laser} are defined as follows

$$\omega_{\text{laser}} = \frac{2\pi}{T_{\text{laser}}} = \frac{c2\pi}{\lambda_{\text{laser}}}, \quad \lambda_{\text{laser}} = 0.35\mu\text{m}, \quad c = 2,99.10^8 \text{ m.s}^{-1}, \quad T_{\text{laser}} = \frac{2\pi}{\omega_{\text{laser}}}. \quad (4.38)$$

These parameters enable us to define dimensionless parameters marked with tilde. The time is normalized to the laser period T_{laser} , distance to the laser wavelength λ_{laser} and velocity with the speed of light c . Electric field, magnetic field and distribution function are respectively normalized such as

$$\tilde{\mathbf{E}} = \frac{e \mathbf{E}}{m_e c \omega_{\text{laser}}}, \quad \tilde{\mathbf{B}} = \frac{e \mathbf{B}}{m_e \omega_{\text{laser}}}, \quad \tilde{f} = f \frac{c^3}{n_0}, \quad \tilde{j} = j \frac{1}{e n_c c}. \quad (4.39)$$

In these units, the Vlasov-Maxwell system becomes

$$\left\{ \begin{array}{l} \frac{\partial \tilde{f}}{\partial t} + \nabla_{\mathbf{x}} \cdot (\mathbf{v} \tilde{f}) - \nabla_{\mathbf{v}} \cdot ((\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tilde{f}) = 0, \\ \frac{\partial \tilde{\mathbf{E}}}{\partial t} - \nabla_{\mathbf{x}} \times \tilde{\mathbf{B}} = (2\pi)^2 n \mathbf{u}, \\ \frac{\partial \tilde{\mathbf{B}}}{\partial t} + \nabla_{\mathbf{x}} \times \tilde{\mathbf{E}} = 0, \\ \nabla_{\mathbf{x}} \cdot \tilde{\mathbf{E}} = \left(\frac{\omega_{pe}}{\omega_{\text{laser}}} \right)^2 (1 - n), \\ \nabla_{\mathbf{x}} \cdot \tilde{\mathbf{B}} = 0, \end{array} \right. \quad (4.40)$$

where ν is the ratio given by

$$\nu = \frac{Z n_0 e^4 \ln \Lambda}{8\pi \epsilon_0^2 m_e^2 c^3}.$$

The zero and first order moments of the distribution function are

$$\left\{ \begin{array}{l} n(t, \mathbf{x}) = \int_{\mathbb{R}^3} \tilde{f}(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}, \\ \mathbf{u}(t, \mathbf{x}) = \frac{1}{n(t, \mathbf{x})} \int_{\mathbb{R}^3} \tilde{f}(t, \mathbf{x}, \mathbf{v}) \mathbf{v} d\mathbf{v}. \end{array} \right.$$

4.9 Appendix C: Scaling with an electron-ion frequency

For the analysis of collisional processes, a new scaling is introduced here which allows time steps to be of the order of the electron-ion collision time. In order to take into account transport phenomena occurring at the collision time scale, several parameters are required. It concerns the electron-ion collision frequency $\nu_{e,i}$, the associated mean free path $\lambda_{e,i}$, the thermal velocity v_{th} , and the cyclotron frequency ω_{ce}

$$\nu_{e,i} = \frac{Z n_0 e^4 \ln \Lambda}{8 \pi \epsilon_0^2 m_e^2 v_{th}^3}, \quad \lambda_{e,i} = \frac{v_{th}}{\nu_{e,i}}, \quad v_{th} = \sqrt{\frac{\kappa_B T_0}{m_e}}, \quad \omega_{ce} = \frac{eB}{m_e}. \quad (4.41)$$

These parameters enable us to define the dimensionless parameters marked with a tilde. Dimensionless time, space and velocity, are respectively given by

$$\tilde{t} = \nu_{e,i} t, \quad \tilde{x} = x / \lambda_{e,i}, \quad \tilde{v} = v / v_{th}. \quad (4.42)$$

Dimensionless electric field $\tilde{\mathbf{E}}$, magnetic field $\tilde{\mathbf{B}}$, and distribution function $\tilde{f}$, are respectively expressed as

$$\tilde{\mathbf{E}} = \frac{e\mathbf{E}}{m_e \nu_{th} \nu_{e,i}}, \quad \tilde{\mathbf{B}} = \frac{e\mathbf{B}}{m_e \nu_{e,i}} = \frac{\omega_{ce}}{\nu_{e,i}}, \quad \tilde{f} = f \frac{v_{th}^3}{n_0}. \quad (4.43)$$

This leads to the following dimensionless equations

$$\left\{ \begin{array}{l} \frac{\partial f}{\partial t} + \nabla_{\mathbf{x}} \cdot (\mathbf{v} f) - \nabla_{\mathbf{v}} \cdot ((\mathbf{E} + \mathbf{v} \times \mathbf{B}) f) = \frac{1}{Z} C_{e,e}(f, f) + C_{e,i}(f), \\ \frac{\partial \mathbf{E}}{\partial t} - \frac{1}{\beta^2} \nabla_{\mathbf{x}} \times \mathbf{B} = \frac{1}{\alpha^2} n \mathbf{u}, \\ \frac{\partial \mathbf{B}}{\partial t} + \nabla_{\mathbf{x}} \times \mathbf{E} = 0, \\ \nabla_{\mathbf{x}} \cdot \mathbf{E} = \frac{1}{\alpha^2} (1 - n), \\ \nabla_{\mathbf{x}} \cdot \mathbf{B} = 0. \end{array} \right. \quad (4.44)$$

where $\alpha = \nu_{e,i} / \omega_{pe}$ and $\beta = v_{th} / c$. The collision terms $C_{e,e}(f, f)$ and $C_{e,i}(f)$ are given in (4.37).

4.10 Appendix D: Description of the Particle-In-Cell code called *PICLS*

The essential feature of Particle-In-Cell (*PIC*) codes is that the plasma is described as a large number of discrete particles moving in their self-consistent electromagnetic fields. *PIC* methods

give details of wave motions in the plasma and of the particle distribution function. Ions and electrons are normally simulated as two distinct species but for brevity the discussion here will only be of the electron equations of motion.

Particle-In-Cell methods (*PIC*) consist to repeatedly solve Vlasov equation (2.2) coupled with Maxwell equations (2.6) for each particle specie using a characteristics method. The particles are grouped in sets called macro-particle whose behavior is given by the equation of motion

$$\begin{cases} \frac{d\mathbf{p}}{dt} = Ze \left(\mathbf{E} + \frac{\mathbf{p}}{\gamma} \times \mathbf{B} \right), \\ \frac{d\mathbf{x}}{dt} = \frac{\mathbf{p}}{\gamma}, \end{cases} \quad (4.45)$$

where $\mathbf{p}$ is the momentum of a macro-particle and $\mathbf{x}$ its position. Here γ is the Lorentz factor and it is given by

$$\gamma = \sqrt{1 + \frac{|\mathbf{p}|^2}{mc^2}}.$$

$\mathbf{E}$ and $\mathbf{B}$ denote respectively electric and magnetic field. Z represents the charge number of the macro-particle and m its mass. Each macro-particle is not considered as a ponctual particle but like a volume as big as the space domain step Δx centered on the position $\mathbf{x}$ of the marco-particle. The typical time scale of the simulation is the plasma period and in general *PIC* codes are unsuitable for description of the plasma on a hydrodynamic time scale. The Particle-In-Cell code used for comparison is *PICLS* introduced in [13]. This code solves multi-dimensional problems by using a numerical dispersion-free Maxwell solver with a directional splitting scheme ([2]). Moreover to supress the numerical instability causing under-resolving the Debye length, *PICLS* has a fourth order current/force interpolation of the sources ρ and $\mathbf{j}$ instead of other *PIC* code. Then a current-conserving integration scheme is implemented (see [7] for more details). Besides binary collisions are enclosed in *PICLS* based on the model developped in [13].

CHAPTER 5

FROM KINETIC TO MESOSCOPIC: AN ENTROPIC SCHEME FOR REDUCED MOMENT PROBLEM

This chapter is submitted for publication in [37].

5.1 Introduction

As seen before, a variety of models are used to describe electrons transport, some hydrodynamic ones ([13, 11, 12]) and some kinetic ones ([5, 6, 14, 21, 22, 43]). Here the standard physical model is studied. Let us consider f^0 as the isotropic part of the distribution function f and f^1 the anisotropic part of the distribution function. The usual model employed in plasma physics can be expressed such as

$$\begin{cases} \partial_t f^0 = Q_{ee}^0(f^0), \\ \partial_t f^1 = -\alpha f^1, \end{cases} \quad (5.1)$$

where $\alpha \in \mathbb{R}^+$. Here $Q_{ee}^0(f^0)$ represents an approximation for the electron-electron collision operator. In plasma physics, classical approximations lead to consider that the main contribution for the electron-electron collision operator comes from the isotropic part, that is why here Q_{ee}^0 depends only on f^0 . This model allows mainly to conserve fundamental properties such as the mass and energy conservation and the entropy dissipation. However this model is ill-posed. Indeed whatever the closure chosen, this model does not preserve the realizability domain.

Of course, apart from this model, some kinetic models like Fokker-Planck model exist but computations are too much expensive and mainly kinetic codes can not run many collisional times. Most of them are restricted to some tens of collisional time in practice. A relevant example is the kinetic code *KETS* described in chapter 3 and cross-validated with an implicit Vlasov-Fokker-Planck code called *IMPACT* developed by Kingham and Bell [31]. Beyond a few tens of collisional time, the *KETS* code becomes unstable in the non-locale regime. To reduce the computational time, the plasma can be described by fluid models. For example in [11, 12, 13] a bi-fluid compressible Euler model coupled with the Poisson equation is considered. However, for the new high energy target drivers, the kinetic effects are too important to neglect them and replace kinetic computation by usual macroscopic Euler Models with T_i - T_e temperatures.

This is why to preserve the realizability domain, we consider in this chapter a new approximation of the electron-electron collision operator to correct the model 5.1. This is the originality of this work. This new continuous model satisfies fundamental properties (i.e. conservation laws and entropy dissipation). Then we will consider an intermediate description between the fluid and kinetic levels. The microscopic velocity is written in spherical coordinates and the model is written by considering the moment system with respect to the angular variable. But the choice of the closure on the angle variable is crucial to guarantee reasonable properties for the resulting model. For example, the moments model P_N proposed in [38] does not satisfy positivity of the underlying distribution function of electrons and entropy dissipation. This comes from the closure which leads to define the distribution function by truncated spherical harmonic

expansion in the angular variable. For the P_N model, modifications are proposed on the closure ([29]) to obtain a non-negative distribution function. Nevertheless, this model does not satisfy some other fundamental properties such as the entropy dissipation. Moreover there is no general theory to describe boundary conditions although some idea are proposed. For example, in [33, 34], authors propose an asymptotic limit approximation to prescribe boundary conditions for the neutrons transport equation.

For this reason there is a considerable motivation to develop an other moment closure. That is why we will derive a new reduced kinetic model called the M_1 model whose construction is based on an entropy minimization principle. This model has been first introduced in [20] to find an approximation of radiative transfer and energy evolution equation. The system of conservation laws that is obtained is closed by minimizing the radiative entropy (see also [25] for extension). This approach has been extended in [4] by proposing numerical methods with high order reconstruction which provides positivity of the distribution function. In order to have accurate results and costless numerical schemes, a convenient change of variables has been proposed to eliminate the space dependence in the flux function. In kinetic theory, the problem of minimization under moments constraints has been first introduced by Levermore in [36]. The main motivation was to construct BGK models ([2]) which could fit transport coefficients at the hydrodynamic limit. Moreover the well-posedness of such problems has been clarified by Junk and Schneider in [30, 41]. But in the present case, the aim is different. Here the energy of particles constitutes a free parameter. Then we integrate only the kinetic equation with respect to the angle variable and we return only the energy of particles as kinetic variable. By using a closure defined from a minimization entropy principle, we obtain a closed system.

The contribution of this chapter is first the derivation of a new moments system from the Maxwell-Fokker-Planck-Landau system. This model is next shown to be entropic and to preserve the realizability domain. In order to perform numerical simulations, an entropic semi-discretization in the energy variable of the model is proposed. The cornerstone of the construction of the scheme is an entropic average used to define the quantities on the interface (Theorem 16). This definition implies that the Rosenbluth and Landau form of the discretized collision operator are still equivalent. This entropic average has been used for a kinetic equation by Delacherie in [8, 7, 16, 17, 18].

This chapter is organized as follows. The principle of the angular moment closure is explained in section 5.2 on a simple kinetic BGK model. In section 5.3, the moment model is generalized for the Maxwell-Fokker-Planck-Landau kinetic equation. Here the distribution function is solution of the system constituted by the Fokker-Planck-Landau kinetic equation coupled with the Maxwell equations which describe the time evolution of the electromagnetic field. The system is derived by considering the angular moments defined in section 5.2. In section 5.4 the angular moment system is proved to be entropic (Theorem 13). Section 5.6 is devoted to the derivation of an entropic semi-discrete scheme for the space homogeneous moments system. The time is kept as a continuous variable whereas the energy variable is discrete. Next several numerical

test cases for the full discrete scheme are performed in section 5.8. Test cases show that the full discrete scheme satisfies the properties of the semi-discrete scheme shown in section 5.6.

5.2 Principle of the angular moment closure

In this section an original derivation of moment closure is proposed. The main difference with other existing moment model is that the final equation is semi-kinetic. Indeed the system is obtained by an angle variable integration and the system remains kinetic in energy.

For the sake of simplicity, we assume a BGK kernel-collision operator to derive the closure. The time evolution of the electron distribution function $f(t, x, v)$, where $x \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$, is governed by the following kinetic equation,

$$\partial_t f(t, x, v) + v \cdot \nabla_x f(t, x, v) = C(f), \quad (5.2)$$

where,

$$C(f) = \nu(|v|)(M_f - f),$$

with

$$M_0 = n_e \left(\frac{m_e}{2\pi k_B T_e} \right)^{3/2} \exp \left(-\frac{m_e v^2}{2k_B T_e} \right), \quad (5.3)$$

and $\nu(|v|)$ being the collision frequency.

Next we explain the construction of the M_1 model on the BGK model (5.2, 5.3). This system is constructed by using a minimum entropy principle for angle variable of particle by keeping the energy of particles as kinetic variable. If S_2 is the unit sphere, $\Omega = v/|v|$ represents the direction of propagation of the particles. In this case, by setting $\zeta = |v|$, the three first moments with respect to Ω are defined as

$$f^0(\zeta) = \zeta^2 \int_{S_2} f(v) d\Omega, \quad f^1(\zeta) = \zeta^2 \int_{S_2} \Omega f(v) d\Omega, \quad f^2(\zeta) = \zeta^2 \int_{S_2} \Omega \otimes \Omega f(v) d\Omega. \quad (5.4)$$

Remark that f^0 is a scalar, f^1 is a vector and f^2 is a tensor. By integrating the system (5.2) with respect to Ω it holds that

$$\begin{aligned} \partial_t f^0(t, x, \zeta) + \nabla_x \cdot (\zeta f^1(t, x, \zeta)) &= \nu(\zeta)(4\pi\zeta^2 M_f(t, x, \zeta) - f^0(t, x, \zeta)), \\ \partial_t f^1(t, x, \zeta) + \nabla_x \cdot (\zeta f^2(t, x, \zeta)) &= -\nu(\zeta)f^1(t, x, \zeta). \end{aligned}$$

Of course, the open problem remains the computation of moment f^2 as a function of f^0 and f^1 . The minimum entropy principle [36, 40] with respect to the moments $(1, \Omega)$ implies that f should be taken under the following form,

$$f = \rho_0(\zeta) \exp(-\Omega \cdot \alpha(\zeta)), \quad (5.5)$$

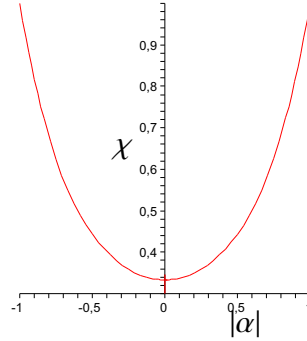


Figure 5.1: Eddington factor χ versus anisotropy parameter modulus $|\alpha|$.

where ρ_0 is a non negative scalar ($\rho_0 \geq 0$), and α is a three components real valued vector. An important parameter is the anisotropy parameter a ,

$$a = f^1 / f^0 ,$$

which satisfies by construction $|a| \leq 1$. The computation of the different moments of the distribution function (5.5) gives

$$f^0 = 4\pi\rho_0 \frac{\sinh(|\alpha|)}{|\alpha|}, \quad f^1 = 4\pi\rho_0 \frac{\sinh(|\alpha|)(1 - |\alpha| \coth(|\alpha|))}{|\alpha|^3} \alpha. \quad (5.6)$$

In fact, these relations can be combined to give,

$$a = \frac{1 - |\alpha| \coth(|\alpha|)}{|\alpha|^2} \alpha, \quad (5.7)$$

or by taking the modulus of a ,

$$|a| = \frac{|\alpha| \coth(|\alpha|) - 1}{|\alpha|}. \quad (5.8)$$

The relation (5.8) cannot be inverted explicitly by hand. However, this relation determines a unique solution which can be computed and tabulated numerically. a is assumed to be known, f^2 can also be computed

$$f^2 = f^0 \left(\frac{1 - \chi}{2} I + \frac{3\chi - 1}{2} \frac{f^1}{|f^1|} \otimes \frac{f^1}{|f^1|} \right), \quad (5.9)$$

where $\chi = \frac{|\alpha|^2 - 2|\alpha| \coth(|\alpha|) + 2}{|\alpha|^2}$. The χ factor is plotted on figure 5.1, and compared with the radiative Eddington factor on figure 5.2.

In [23, 26], several possibilities are given to approach the χ factor. In particular a pre-computed

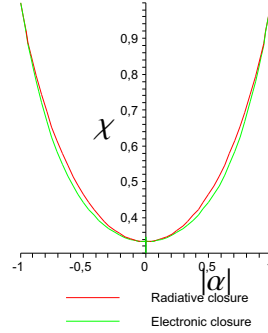


Figure 5.2: Comparison of Eddington factors χ from radiative transfer closure and electronic closure.

table gives the factor χ as a function of α . By considering many moments, this approach becomes too expensive. Another possibility used here consists to approximate χ by a suitable function depending on χ properties ([1]) given by

$$\chi(0) = 1/3, \chi(1) = \chi(-1) = 1 \text{ and } \chi'(1) = -\chi'(-1) = 2. \quad (5.10)$$

For concrete applications, the χ factor can be tabulated by a very simple rational approximation,

$$\chi \approx \frac{a_6 a^6 + a_4 a^4 + a_2 a^2 + a_0}{a^4 + b_2 a^2 + b_0}, \quad (5.11)$$

which is very accurate (the difference with the exact curve is about 10^{-15}). The coefficients a and b (used in [23]) are given by

$$\begin{aligned} a_0 &= 0.762066949972264, & b_0 &= 2.28620084991677, \\ a_2 &= 0.219172080193380, & b_2 &= -2.10758208969840, \\ a_4 &= -0.259725400168378, \\ a_6 &= 0.457105130221120. \end{aligned}$$

Remark 11. Notice that with $\chi = 1/3$ we find the P_1 model ([27]) described in the introduction of this chapter.

5.3 Moment model for the classical case

In this section we recall some backgrounds about the Maxwell-Fokker-Planck-Landau model. In a second subsection the M_1 model associated to Maxwell-Fokker-Planck-Landau is then presented.

5.3.1 Classical Maxwell-Fokker-Planck-Landau kinetic equation

The time evolution of the electron distribution function $f(t, \mathbf{x}, \mathbf{v})$ is governed by the Fokker-Planck-Landau kinetic equation coupled with the Maxwell equations which describe the time evolution of the electric field $\mathbf{E}(t, \mathbf{x})$ and magnetic field $\mathbf{B}(t, \mathbf{x})$. The resulting equations ([10, 15, 35]) can be written as,

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = C_{ee}(f, f) + C_{ei}(f), \quad (5.12)$$

$$\partial_t \mathbf{E} - c^2 \nabla_{\mathbf{x}} \wedge \mathbf{B} = -\frac{\mathbf{j}}{\epsilon_0}, \quad (5.13)$$

$$\partial_t \mathbf{B} + \nabla_{\mathbf{x}} \wedge \mathbf{E} = 0, \quad (5.14)$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad (5.15)$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{B} = 0, \quad (5.16)$$

where C_{ee} and C_{ei} are the electron-electron and electron-ion collision operators, $\rho = q(n_e - n_0)$ is the plasma electric charge, $q = -e$ is the electron charge, Z is the ion charge number and ϵ_0 is the vacuum permittivity. The electron density $n_e(t, \mathbf{x}) = \int f d\mathbf{v}$ and the current $\mathbf{j}(t, \mathbf{x}) = q \int f \mathbf{v} d\mathbf{v}$ are given from the electron distribution function of electrons.

Equation (5.15) is redundant because (5.13) and (5.15) are equivalent from the charge conservation if Poisson equation (5.15) is verified at initial time.

The electron-electron and electron-ion collision operators are given by definition (2.7, 2.10) and present different interesting properties describe in chapter 2, mainly conservation laws and equilibrium states.

5.3.2 Moment closure for the classical kinetic equation

As the main part of angular closure is given in section 5.2, we give only the final result of the closure on Maxwell-Fokker-Planck-Landau system (5.12 - 5.16) (see appendix 5.12 for computations). For f^0 , f^1 and f^2 defined as in (5.4), we obtain

$$\partial_t f^0 + \nabla_{\mathbf{x}} \cdot (\zeta f^1) + \partial_{\zeta} \left(\frac{q}{m} E \cdot f^1 \right) = Q_{ee}^0(f^0), \quad (5.17)$$

$$\begin{aligned} \partial_t f^1 + \nabla_{\mathbf{x}} \cdot (\zeta f^2) + \partial_{\zeta} \left(\frac{q}{m} f^2 E \right) - \frac{q}{m\zeta} (f^0 E + \zeta f^1 \wedge B - f^2 E) \\ = Q_{ee}^1(f^1) + Q_{ei}^1(f^1), \end{aligned} \quad (5.18)$$

$$\partial_t E - c^2 \nabla_{\mathbf{x}} \wedge B = -\frac{j}{\epsilon_0}, \quad (5.19)$$

$$\partial_t B + \nabla_{\mathbf{x}} \wedge E = 0, \quad (5.20)$$

$$\nabla_{\mathbf{x}} \cdot E = \frac{\rho}{\epsilon_0}, \quad (5.21)$$

$$\nabla_{\mathbf{x}} \cdot B = 0, \quad (5.22)$$

where the collision operators Q^0 and Q^1 are given by,

$$Q_{ee}^i(f^i) = \frac{2\alpha_{ee}}{3} \partial_\zeta \left(\zeta^2 A(\zeta) \partial_\zeta \left(\frac{f^i}{\zeta^2} \right) - \zeta B(\zeta) f^i \right), \quad i = 0, 1, \quad (5.23)$$

$$Q_{ei}^1(f^1) = -\frac{2\alpha_{ei}}{\zeta^3} f^1. \quad (5.24)$$

The coefficients α_{ee} and α_{ei} are introduced in the formula (2.9) and (2.11), and the coefficients $A(\zeta)$ and $B(\zeta)$ are given by

$$A(\zeta) = \int_0^\infty \min\left(\frac{1}{\zeta^3}, \frac{1}{w^3}\right) w^2 f^0(w) dw, \quad (5.25)$$

$$B(\zeta) = \int_0^\infty \min\left(\frac{1}{\zeta^3}, \frac{1}{w^3}\right) w^3 \partial_w \left(\frac{f^0(w)}{w^2} \right) dw. \quad (5.26)$$

The expression of coefficient $B(\zeta)$ (see annex 5.11), can be simplified as

$$B(\zeta) = -\frac{3}{\zeta^3} \int_0^\zeta f^0(w) dw. \quad (5.27)$$

Equation (5.17) represents the charge balance and involves the electron-electron collision kernel C_{ee} , whereas equation (5.18) represents the momentum balance and involves the electron-ion collision kernel C_{ei} .

The charge ρ and the current j are given by

$$\rho = q \int_0^\infty f^0(\zeta) d\zeta, \quad j = q \int_0^\infty \zeta f^1(\zeta) d\zeta,$$

and the closure of f^2 is already established in formula (5.9).

5.4 Entropic properties of the continuous model

First we prove that the realizability domain is preserved by the continuous model (5.30, 5.31). Next the model is shown to satisfy a H theorem.

5.4.1 Introduction

From now, we chose a one dimensional direction of propagation. Therefore we take $\mu \in [-1, 1]$ as the direction of propagation variable instead of Ω . In that case the formula (5.5) writes

$$f = \rho \exp(-\alpha\mu), \quad \alpha \in \mathbb{R}, \quad \mu \in [-1, 1], \quad (5.28)$$

and the closure relation writes $f^2 = f^0 \chi$. Next, we set

$$F^i(\zeta) = \frac{f^i(\zeta)}{\zeta^2} = \int_{-1}^1 \mu^i \rho(\zeta) \exp(-\alpha(\zeta)\mu) d\mu. \quad (5.29)$$

In this part, equations (5.17, 5.18) are considered with electromagnetic fields equal to zero leading to the system

$$\partial_t f^0 + \nabla \cdot (\zeta f^1) = Q_{ee}^0(f^0), \quad (5.30)$$

$$\partial_t f^1 + \nabla \cdot (\zeta f^2) = Q_{ee}^1(f^1) + Q_{ei}^1(f^1), \quad (5.31)$$

where

$$\begin{cases} Q_{ee}^0(f^0) = \partial_\zeta \left(\zeta \int_0^\infty \tilde{J}(\zeta, \zeta') \left[F^0(\zeta') \frac{1}{\zeta} \partial_\zeta (F^0(\zeta)) - F^0(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} F^0(\zeta') \right] \zeta'^2 d\zeta' \right), \\ Q_{ee}^1(f^1) = \partial_\zeta \left(\zeta \int_0^\infty \tilde{J}(\zeta, \zeta') \left[F^0(\zeta') \frac{1}{\zeta} \partial_\zeta (F^1(\zeta)) - F^1(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} F^0(\zeta') \right] \zeta'^2 d\zeta' \right), \\ Q_{ei}^1(f^1) = -\frac{2\alpha_{ei}}{\zeta^3} f^1, \end{cases} \quad (5.32)$$

with $\tilde{J}(\zeta, \zeta') = \frac{2\alpha_{ee}}{3} \inf\left(\frac{1}{\zeta^3}, \frac{1}{\zeta'^3}\right) \zeta'^2 \zeta^2$.

5.4.2 Realizability domain

To solve the minimization problem, we introduce the realizability domain $\mathcal{A}$ defined by

$$\mathcal{A} = \left\{ \bar{g} = \begin{pmatrix} g^0 \\ g^1 \end{pmatrix} \in \mathbb{R}^2 / \exists g \in L^1([-1, 1]) : [-1, 1] \rightarrow \mathbb{R}^+ \text{ such that } f^i = \zeta^2 < \mu^i g(\alpha, \mu) > \forall i = 0, 1 \right\}.$$

Remark 12. In the particular case of moments up to order 1, the realizability domain is shown to be equal to $\mathcal{B}$ defined by

$$\mathcal{B} = \left\{ \bar{g} = \begin{pmatrix} g^0 \\ g^1 \end{pmatrix} \in \mathbb{R}^2, g^0 > 0 \text{ and } |g^1| < g^0 \right\} \cup \{(0, 0)\}.$$

Proposition 8. By using the continuous problem defined by equations (5.30, 5.31, 5.32) and under the assumption $\bar{f}(t = 0, \zeta) \in \mathcal{A}$, the realizability domain is conserved with respect to time, i.e. $\bar{f}(t, \zeta) \in \mathcal{A}$ for any $t \geq 0$.

Proof. We first show that the positivity of F^0 is preserved. Consider here (5.17) in a homogeneous space situation

$$\partial_t f^i = \frac{2\alpha_{ee}}{3} \partial_\zeta \left(\zeta^2 A(f^0, \zeta) \partial_\zeta \left(\frac{f^i}{\zeta^2} \right) - \zeta B(f^0, \zeta) f^i \right), \quad i = 0, 1, \quad (5.33)$$

where $A(f^0, \zeta)$ and $B(f^0, \zeta)$ are defined in (5.25, 5.26). (5.33) can be rewritten as

$$\zeta^2 \partial_t F^i = \frac{2\alpha_{ee}}{3} \partial_\zeta \left(\zeta^2 A(f^0, \zeta) \partial_\zeta F^i - \zeta^3 B(f^0, \zeta) F^i \right).$$

By setting

$$\widetilde{A}(F^0, \zeta) = \int_0^\infty \min\left(\frac{1}{\zeta^3}, \frac{1}{w^3}\right) w^4 F^0(w) dw \quad \text{and} \quad \widetilde{B}(F^0, \zeta) = 3 \int_0^\zeta w^2 F^0(w) dw,$$

we obtain

$$\zeta^2 \partial_t F^i = \frac{2\alpha_{ee}}{3} \partial_\zeta \left(\zeta^2 \widetilde{A}(F^0, \zeta) \partial_\zeta F^i + \widetilde{B}(F^0, \zeta) F^i \right), \quad i = 0, 1. \quad (5.34)$$

Let H be a convex positive function such as $H \leq 0$ and defined by

$$H(x) = \begin{cases} -Cx & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}, \quad C \in \mathbb{R}^+.$$

We consider now equation (5.34) with $i=0$. By multiplying equation (5.34) taken for $i = 0$ by $H'(F^0)$ and integrating over ζ , we get

$$\begin{aligned} \int_0^\infty \zeta^2 \partial_t H(F^0) d\zeta &= \int_0^\infty \frac{2\alpha_{ee}}{3} \partial_\zeta \left(\zeta^2 \widetilde{A}(F^0, \zeta) \partial_\zeta F^0 \right) H'(F^0) d\zeta \\ &+ \int_0^\infty \frac{2\alpha_{ee}}{3} \partial_\zeta \left(\widetilde{B}(F^0, \zeta) F^0 \right) H'(F^0) d\zeta. \end{aligned} \quad (5.35)$$

By using the Green formula on the first term of the right-hand side of (5.35), convexity of H implies

$$\int_0^\infty \frac{2\alpha_{ee}}{3} \partial_\zeta \left(\zeta^2 \widetilde{A}(F^0, \zeta) \partial_\zeta F^0(\zeta) \right) H'(F^0) d\zeta \leq 0.$$

Moreover the second term of (5.35) can be simplified as

$$\begin{aligned} \int_0^\infty \frac{2\alpha_{ee}}{3} \partial_\zeta \left(\widetilde{B}(F^0, \zeta) F^0 \right) H'(F^0) d\zeta &= \int_0^\infty 2\alpha_{ee} H'(F^0) \partial_\zeta F^0 \int_0^\zeta w^2 F^0(w) dw d\zeta \\ &+ \int_0^\infty 2\alpha_{ee} H'(F^0) \left(\zeta F^0 \right)^2 d\zeta. \end{aligned}$$

By using an integration by part, we get

$$\int_0^\infty \frac{2\alpha_{ee}}{3} \partial_\zeta \left(\widetilde{B}(F^0, \zeta) F^0 \right) H'(F^0) d\zeta = \int_0^\infty 2\alpha_{ee} \zeta^2 F^0 \left(-H(F^0) + H'(F^0) F^0 \right) d\zeta.$$

Moreover, by construction of H , $\forall x$, $(-H(x) + H'(x)x)x = 0$. Therefore, (5.35) gives

$$\int_0^\infty \zeta^2 \partial_t H(F^0) d\zeta \leq 0.$$

After integrating the previous inequality, we get

$$\int_0^\infty \zeta^2 H(F^0, t) d\zeta \leq \int_0^\infty \zeta^2 H(F^0, t=0) d\zeta.$$

By using the definition of H , $\int_0^\infty H(F^0) d\zeta = 0$ for $F^0 \geq 0$. Then $H(F^0) = 0$ a.e. and the positivity of F^0 is preserved.

Notice that F^1 satisfies (5.34). So by arguing in the same way as previously, we can deduce that the positivity of F^1 is conserved. By linearity of (5.34), $\Phi_1 = F^1 + F^0$ and $\Phi_2 = F^1 - F^0$ also satisfies equation (5.34). Then the positivity of Φ_1 and Φ_2 is preserved. So the realizability domain is conserved in time. $\square$

5.4.3 H Theorem

The main result of this subsection is the following Theorem proving that (5.30, 5.31, 5.32) is entropic.

Theoreme 13. $E = \ln(\rho)f^0 - \alpha f^1$ is an entropy for the system (5.30, 5.31, 5.32). More precisely, we have

$$\partial_t E + \nabla \cdot F \leq 0,$$

where F is the entropic flux given by $F = \ln(\rho)f^1 - \alpha f^2$.

Proof. Multiply equation (5.30) by $\ln(\rho)$ and equation (5.31) by $-\alpha$ and add the two equalities. In that case, the left-hand side is equal to $\partial_t E + \nabla \cdot F$. Therefore it remains to prove that

$$\int_0^{+\infty} \ln(\rho) Q_{ee}^0(f^0) d\zeta - \int_0^{+\infty} \alpha(\zeta) Q_{ee}^1(f^1) d\zeta - \int_0^{+\infty} \alpha(\zeta) Q_{ei}^1(f^1) d\zeta \leq 0. \quad (5.36)$$

By using the definition of $Q_{ei}^1(f^1)$, equation (5.7) and the fact that $1 - |x| \coth(|x|) \leq 0$ for any $x \in \mathbb{R}$, it comes that

$$- \int_0^{+\infty} \alpha(\zeta) Q_{ei}^1(f^1) d\zeta \leq 0.$$

So it remains to prove that

$$\begin{aligned} \int_0^{+\infty} \ln(\rho) Q_{ee}^0(f^0) d\zeta = \\ \int_0^{+\infty} \partial_\zeta \left[\zeta \int_0^{+\infty} \tilde{J}(\zeta, \zeta') \left(\frac{f^0(\zeta')}{\zeta'^2} \frac{1}{\zeta} \partial_\zeta \left(\frac{f^0(\zeta)}{\zeta^2} \right) - \frac{f^0(\zeta)}{\zeta^2} \frac{1}{\zeta'} \partial_{\zeta'} \left(\frac{f^0(\zeta')}{\zeta'^2} \right) \right) \right. \\ \left. (\zeta')^2 d\zeta' \right] \ln(\rho) d\zeta \leq 0. \end{aligned} \quad (5.37)$$

By using a Green formula in the expression of $\int_0^{+\infty} \ln(\rho) Q_{ee}^0(f^0) d\zeta$, we get

$$\begin{aligned} & \int_0^{+\infty} \partial_\zeta \left[\zeta \int_0^{+\infty} \tilde{J}(\zeta, \zeta') \left(\frac{f^0(\zeta')}{\zeta'^2} \frac{1}{\zeta} \partial_\zeta \left(\frac{f^0(\zeta)}{\zeta^2} \right) - \frac{f^0(\zeta)}{\zeta^2} \frac{1}{\zeta'} \partial_{\zeta'} \left(\frac{f^0(\zeta')}{\zeta'^2} \right) \right) \right. \\ & \quad \left. (\zeta')^2 d\zeta' \right] \ln(\rho) d\zeta \\ &= - \int_0^{+\infty} \int_0^{+\infty} \tilde{J}(\zeta, \zeta') \left(\frac{1}{\zeta} F^0(\zeta') \partial_\zeta F^0(\zeta) - \frac{1}{\zeta'} F^0(\zeta) \partial_{\zeta'} F^0(\zeta') \right) \frac{\partial_\zeta \rho}{\rho} \\ & \quad \frac{1}{\zeta} (\zeta')^2 d\zeta d\zeta'. \end{aligned} \quad (5.38)$$

Next we compute $\frac{1}{\zeta} F^0(\zeta') \partial_\zeta F^0(\zeta) - \frac{1}{\zeta'} F^0(\zeta) \partial_{\zeta'} F^0(\zeta')$. From (5.29), we get the relation

$$\partial_\zeta F^0(\zeta) = \int_{-1}^1 \partial_\zeta \rho(\zeta) \exp(-\alpha(\zeta)\mu) d\mu - \int_{-1}^1 \mu \partial_\zeta \alpha(\zeta) \rho(\zeta) \exp(-\alpha(\zeta)\mu) d\mu. \quad (5.39)$$

By using the expressions of F^0 and $\partial_\zeta F^0$, it holds that

$$\begin{aligned} & \frac{1}{\zeta} F^0(\zeta') \partial_\zeta F^0(\zeta) - \frac{1}{\zeta'} F^0(\zeta) \partial_{\zeta'} F^0(\zeta') = \\ & \int_{-1}^1 \int_{-1}^1 \left[\rho(\zeta') \exp(-\alpha(\zeta')\mu') \left(\frac{\partial_\zeta \rho(\zeta)}{\zeta} \exp(-\alpha(\zeta)\mu) - \frac{\mu}{\zeta} \partial_\zeta \alpha(\zeta) \rho(\zeta) \exp(-\alpha(\zeta)\mu) \right) \right. \\ & \quad \left. - \rho(\zeta) \exp(-\alpha(\zeta)\mu) \left(\frac{\partial_{\zeta'} \rho(\zeta')}{\zeta'} \exp(-\alpha(\zeta')\mu') - \frac{\mu'}{\zeta'} \partial_{\zeta'} \alpha(\zeta') \rho(\zeta') \exp(-\alpha(\zeta')\mu') \right) \right] d\mu d\mu'. \end{aligned}$$

So

$$\begin{aligned} & \frac{1}{\zeta} F^0(\zeta') \partial_\zeta F^0(\zeta) - \frac{1}{\zeta'} F^0(\zeta) \partial_{\zeta'} F^0(\zeta') = \int_{-1}^1 \int_{-1}^1 \rho(\zeta) \exp(-\alpha(\zeta)\mu) \rho(\zeta') \exp(-\alpha(\zeta')\mu') \\ & \quad \left(\frac{\partial_\zeta \rho(\zeta)}{\rho(\zeta)\zeta} - \frac{\mu}{\zeta} \partial_\zeta \alpha(\zeta) - \frac{\partial_{\zeta'} \rho(\zeta')}{\rho(\zeta')\zeta'} + \frac{\mu'}{\zeta'} \partial_{\zeta'} \alpha(\zeta') \right) d\mu d\mu'. \end{aligned} \quad (5.40)$$

Next we set

$$K(\zeta, \zeta', \mu, \mu') = \tilde{J}(\zeta, \zeta') \zeta^2 \zeta'^2 \rho(\zeta) \exp(-\alpha(\zeta)\mu) \rho(\zeta') \exp(-\alpha(\zeta')\mu'), \quad (5.41)$$

$$\delta(\zeta) = \frac{\partial_\zeta \rho(\zeta)}{\rho(\zeta)\zeta} \frac{1}{\zeta}, \quad \beta(\zeta) = \frac{\partial_\zeta \alpha(\zeta)}{\zeta}. \quad (5.42)$$

Therefore by using equality (5.40) in (5.38) we get

$$\begin{aligned} & - \int_0^{+\infty} \zeta \int_0^{+\infty} \tilde{J}(\zeta, \zeta') \left(\frac{1}{\zeta} F^0(\zeta') \partial_\zeta F^0(\zeta) - \frac{1}{\zeta'} F^0(\zeta) \partial_{\zeta'} F^0(\zeta') \right) \frac{\partial_\zeta \rho}{\rho \zeta} d\zeta d\zeta' \\ &= - \int_0^{+\infty} \int_0^{+\infty} \int_{-1}^1 \int_{-1}^1 K(\zeta, \zeta', \mu, \mu') (\delta(\zeta) - \delta(\zeta')) \delta(\zeta) d\zeta d\zeta' d\mu d\mu' \\ &+ \int_0^{+\infty} \int_0^{+\infty} \int_{-1}^1 \int_{-1}^1 K(\zeta, \zeta', \mu, \mu') (\mu \beta(-\zeta) - \mu' \beta(\zeta')) \delta(\zeta) d\zeta d\zeta' d\mu d\mu'. \end{aligned}$$

The transform $(\zeta, \zeta') \mapsto (\zeta', \zeta)$ leads to

$$\begin{aligned}
& - \int_0^{+\infty} \zeta \int_0^{+\infty} \tilde{J}(\zeta, \zeta') \left(\frac{1}{\zeta} F^0(\zeta') \partial_\zeta F^0(\zeta) - \frac{1}{\zeta'} F^0(\zeta) \partial_{\zeta'} F^0(\zeta') \right) \frac{\partial_\zeta \rho}{\rho \zeta} d\zeta d\zeta' \\
& = -\frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \int_{-1}^1 \int_{-1}^1 K(\zeta, \zeta', \mu, \mu') (\delta(\zeta) - \delta(\zeta'))^2 d\zeta d\zeta' d\mu d\mu' \\
& + \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \int_{-1}^1 \int_{-1}^1 K(\zeta, \zeta', \mu, \mu') (\mu\beta(\zeta) - \mu'\beta(\zeta')) (\delta(\zeta) - \delta(\zeta')) d\zeta d\zeta' d\mu d\mu'.
\end{aligned} \tag{5.43}$$

Next, for the remaining term

$$\begin{aligned}
\int_0^{+\infty} \mathcal{Q}_{ee}^1(f^1)(-\alpha(\zeta)) d\zeta & = \int_0^{+\infty} \int_0^{+\infty} \tilde{J}(\zeta, \zeta') \left(\frac{1}{\zeta} F^0(\zeta') \partial_\zeta F^1(\zeta) - \frac{1}{\zeta'} F^1(\zeta) \partial_{\zeta'} F^0(\zeta') \right) \\
& \quad \frac{\partial_\zeta \alpha}{\zeta} (\zeta')^2 d\zeta d\zeta',
\end{aligned}$$

we proceed as previously. The expression of F^1 given in (5.29) leads to

$$\partial_\zeta F^1(\zeta) = \int_{-1}^1 \mu \partial_\zeta \rho(\zeta) \exp(-\alpha(\zeta)\mu) d\mu - \int_{-1}^1 \mu^2 \partial_\zeta \alpha(\zeta) \rho(\zeta) \exp(-\alpha(\zeta)\mu) d\mu. \tag{5.44}$$

Therefore by using expressions (5.39, 5.44), we get

$$\begin{aligned}
& \int_0^{+\infty} \int_0^{+\infty} \tilde{J}(\zeta, \zeta') \zeta^2 \zeta'^2 \left(F^0(\zeta') \frac{1}{\zeta} \partial_\zeta F^1(\zeta) - F^1(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} F^0(\zeta') \right) \frac{\partial_\zeta \alpha}{\zeta} d\zeta d\zeta' \\
& = \int_0^{+\infty} \int_0^{+\infty} \int_{-1}^1 \int_{-1}^1 \frac{2\alpha_{ee}}{3} K(\zeta, \zeta', \mu, \mu') (\delta(\zeta) - \delta(\zeta')) \mu\beta(\zeta) d\zeta d\zeta' d\mu d\mu' \\
& + \int_0^{+\infty} \int_0^{+\infty} \int_{-1}^1 \int_{-1}^1 K(\zeta, \zeta', \mu, \mu') (\beta(\zeta')\mu' - \beta(\zeta)\mu) \mu\beta(\zeta) d\zeta d\zeta' d\mu d\mu'.
\end{aligned}$$

Then by using again the transform $(\zeta, \zeta') \mapsto (\zeta', \zeta)$ it comes that

$$\begin{aligned}
& \int_0^{+\infty} \int_0^{+\infty} \tilde{J}(\zeta, \zeta') \zeta^2 \zeta'^2 \left(F^0(\zeta') \frac{1}{\zeta} \partial_\zeta F^1(\zeta) - F^1(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} F^0(\zeta') \right) \frac{\partial_\zeta \alpha}{\zeta} d\zeta d\zeta' \\
& = \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \int_{-1}^1 \int_{-1}^1 K(\zeta, \zeta', \mu, \mu') (\delta(\zeta) - \delta(\zeta')) (\mu\beta(\zeta) - \mu'\beta(\zeta')) d\zeta d\zeta' d\mu d\mu' \\
& \quad - \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \int_{-1}^1 \int_{-1}^1 K(\zeta, \zeta', \mu, \mu') (\beta(\zeta')\mu' - \beta(\zeta)\mu)^2 d\zeta d\zeta' d\mu d\mu'.
\end{aligned} \tag{5.45}$$

Next in order to end the proof, we add the right-hand sides of (5.43) and (5.45). We obtain the

term

$$\begin{aligned} & \int_0^{+\infty} \int_0^\infty \int_{-1}^1 \int_{-1}^1 K(\zeta, \zeta', \mu, \mu') (\delta(\zeta) - \delta(\zeta')) (\mu\beta(\zeta) - \beta(\zeta')) d\zeta d\zeta' d\mu d\mu' \\ & - \frac{1}{2} \int_0^{+\infty} \int_0^\infty \int_{-1}^1 \int_{-1}^1 K(\zeta, \zeta', \mu, \mu') (\beta(\zeta')\mu' - \beta(\zeta)\mu)^2 d\zeta d\zeta' d\mu d\mu' \\ & - \frac{1}{2} \int_0^{+\infty} \int_0^\infty \int_{-1}^1 \int_{-1}^1 K(\zeta, \zeta', \mu, \mu') (\delta(\zeta) - \delta(\zeta'))^2 d\zeta d\zeta' d\mu d\mu'. \end{aligned}$$

Moreover, the inequality

$$(\delta(\zeta) - \delta(\zeta'))(\beta(\zeta)\mu - \beta(\zeta')\mu') \leq \frac{1}{2}((\delta(\zeta) - \delta(\zeta'))^2 + (\beta(\zeta)\mu - \beta(\zeta')\mu')^2),$$

gives

$$\begin{aligned} & \int_0^{+\infty} \int_0^\infty \int_{-1}^1 \int_{-1}^1 K(\zeta, \zeta', \mu, \mu') (\delta(\zeta) - \delta(\zeta')) (\mu\beta(\zeta) - \mu'\beta(\zeta')) d\zeta d\zeta' d\mu d\mu' \\ & \leq \frac{1}{2} \int_0^{+\infty} \int_0^\infty \int_{-1}^1 \int_{-1}^1 K(\zeta, \zeta', \mu, \mu') (\delta(\zeta) - \delta(\zeta'))^2 d\zeta d\zeta' d\mu d\mu' \\ & + \frac{1}{2} \int_0^{+\infty} \int_0^\infty \int_{-1}^1 \int_{-1}^1 K(\zeta, \zeta', \mu, \mu') (\mu\beta(\zeta) - \mu'\beta(\zeta'))^2 d\zeta d\zeta' d\mu d\mu'. \end{aligned}$$

This gives the non positivity of (5.36) and the theorem follows. $\square$

5.5 Inverse Bremsstrahlung effect

To describe laser-plasma interaction, an other collision operator (Langdon operator) can be added to the equation (5.12). Indeed, to understand the energy absorption of laser radiation deposited in the matter, we need to know the mechanisms involved. Physically, the laser wave penetrating into the matter provides energy to the electrons which oscillate at a certain frequency. These last one take energy getting in collision with ions and after deflection, they absorb a photon by bremsstrahlung inverse (BI) effect. This is Langdon [32] who manifests first an interest to this heated phenomenon in suggesting an operator acting on distribution functions. The bremsstrahlung inverse phenomenon characterizes inelastic collisions between a photon and an electron in ion field. The photon disappears and its impulsion and its energy are transferred irreversibly to the electron.

First the extraction moment of Langdon operator is explained. Then we describe the Epperlein formalism ([24]) to express the electric field and the heat flux in such configuration.

In this work, we focus on electron-electron and electron-ion collision operator. This subsection is just an introduction of the Langdon operator and its relevance to describe all physics phenomena. We do not use this term in the next simulations. It will be interesting to use this Langdon term in a forthcoming work.

5.5.1 Moment closure for the classical kinetic equation with the Langdon operator

In this subsection, we consider the classical Maxwell-Fokker-Planck-Landau system. Then we add the Langdon operator to the collision part. It is also called the collisional absorption operator. The system obtained is given by

$$\left\{ \begin{array}{l} \partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = C_{ee}(f, f) + C_{ei}(f) + C_{lang}(f), \\ \partial_t \mathbf{E} - c^2 \nabla_{\mathbf{x}} \wedge \mathbf{B} = -\frac{\mathbf{j}}{\epsilon_0}, \\ \partial_t \mathbf{B} + \nabla_{\mathbf{x}} \wedge \mathbf{E} = 0, \\ \nabla_{\mathbf{x}} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \\ \nabla_{\mathbf{x}} \cdot \mathbf{B} = 0. \end{array} \right. \quad (5.46)$$

Here $C_{lang}(f)$ denotes the Langdon operator ([32]) and it is given by

$$C_{lang}(f) = A \operatorname{div}_{\mathbf{v}} \left(\frac{g(v)}{v^5} (\mathbf{v} \otimes \mathbf{v}) \nabla_{\mathbf{v}} f \right), \quad (5.47)$$

with $A = 2\pi n_e Z \frac{e^4}{m_e^2} \ln \lambda \frac{v_0^2}{3}$ and v_0 is the peak velocity of oscillation of the electrons in the high-

frequency electric field. g is a function defined such as $g(v) = \left(1 + \omega^{-2} \tau_{ei}^{-2}(v)\right)^{-1} = \left(1 + \frac{v_{\omega}^6}{v^6}\right)^{-1}$.

Here $\tau_{ei}^{-1}(v) \equiv \frac{2A}{v^3}$ is the electron-ion scattering rate and v_{ω} is defined by $\omega \tau_{ei}(v_{\omega}) = 1$. We consider here that the laser pulse is very large compared to other oscillatory phenomenon so we can approximate g such as $g(v) = 1$.

Then by extracting moments of (5.46) with respect to μ^i , we get

$$\left\{ \begin{array}{l} \partial_t f^0 + \nabla_x \cdot (\zeta f^1) + \partial_\zeta \left(\frac{q}{m} E \cdot f^1 \right) = Q_{ee}^0(f^0) + Q_{lang}^0(f^0), \\ \partial_t f^1 + \nabla_x \cdot (\zeta f^2) + \partial_\zeta \left(\frac{q}{m} f^2 E \right) - \frac{q}{m\zeta} (f^0 E + \zeta f^1 \wedge B - f^2 E) = Q_{ee}^1(f^1) + Q_{ei}^1(f^1) + Q_{lang}^1(f^1), \\ \partial_t E - c^2 \nabla_x \wedge B = -\frac{j}{\epsilon_0}, \\ \partial_t B + \nabla_x \wedge E = 0, \\ \nabla_x \cdot E = \frac{\rho}{\epsilon_0}, \\ \nabla_x \cdot B = 0. \end{array} \right. \quad (5.48)$$

Recall that the collision operators Q_{ee}^i and Q_{ei}^1 are given by (5.32). Here Q_{lang}^0 and Q_{lang}^1 represent the Langdon operator and are expressed as

$$Q_{lang}^0(f^0) = A \frac{\partial}{\partial \zeta} \left(\frac{g(\zeta)}{\zeta^3} \frac{\partial f^0}{\partial \zeta} - \frac{2g(\zeta)}{\zeta^4} f^0 \right), \quad (5.49)$$

$$Q_{lang}^1(f^1) = A \frac{\partial}{\partial \zeta} \left(f^1 \frac{2g(\zeta)}{\zeta^4} - \frac{\partial f^1}{\partial \zeta} \frac{g(\zeta)}{\zeta^3} \right). \quad (5.50)$$

Moment closure for the Langdon operator is detailed in appendix 5.13.

5.5.2 Analytical solution: Ridgers transport theory

In [42], Ridgers *et al.* have proposed to develop transport equations in the presence of inverse bremsstrahlung (IB) heating and magnetic fields. They revisit the transport theory and determine the effects of IB heating on transport by deriving a transport theory for a super gaussian electron distribution function in the presence of magnetic fields.

From the formalism of Epperlein [24], the resulting transport relations (classical Ohm's law and heat flow equation) are given by

$$\left\{ \begin{array}{l} q = -\kappa_{ri} \cdot \nabla_x T_e - T_e \psi_{ri} \cdot \frac{\mathbf{j}}{e} - \phi_{ri} \cdot \nabla_x p, \\ \mathbf{E} = \frac{1}{en_e} \left(-\gamma_{ri} \nabla_x p + \mathbf{j} \times \mathbf{B} - \frac{\alpha_{ri} \mathbf{j}}{n_e e} - n_e \beta_{ri} \cdot \nabla_x T_e \right). \end{array} \right. \quad (5.51)$$

As in subsection 4.5, the heat flux and the electric field can then be rewritten in terms of dimensionless quantities, at a reference point denoted by the subscript R, for the particular 1D geometry of our temperature gradient configuration. The scaling using a collision frequency is used.

We obtain

$$\left\{ \begin{array}{l} q_1 = -\frac{3}{7} \sqrt{\frac{\pi}{2}} \kappa_{ri,\perp}^c \nabla_{x_1} T_e^{\frac{7}{2}} - T_e (\beta_{ri,\perp}^c j_1 - \beta_{ep,\wedge}^c j_2) - \frac{3}{2} \sqrt{\frac{\pi}{2}} n_e^{-1} T_e^{\frac{3}{2}} \phi_{ri,\perp}^c \nabla_{x_1} p, \\ q_2 = -\frac{3}{7} \sqrt{\frac{\pi}{2}} \kappa_{ri,\wedge}^c \nabla_{x_1} T_e^{\frac{7}{2}} - T_e (\beta_{ri,\perp}^c j_2 - \beta_{ep,\wedge}^c j_1) - \frac{3}{2} \sqrt{\frac{\pi}{2}} n_e^{-1} T_e^{\frac{3}{2}} \phi_{ri,\wedge}^c \nabla_{x_1} p, \\ E_1 = -n_e^{-1} \gamma_{ri,\perp}^c \nabla_{x_1} p + n_e^{-1} j_2 B_3 - \frac{2}{3} \sqrt{\frac{2}{\pi}} (\alpha_{ri,\perp}^c j_1 + \alpha_{ri,\wedge}^c j_2) - \beta_{ri,\perp}^c \nabla_{x_1} T_e, \\ E_2 = -n_e^{-1} \gamma_{ri,\wedge}^c \nabla_{x_1} p + n_e^{-1} j_1 B_3 - \frac{2}{3} \sqrt{\frac{2}{\pi}} (\alpha_{ri,\perp}^c j_2 - \alpha_{ri,\wedge}^c j_1) - \beta_{ri,\wedge}^c \nabla_{x_1} T_e. \end{array} \right. \quad (5.52)$$

where $\alpha_{ri}^c, \beta_{ri}^c, \kappa_{ri}^c, \gamma_{ri}^c$ and ϕ_{ri}^c are the dimensionless transport coefficients which are functions of Z and the hall parameters $\chi = \omega_c \tau$ only.

They are defined as

$$\left\{ \begin{array}{l} \text{dimensioned quantities} \rightarrow \text{dimensionless quantities} \\ \alpha_{ri} \left(\frac{\tau}{m_e n_e} \right) \Big|_R \rightarrow \alpha_{ri} = \frac{n_e m_e}{\tau} \alpha_{ri}^c, \\ \beta_{ri} \rightarrow \beta_{ri} = \beta_{ri}^c, \\ \kappa_{ri} \left(\frac{m_e}{n_e \tau T_e} \right) \Big|_R \rightarrow \kappa_{ri} = \frac{n_e T_e \tau}{m_e} \kappa_{ri}^c, \\ \gamma_{ri} \rightarrow \gamma_{ri} = \gamma_{ri}^c, \\ \phi_{ri} \left(\frac{m_e}{T_e \tau} \right) \Big|_R \rightarrow \phi_{ri} = \frac{T_e \tau}{m_e} \phi_{ri}^c. \end{array} \right. \quad (5.53)$$

$$\text{where } \tau = \frac{12\pi^2 \epsilon_0^2 \sqrt{m_e} T_e^{\frac{3}{2}}}{\sqrt{2\pi} n_i Z^2 e^4 \ln \Lambda} = \frac{3}{2} \sqrt{\frac{\pi}{2}} \frac{1}{\nu_{ei}}.$$

5.6 Semi-discretization of the problem

In this section we propose an energy discretization such that the Landau form and the Rozenbluth form of the discretized operators are the same (see Remark 14). The key point of the

scheme construction is the approximation of F^0 and F^1 on the interface through an entropic average (Theorem 16). This allows to reproduce in Theorem 17 the proof of Theorem 13 in a discrete setting showing that the scheme is entropic.

5.6.1 Introduction

Let us define the primal mesh $\mathcal{M}$, for the velocity variable ζ , decomposed into a family of rectangles $\mathcal{M}_{j+\frac{1}{2}} =]\zeta_{j-1}, \zeta_j[$, $\forall j \in [1, m]$, where $\zeta_j = j\Delta\zeta_j$ and $m \in \mathbb{N}$ the number of points which discretize the energy domain. $\Delta\zeta_j$ represents the step of discretization which can be variable. We denote by $\mathcal{D}$ its associated dual mesh consisting of cells $\mathcal{D}_j =]\zeta_{j-\frac{1}{2}}, \zeta_{j+\frac{1}{2}}[$ where $\zeta_{j-\frac{1}{2}} = (j - \frac{1}{2})\Delta\zeta_j$. Let h_j be an approximation of $h(\zeta_j)$ for all distribution function h and $h_{j+\frac{1}{2}}$ an approximation of $h(\zeta_{j+\frac{1}{2}})$. For the sake of clarity, we define

$$F_j^i(\zeta) = \frac{f_j^i(\zeta)}{\zeta_j^2}, \quad i = 0, 1. \quad (5.54)$$

To solve this minimization problem, let us introduce the realizability domain $\mathcal{A}_d$ defined by

$$\mathcal{A}_d = \left\{ G = \begin{pmatrix} G_j^0 \\ G_j^1 \end{pmatrix} \in \mathbb{R}^2 / \exists g_j \text{ as } F_j^i = \langle \mu^i g_j(\alpha, \mu) \rangle \forall i = 0, 1, \forall j \in [1, m] \right\}.$$

The aim of this section is to propose an entropic discretization of (5.30) and (5.31) in a space homogeneous situation

$$\begin{cases} \partial_t f_j^0 = Q_{ee,j}^0, \\ \partial_t f_j^1 = Q_{ee,j}^1 + Q_{ei,j}^1, \end{cases} \quad (5.55)$$

where the operators $Q_{ee,j}^0$, $Q_{ee,j}^1$ and $Q_{ei,j}^1$ are respectively the discrete forms of $Q_{ee}(f^0)$, $Q_{ee}(f^1)$ and $Q_{ei}(f^1)$ defined in (5.32). The expressions of $Q_{ee,j}^0$, $Q_{ee,j}^1$ and $Q_{ei,j}^1$ are given in (5.56, 5.57).

Definition 1. The collision operators $Q_{ee,j}^0$, $Q_{ee,j}^1$ and $Q_{ei,j}^1$ involved in (5.55) are respectively given by

$$\begin{cases} Q_{ee,j}^0 = \frac{G_{j+\frac{1}{2}}^0 - G_{j-\frac{1}{2}}^0}{\Delta\zeta_j}, \\ Q_{ee,j}^1 = \frac{G_{j+\frac{1}{2}}^1 - G_{j-\frac{1}{2}}^1}{\Delta\zeta_j}, \\ Q_{ei,j}^1 = -\frac{2\alpha_{ei}}{\zeta_j^3} f_j^1, \end{cases} \quad (5.56)$$

$$\left\{ \begin{aligned} G_{j+\frac{1}{2}}^0 &= \zeta_{j+\frac{1}{2}} \sum_k \tilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) \left[F_{k+\frac{1}{2}}^0 \frac{1}{\zeta_{j+\frac{1}{2}}} \frac{F_{j+1}^0 - F_j^0}{\Delta \zeta_{j+\frac{1}{2}}} - \frac{1}{\zeta_{k+\frac{1}{2}}} F_{j+\frac{1}{2}}^0 \frac{F_{k+1}^0 - F_k^0}{\Delta \zeta_{k+\frac{1}{2}}} \right] \\ &\quad \zeta_{k+\frac{1}{2}}^2 \Delta \zeta_{k+\frac{1}{2}} \ , \\ G_{j+\frac{1}{2}}^1 &= \zeta_{j+\frac{1}{2}} \sum_k \tilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) \left[F_{k+\frac{1}{2}}^0 \frac{1}{\zeta_{j+\frac{1}{2}}} \frac{F_{j+1}^1 - F_j^1}{\Delta \zeta_{j+\frac{1}{2}}} - \frac{1}{\zeta_{k+\frac{1}{2}}} F_{j+\frac{1}{2}}^1 \frac{F_{k+1}^0 - F_k^0}{\Delta \zeta_{k+\frac{1}{2}}} \right] \\ &\quad \zeta_{k+\frac{1}{2}}^2 \Delta \zeta_{k+\frac{1}{2}} \ , \end{aligned} \right. \quad (5.57)$$
$$f_{j+\frac{1}{2}} = \frac{f_{j+1} - f_j}{\ln(f_{j+1}) - \ln(f_j)} \text{ if } f_{j+1} \neq f_j \text{ and } f_{j+1} \text{ else.}$$

Hence in the present chapter, we generalize this approach to a system.

More precisely we aim to give a discretization of (5.39) and (5.44) such that

$$\begin{cases} \frac{F_{j+1}^0 - F_j^0}{\Delta \zeta_{j+\frac{1}{2}}} = \frac{\Delta \ln(\rho_j)}{\Delta \zeta_{j+\frac{1}{2}}} F_{j+\frac{1}{2}}^0 - \frac{\Delta \alpha_j}{\Delta \zeta_{j+\frac{1}{2}}} F_{j+\frac{1}{2}}^1, \\ \frac{F_{j+1}^1 - F_j^1}{\Delta \zeta_{j+\frac{1}{2}}} = \frac{\Delta \ln(\rho_j)}{\Delta \zeta_{j+\frac{1}{2}}} F_{j+\frac{1}{2}}^1 - \frac{\Delta \alpha_j}{\Delta \zeta_{j+\frac{1}{2}}} F_{j+\frac{1}{2}}^2, \end{cases} \quad (5.58)$$

where $\Delta \ln(\rho_j) = \ln(\rho_{j+1}) - \ln(\rho_j)$ and $\Delta \alpha_j = \alpha_{j+1} - \alpha_j$. ρ_j and α_j represent here the approximations of $\rho(\zeta_j)$ and $\alpha(\zeta_j)$.

That is why (F_{j+1}^0, F_j^0) , (F_{j+1}^1, F_j^1) , $\Delta \ln(\rho_j)$, $\Delta \alpha_j$ being given, $F_{j+\frac{1}{2}}^0$, $F_{j+\frac{1}{2}}^1$, $F_{j+\frac{1}{2}}^2$ are researched such that

$$\begin{cases} F_{j+1}^0 - F_j^0 = \Delta \ln(\rho_j) F_{j+\frac{1}{2}}^0 - \Delta \alpha_j F_{j+\frac{1}{2}}^1, \\ F_{j+1}^1 - F_j^1 = \Delta \ln(\rho_j) F_{j+\frac{1}{2}}^1 - \Delta \alpha_j F_{j+\frac{1}{2}}^2, \end{cases} \quad (5.59)$$

is satisfied.

5.6.2 Realizability domain

Proposition 9. By using the semi-discretized problem defined by equations (5.54, 5.55, 5.56, 5.57) and under the assumption that $\bar{f}_j(t=0) \in \mathcal{A}_d$, the realizability domain is conserved with respect to time, i.e. $\bar{f}_j(t) \in \mathcal{A}_d$ for any $t \geq 0$.

Proof. Let F^0 such as $F_j^0(t=0) > 0$ for any j . Hence there is $\tau > 0$, such that $F_j^0(t) > 0$ for any $t \in [0, \tau[$ and any j . Suppose that there exist j_0 such that $F_{j_0}^0(\tau) = 0$. By evaluating the time evolution of f_j^0 in ζ_{j_0} , we get

$$\begin{aligned} \partial_t f_{j_0}^0(\tau) &= \frac{1}{\Delta t} \left(A_{j_0+\frac{1}{2}} \frac{F_{j_0+1}^0(\tau) - F_{j_0}^0(\tau)}{\Delta \zeta} - A_{j_0-\frac{1}{2}} \frac{F_{j_0}^0(\tau) - F_{j_0-1}^0(\tau)}{\Delta \zeta} \right) \\ &+ \frac{1}{\Delta \zeta} \left(B_{j_0+\frac{1}{2}} F_{j_0+\frac{1}{2}}^0(\tau) - B_{j_0-\frac{1}{2}} F_{j_0-\frac{1}{2}}^0(\tau) \right), \end{aligned} \quad (5.60)$$

where

$$A_{j_0+\frac{1}{2}} = \sum_j \tilde{J}(\zeta_{j_0+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) F_{k+\frac{1}{2}}^0(\tau) \Delta \zeta_{k+\frac{1}{2}},$$

and

$$B_{j_0+\frac{1}{2}} = \zeta_{j_0+\frac{1}{2}} \sum_k \tilde{J}(\zeta_{j_0+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) \frac{F_{k+1}^0(\tau) - F_k^0(\tau)}{\Delta \zeta_{k+\frac{1}{2}}} \Delta \zeta_{k+\frac{1}{2}}.$$

From the definition of $F_{j_0+\frac{1}{2}}^0$ through the entropic average in ζ_{j_0} , it holds that

$$F_{j_0+\frac{1}{2}}^0 = \frac{F_{j_0+1}^0 - F_{j_0}^0}{\log(F_{j_0+1}^0) - \log(F_{j_0}^0)} \xrightarrow{F_{j_0}^0 \rightarrow 0} 0.$$

So (5.60) can be rewritten as

$$\partial_t f_{j_0}(\tau) = \frac{1}{\Delta \zeta^2} \left(A_{j_0+\frac{1}{2}} F_{j_0+1}^0(\tau) + A_{j_0-\frac{1}{2}} F_{j_0-1}^0(\tau) \right). \quad (5.61)$$

As $A_{j_0+\frac{1}{2}} \geq 0, A_{j_0-\frac{1}{2}} \geq 0$, equation (5.61) gives $\partial_t f_{j_0}(\tau) \geq 0$. F^0 satisfy for Δt small enough

$$F_{j_0}^0(\tau - \Delta t) = F_{j_0}^0(\tau) - \Delta t \partial_t F_{j_0}^0(\tau) + O(\Delta t^2) = -\Delta t \partial_t F_{j_0}^0(\tau) + O(\Delta t^2),$$

because $F_{j_0}^0(\tau) = 0$. If we consider that $\partial_t f_{j_0}(\tau) > 0$, we get $F_{j_0}^0(\tau - \Delta t) < 0$ for small Δt . Therefore we have

$$\frac{1}{\Delta \zeta^2} \left(A_{j_0+\frac{1}{2}} F_{j_0+1}^0(\tau) + A_{j_0-\frac{1}{2}} F_{j_0-1}^0(\tau) \right) = 0.$$

If $A_{j_0+\frac{1}{2}} = 0$ or $A_{j_0-\frac{1}{2}} = 0$, the definition of $A_{j_0+\frac{1}{2}}$ gives $\forall j, F_j^0(\tau) = 0$. In the other case, we get $F_{j_0+1}^0(\tau) = F_{j_0-1}^0(\tau) = 0$. By induction on j we still obtain that $\forall j, F_j(\tau) = 0$. Therefore by using the Cauchy-Lipschitz to the Cauchy problem (5.61) with the initial conditions $\forall j, F_j(\tau) = 0$, we get that $F_j(t) = 0, \forall j, \forall t > \tau$.

If for any $j, F_j(0) = 0$, we get $F_j(t) = 0$ by using the Cauchy-Lipschitz theorem, for $t \geq 0$ is a global solution. Finally consider the case where $F_{j_0}(0) = 0$ and $F_{j_0+1}(0) > 0$. If there is $\tau > 0$ s.t. $F_{j_0} \geq 0$ on $]0, \tau]$, we are in the same situation as the first one. If $F_{j_0} < 0$ on $]0, \tau]$, we have for Δt small enough, $F_{j_0}(\Delta t) = \partial_t F_{j_0} \Delta t + O(\Delta t^2)$. Then $\partial_t F_{j_0} \leq 0$. As $\partial_t F_{j_0}(0) < 0$, is impossible because of equation (5.61), $\partial_t F_{j_0}(0) = 0$. But this situation leads to $F_{j_0+1}(0) = 0$ which contradicts our assumption. The case where $\partial_t F_{j_0-1}(0) < 0$ and $\partial_t F_{j_0}(0) = 0$ being treated analogously, the positivity of the initial condition is preserved for f^0 .

Next we prove in the same way that the positivity of f^1 is also preserved. By arguing like for the continuous model we conclude that the realizability domain is preserved. $\square$

5.6.3 Boundary conditions

We have shown in the previous subsection that the semi-discretized model (5.54, 5.55, 5.56, 5.57) conserves the realizability domain. So we have to choose boundary conditions adapted to the conservation of the realizability domain.

Consider here the space homogeneous system (5.30, 5.31, 5.32). The conservation laws for electron-electron collision operator (2.12) read in spherical coordinates

$$\int_{S_2} \int_0^\infty C_{ee}(f) \begin{pmatrix} 1 \\ \frac{\zeta \Omega}{\zeta^2} \\ \frac{\zeta}{2} \end{pmatrix} d\Omega \zeta^2 d\zeta = 0, \quad t \geq 0. \quad (5.62)$$

Therefore, the conservations of mass, momentum and energy write

$$\int_0^\infty Q_{ee}^0(f^0) \begin{pmatrix} 1 \\ \frac{\zeta^2}{2} \end{pmatrix} \zeta^2 d\zeta = 0 \quad \text{and} \quad \int_0^\infty Q_{ee}^1(f^1) \zeta \zeta^2 d\zeta = 0.$$

After integrating over ζ_j the first equation of (5.55), mass conservation imposes the boundary condition $G_{-\frac{1}{2}}^0 = 0$.

After multiplying by ζ_j and integrating over ζ_j the second equation of (5.55), we obtain

$$\sum_{j=0}^\infty \zeta_j^3 \frac{\partial F_j^1}{\partial t} \Delta \zeta_j = \sum_{j=0}^\infty \zeta_j Q_j^1 \Delta \zeta_j = G_{-\frac{1}{2}}^1.$$

However, to satisfy the proof of Proposition 8 on the realizability domain, we must have the same conditions on G^0 and G^1 . So we use an approximation for f^1 , i.e. we take $G_{-\frac{1}{2}}^1 = 0$.

Of course, we lost the momentum conservation but this conservation is not satisfied by the operator C_{ei} and also therefore by $C_{ee} + C_{ei}$.

5.6.4 Entropic property of the scheme

Next the following theorem proves that the system (5.59) has a solution in the realizability domain $\mathcal{A}_d$.

Theoreme 16. $(F_j^0, F_{j+1}^0, F_j^1, F_{j+1}^1)$ being given, there are $(F_{j+\frac{1}{2}}^0, F_{j+\frac{1}{2}}^1) \in \mathcal{A}_d$ such that

$$\begin{cases} F_{j+1}^0 - F_j^0 = \Delta \ln(\rho_j) F_{j+\frac{1}{2}}^0 - \Delta \alpha_j F_{j+\frac{1}{2}}^1, \\ F_{j+1}^1 - F_j^1 = \Delta \ln(\rho_j) F_{j+\frac{1}{2}}^1 - \Delta \alpha_j F_{j+\frac{1}{2}}^2. \end{cases} \quad (5.63)$$

Proof. By setting $\bar{\alpha} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$ and $\bar{\mu} = \begin{pmatrix} \mu^0 \\ \mu^1 \end{pmatrix}$ vectors defining $f = \rho \exp(\bar{\alpha} \cdot \bar{\mu})$, F_i can be written as

$$\mathcal{F}(\bar{\alpha}) = \begin{pmatrix} F^0 \\ F^1 \end{pmatrix} = \langle \exp(\bar{\alpha} \cdot \bar{\mu}) \bar{\mu} \rangle,$$

where $\langle . \rangle$ is defined for any function Ψ by $\langle \Psi \rangle = \frac{1}{2} \int_{-1}^1 \Psi(\mu) d\mu$.

Consider the function ϕ_i such that $\phi_i(0) = 0$, with $i = 0, 1$, defined by

$$\phi_i : \begin{cases} [0, 1] & \rightarrow \mathbb{R}, \\ t & \mapsto \phi_i(t) = F^i(t\bar{\alpha}_d + (1-t)\bar{\alpha}_g) - F^i(\bar{\alpha}_g), \end{cases}$$

with $\bar{\alpha}_t = t\bar{\alpha}_d + (1-t)\bar{\alpha}_g$ and $\bar{\alpha}_d, \bar{\alpha}_g$ two different given values of $\bar{\alpha}$. As, $\phi_i(0) = 0$ we have

$$\phi_i(1) = \int_0^1 \phi'_i(s) ds,$$

and

$$F^i(\bar{\alpha}_d) - F^i(\bar{\alpha}_g) = \int_0^1 \nabla_{\bar{\alpha}} F^i(t\bar{\alpha}_d + (1-t)\bar{\alpha}_g) dt \cdot (\bar{\alpha}_d - \bar{\alpha}_g). \quad (5.64)$$

Since $\nabla_{\bar{\alpha}} F^i = \langle \exp(\bar{\alpha} \cdot \bar{\mu}) \bar{\mu} \mu^i \rangle$, (5.64) becomes

$$F^i(\bar{\alpha}_d) - F^i(\bar{\alpha}_g) = \int_0^1 \langle \exp(\bar{\alpha}_t \cdot \bar{\mu}) \bar{\mu} \mu^i \rangle \cdot (\bar{\alpha}_d - \bar{\alpha}_g). \quad (5.65)$$

The relation (5.65) can be written in the algebraic form

$$\mathcal{F}(\bar{\alpha}_d) - \mathcal{F}(\bar{\alpha}_g) = A(\bar{\alpha}_g, \bar{\alpha}_d)(\bar{\alpha}_d - \bar{\alpha}_g),$$

with

$$A(\bar{\alpha}_g, \bar{\alpha}_d) = \int_0^1 \langle \exp(\bar{\alpha}_t \cdot \bar{\mu}) \bar{\mu} \otimes \bar{\mu} \rangle dt.$$

By using the Fubini theorem, we can simplified the matrix A into

$$A(\bar{\alpha}_g, \bar{\alpha}_d) = \langle \int_0^1 \exp(\bar{\alpha}_t \cdot \bar{\mu}) dt \bar{\mu} \otimes \bar{\mu} \rangle = \langle \Psi(\bar{\mu}) \bar{\mu} \otimes \bar{\mu} \rangle,$$

with $\Psi(\bar{\mu}) = \int_0^1 \exp(\bar{\alpha}_t \cdot \bar{\mu}) dt$. Since $\Psi \geq 0$ and $\bar{\mu} \otimes \bar{\mu}$ is a positive bi-linear form, the matrix A is symmetric and positive defined. So there is a matrix A which contains the average values. This matrix is so invertible, thus the function $\bar{\alpha} \mapsto A\Delta\bar{\alpha}$ realizes a bijection.

We can conclude that there exists solutions for the system (5.63) in the realizability domain $\mathcal{A}_d$. $\square$

Next Theorem 16 is applied for the moments problem with $\bar{\alpha} = \begin{pmatrix} \ln(\rho) \\ -\alpha \end{pmatrix}$ and

$$\begin{cases} F_{j+1}^0 - F_j^0 &= A_{(0,0)}\Delta \ln(\rho_j) - A_{(0,1)}\Delta \alpha_j, \\ F_{j+1}^1 - F_j^1 &= A_{(1,0)}\Delta \ln(\rho_j) - A_{(1,1)}\Delta \alpha_j, \end{cases} \quad (5.66)$$

where

$$A = \begin{pmatrix} \langle \int_0^1 \exp(\bar{\alpha} \cdot \bar{\mu}) dt \rangle & \langle \int_0^1 \exp(\bar{\alpha} \cdot \bar{\mu}) dt \mu \rangle \\ \langle \int_0^1 \exp(\bar{\alpha} \cdot \bar{\mu}) dt \mu \rangle & \langle \int_0^1 \exp(\bar{\alpha} \cdot \bar{\mu}) dt \mu^2 \rangle \end{pmatrix} = \begin{pmatrix} F_{j+\frac{1}{2}}^0 & F_{j+\frac{1}{2}}^1 \\ F_{j+\frac{1}{2}}^1 & F_{j+\frac{1}{2}}^2 \end{pmatrix}.$$

Finally we prove that the scheme is entropic.

Theoreme 17. *For G defined in formula (5.57),*

$$E = \sum_j \frac{G_{j+\frac{1}{2}}^0 - G_{j-\frac{1}{2}}^0}{\Delta \zeta_j} \Delta \zeta_j \ln(\rho_j) - \sum_j \frac{G_{j+\frac{1}{2}}^1 - G_{j-\frac{1}{2}}^1}{\Delta \zeta_j} \Delta \zeta_j \alpha_j - \frac{2\alpha_{ei}}{\zeta_j^3} f_j^1, \quad (5.67)$$

is an entropy for the system (5.55, 5.56, 5.57).

Proof. Firstly the third term defining E in (5.67), in the same way as in continuous model, we get

$$-\frac{2\alpha_{ei}}{\zeta_j^3} f_j^1 \leq 0.$$

Next the first term in (5.67) can be rewritten

$$\sum_j \frac{G_{j+\frac{1}{2}}^0 - G_{j-\frac{1}{2}}^0}{\Delta \zeta_j} \Delta \zeta_j \ln(\rho_j) = - \sum_j G_{j+\frac{1}{2}}^0 \ln\left(\frac{\rho_{j+1}}{\rho_j}\right). \quad (5.68)$$

By using the discrete form of the collision operator electron-electron $G^0(f^0)$ given by (5.57), the right-hand side of (5.68) reads

$$- \sum_{j,k} \zeta_{j+\frac{1}{2}} \tilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) \left[F_{k+\frac{1}{2}}^0 \frac{1}{\zeta_{j+\frac{1}{2}}} \frac{F_{j+1}^0 - F_j^0}{\Delta \zeta_{j+\frac{1}{2}}} - \frac{1}{\zeta_{k+\frac{1}{2}}} F_{j+\frac{1}{2}}^0 \frac{F_{k+1}^0 - F_k^0}{\Delta \zeta_{k+\frac{1}{2}}} \right] \ln\left(\frac{\rho_{j+1}}{\rho_j}\right) \\ \zeta_{k+\frac{1}{2}}^2 \Delta \zeta_{k+\frac{1}{2}} \Delta \zeta_{j+\frac{1}{2}}. \quad (5.69)$$

By setting

$$\begin{cases} \delta_{j+\frac{1}{2}} &= \frac{1}{\zeta_{j+\frac{1}{2}}} \frac{\ln(\rho_{j+1}) - \ln(\rho_j)}{\Delta \zeta_{j+\frac{1}{2}}}, \\ \beta_{j+\frac{1}{2}} &= \frac{1}{\zeta_{j+\frac{1}{2}}} \frac{\alpha_{j+1} - \alpha_j}{\Delta \zeta_{j+\frac{1}{2}}}, \end{cases} \quad (5.70)$$

which represent the discrete version of (5.42), (5.69) can be simplified into

$$- \sum_{j,k} \zeta_{j+\frac{1}{2}}^2 \zeta_{k+\frac{1}{2}}^2 \tilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) \left[F_{k+\frac{1}{2}}^0 \frac{1}{\zeta_{j+\frac{1}{2}}} \frac{F_{j+1}^0 - F_j^0}{\Delta \zeta_{j+\frac{1}{2}}} - \frac{1}{\zeta_{k+\frac{1}{2}}} F_{j+\frac{1}{2}}^0 \frac{F_{k+1}^0 - F_k^0}{\Delta \zeta_{k+\frac{1}{2}}} \right] \delta_{j+\frac{1}{2}} \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}} .$$

Next by using Theorem 16, there are $(F_{j+\frac{1}{2}}^0, F_{j+\frac{1}{2}}^1)$ such that

$$\begin{cases} \frac{1}{\zeta_{j+\frac{1}{2}}} \frac{F_{j+1}^0 - F_j^0}{\Delta \zeta_{j+\frac{1}{2}}} = \delta_{j+\frac{1}{2}} F_{j+\frac{1}{2}}^0 - \beta_{j+\frac{1}{2}} F_{j+\frac{1}{2}}^1 , \\ \frac{1}{\zeta_{j+\frac{1}{2}}} \frac{F_{j+1}^1 - F_j^1}{\Delta \zeta_{j+\frac{1}{2}}} = F_{j+\frac{1}{2}}^1 \delta_{j+\frac{1}{2}} - F_{j+\frac{1}{2}}^2 \beta_{j+\frac{1}{2}} . \end{cases} \quad (5.71)$$

Therefore, we obtain

$$\begin{aligned} & - \sum_j G_{j+\frac{1}{2}}^0 \ln \left(\frac{\rho_{j+1}}{\rho_j} \right) \\ &= - \sum_{j,k} \zeta_{j+\frac{1}{2}}^2 \zeta_{k+\frac{1}{2}}^2 \tilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) F_{j+\frac{1}{2}}^0 F_{k+\frac{1}{2}}^0 (\delta_{j+\frac{1}{2}} - \delta_{k+\frac{1}{2}}) \delta_{j+\frac{1}{2}} \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}} \\ &+ \sum_{j,k} \zeta_{j+\frac{1}{2}}^2 \zeta_{k+\frac{1}{2}}^2 \tilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) (F_{j+\frac{1}{2}}^1 F_{k+\frac{1}{2}}^0 \beta_{j+\frac{1}{2}} - F_{j+\frac{1}{2}}^0 F_{k+\frac{1}{2}}^1 \beta_{k+\frac{1}{2}}) \delta_{j+\frac{1}{2}} \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}} . \end{aligned} \quad (5.72)$$

Next we introduce

$$K_{j+\frac{1}{2}, k+\frac{1}{2}}(\mu, \mu') = \zeta_{j+\frac{1}{2}}^2 \zeta_{k+\frac{1}{2}}^2 \tilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) f(\zeta_{j+\frac{1}{2}}, \mu) f(\zeta_{k+\frac{1}{2}}, \mu') , \quad (5.73)$$

which corresponds to the discrete form of (5.41). Hence (5.72) can be rewritten as

$$\begin{aligned} & - \sum_j G_{j+\frac{1}{2}}^0 \ln \left(\frac{\rho_{j+1}}{\rho_j} \right) \\ &= - \int_{-1}^1 \int_{-1}^1 \sum_{i,j} K_{j+\frac{1}{2}, k+\frac{1}{2}}(\mu, \mu') (\delta_{j+\frac{1}{2}} - \delta_{k+\frac{1}{2}}) \delta_{j+\frac{1}{2}} \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}} d\mu d\mu' \\ &+ \int_{-1}^1 \int_{-1}^1 \sum_{i,j} K_{j+\frac{1}{2}, k+\frac{1}{2}}(\mu, \mu') (\beta_{j+\frac{1}{2}} - \beta_{k+\frac{1}{2}}) \delta_{j+\frac{1}{2}} \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}} d\mu d\mu' . \end{aligned} \quad (5.74)$$

By exchanging the indexes j and k and by using the symmetry of the operator K , the first term of the right-hand side of (5.74) writes

$$\begin{aligned} & - \int_{-1}^1 \int_{-1}^1 \sum_{j,k} K_{j+\frac{1}{2},k+\frac{1}{2}}(\mu, \mu') (\delta_{j+\frac{1}{2}} - \delta_{k+\frac{1}{2}}) \delta_{j+\frac{1}{2}} \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}} d\mu d\mu' \\ & = -\frac{1}{2} \int_{-1}^1 \int_{-1}^1 \sum_{j,k} K_{j+\frac{1}{2},k+\frac{1}{2}}(\mu, \mu') (\delta_{j+\frac{1}{2}} - \delta_{k+\frac{1}{2}})^2 \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}} d\mu d\mu' , \end{aligned} \quad (5.75)$$

and the second one reads

$$\begin{aligned} & \int_{-1}^1 \int_{-1}^1 \sum_{j,k} K_{j+\frac{1}{2},k+\frac{1}{2}}(\mu, \mu') (\mu \beta_{j+\frac{1}{2}} - \mu' \beta_{k+\frac{1}{2}}) \delta_{j+\frac{1}{2}} \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}} d\mu d\mu' \\ & = \frac{1}{2} \int_{-1}^1 \int_{-1}^1 \sum_{j,k} K_{j+\frac{1}{2},k+\frac{1}{2}}(\mu, \mu') (\delta_{j+\frac{1}{2}} - \delta_{k+\frac{1}{2}}) (\mu \beta_{j+\frac{1}{2}} - \mu' \beta_{k+\frac{1}{2}}) \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}} d\mu d\mu' . \end{aligned} \quad (5.76)$$

Therefore, by adding (5.75) and (5.76), we get

$$\begin{aligned} & - \sum_j G_{j+\frac{1}{2}}^0 \ln \left(\frac{\rho_{j+1}}{\rho_j} \right) \\ & = -\frac{1}{2} \int_{-1}^1 \int_{-1}^1 \sum_{j,k} K_{j+\frac{1}{2},k+\frac{1}{2}}(\mu, \mu') (\delta_{j+\frac{1}{2}} - \delta_{k+\frac{1}{2}})^2 \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}} d\mu d\mu' \\ & + \frac{1}{2} \int_{-1}^1 \int_{-1}^1 \sum_{j,k} K_{j+\frac{1}{2},k+\frac{1}{2}}(\mu, \mu') (\delta_{j+\frac{1}{2}} - \delta_{k+\frac{1}{2}}) (\mu \beta_{j+\frac{1}{2}} - \mu' \beta_{k+\frac{1}{2}}) \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}} d\mu d\mu' . \end{aligned}$$

Next it remains to study the term

$$\sum_j \frac{G_{j+\frac{1}{2}}^1 - G_{j-\frac{1}{2}}^1}{\Delta \zeta_j} \Delta \zeta_j \alpha_j = - \sum_j G_{j+\frac{1}{2}}^1 (\alpha_{j+1} - \alpha_j) . \quad (5.77)$$

By using the discretized form of the collision operator $Q_{ee}^1(f^1)$ given by (5.57), the right-hand side of (5.77) can be expressed as

$$\begin{aligned} & - \sum_{j,k} \zeta_{j+\frac{1}{2}} \tilde{\mathcal{J}}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) \left[F_{k+\frac{1}{2}}^0 \frac{1}{\zeta_{j+\frac{1}{2}}} \frac{F_{j+1}^1 - F_j^1}{\Delta \zeta_{j+\frac{1}{2}}} - \frac{1}{\zeta_{k+\frac{1}{2}}} F_{j+\frac{1}{2}}^1 \frac{F_{k+1}^0 - F_k^0}{\Delta \zeta_{k+\frac{1}{2}}} \right] (\alpha_{j+1} - \alpha_j) \\ & \quad \zeta_{k+\frac{1}{2}}^2 \Delta \zeta_{k+\frac{1}{2}} \Delta \zeta_{j+\frac{1}{2}} . \end{aligned} \quad (5.78)$$

For the sake of clarity, recall that $\delta_{j+\frac{1}{2}}$ and $\beta_{j+\frac{1}{2}}$ are given by (5.70). So the expression (5.78) writes

$$-\sum_{j,k} \zeta_{j+\frac{1}{2}}^2 \zeta_{k+\frac{1}{2}}^2 \tilde{\mathcal{J}}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) \left[F_{k+\frac{1}{2}}^0 \frac{1}{\zeta_{j+\frac{1}{2}}} \frac{F_{j+1}^1 - F_j^1}{\Delta \zeta_{j+\frac{1}{2}}} - \frac{1}{\zeta_{k+\frac{1}{2}}} F_{j+\frac{1}{2}}^1 \frac{F_{k+1}^0 - F_k^0}{\Delta \zeta_{k+\frac{1}{2}}} \right] \beta_{j+\frac{1}{2}} \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}}. \quad (5.79)$$

By using expressions of $F_{j+\frac{1}{2}}^0$ and $F_{j+\frac{1}{2}}^1$ given in (5.71), we obtain

$$\begin{aligned} & -\sum_j G_{j+\frac{1}{2}}^1 (\alpha_{j+1} - \alpha_j) \\ &= -\int_{-1}^1 \int_{-1}^1 \sum_{j,k} \tilde{\mathcal{J}}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) \left[F_{j+\frac{1}{2}}^1 F_{k+\frac{1}{2}}^0 (\delta_{j+\frac{1}{2}} - \delta_{k+\frac{1}{2}}) \right] \beta_{j+\frac{1}{2}} \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}} d\mu d\mu' \\ &+ \int_{-1}^1 \int_{-1}^1 \sum_{j,k} \tilde{\mathcal{J}}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) \left[F_{j+\frac{1}{2}}^2 F_{k+\frac{1}{2}}^0 \beta_{j+\frac{1}{2}} - F_{j+\frac{1}{2}}^1 F_{k+\frac{1}{2}}^1 \beta_{k+\frac{1}{2}} \right] \beta_{j+\frac{1}{2}} \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}} d\mu d\mu'. \end{aligned}$$

Hence, by using the definition of $K_{j+\frac{1}{2}, k+\frac{1}{2}}$ given in (5.73), it holds that

$$\begin{aligned} & -\sum_j G_{j+\frac{1}{2}}^1 (\alpha_{j+1} - \alpha_j) \\ &= -\int_{-1}^1 \int_{-1}^1 \sum_{j,k} K_{j+\frac{1}{2}, k+\frac{1}{2}}(\mu, \mu') (\delta_{j+\frac{1}{2}} - \delta_{k+\frac{1}{2}}) \mu \beta_{j+\frac{1}{2}} \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}} d\mu d\mu' \\ &+ \int_{-1}^1 \int_{-1}^1 \sum_{j,k} K_{j+\frac{1}{2}, k+\frac{1}{2}}(\mu, \mu') (\mu \beta_{j+\frac{1}{2}} - \mu' \beta_{k+\frac{1}{2}}) \mu \beta_{j+\frac{1}{2}} \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}} d\mu d\mu'. \end{aligned}$$

By proceeding as in the proof of (5.77), we get

$$\begin{aligned} & \sum_j G_{j+\frac{1}{2}}^1 (\alpha_{j+1} - \alpha_j) \\ &= -\frac{1}{2} \int_{-1}^1 \int_{-1}^1 \sum_{j,k} K_{j+\frac{1}{2}, k+\frac{1}{2}}(\mu, \mu') (\mu \beta_{j+\frac{1}{2}} - \mu' \beta_{k+\frac{1}{2}}) (\delta_{j+\frac{1}{2}} - \delta_{k+\frac{1}{2}}) \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}} d\mu d\mu' \\ &+ \frac{1}{2} \int_{-1}^1 \int_{-1}^1 \sum_{j,k} K_{j+\frac{1}{2}, k+\frac{1}{2}}(\mu, \mu') (\mu \beta_{j+\frac{1}{2}} - \mu' \beta_{k+\frac{1}{2}})^2 \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}} d\mu d\mu'. \quad (5.80) \end{aligned}$$

Then, by adding the two right-hand sides of (5.77) and (5.80) and by using that

$$(\delta_{j+\frac{1}{2}} - \delta_{k+\frac{1}{2}}) (\mu \beta_{j+\frac{1}{2}} - \mu' \beta_{k+\frac{1}{2}}) \leq \frac{1}{2} \left((\delta_{j+\frac{1}{2}} - \delta_{k+\frac{1}{2}})^2 + (\mu \beta_{j+\frac{1}{2}} - \mu' \beta_{k+\frac{1}{2}})^2 \right),$$

we get the non positivity of E. □

5.7 Numerical scheme for the free transport

To discretize the free transport part in equation (5.2), a Finite Volume method based on *HLL* method ([28]) is derived .

The so-called *HLL* scheme has been developed by Harten, Lax and Van Leer [28] with the purpose of computing the Godunov flux in order to obtain an approximate solution for the Riemann problem. With this approach, the inter-cell numerical flux is directly approximated. Starting from a two waves configuration separating three constant states, and assuming the wave speeds are known, the application of the integral form of the conservation laws allows to approximate the flux. The resulting *HLL* Riemann solvers constitute very efficient and robust approximate methods ([3]). We will sketch here its principle in order to set the starting point of the study that follows.

For the sake of simplicity, we omit collision terms and let us consider here only the free transport equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \mathcal{F}(u) = 0 , \quad (5.81)$$

where $\mathcal{F}$ stands for the flux in the x space direction.

Let us define the mesh $\mathcal{M}$, for the space variable x , decomposed into a family of rectangles $\mathcal{M}_j =]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[$, $\forall j \in [0, n_x]$, where $x_{j+\frac{1}{2}} = (j + \frac{1}{2})\Delta x$ and $n_x \in \mathbb{N}$ the number of points which discretize the space domain (O, L) . Δx represents the uniform step of discretization. Let Δt the time step and $t^n = n\Delta t$.

As usual, we consider known a piecewise constant approximation $u^h(x, t^n)$ defined by $u^h(x, t^n) = u_j^n$ for $x \in \mathcal{M}_j$, $\forall j \in \mathbb{Z}$.

In the *HLL* approach, the exact Riemann solver solution is substituted by a single approximate state (see figure 5.7). Here b_L and b_R stand for the wave speed, i.e. the eigenvalue of the flux jacobian matrix and verify $b_L < 0 < b_R$.

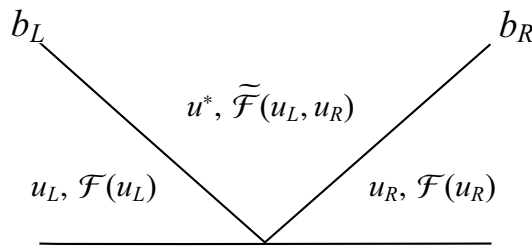


Figure 5.3: Structure of the *HLL* scheme - Riemann problem.

The approximate solution of the Riemann problem is given by

$$u_{HLL}(x, t) = \begin{cases} u_L & \text{if } \frac{x}{t} < b_L , \\ u^* & \text{if } b_L \leq \frac{x}{t} \leq b_R , \\ u_R & \text{if } b_R < \frac{x}{t} . \end{cases} \quad (5.82)$$

The *HLL* Riemann solver involves a “star” region that consists of a single constant state. Besides the search of weak solutions lead to the Rankine-Hugoniot jump condition

$$\begin{cases} -b_L(u^* - u_L) + (\tilde{\mathcal{F}}(u_L, u_R) - \mathcal{F}(u_L)) = 0, \\ -b_R(u_R - u^*) + (\mathcal{F}(u_R) - \tilde{\mathcal{F}}(u_L, u_R)) = 0. \end{cases} \quad (5.83)$$

From (5.83), expression for u^* and $\tilde{\mathcal{F}}(u_L, u_R)$ are expressed as

$$\begin{aligned} u^* &= \frac{b_R u_R - b_L u_L - (\mathcal{F}(u_L) - \mathcal{F}(u_R))}{b_R - b_L}, \\ \tilde{\mathcal{F}}(u_L, u_R) &= \frac{b_R \mathcal{F}(u_L) - b_L \mathcal{F}(u_R) + b_R b_L (u_R - u_L)}{b_R - b_L}. \end{aligned} \quad (5.84)$$

The *HLL* approximate Riemann solver (5.82, 5.84) is impose at each interface $x_{i+\frac{1}{2}}$ assuming CFL condition (5.85) ensuring that the Riemann solvers do not interact in the case where $b_{L,j+\frac{1}{2}} < 0$ and $b_{R,j-\frac{1}{2}} < 0$

$$\frac{\Delta t}{\Delta x} \leq \frac{b_{L,j+\frac{1}{2}} b_{R,j-\frac{1}{2}}}{b_{L,j+\frac{1}{2}} - b_{R,j-\frac{1}{2}}}. \quad (5.85)$$

The updated approximation at time t^{n+1} can be expressed as

$$u_j^{n+1} = \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^h(x, t^n + \Delta t) dx.$$

Integrating the equation (5.81) on a control volume $C_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times (t^n, t^{n+1})$, it yields

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} \left(\mathcal{F}_{j+\frac{1}{2}}^n - \mathcal{F}_{j-\frac{1}{2}}^n \right),$$

where

$$\mathcal{F}_{j+\frac{1}{2}}^n(u_j^n, u_{j+1}^n) = \begin{cases} \mathcal{F}(u_j^n) & \text{if } 0 < b_{L,j+\frac{1}{2}}, \\ \tilde{\mathcal{F}}_{j+\frac{1}{2}}^n(u_j^n, u_{j+1}^n) & \text{if } b_{L,j+\frac{1}{2}} \leq 0 \leq b_{R,j+\frac{1}{2}}, \\ \mathcal{F}(u_{j+1}^n) & \text{if } b_{R,j+\frac{1}{2}} < 0. \end{cases} \quad (5.86)$$

5.8 Numerical tests for validation

Now, we turn considering numerical applications to illustrate the properties of the previous method. For this numerical application, we chose a one dimensional direction of propagation. Therefore we take $\mu \in [-1, 1]$ as the direction of propagation variable instead of Ω . The two first subsections are devoted to the the full discrete model

$$\begin{cases} \frac{1}{\Delta t} (f_j^{0,N+1} - f_j^{0,N}) = \mathcal{Q}_{ee,j}^0, \\ \frac{1}{\Delta t} (f_j^{1,N+1} - f_j^{1,N}) = \mathcal{Q}_{ee,j}^1 + \mathcal{Q}_{ei,j}^1, \end{cases} \quad (5.87)$$

where the operators $\mathcal{Q}_{ee,j}^0$, $\mathcal{Q}_{ee,j}^1$ and $\mathcal{Q}_{ei,j}^1$ are defined in (5.56). The scaling used here gives time steps Δt to be of the order of the electron-ion collision time.

In the first subsection we consider the convergence from an non-centered Maxwellian to a centered one. Next the second test case illustrates the convergence from a bi-Maxwellian distribution function to a centered one.

5.8.1 Convergence to the steady state with an non-centered Maxwellian

This test case is devoted to study the evolution of an non-centered distribution function to the steady state thanks to the collision effects.

The initial distribution function f is a Maxwellian distribution function centered around $\zeta_d = 2$ with a density $n = n_c$ and a temperature $T = 1 \text{ KeV}$. More precisely f is defined by

$$f(t = 0, \zeta) = \left(\frac{m_e}{2\pi}\right)^{3/2} \exp\left(-\frac{(\zeta - \zeta_d)^2}{2}\right).$$

The energy domain is discretized with 64 points and the length of the energy space is denoted by $\zeta_{\max} = 8$ (for the 4 thermal velocity). We choose a time step satisfying the CFL condition. The scaling used here gives time steps to be of the order of the electron-electron collision time. The numerical results are computed at time $t = 100$. For the validation procedure, the numerical results are compared with the exact solution, i.e. the centered Maxwellian distribution with the same mass and energy than $f(t = 0, \zeta)$.

The boundary conditions for the energy domain satisfies also naturally the positivity constraints of f^0 and the conservation of the mass, i.e. at the left boundary $f_{-1}^0 = f_0^0$ and $f_{-1}^1 = -f_0^1$.

Isotropic case

For this test case, the initial data is chosen in the class of isotropic functions, i.e.

$$\begin{cases} F^0(t = 0, \zeta) = \exp(-(\zeta - 2)^2), \\ F^1(t = 0, \zeta) = 0, \end{cases}$$

In Figure 5.4 (a) and figure 5.4 (b), we have represented the evolution of mass and energy in function of time. We observe that mass and energy are conserved. This result is consistent with the conservation laws given in subsection 5.3.1.

Besides, the evolution of the entropy (figure 5.4 (c)) induced by the scheme is decreasing in time as shown in Theorem 13.

Notice also on figures 5.5 (a) and 5.5 (b) that the realizability domain is preserved in time, i.e. F^0 remains positive and $|F^1| \leq F^0$. Moreover $F^1(t, \zeta)$ remains equal to 0 and the isotropic part

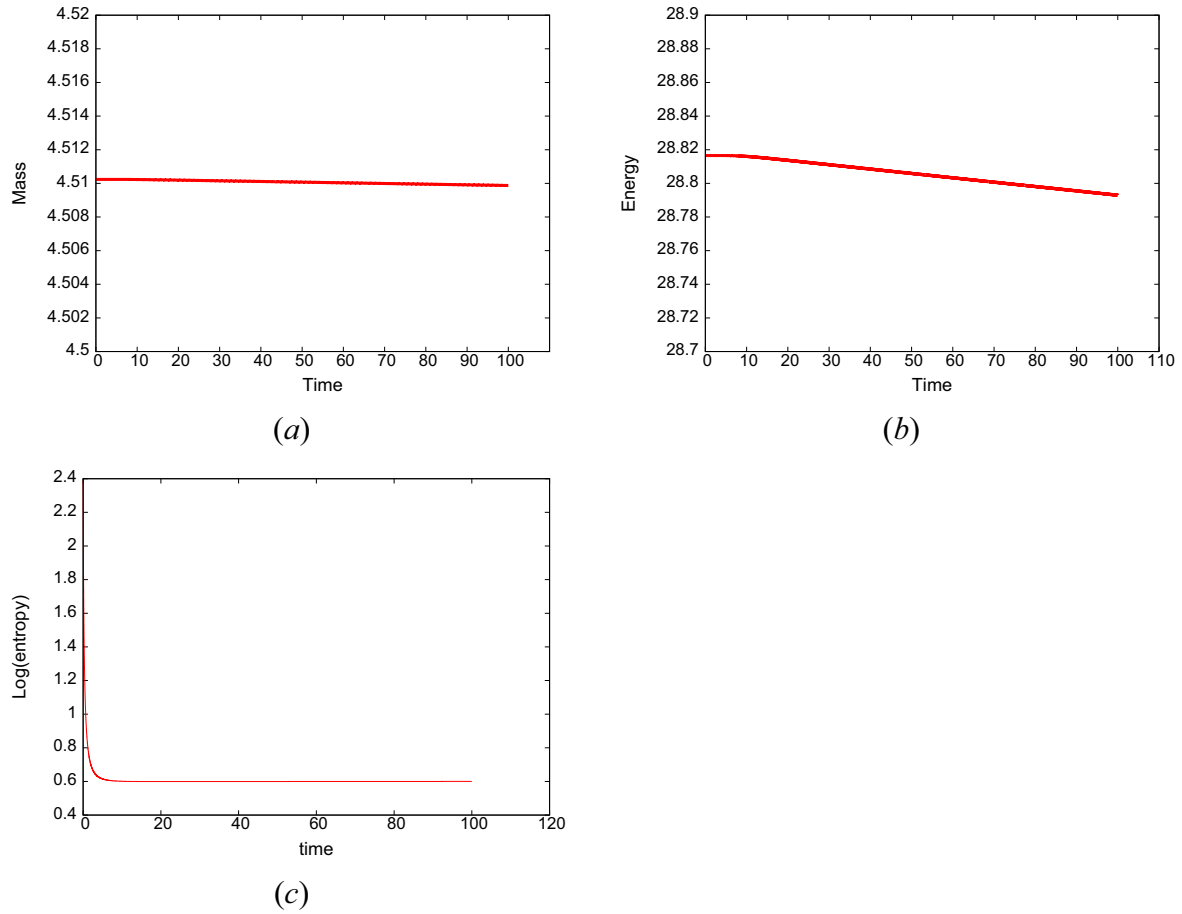


Figure 5.4: Evolution of the mass (a), energy in time (b) and the entropy (c) with respect to time in the case of $\zeta_{\max} = 8$ with 64 points for isotropic initial data.

$F^0(t, \zeta)$ converges to a centered Maxwellian. As shown on figure 5.5 (c), the scheme has converged quickly from $t = 40$.

Figure 5.5 (d) shows the evolution of the distribution function F with $\mu = 1$, defined by its isotropic part F^0 and its anisotropic part F^1 . With the anisotropic coefficient a , we determine then α (5.7) and therefore F given by formula (5.5). We remark that the distribution function F converges to a centered Maxwellian distribution which is close to the exact equilibrium state with an L_2 relative error about 10^{-4} . This enhances the good accuracy of the scheme.

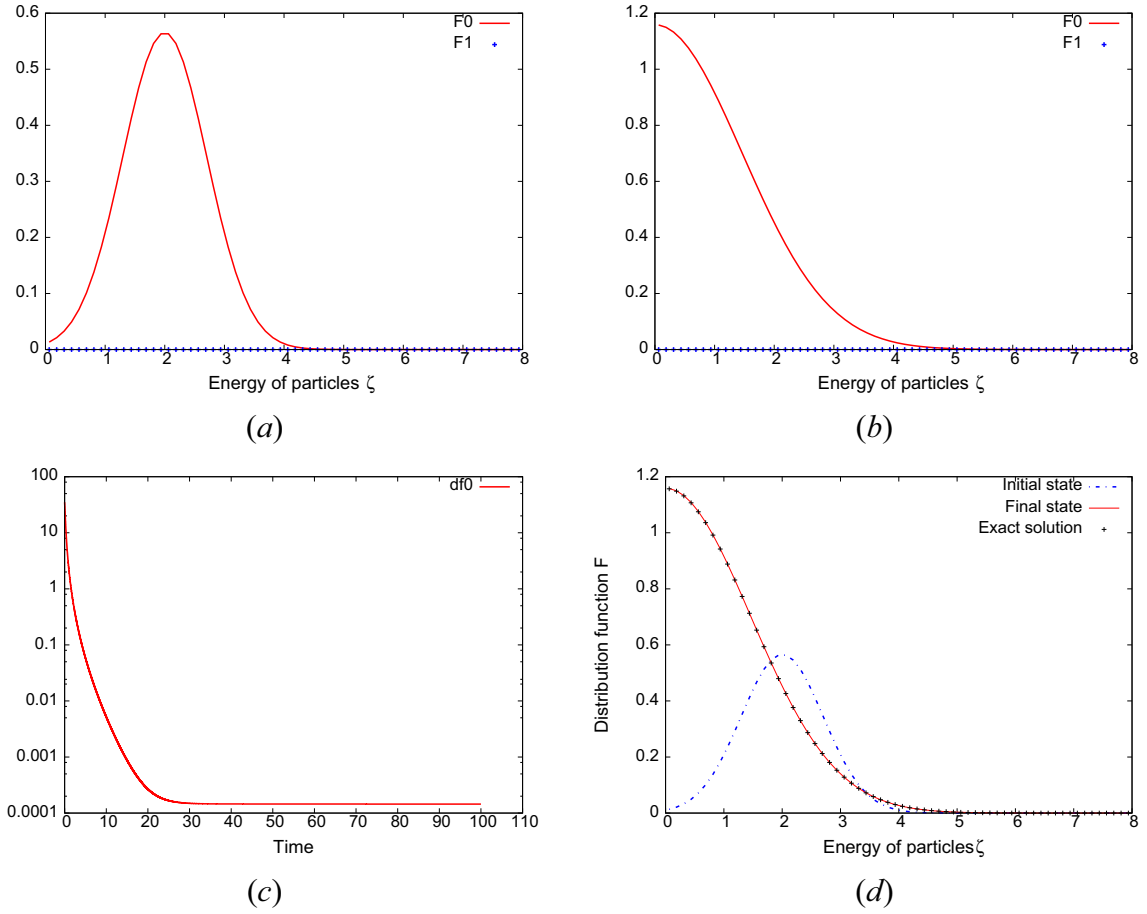


Figure 5.5: Description of the moments F^0 and F^1 with respect to the energy ζ at initial time (a) and at final time (b), evolution of the convergence in logarithmic scale i.e. $df0 = \sum_j |F_j^{0,N+1} - F_{j+1}^{0,N}|$ and $df1 = \sum_j |F_j^{1,N+1} - F_{j+1}^{1,N}|$ (c) and evolution of the distribution function F in time with $\mu = 1$ (d) in the case of $\zeta_{\max} = 8$ with 64 points.

Anisotropic case

In this subsection, an anisotropic initial function is chosen such as

$$\begin{cases} F^0(t=0, \zeta) = \exp(-(\zeta - 2)^2), \\ F^1(t=0, \zeta) = \frac{F^0(t=0, \zeta)}{3}. \end{cases}$$

Figures 5.6 (a) and 5.6 (b) show the conservation of mass and energy in time which is consistent with the conservation laws given in subsection 5.3.1. Moreover, the evolution of the entropy induced by the scheme is decreasing in time (5.6 (c)) as shown by Theorem 13.

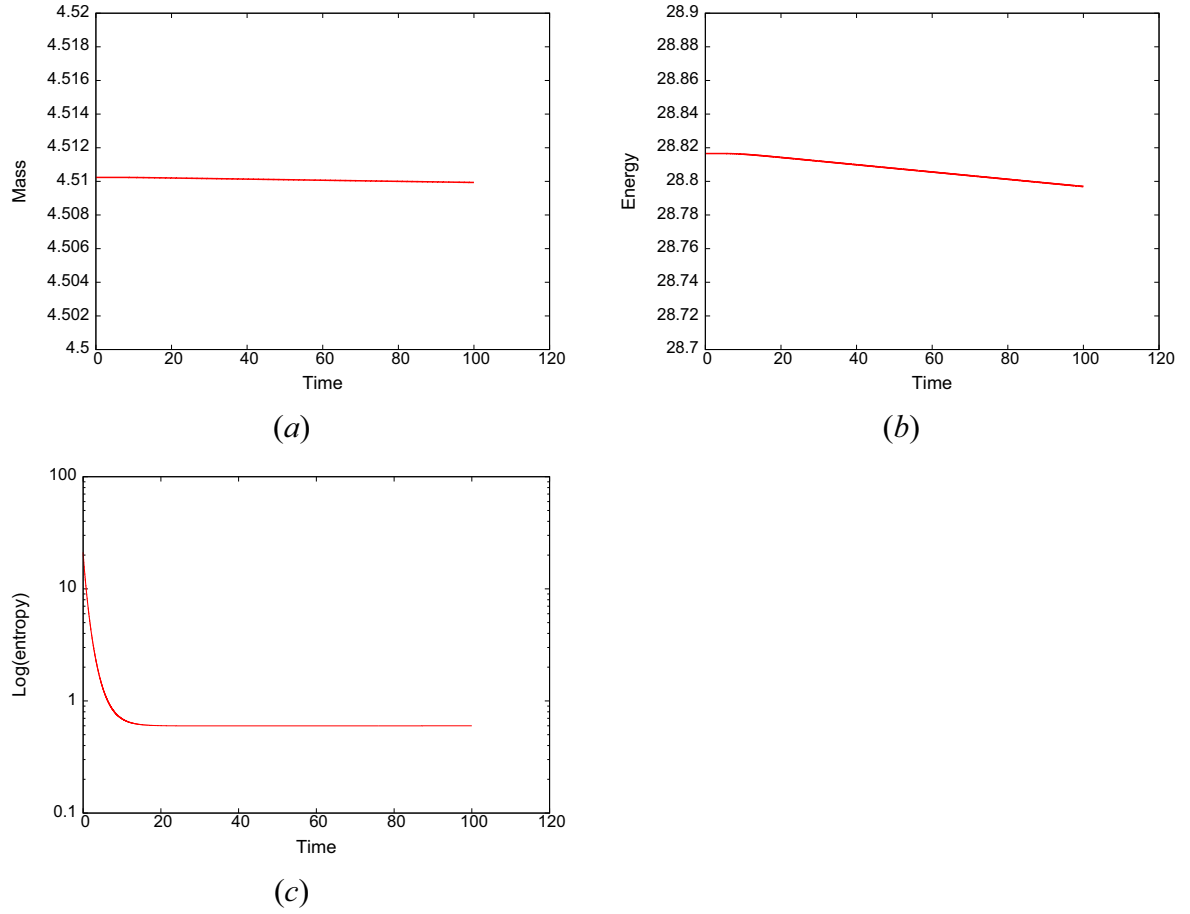


Figure 5.6: Evolution of the mass (a), energy in time (b) and the entropy (in log scale) (c) with respect to time in the case of $\zeta_{\max} = 8$ with 64 points for anisotropic initial data.

From figures 5.7 (a) and 5.7 (b) the realizability domain is still preserved during the simulation. Moreover the isotropic part F^0 converges to a centered Maxwellian. This fact can be explained by the electron-electron collision operator effect whereas the anisotropic part F^1 converges to null state because of the electron-ion collision operator effect. From figure 5.7 (c), the scheme has converged quickly from $t = 80$.

Figure 5.7 (d) represents the evolution of the distribution function F with $\mu = 1$, reconstructed from its isotropic part F^0 and its anisotropic part F^1 . We remark that the distribution function F converges to a centered Maxwellian.

In order to compare the M_1 model with the P_1 model ([27]), let us consider the initial data

$$F^1(t = 0, \zeta) = F^0(t = 0, \zeta)/2$$

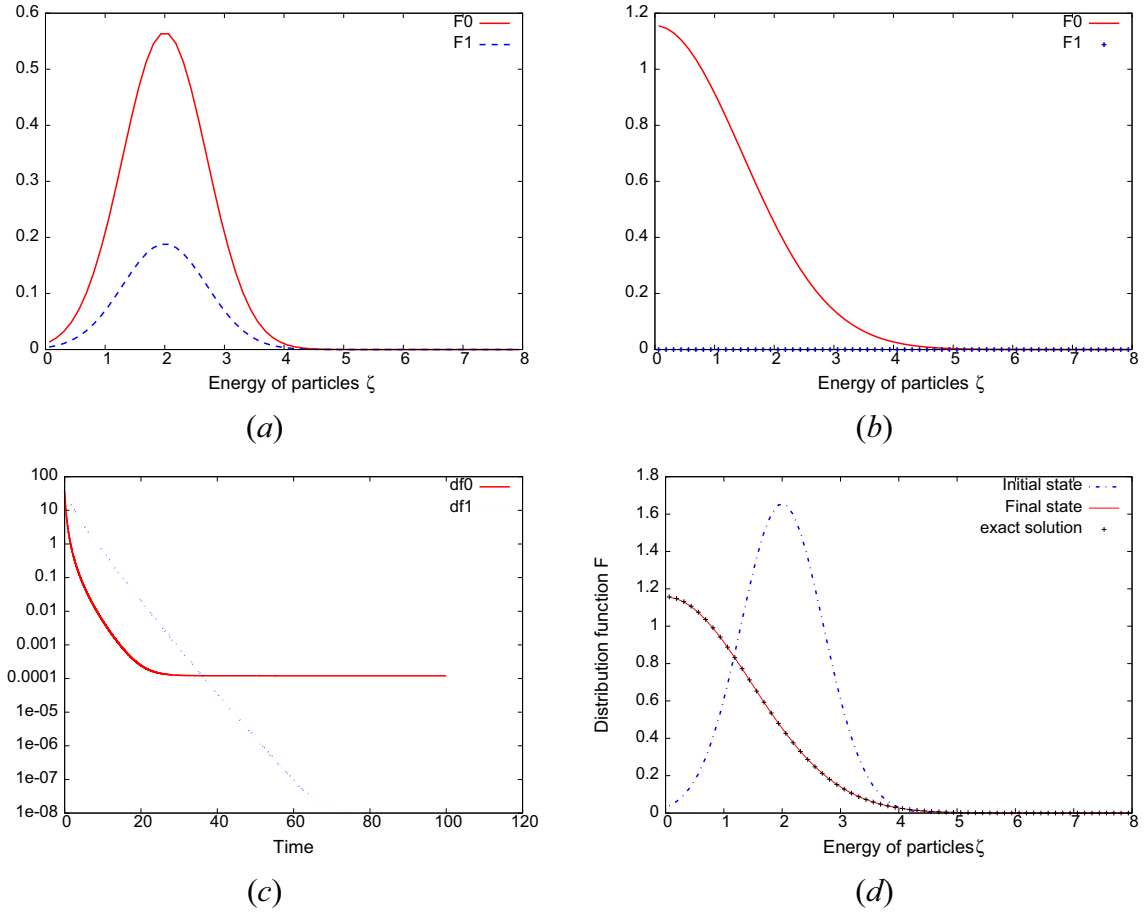


Figure 5.7: Description of the moments F^0 and F^1 with respect to the energy ζ at initial time (a) and at final time (b), evolution of the convergence in logarithmic scale i.e. $df^0 = \sum_j |F_j^{0,N+1} - F_{j+1}^{0,N}|$ and $df^1 = \sum_j |F_j^{1,N+1} - F_{j+1}^{1,N}|$ (c) and evolution of the distribution function F in time with $\mu = 1$ (d) in the case of $\zeta_{\max} = 8$ with 64 points for anisotropic data.

instead of

$$F^1(t=0, \zeta) = F^0(t=0, \zeta)/3 .$$

The M_1 model still shows the convergence of the distribution function to the steady state while preserving realizability domain.

Remark that, with this initial distribution function, the P_1 model does not conserve the realizability domain from first iterations. Indeed from figure (5.8), at time $t = 0.15$, F^0 becomes negative and F^1 is greater than F^0 . This highlights the advantage of the M_1 model which conserves the realizability domain.

Now to evaluate the computational time of the M_1 model, we compare M_1 results with results obtained with the kinetic code *KETS* (presented in chapter 3 and see also [21, 22]) and validated in chapter 4 and with some comparison with results obtained with an implicit Vlasov-

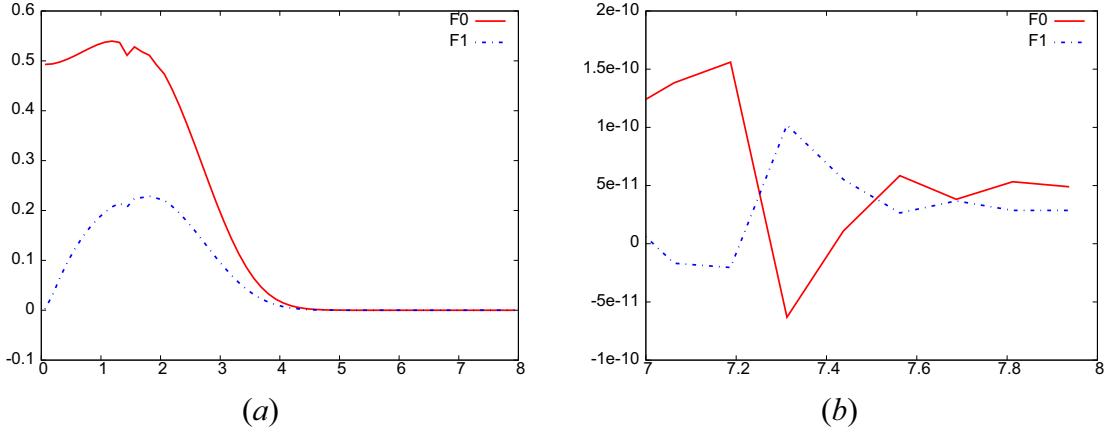


Figure 5.8: Description of the moments F^0 and F^1 with respect to the energy ζ at time $t = 0.15$ with $\zeta \in [0 : 8]$ for the M_1 model (a) and with $\zeta \in [0 : 8]$ for the P_1 model (b).

Fokker-Planck code called IMPACT developed by Kingham and Bell [31]. For the simulation with the kinetic model, each component of the velocity domain is discretized with 16 points. The numerical test is performed with $v_{x,\max} = v_{y,\max} = v_{z,\max} = 8$. M_1 model run during 305 s whereas the CPU time describing *KETS* simulation is about 8 717 s. Therefore, M_1 simulation runs 28 times faster than the *KETS* one. This enhances the good computational time of the M_1 model.

We have also check that for the isotropic case, i.e. $F^1(t = 0, \zeta) = 0$, the conservation laws are preserved. Indeed, we observe that mass and energy are conserved. Besides, the evolution of the entropy induced by the scheme is decreasing in time. Moreover, by plotting the evolution moments F^0 and F^1 , we observe that $F^1(t, \zeta)$ remains equal to 0 and the isotropic part $F^0(t, \zeta)$ converges to the same centered Maxwellian than for the anisotropic case. Notice that the scheme has converged quickly from $t = 40$. We remark also that the distribution function F (defined with $F^0(t, \zeta)$ and $F^1(t, \zeta)$) converges to a centered Maxwellian distribution which is close to the exact equilibrium state with an L^2 relative error about 10^{-4} . This enhances the good accuracy of the scheme.

5.8.2 Convergence to the steady state with a bi-Maxwellian distribution function

The initial data is now a bi-Maxwellian distribution function i.e. a sum of two Maxwellian functions with a density $n = 1 \text{ m}^{-3}$ and $T = 1 \text{ KeV}$. The first one is centered around $\zeta_{d1} = 2$

and the second one around $\zeta_{d2} = 6$. So the bi-Maxwellian function is defined as

$$\begin{cases} F^0(t=0, \zeta) = \left(\frac{m_e}{2\pi}\right)^{3/2} \left(\exp\left(-\frac{(\zeta - \zeta_{d1})^2}{2}\right) + \exp\left(-\frac{(\zeta - \zeta_{d2})^2}{2}\right) \right), \\ F^1(t=0, \zeta) = \frac{F^0(t=0, \zeta)}{3}. \end{cases}$$

For this test case, the energy domain is discretized with 64 points. The numerical test is performed with $\zeta_{\max} = 20$ which represents the length of the energy space. The scaling used here gives time steps to be of the order of the electron-electron collision time as previously. We choose the time step in order to satisfy the CFL condition and the results are computed at time $t = 100$. The boundary conditions for the energy domain satisfies also naturally the positivity constraints of F^0 and the conservation of the mass, i.e. at the left boundary $F_{-1}^0 = F_0^0$ and $F_{-1}^1 = -F_0^1$.

Figures 5.9 (a) and 5.9 (b) show the mass and the energy conservation in time. The evolution of the entropy induced by the scheme is still decreasing in time as shown by Theorem 13.

Figures 5.10 (a) and 5.10 (b) confirm that the realizability domain is preserved in time. Moreover the isotropic part F^0 converges to a centered Maxwellian whereas the anisotropic part F^1 converges to null state. As shown on figure 5.10 (c), the scheme has converged quickly from $t = 200$.

Figure 5.10 (d) shows the evolution of the distribution function F with $\mu = 1$. We remark that F converges to a centered Maxwellian distribution function with an L_2 relative error about 10^{-4} .

5.9 Comparison with kinetic model

This section is devoted to study the good behavior and the accuracy of the M_1 method by using some comparisons with results obtained by kinetic model. Indeed to evaluate the accuracy of the M_1 model, we compare M_1 results with results obtained with the kinetic code *KETS* described in chapter 3 (see also [21, 22]).

In a first subsection, a collisional regime is considered. The propagation of the electron beam in vacuum which converges to a steady state is illustrated. Then the second subsection is devoted to study the good behavior and the accuracy of the M_1 method by studying the interaction of two electrons beams described in chapter 4. Indeed to compare the accuracy, a collisionless regime is considered and the evolution of the distribution function is given by the Vlasov dimensionless equation (3.63) and Maxwell system.

In both test cases to discretize the transport equation, we use a Finite Volume method coming from the HLL method introduced by Harten *et al.* ([28]) and explained in section 5.7.

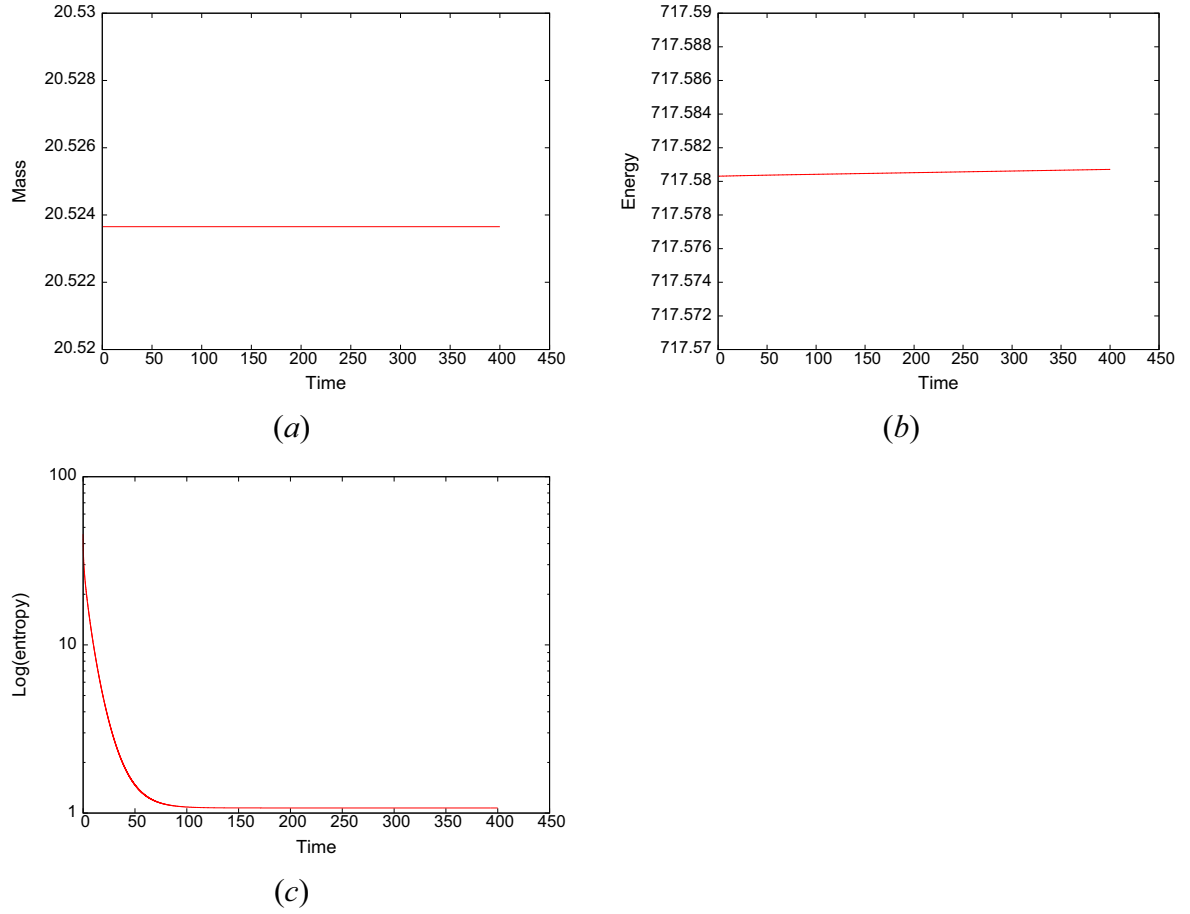


Figure 5.9: Evolution of the mass (a), energy in time (b) and the entropy(c) with respect to time in the case of $\zeta_{\max} = 20$ with 64 points for bi-Maxwellian initial data.

5.9.1 Evolution of an electron beam

Electron transport is an important process in plasma physics which impacts on many areas including laser-plasma. That is why now we turn considering the full discrete model including transport part and collision part to illustrate the good behavior and the accuracy of the method on a one space dimension domain. To discretize the transport equation, we use a Finite Volume method coming from the HLL method introduced by Harten *et al.* ([28]) and explained in section 5.7. The full discrete model can be written such as

$$\begin{cases} \frac{1}{\Delta t} (f_j^{0,N+1} - f_j^{0,N}) + \frac{1}{\Delta x} (\mathcal{F}_{j+\frac{1}{2}}^N - \mathcal{F}_{j-\frac{1}{2}}^N) = \mathcal{Q}_{ee,j}^0, \\ \frac{1}{\Delta t} (f_j^{1,N+1} - f_j^{1,N}) + \frac{1}{\Delta x} (\mathcal{G}_{j+\frac{1}{2}}^N - \mathcal{G}_{j-\frac{1}{2}}^N) = \mathcal{Q}_{ee,j}^1 + \mathcal{Q}_{ei,j}^1, \end{cases}$$

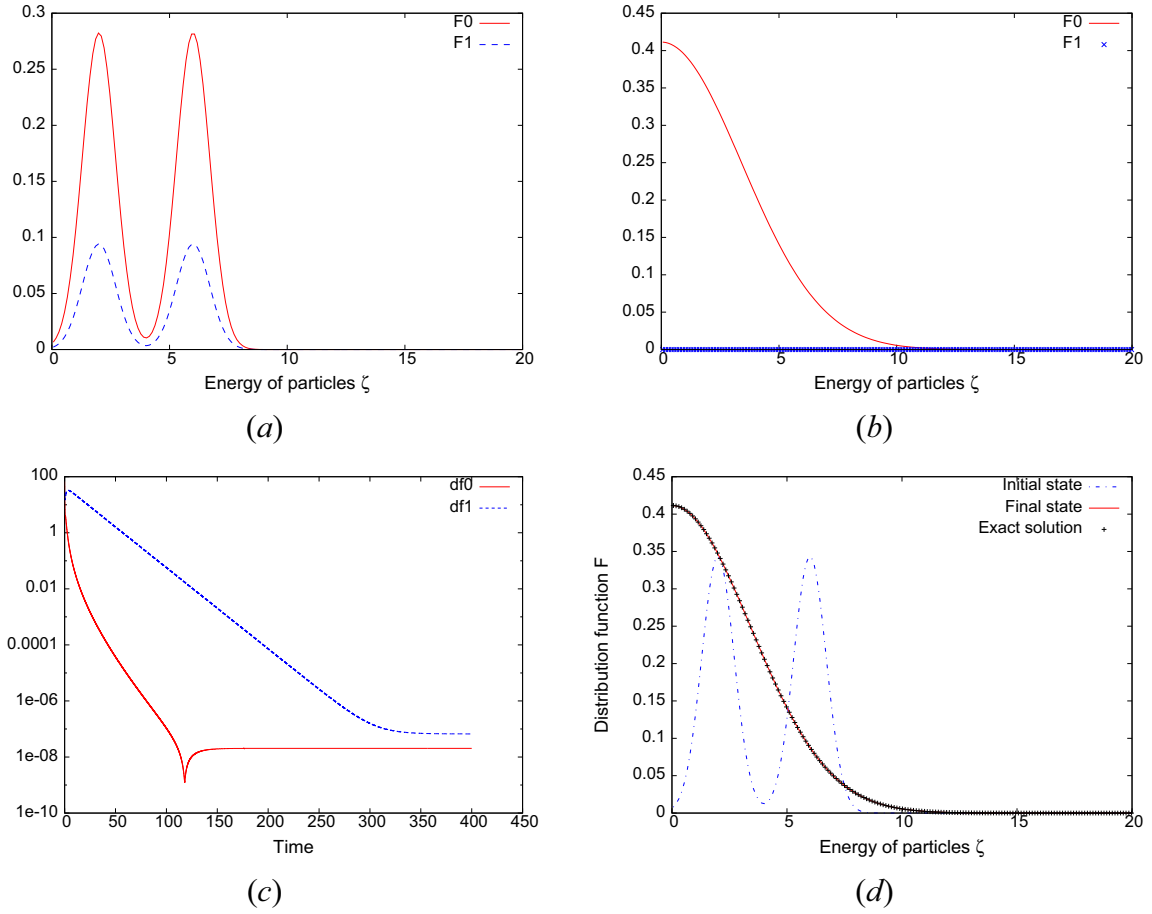


Figure 5.10: Description of the moments F^0 and F^1 with respect to the energy ζ at initial time (a) and at final time (b), evolution of the convergence in logarithmic scale i.e. $df0 = \sum_i |F_i^{0,N+1} - F_{i+1}^{0,N}|$ and $df1 = \sum_i |F_i^{1,N+1} - F_{i+1}^{1,N}|$ (c) and evolution of the distribution function F in time with $\mu = 1$ (d) in the case of $\zeta_{\max} = 8$ with 64 points for bi-Maxwellian data.

where $\mathcal{F}$, $\mathcal{G}$ are HLL flux and the operators $Q_{ee,j}^0$, $Q_{ee,j}^1$ and $Q_{ei,j}^1$ are defined in (5.56).

This test case consists to study the convergence of an electron beam, which atomic mass is $Z = 10$, in the vacuum to a steady state. The initial distribution function in the one dimension space domain is null. The boundary condition on the right side is a Dirichlet condition chosen in order to satisfy realizability domain and to create an electron beam

$$F_{-1}^0 = \left(\frac{m}{2\pi}\right)^{\frac{3}{2}} \exp\left(-\frac{(\zeta - \zeta_d)^2}{2}\right) \text{ and } F_{-1}^1 = \frac{f_{-1}^0}{3}.$$

For this test case, the energy domain is discretized with 64 points and the space domain is discretized with 100 points. The numerical test is performed with $\zeta_{\max} = 8$ and $x_{\max} = 5$. The scaling used here gives time steps to be of the order of the electron-ion collision time. We

choose a time step satisfying the CFL condition. The numerical results are computed at time $t = 300$.

Figures 5.12 (a) – (f) show the evolution of the electron beam at different time steps. After few iterations, at time $t = 0.015$, the electron beam spreads through the space domain (figures 5.12 (a) and 5.12 (b)). Then at time $t = 0.15$, figures 5.12 (c) and 5.12 (d) show that the isotropic part F^0 begins to relax to a Maxwell function centered around $\zeta = 0$ while the anisotropic part F^1 is close to 0 at the end of the space domain. At the end of the computation, at time $t = 300$, the isotropic part has converged to a Maxwellian function centered around 0. This Maxwellian is not uniform in space (figure 5.11). This phenomenon can be explained by the fact that the electron beam is still injected at the beginning of the space domain. Concerning the anisotropic part, F^1 is null except at the beginning of the space domain (5.11) for the same reason as for the isotropic part. The results obtained transcribe the convergence to the steady state due to the collision operators. Notice also that the realizability domain is still preserved in time.

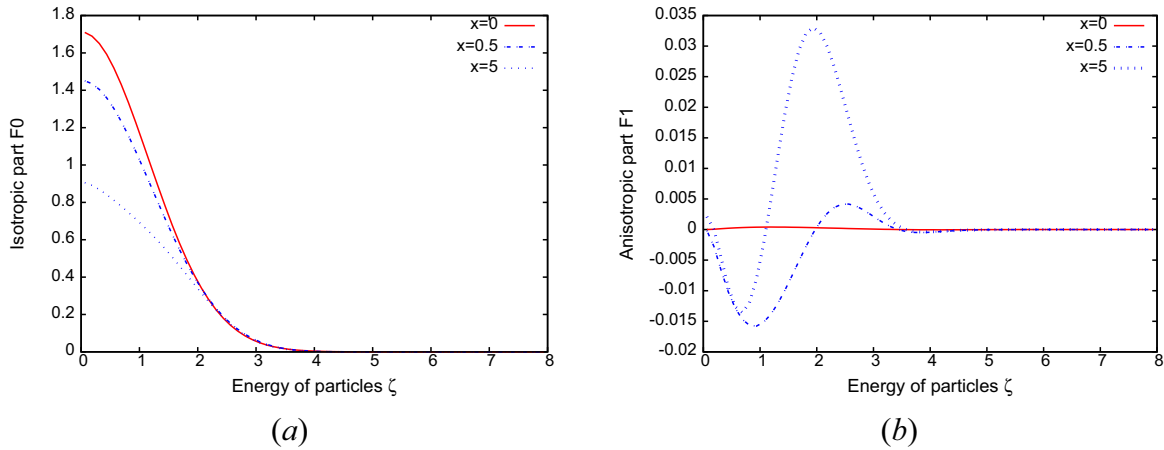
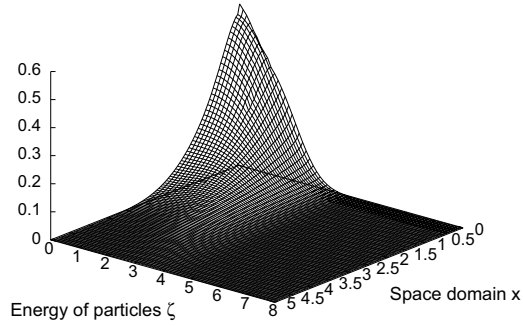


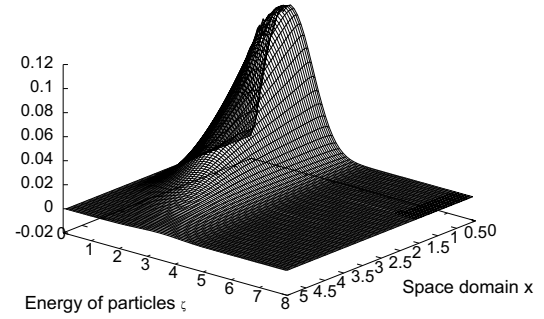
Figure 5.11: Description of moment F^0 (a) and F^1 (b) with respect to the energy ζ at time $t = 100$ for $x = 0$, $x = 0.5$ and $x = 5$ in the case of $\zeta_{\max} = 8$ with 64 points and $x_{\max} = 5$ with 100 points for electron beam data.

To evaluate the good behavior and the accuracy of the M_1 model, we compare M_1 results with results obtained with the kinetic code *KETS*. For the simulation with the kinetic model, each component of the velocity domain is discretized with 16 points and the space domain is discretized with 100 points. The numerical test is performed with $v_{x,\max} = v_{y,\max} = v_{z,\max} = 8$ and $x_{\max} = 5$. We choose a time step satisfying the CFL condition. The numerical results are computed at time $t = 300$.

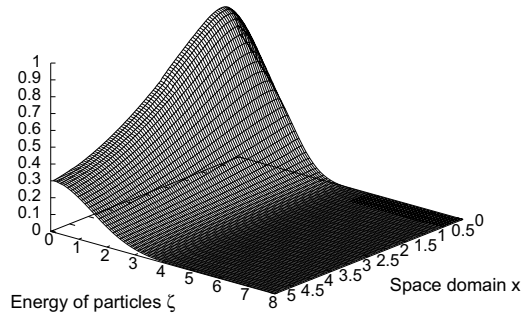
Figure 5.13 represents the mass with respect to the space domain computed with the present M_1 model and the *KETS* code. We observe that the two masses obtained with the two codes lead to constant distribution function at the end of the space computation domain. This fact is



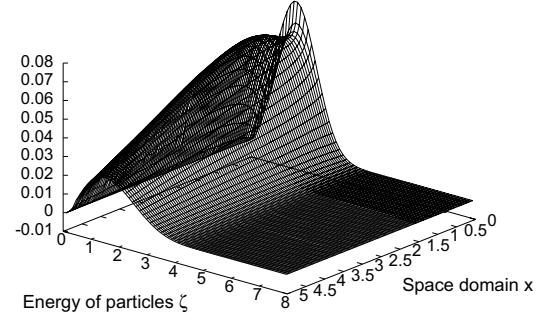
(a)



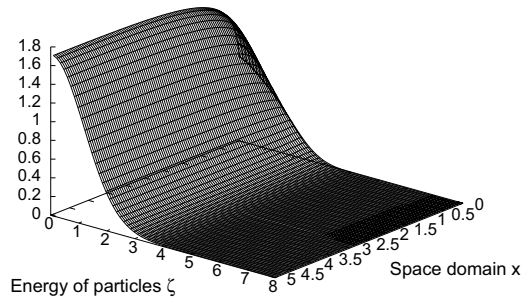
(b)



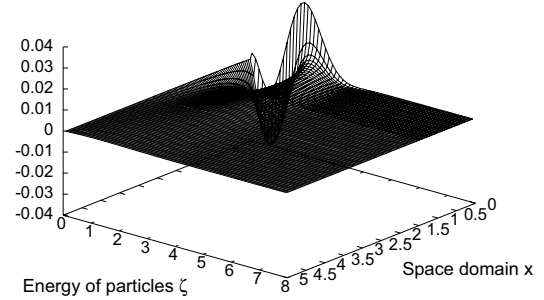
(c)



(d)



(e)



(f)

Figure 5.12: Description of moments F^0 and F^1 with respect to the energy ζ at time $t = 0.015$ (a) and (b), at time $t = 0.15$ (c) and (d) and at time $t = 100$ (e) and (f) in the case of $\zeta_{\max} = 8$ with 64 points and $x_{\max} = 5$ with 100 points for electron beam data.

explained by the convergence of the distribution function toward a centered Maxwellian. Moreover it can be shown that the M_1 model creates a boundary layer of length $\frac{1}{3}$ because of incompatibility data between the beam injected and the converged distribution function. This phenomenon is not reproduced with the *KETS* code which has no difficulties to take into account this situation. That is why the two results are different on $[0, \frac{1}{3}]$. On the remaining area, results are observed to be close. Moreover the sameness between two results can be more accentuated by using more discretization points but this approach need too much resources and time for the kinetic code *KETS*.

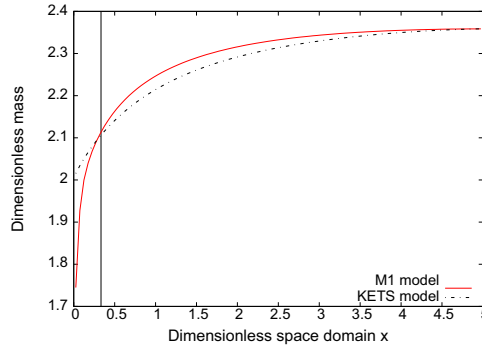


Figure 5.13: Description of the mass ρ in space domain at final time obtained with the M_1 model and *KETS* model in the case of $\zeta_{\max} = 8$ with 64 points and $x_{\max} = 5$ with 100 points for electron beam data.

Moreover an important remark on the M_1 model is interesting. Indeed, for this test case, M_1 model run during 175000s whereas the CPU time used by *KETS* code is around 530000s. Therefore, M_1 model run 3 times faster than *KETS* code by having the same precision.

5.9.2 1D two-stream instability

Here the test case of section 4.2 chapter (4) is used for the M_1 model in order to study the accuracy. Indeed this subsection is devoted to reproduced the interaction of two electron beams of equal density with M_1 method. Recall that the two counter-streaming beams that are initially well separated in the phase space, start to mix together and create finally a complicated vortex structure as shown on the figure 5.14. Vlasov-Maxwell system is used to describe the time evolution of the distribution function. We choose a scaling with the plasma frequency detailed

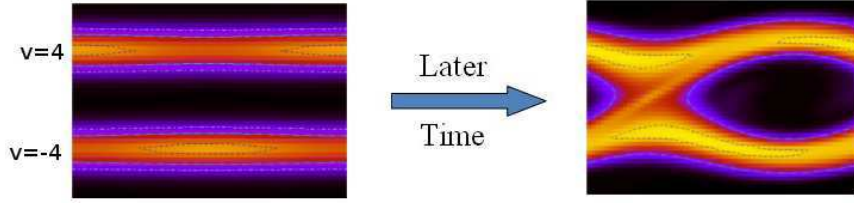


Figure 5.14: *Evolution of the two counter-streaming beams in time*

in annex 4.7. The Vlasov-Maxwell system writes in these units

$$\left\{ \begin{array}{l} \frac{\partial f}{\partial t} + \nabla_{\mathbf{x}} \cdot (\mathbf{v}f) - \nabla_{\mathbf{v}} \cdot ((\mathbf{E} + \mathbf{v} \times \mathbf{B})f) = 0, \\ \frac{\partial \mathbf{E}}{\partial t} - \frac{1}{\beta^2} \nabla_{\mathbf{x}} \times \mathbf{B} = n \mathbf{u}, \\ \frac{\partial \mathbf{B}}{\partial t} + \nabla_{\mathbf{x}} \times \mathbf{E} = 0, \\ \nabla_{\mathbf{x}} \cdot \mathbf{E} = (1 - n), \quad \nabla_{\mathbf{x}} \cdot \mathbf{B} = 0, \end{array} \right. \quad (5.88)$$

where the density and the current are expressed such as

$$v = \frac{Z n_0 e^4 \ln \Lambda}{8 \pi \epsilon_0^2 m_e^2 v_{th}^3 \omega_{pe}} = \frac{Z \ln \Lambda}{8 \pi n_0 \lambda_D^3} = \frac{v_{e,i}}{\omega_{pe}} \quad \text{with} \quad v_{e,i} = \frac{Z n_0 e^4 \ln \Lambda}{8 \pi \epsilon_0^2 m_e^2 v_{th}^3}.$$

Regarding the kinetic *KETS* code, as in section 4.2, we consider here the phase space with one dimension in space and one dimension in velocity space. The initial distribution function and electric field are given by

$$\left\{ \begin{array}{l} f(t=0, x, v) = \frac{1}{2} [(1 + A \cos(kx)) \mathcal{M}_{v_d}(v) + (1 - A \cos(kx)) \mathcal{M}_{-v_d}(v)], \\ E(t=0, x) = 0, \end{array} \right. \quad (5.89)$$

where

$$\mathcal{M}_{v_d}(\mathbf{v}) = n_e \left(\frac{m_e}{2\pi k_B T_e} \right)^{\frac{3}{2}} \exp \left(-\frac{m_e (\mathbf{v} - v_d)^2}{2k_B T_e} \right), \quad (5.90)$$

is the Maxwellian distribution function centered around v_d . Here the temperature of the plasma is $T_e = 1 \text{ KeV}$ and the density is $n_e = n_c$.

Boundary conditions in space are chosen as periodic conditions on each side. The physical parameters are chosen such as the initial beams mean velocity is $\pm v_d = \pm 4$. The periodic space domain which length is $x_{\max} = 25$ is discretized with 64 points. The velocity space $[-v_{\max}; v_{\max}]$

is also discretized with 64 points and the truncation of the velocity space is $v_{\max} = 12$. In order to respect the CFL condition of the scheme, we choose a time step $\Delta t = 1/200$.

Concerning the M_1 model, the energy domain is discretized with 64 points and the length of the energy space is denoted by $\zeta_{\max} = 12$. We chose a time step satisfying the CFL condition. The scaling used here is defined in appendix 4.7. The numerical results are computed at time $t = 30$.

To describe the interaction between both streams, we choose two groups of particles, one for each beam. From 5.5 and the definition of the Maxwellian (5.90), we get by identification

$$\begin{cases} \rho_0(\zeta) = n_e \left(\frac{m_e}{2\pi k_B T_e} \right)^{\frac{3}{2}} \exp \left(-\frac{m_e(\zeta_{g1,d}^2 + \zeta^2)}{2k_B T_e} \right), \\ \alpha(\zeta) = \frac{m_e \zeta \zeta_{g1,d}}{2k_B T_e}. \end{cases} \quad (5.91)$$

Therefore, the initial data for the first group of particles is chosen such as

$$\begin{cases} f_{g1}^0 = \frac{1}{2}(1 + A \cos(kx)) \times 4\pi\rho_0 \frac{\sinh(|\alpha|)}{|\alpha|}, \\ f_{g1}^1 = \frac{1}{2}(1 + A \cos(kx)) \times 4\pi\rho_0 \frac{\sinh(|\alpha|)(1 - |\alpha| \coth(|\alpha|))}{|\alpha|^3} \alpha, \end{cases} \quad (5.92)$$

with ρ_0 and α defined in (5.91). The initial data for the second group of particles is chosen such as the first one but with an opposite drift energy, i.e. (f_{g2}^0, f_{g2}^1) are given by (5.91, 5.92) with $\zeta_{g2,d} = -\zeta_{g1,d}$.

We have already seen the good behavior of M_1 model and M_1 reproduced here the good behavior of the two stream mixing. To evaluate the accuracy of the M_1 model, figure 5.15 shows the evolution of the dimensionless electrostatic energy with respect to time for both models. By using a perturbation amplitude of $A = 0.001$, we can observe that the electrostatic energy from M_1 and *KETS* are very close which is relevant for the accuracy of M_1 . However by using a perturbation amplitude of $A = 0.1$, results presents some slight differences. In the linear regime obtained by both codes are very close with a strong perturbation $A = 0.1$ but in non-linear regime, M_1 model develops more non linear instabilities.

5.10 Conclusion

We have developed a new model for electron transport in plasma taking into account the collisions between electrons and between electrons and ions. Its construction is essentially based on

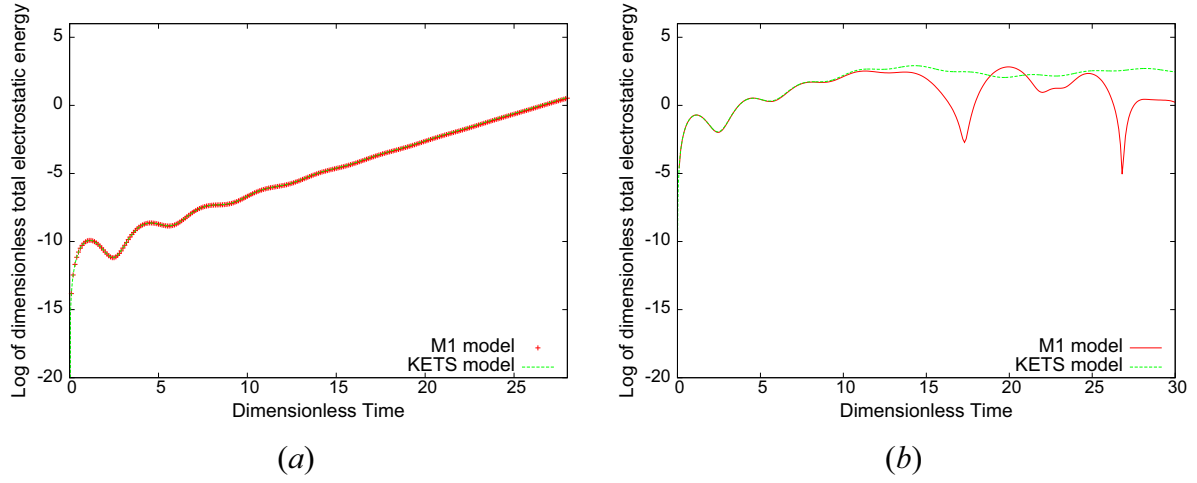


Figure 5.15: *Evolution of the electrostatic energy with respect to time with a weak perturbation amplitude $A = 0.001$ (a) and a strong perturbation amplitude $A = 0.1$ (b) for M_1 and KETS codes.*

an entropy minimization principle and an integration of the kinetic model w.r.t the angle variable of the microscopic velocity. Next an entropic semi-discretization in energy of the model based on an entropic average is proposed. Finally numerical simulation in the context of the full discretized model are performed. The results are in a very good agreement with the properties proved on the semi-discrete model and results obtained with kinetic model.

Several extension and application of this work can be considered. For instance, the generalization of the present results to a model which uses N moments is proposed in the next chapter.

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5.11 Appendix A: computation of the B coefficient

The coefficient B (5.26) can be computed as

$$\begin{aligned}
 B(\zeta) &= \int_0^\infty \min\left(\frac{1}{\zeta^3}, \frac{1}{w^3}\right) w^3 \partial_w \left(\frac{f^0(w)}{w^2} \right) dw, \\
 &= \int_0^\zeta \frac{w^3}{\zeta^3} \partial_w \left(\frac{f^0(w)}{w^2} \right) dw + \int_\zeta^\infty \partial_w \left(\frac{f^0(w)}{w^2} \right) dw, \\
 &= \left[\frac{w^3}{\zeta^3} \frac{f^0(w)}{w^2} \right]_0^\zeta - \int_0^\zeta \frac{3w^2}{\zeta^3} \frac{f^0(w)}{w^2} dw + \left[\frac{f^0(w)}{w^2} \right]_\zeta^\infty, \\
 &= -\frac{3}{\zeta^3} \int_0^\zeta f^0(w) dw.
 \end{aligned}$$

5.12 Appendix B: moment closure for Maxwell-Fokker-Planck-Landau system

To establish this closure we will proceed by step. First the closure for the collisionless kinetic equation is determined. Then the moment closure for homogeneous system will be computed.

5.12.1 Derivation of the equations about f_0 and f_1 free-streaming

First, the moment extraction of (5.17, 5.18) without collision is considered.

Let us denote by φ some test function, we get

$$\int_{v \in \mathbb{R}^3} (\partial_t f + v \cdot \nabla_x f + q(E + v \wedge B) \cdot \nabla_v f) \varphi(\zeta) dv = 0. \quad (5.93)$$

Recall that $v = \zeta \Omega$ and $dv = \zeta^2 d\zeta d\Omega$. Since $\nabla_v \varphi(\zeta) = \varphi'(\zeta) \Omega$, after using Green formula, equation (5.93) can be expressed in spherical coordinate (ζ, μ, φ) as

$$\begin{aligned}
 \int_0^\infty \partial_t \left(\zeta^2 \int_{S_2} f d\Omega \right) \varphi(\zeta) d\zeta + \int_0^\infty \nabla_x \cdot \left(\zeta \zeta^2 \int_{S_2} \Omega f d\Omega \right) \varphi(\zeta) d\zeta \\
 - \int_0^\infty \varphi'(\zeta) q E \cdot \left(\zeta^2 \int_{S_2} \Omega f d\Omega \right) d\zeta = 0.
 \end{aligned}$$

Therefore we get for all test functions φ

$$\int_0^\infty \left(\partial_t f_0 + \nabla_x \cdot (\zeta f_1) + \partial_\zeta (q E \cdot f_1) \right) \varphi(\zeta) d\zeta.$$

Now to obtain the equation on the evolution of f^1 , for all test function φ , we get

$$\int_{v \in \mathbb{R}^3} (\partial_t f + v \cdot \nabla_x f + q(E + v \wedge B) \cdot \nabla_v f) \Omega \varphi(\zeta) dv = 0 .$$

By using Green formula, we get

$$\begin{aligned} & \int_{v \in \mathbb{R}^3} (\partial_t f + v \cdot \nabla_x f) \Omega \varphi(\zeta) dv - \int_{v \in \mathbb{R}^3} \varphi'(\zeta) \Omega \cdot (q(E + v \wedge B) f) \Omega dv \\ & - \int_{v \in \mathbb{R}^3} \frac{1}{\zeta} \varphi(\zeta) q(E + v \wedge B) f dv + \int_{v \in \mathbb{R}^3} \frac{1}{\zeta} \varphi(\zeta) \Omega \otimes \Omega (q(E + v \wedge B) f) dv = 0 , \end{aligned}$$

because $\nabla_v \Omega = \frac{1}{\zeta} (I - \Omega \otimes \Omega)$. Since $\Omega \otimes \Omega (v \wedge B) = 0$, we obtain

$$\begin{aligned} & \int_0^\infty (\partial_t f_1 + \nabla_x \cdot (\zeta f_2)) \varphi(\zeta) d\zeta - \int_{v \in \mathbb{R}^3} \varphi'(\zeta) \Omega \cdot (qE f) \Omega dv - \int_0^\infty \frac{q}{\zeta} \varphi(\zeta) (\zeta^2 \\ & \int_{S_2} f d\Omega E + \zeta^3 \int_{S_2} \Omega f d\Omega \wedge B) d\zeta + \int_0^\infty \frac{q}{\zeta} \varphi(\zeta) \left(\zeta^2 \int_{S_2} \Omega \otimes \Omega f d\Omega E \right) d\zeta = 0 . \end{aligned}$$

Finally, previous equation can be rewritten for all test functions φ

$$\int_0^\infty \left(\partial_t f_1 + \nabla_x \cdot (\zeta f_2) + q \left(\partial_\zeta (f_2 E) - \frac{1}{\zeta} (f_0 E + \zeta f_1 \wedge B - f_2 E) \right) \right) \varphi(\zeta) d\zeta = 0 ,$$

Hence system (5.17, 5.18) without collision is verified.

5.12.2 Moment closure for the collision operators C_{ee} and C_{ei}

We begin by considering the electron-ion collision operator C_{ei} which is much simpler to compute.

Moment closure for the electron-ion collisions

To establish this closure we consider only the homogeneous case with the collision operator C_{ei} ,

$$\partial_t f = C_{ei}[f] = \alpha_{ei} \nabla_v \cdot \left[\frac{|v|^2 I - v \otimes v}{|v|^3} \nabla_v f(v) \right] = \alpha_{ei} \nabla_v \cdot [S(v) \nabla_v f(v)] , \quad (5.94)$$

where $S(v)$ denotes the projection operator on the orthogonal plane to the vector v ($S(v)v = 0$)

$$S(v) = \frac{|v|^2 I - v \otimes v}{|v|^3} = \frac{1}{\zeta} (I - \Omega \otimes \Omega) . \quad (5.95)$$

Let us denote by φ some test function, the duality approach leads to

$$\int_{v \in \mathbb{R}^3} (\partial_t f - \alpha_{ei} \nabla_v \cdot [S(v) \nabla_v f]) \varphi(\zeta) dv = 0 . \quad (5.96)$$

By using Green formula, equation (5.96) writes in spherical coordinate (ζ, μ, φ) such as

$$\int_{v \in \mathbb{R}^3} \partial_t f \varphi(\zeta) dv + \int_{v \in \mathbb{R}^3} \alpha_{ei} \frac{\partial \varphi(\zeta)}{\partial v} \cdot [S(v) \nabla_v f] dv = 0 ,$$

wich can be rewritten such as

$$\int_{v \in \mathbb{R}^3} \partial_t f \varphi(\zeta) dv + \int_{v \in \mathbb{R}^3} \alpha_{ei} \varphi'(\zeta) \frac{v}{\zeta} \cdot [S(v) \nabla_v f] dv = 0 . \quad (5.97)$$

Since $S(v)$ is symmetric, equation (5.97) can be simplified into

$$\int_{v \in \mathbb{R}^3} \partial_t f \varphi(\zeta) dv + \int_{v \in \mathbb{R}^3} \alpha_{ei} \varphi'(\zeta) \left(S(v) \frac{v}{\zeta} \right) \cdot \nabla_v f dv = 0 .$$

Since $S(v)v = 0$, we get $\partial_t f_0 = 0$ and the C_{ei} operator has no influence on the equation for f_0 .

After extracting the zeroth moment order, consider now the first order moment extraction. The duality approach gives

$$\int_{v \in \mathbb{R}^3} (\partial_t f - \alpha_{ei} \nabla_v \cdot [S(v) \nabla_v f]) \Omega \varphi(\zeta) dv = 0 . \quad (5.98)$$

By using Green formula, equation (5.98) writes

$$\begin{aligned} \int_{v \in \mathbb{R}^3} \partial_t f \Omega \varphi(\zeta) dv + \alpha_{ei} \int_{v \in \mathbb{R}^3} \varphi'(\zeta) \Omega \cdot [S(v) \nabla_v f] \Omega dv \\ + \alpha_{ei} \int_{v \in \mathbb{R}^3} \varphi(\zeta) S(v) [S(v) \nabla_v f] dv = 0 . \end{aligned}$$

Since $S(v)$ is symmetric, $S(v)S(v) = \frac{1}{\zeta} S(v)$. So

$$\int_{v \in \mathbb{R}^3} \partial_t f \Omega \varphi(\zeta) dv - \alpha_{ei} \int_{v \in \mathbb{R}^3} \frac{\varphi(\zeta)}{\zeta^4} \nabla_v \cdot [\zeta^2 I - v \otimes v] f dv = 0 . \quad (5.99)$$

Moreover $\nabla_v \cdot [|\mathbf{v}|^2 I - \mathbf{v} \otimes \mathbf{v}] = -2\mathbf{v} = -2\zeta \Omega$, therefore equation (5.99) can be expressed in spherical coordinate (ζ, μ, φ) as

$$\partial_t f_1 = -\frac{2\alpha_{ei}}{\zeta^3} f_1 .$$

We remark that the electron-ion collisions are only relevant for the f_1 behavior, not the f_0 .

Moment closure for the electron-electron collisions

To establish this closure we consider only the homogeneous case with the collision operator C_{ee} ,

$$\partial_t f = C_{ee}(f, f) = \alpha_{ee} \nabla_v \cdot \int_{v' \in \mathbb{R}^3} S(u) [f(v') \nabla_v f(v) - f(v) \nabla_{v'} f(v')] dv', \quad (5.100)$$

where $S(u)$ is given in (5.95) and $u = v' - v$. To simplify the moment extraction of this operator, we will assume that the main contribution from this operator comes from the isotropic part of f . In the remainder of this subsection, we will assume that the distribution function f is isotropic, i.e. $f = f(|v|) = f(\zeta)$.

We need introduce the notation $\Gamma(v)$ which is the term into the divergence in v ,

$$\Gamma(v) = \int_{v' \in \mathbb{R}^3} S(u) [f(v') \nabla_v f(v) - f(v) \nabla_{v'} f(v')] dv'.$$

From (5.100), $C_{ee}(f, f) = \alpha_{ee} \nabla_v \cdot \Gamma(v)$. Moreover if the distribution function f is also isotropic, $\Gamma(v)v$ is isotropic. So for some test function φ ,

$$\int_0^\infty \left(\partial_t f^0 - 4\pi\alpha_{ee} \partial_\zeta (\zeta \Gamma(v) \cdot v) \right) \varphi(\zeta) d\zeta = 0,$$

which implies that

$$\partial_t f^0 = 4\pi\alpha_{ee} \partial_\zeta (\zeta \Gamma(v) \cdot v). \quad (5.101)$$

The term $\Gamma(v) \cdot v$ can be simplified into

$$\Gamma(v) \cdot v = \int_0^\infty \int_{\Omega' \in S_2} S(u) v \cdot \left[f(\zeta') \frac{v}{\zeta} \partial_\zeta f(\zeta) - f(\zeta) \Omega' \partial_{\zeta'} f(\zeta') \right] d\Omega' \zeta'^2 d\zeta'.$$

Since $\Gamma(v) \cdot v$ involves the $S(u)v$ matrix-vector product, we obtain

$$S(u)v = \frac{(\zeta'^2 - \zeta'v \cdot \Omega')v - (\zeta'^2(\Omega' \cdot v) - \zeta^2 \zeta')\Omega'}{(\zeta^2 + \zeta'^2 - 2\zeta\zeta'v \cdot \Omega')^{\frac{3}{2}}} = \frac{\zeta^2 \zeta'^2 (1 - \mu^2)}{(\zeta^2 + \zeta'^2 - 2\zeta\zeta'\mu)^{\frac{3}{2}}}.$$

As,

$$\int_{-1}^{+1} \frac{\zeta^2 \zeta'^2 (1 - \mu^2)}{(\zeta^2 + \zeta'^2 - 2\zeta\zeta'\mu)^{\frac{3}{2}}} d\mu = \frac{4}{3} \inf\left(\frac{1}{\zeta^3}, \frac{1}{\zeta'^3}\right),$$

we obtain finally the expression for the collision operator (5.23).

5.13 Appendix C: moment extraction for Langdon operator

The moment closure for the Langdon operator is now detailed. Let us consider the space homogeneous operator having the general form

$$\frac{\partial f}{\partial t} = A \operatorname{div}_v \left(\frac{g(v)}{v^5} (v \otimes v) \nabla_v f \right). \quad (5.102)$$

Let us denote by φ some test function , the weak form of equation (5.102) can be written as

$$\int_{\mathbb{R}^3} \frac{\partial f}{\partial t} \varphi(\zeta) d\mathbf{v} = A \int_{\mathbb{R}^3} \operatorname{div}_{\mathbf{v}} \left(\frac{g(v)}{v^5} \mathbf{v} \otimes \mathbf{v} \nabla_{\mathbf{v}} f \right) \varphi(v) d\mathbf{v} . \quad (5.103)$$

Equation (5.103) can be simplified into

$$\int_{\mathbb{R}^3} \frac{\partial f}{\partial t} \varphi(v) d\mathbf{v} = -A \int_{\mathbb{R}^3} \left(\left(\frac{g(v)}{v^5} \mathbf{v} \otimes \mathbf{v} \right) \nabla_{\mathbf{v}} f \right) \cdot \nabla_{\mathbf{v}} \varphi(v) d\mathbf{v} . \quad (5.104)$$

Since $\nabla_{\mathbf{v}} \varphi(v) = \varphi'(v) \nabla_{\mathbf{v}}(v) = \varphi'(v) \mathbf{\Omega}$, equation (5.104) rewrites

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{\partial f}{\partial t} \varphi(v) d\mathbf{v} &= -A \int_{\mathbb{R}^3} \frac{g(v)}{v^5} \varphi'(v) ((\mathbf{v} \otimes \mathbf{v}) \nabla_{\mathbf{v}} f \cdot \mathbf{\Omega}) dv \\ &= -A \int_{\mathbb{R}^3} \frac{g(v)}{v^3} \varphi'(v) \nabla_{\mathbf{v}} f \cdot (\mathbf{\Omega} \otimes \mathbf{\Omega}) \mathbf{\Omega} dv \end{aligned} \quad (5.105)$$

$$(5.106)$$

Since $(\mathbf{\Omega} \otimes \mathbf{\Omega}) \mathbf{\Omega} = \mathbf{\Omega}' \mathbf{\Omega} \mathbf{\Omega} = \mathbf{\Omega}$, (5.105) can be expressed as

$$\int_{\mathbb{R}^3} \frac{\partial f}{\partial t} \varphi(v) d\mathbf{v} = -A \int_{\mathbb{R}^3} \frac{g(v)}{v^3} \varphi'(v) \nabla_{\mathbf{v}} f \cdot \mathbf{\Omega} d\mathbf{v} ,$$

Then by using green formula, we get

$$\int_{\mathbb{R}^3} \frac{\partial f}{\partial t} \varphi(v) d\mathbf{v} = A \int_{\mathbb{R}^3} f(\mathbf{v}) \operatorname{div}_{\mathbf{v}} \left(\frac{g(v)}{v^3} \varphi'(v) \cdot \mathbf{\Omega} \right) d\mathbf{v} . \quad (5.107)$$

Equation (5.107) can be simplified into

$$\int_{\mathbb{R}^3} \frac{\partial f}{\partial t} \varphi(v) d\mathbf{v} = A \int_{\mathbb{R}^3} f(\mathbf{v}) \left[\nabla_{\mathbf{v}} \left(\frac{g(v)}{v^3} \varphi'(v) \right) \cdot \mathbf{\Omega} + (\operatorname{div}_{\mathbf{v}} \mathbf{\Omega}) \frac{g(v)}{v^3} \varphi'(v) \right] d\mathbf{v} . \quad (5.108)$$

Moreover $\nabla_{\mathbf{v}} v = \mathbf{\Omega}$ and $(\operatorname{div}_{\mathbf{v}} \mathbf{\Omega}) = \frac{1}{v} \operatorname{div}_{\mathbf{v}} \mathbf{v} + \nabla_{\mathbf{v}} \left(\frac{1}{v} \right) \cdot \mathbf{v} = \frac{3}{v} - \frac{1}{v} = \frac{2}{v}$, thus equation (5.108) can be written as

$$\int_{\mathbb{R}^3} \frac{\partial f}{\partial t} \varphi(v) d\mathbf{v} = A \int_{\mathbb{R}^3} f(\mathbf{v}) \left[\frac{\partial}{\partial v} \left(\frac{g(v)}{v^3} \varphi'(v) \right) \mathbf{\Omega} \cdot \mathbf{\Omega} + \frac{2}{v} \frac{g(v)}{v^3} \varphi'(v) \right] d\mathbf{v} . \quad (5.109)$$

Equation (5.109) writes in spherical coordinates such as

$$\int_0^{+\infty} \frac{\partial f^0}{\partial t} \varphi(\zeta) d\zeta = A \int_0^{+\infty} \left[\frac{\partial}{\partial \zeta} \left(\frac{g(\zeta)}{\zeta^3} \varphi'(\zeta) \right) + \frac{2}{\zeta} \frac{g(\zeta)}{\zeta^3} \varphi'(\zeta) \right] f^0 d\zeta . \quad (5.110)$$

By using some integration by parts, equation (5.110) becomes

$$\int_0^{+\infty} \frac{\partial f^0}{\partial t} \varphi(\zeta) d\zeta = A \int_0^{+\infty} \frac{\partial}{\partial \zeta} \left(\frac{g(\zeta)}{\zeta^3} \frac{\partial f^0}{\partial \zeta} - \frac{2g(\zeta)}{\zeta^4} f^0 \right) \varphi(\zeta) d\zeta, \quad (5.111)$$

because φ is a smooth test function on compact support. Finally we obtain the equivalence in the dual space

$$\frac{\partial f^0}{\partial t} = A \frac{\partial}{\partial \zeta} \left(\frac{g(\zeta)}{\zeta^3} \frac{\partial f^0}{\partial \zeta} - \frac{2g(\zeta)}{\zeta^4} f^0 \right)$$

Now to obtain the equation on the evolution of f^1 , for all test function φ , we get

$$\int_{\mathbb{R}^3} \frac{\partial f}{\partial t} \varphi(v) \Omega dv = A \int_{\mathbb{R}^3} \operatorname{div}_v \left(\frac{g(v)}{v^5} v \otimes v \nabla_v f \right) \varphi(v) \Omega dv. \quad (5.112)$$

Moreover we get

$$\left(\int \operatorname{div} A \alpha \right)_i = - \int \sum_j A_j \frac{\partial \alpha_i}{\partial x_j} = - \int \sum_j \frac{\partial \alpha_i}{\partial x_j} A_j = - \left(\int \nabla \alpha A \right)_i,$$

with A a vector and α a scalar. Therefore equation (5.112) can be simplified into

$$\int_{\mathbb{R}^3} \frac{\partial f}{\partial t} \varphi(v) \Omega dv = -A \int_{\mathbb{R}^3} \nabla_v (\varphi(v) \Omega) \frac{g(v)}{v^5} (v \otimes v) \nabla_v f dv. \quad (5.113)$$

Since $\nabla_v (\varphi(v) \Omega) = \varphi(v) \nabla_v \Omega + \varphi'(v) \Omega \otimes \Omega$, equation (5.113) becomes

$$\int_{\mathbb{R}^3} \frac{\partial f}{\partial t} \varphi(v) \Omega dv = -A \int_{\mathbb{R}^3} \frac{g(v)}{v^3} \left[\varphi(v) \frac{1}{v} (Id - \Omega \otimes \Omega) + \varphi'(v) \Omega \otimes \Omega \right] \Omega \otimes \Omega \nabla_v f dv.$$

Moreover we have $(\Omega \otimes \Omega)(\Omega \otimes \Omega) = \Omega' \Omega \Omega' \Omega = \Omega' \Omega = \Omega \otimes \Omega$. Therefore we obtain by using a green formula

$$\int_{\mathbb{R}^3} \frac{\partial f}{\partial t} \varphi(v) \Omega dv = A \int_{\mathbb{R}^3} f \left[\frac{g(v)}{v^3} \varphi'(v) \operatorname{div}(\Omega \otimes \Omega) + (\Omega \otimes \Omega) \nabla_v \left(\frac{g(v)}{v^3} \varphi'(v) \right) \right]. \quad (5.114)$$

Since

$$\begin{aligned} \operatorname{div}(\Omega \otimes \Omega) &= (\nabla_v \Omega) \Omega + (\operatorname{div}_v \Omega) \Omega, \\ &= \frac{1}{v} (Id - \Omega \otimes \Omega) \Omega + \frac{2}{v} \Omega = \frac{2}{v} \Omega, \end{aligned}$$

equation (5.114) becomes

$$\int_{\mathbb{R}^3} \frac{\partial f}{\partial t} \varphi(v) \Omega dv = A \int_{\mathbb{R}^3} f \left[\frac{g(v)}{v^3} \varphi'(v) \frac{2}{v} \Omega + (\Omega \otimes \Omega) \Omega \frac{\partial}{\partial v} \left(\frac{g(v)}{v^3} \varphi'(v) \right) \right]. \quad (5.115)$$

In spherical coordinates, equation (5.115) writes

$$\int_0^{+\infty} \frac{\partial f^1}{\partial t} \varphi(\zeta) d\zeta = A \int_0^{+\infty} \left[\frac{g(\zeta)}{\zeta^3} \varphi'(\zeta) \frac{2}{\zeta} + \frac{\partial}{\partial \zeta} \left(\frac{g(\zeta)}{\zeta^3} \varphi'(\zeta) \right) \right] f^1 d\zeta . \quad (5.116)$$

By using some integration by parts, equation (5.110) becomes

$$\int_0^{+\infty} \frac{\partial f^1}{\partial t} \varphi(\zeta) d\zeta = -A \int_0^{+\infty} \left[\frac{\partial}{\partial \zeta} \left(f^1 \frac{2g(\zeta)}{\zeta^4} - \frac{\partial f^1}{\partial \zeta} \frac{g(\zeta)}{\zeta^3} \right) \right] \varphi(\zeta) d\zeta , \quad (5.117)$$

because φ is a smooth test function on compact support. We can conclude that $\forall \varphi \in \mathcal{D}(]0, \infty[)$, we obtain

$$\int_0^{+\infty} \frac{\partial f^1}{\partial t} \varphi(\zeta) d\zeta = A \int_0^{+\infty} \left[\frac{\partial}{\partial \zeta} \left(\frac{\partial f^1}{\partial \zeta} \frac{g(\zeta)}{\zeta^3} \right) - \frac{\partial}{\partial \zeta} \left(f^1 \frac{2g(\zeta)}{\zeta^4} \right) \right] \varphi(\zeta) d\zeta ,$$

definition (5.50) follows.

CHAPTER 6

GENERAL MOMENT SYSTEM FOR PLASMA PHYSICS BASED ON MINIMUM ENTROPY PRINCIPLE

This chapter on generalized moment model is the subject of one publication in [26].

6.1 Introduction

In comparison with the M_1 model derived in chapter 5, the M_N model is a generalization of M_1 model to N given moments. It is performed by introducing a new formalism leading to a concise form of the moment system. Moreover, contrarily to chapter 5 where the moment system was discretized, we chose here to discretize the energy variable to extract the moments. Besides proof presented in the previous chapter are very specific to M_1 model. Indeed theorems and proofs are based on the knowledge of known coefficients to define the distribution function f . Here these coefficients are unknown which implies a different approach.

The remainder of the chapter is organized as follows. The kinetic continuous equation in spherical domain is described in section 6.2. Moreover, a new model is chosen for collision in order to preserve collision properties. Section 6.3 is devoted to the derivation of a semi-discrete scheme. The time is kept as a continuous variable whereas the energy variable is discrete. The definition of the approximated solution on the dual mesh through an entropic average allows to prove the entropic dissipation property of collision operator. Next, the N-moments model is derived from the continuous kinetic equation in section 6.4 and the model is shown to preserve previous property. In section 6.5, the semi-discrete scheme associated is proved to be an entropic scheme for the moments system.

6.2 New Kinetic continuous model

The extraction of moments for the classical kinetic equation

$$\partial_t f + v \partial_x f = C(f, f), \quad (6.1)$$

is complicated because of the expression of the electron-electron collision operator. Some approximation has been proposed in [4, 14] but the new model obtained does not satisfy fundamental properties. That is why a new model for collisions will be proposed here to preserve collision properties.

6.2.1 Presentation of the Kinetic continuous equation

Notations

If S_2 is the unit sphere, $\Omega = v/|v|$ represents the direction of propagation of the particles and $\mu = \Omega_x = \cos \theta$, $\theta \in [0, \frac{\pi}{2}]$. In this chapter, we chose a one dimensional direction of propagation, i.e. we take $\mu \in [-1, 1]$ as the direction of propagation variable instead of Ω . By setting $\zeta = |v|$,

the distribution function f can be expressed in spherical coordinates by $f(x, \zeta, \mu)$. The N first moments with respect to μ are defined by

$$f^i = 2\pi\zeta^2 \int_{-1}^1 f(\zeta, \mu) \mu^i d\mu = \zeta^2 \langle f\mu^i \rangle \quad \forall i \in \{0, N\}, \quad (6.2)$$

where $\langle . \rangle$ is defined for any function Ψ by

$$\langle \Psi \rangle = 2\pi \int_{-1}^1 \Psi(\mu) d\mu,$$

and $f(\zeta, \mu)$ denotes the distribution function f in spherical coordinate. Next, we set

$$F^i(\zeta) = \frac{f^i(\zeta)}{\zeta^2}. \quad (6.3)$$

Realizability problem

Consider here the case of a space homogeneous situation with one specie of particle, i.e. the plasma consists only of electrons. In this case, equation (6.1) is reduced to

$$\partial_t f = C_{ee}(f, f). \quad (6.4)$$

The moments extraction for the electron-electron collision operator $C_{ee}(f, f)$ is very complicated because of its nonlinear quadratic formulation. That is why in Plasma physics, classical approximations for the operator lead to consider that the main contribution for the electron-electron collision operator comes from the isotropic part. This approximation denoted by Q_{ee} has been defined in [30] and used in [4, 14] to study the homogeneous Fokker-Planck-Landau equation for isotropic distribution functions.

For the sake of simplicity, consider here the M_1 model constructed from the three first moments. By integrating the equation (6.4) with respect to Ω in a space homogeneous situation, it holds that

$$\begin{cases} \partial_t f^0 = Q_{ee}^0, \\ \partial_t f^1 = 0, \end{cases} \quad (6.5)$$

with

$$Q_{ee}^0 = \partial_\zeta \left(\zeta \int_0^\infty \tilde{J}(\zeta, \zeta') \left[F^0(\zeta') \frac{1}{\zeta} \partial_\zeta (F^0(\zeta)) - F^0(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} F^0(\zeta') \right] \zeta'^2 d\zeta' \right), \quad (6.6)$$

and

$$\tilde{J}(\zeta, \zeta') = \frac{2\alpha_{ee}}{3} \inf\left(\frac{1}{\zeta^3}, \frac{1}{\zeta'^3}\right) \zeta'^2 \zeta^2. \quad (6.7)$$

The definition of $\mathcal{Q}_{ee}^0$ has been established by considering that the main contribution from this operator comes from the isotropic part of f . Its expression (6.6) and the system (6.5) are explained in Appendix 6.7. However, the following example 18 demonstrates that this model does not preserve the realizability domain $\mathcal{A}$ defined by

$$\mathcal{A} = \left\{ \bar{g} = \begin{pmatrix} g^0 \\ \vdots \\ g^N \end{pmatrix} \in \mathbb{R}^N / \exists g \geq 0 \in L^1([-1, 1]) \text{ and } \zeta^2 < \mu^i g(\alpha) = f^i \forall i \in \{0, N\} \right\}.$$

In the particular case of the M_1 model, the realizability domain $\mathcal{A}$ is shown to be equal to $\mathcal{B}$ defined by

$$\mathcal{B} = \left\{ \bar{g} = \begin{pmatrix} g^0 \\ g^1 \end{pmatrix} \in \mathbb{R}^2, g^0 > 0 \text{ and } |g^1| < g^0 \right\} \cup \{(0, 0)\}.$$

Example 18. Let us chose the following initial data for the isotropic and anisotropic part

$$f^0(t=0) = \frac{1}{3}\chi_{[0,3]}(\zeta) \text{ and } f^1(t=0) = \frac{1}{4}\chi_{[0,3]}(\zeta).$$

Because of the electron-electron collision operator effect, the isotropic part converges to a centered Maxwellian. Moreover, the M_1 model preserves mass and energy. So the obtained Maxwellian presents the same mass ($\int_0^\infty f^0 d\zeta = 1$) and the same energy ($\int_0^\infty f^0 \zeta^2 d\zeta = 3$) as the initial distribution function. Therefore the Maxwellian writes

$$f^0 = \sqrt{\frac{2}{\pi}} \exp\left(\frac{-\zeta^2}{2}\right) \zeta^2.$$

Hence figure 6.1 shows that the realizability domain $\mathcal{B}$ is not preserved anymore.

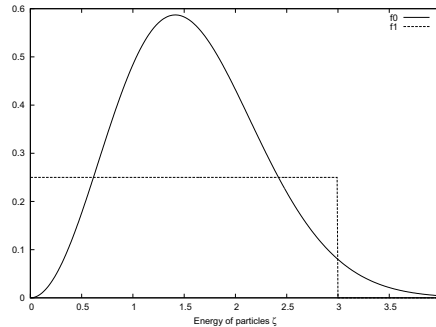


Figure 6.1: Representation of the moments f^0 and f^1 with respect to the energy ζ at the steady state in the case of $\zeta_{\max} = 4$ which denotes the length of the energy space.

New linearized approximation of the collision operator Q_{ee}

Therefore, to preserve the realizability domain (proof in section 6.4.2), we consider that the contribution from the electron-electron operator comes from the isotropic part for distribution function depending on ζ' and the expression of collision operator is given by the following definition.

Definition 2. *The electron-electron collision operator $Q_{ee}(f)$ and the electron-ion collision operator $Q_{ei}(f)$ are given by*

$$Q_{ee}(f) = \frac{1}{\zeta^2} \partial_\zeta \left(\zeta \int_0^\infty \tilde{J}(\zeta, \zeta') \left[F^0(\zeta') \frac{1}{\zeta} \partial_\zeta f(\zeta) - f(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} F^0(\zeta') \right] \zeta'^2 d\zeta' \right), \quad (6.8)$$

and

$$Q_{ei}(f) = \frac{\alpha_{ei}}{\zeta^3} \frac{\partial}{\partial \mu} \left((1 - \mu^2) \frac{\partial f}{\partial \mu} \right), \quad (6.9)$$

where $\tilde{J}(\zeta, \zeta')$ is given in (6.7).

Expression (6.9) of Q_{ei} in spherical coordinate (ζ, μ, φ) is detailed in Appendix 6.8 by considering one direction of propagation. So equation (6.1) can be approached in spherical coordinates by

$$\partial_t f + \zeta \mu \partial_x f = Q(f), \quad (6.10)$$

where

$$Q(f) = Q_{ee}(f) + Q_{ei}(f),$$

and Q_{ee}, Q_{ei} are defined by (6.8, 6.9).

Remark 19. *If we assume that f is isotropic, i.e. $f = f^0$, we obtain*

$$Q(f) = Q(f^0) = \int_{\Omega} C_{ee}(f, f) d\Omega = \int_{\Omega} C_{ee}(f^0, f^0) d\Omega.$$

Now, let us check that the new model conserves fundamental properties.

6.2.2 Properties of the model

In this subsection, we present the fundamental properties satisfied by the new model, that is the conservation laws, a H-theorem and the conservation of the realizability domain.

Proposition 10. *The operator $Q(f)$ satisfies mass and energy conservation properties and the entropy dissipation property, i.e.*

$$\left\langle \int_0^\infty \zeta^2 Q(f) \left(\frac{1}{\zeta^2} \right) d\zeta \right\rangle = 0 \quad \text{and} \quad \left\langle \int_0^\infty \zeta^2 Q(f) \log f d\zeta \right\rangle \leq 0.$$

Proof. Firstly, in order to recover the conservations property, let us consider the quantity

$$\langle \int_0^\infty \zeta^2 Q(f) \left(\frac{1}{\zeta^2} \right) d\zeta \rangle = \int_0^\infty \zeta^2 \langle Q_{ee}(f) \rangle \left(\frac{1}{\zeta^2} \right) d\zeta + \int_0^\infty \zeta^2 \langle Q_{ei}(f) \rangle \left(\frac{1}{\zeta^2} \right) d\zeta . \quad (6.11)$$

For the sake of clarity, we introduce the notation

$$Q_{ee}^0 = \zeta^2 \langle Q_{ee}(f) \rangle .$$

The first term of the right hand side of equation (6.11) can be rewritten as

$$\int_0^\infty Q_{ee}^0 \left(\frac{1}{\zeta^2} \right) d\zeta = \int_0^\infty \partial_\zeta \left(\zeta \int_0^\infty \tilde{J}(\zeta, \zeta') \left[F^0(\zeta') \frac{1}{\zeta} \partial_\zeta f^0(\zeta) - f^0(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} F^0(\zeta') \right] \zeta'^2 d\zeta' \right) \left(\frac{1}{\zeta^2} \right) d\zeta .$$

So, $\int_0^\infty Q_{ee}^0 d\zeta = 0$. Since

$$\begin{aligned} \int_0^\infty Q_{ee}^0 \zeta^2 d\zeta &= -2 \int_0^\infty \int_0^\infty \zeta^2 \zeta'^2 \tilde{J}(\zeta, \zeta') \left[F^0(\zeta') \frac{1}{\zeta} \partial_\zeta f^0(\zeta) \right] d\zeta' d\zeta \\ &+ 2 \int_0^\infty \int_0^\infty \zeta^2 \zeta'^2 \tilde{J}(\zeta, \zeta') \left[f^0(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} F^0(\zeta') \right] d\zeta' d\zeta , \end{aligned}$$

we get from Fubini theorem

$$\int_0^\infty Q_{ee}^0 \zeta^2 d\zeta = 0 .$$

Besides we have $\langle Q_{ei}(f) \rangle = 0$. Therefore the conservation properties follow

$$\langle \int_0^\infty \zeta^2 Q(f) \left(\frac{1}{\zeta^2} \right) d\zeta \rangle = 0 .$$

Now we prove the entropy dissipation property of the collision operator.

$$\begin{aligned} \langle \int_0^\infty \zeta^2 Q(f) \log f d\zeta \rangle &= - \langle \int_0^\infty \int_0^\infty \zeta'^2 \zeta \tilde{J}(\zeta, \zeta') \left[F^0(\zeta') \frac{1}{\zeta} \partial_\zeta f(\zeta) - f(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} F^0(\zeta') \right] \right. \\ &\quad \left. \frac{1}{f} \partial_\zeta f d\zeta d\zeta' \right\rangle . \end{aligned} \quad (6.12)$$

Equation (6.12) can be rewritten in terms of f as

$$\begin{aligned} \langle \int_0^\infty \zeta^2 Q(f) \log f d\zeta \rangle &= - \int_{-1}^1 \int_{-1}^1 \int_0^\infty \int_0^\infty \zeta'^2 \zeta \tilde{J}(\zeta, \zeta') \left[f(\zeta') \frac{1}{\zeta} \partial_\zeta f(\zeta) - f(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} f(\zeta') \right] \\ &\quad \frac{1}{f} \partial_\zeta f d\zeta d\zeta' d\mu d\mu' \\ &= - \int_{-1}^1 \int_{-1}^1 \int_0^\infty \int_0^\infty \zeta'^2 \zeta^2 \tilde{J}(\zeta, \zeta') f(\zeta) f(\zeta') \left[\frac{1}{\zeta} \partial_\zeta (\log f(\zeta)) - \frac{1}{\zeta'} \partial_{\zeta'} (\log f(\zeta')) \right] \\ &\quad \frac{1}{\zeta} \partial_\zeta \log f(\zeta) d\zeta d\zeta' d\mu d\mu' . \end{aligned}$$

Moreover the changing of variables $(\zeta, \zeta') \mapsto (\zeta', \zeta)$ leads to

$$\begin{aligned} \langle \int_0^\infty \zeta^2 Q(f) \log f d\zeta \rangle = & -\frac{1}{2} \int_{-1}^1 \int_{-1}^1 \int_0^\infty \int_0^\infty \zeta'^2 \zeta^2 \tilde{J}(\zeta, \zeta') f(\zeta) f(\zeta') \left[\frac{1}{\zeta} \partial_\zeta (\log f(\zeta)) - \right. \\ & \left. \frac{1}{\zeta'} \partial_{\zeta'} (\log f(\zeta')) \right]^2 d\zeta d\zeta' d\mu d\mu' , \end{aligned}$$

and the entropy dissipation property follows. $\square$

6.3 Semi-discretized kinetic equation

In this section we propose an energy discretization for the new continuous model defined in section 6.2 such that the fundamental properties of collision operator are conserved. In chapter 5, a semi-discretized scheme of the moment system has already been proposed. Here, we discretize the kinetic equation and then extract moments in section 6.5. The key point of the following scheme is the approximation of the distribution function f on the interface from the primal one, through an entropic average as in chapter 5. This average leads to the entropy dissipation property for the discretized collisional operator.

6.3.1 Introduction

Let us define the primal mesh $\mathcal{M}$, for the energy variable ζ , decomposed into a family of rectangles $\mathcal{M}_{j+\frac{1}{2}} =]\zeta_{j-\frac{1}{2}}, \zeta_j[$, $\forall j \in \{1, m\}$, where $\zeta_j = j\Delta\zeta_j$ and $m \in \mathbb{N}$ the number of points which discretize the energy domain. $\Delta\zeta_j$ represents the discretization step, which can be variable. We denote by $\mathcal{D}$ its associated dual mesh consisting of cells $\mathcal{D}_j =]\zeta_{j-\frac{1}{2}}, \zeta_{j+\frac{1}{2}}[$ where $\zeta_{j-\frac{1}{2}} = (j - \frac{1}{2})\Delta\zeta_j$. Let h_j be an approximation of $h(\zeta_j)$ for all distribution function h and $h_{j+\frac{1}{2}}$ an approximation of $f(\zeta_{j+\frac{1}{2}})$.

We denote f_j by

$$f_j = \exp(\overline{\alpha_j} \cdot \overline{\mu}) , \quad (6.13)$$

with

$$\overline{\alpha_j} = \begin{pmatrix} \alpha_{0,j} \\ \vdots \\ \alpha_{n,j} \end{pmatrix} \text{ and } \alpha_{i,j} = \alpha_i(\zeta_j) . \quad (6.14)$$

The discrete form of (6.10) can be expressed as

$$\partial_t f_j + \zeta_j \mu \partial_x f_j = Q_j , \quad (6.15)$$

where

$$Q_j = Q(f_j) = Q_{ee,j} + Q_{ei,j} . \quad (6.16)$$

Definition 3. The collision operators $Q_{ee,j}$ and $Q_{ei,j}$ are respectively given by

$$\begin{cases} Q_{ee,j} = \frac{G_{j+\frac{1}{2}} - G_{j-\frac{1}{2}}}{\zeta_j^2 \Delta \zeta_j}, \\ Q_{ei,j} = \frac{\alpha_{ei}}{\zeta_j^3} \frac{\partial}{\partial \mu} \left((1 - \mu^2) \frac{\partial f_j}{\partial \mu} \right), \end{cases} \quad (6.17)$$

with

$$G_{j+\frac{1}{2}} = \zeta_{j+\frac{1}{2}} \sum_{k=0}^m \widetilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) \left[F_{k+\frac{1}{2}}^0 \frac{1}{\zeta_{j+\frac{1}{2}}} \frac{f_{j+1} - f_j}{\Delta \zeta_{j+\frac{1}{2}}} - \frac{1}{\zeta_{k+\frac{1}{2}}} f_{j+\frac{1}{2}} \frac{F_{k+1}^0 - F_k^0}{\Delta \zeta_{k+\frac{1}{2}}} \right] \zeta_{k+\frac{1}{2}}^2 \Delta \zeta_{k+\frac{1}{2}}, \quad (6.18)$$

where

$$\widetilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) = \frac{2\alpha_{ee}}{3} \inf \left(\frac{1}{\zeta_{j+\frac{1}{2}}^3}, \frac{1}{\zeta_{k+\frac{1}{2}}^3} \right) \zeta_{j+\frac{1}{2}}^2 \zeta_{k+\frac{1}{2}}^2, \quad (6.19)$$

and $f_{j+\frac{1}{2}}$ is defined by the following entropic average

$$\begin{cases} f_{j+\frac{1}{2}} = \frac{f_{j+1} - f_j}{\log |f_{j+1}| - \log |f_j|} & \text{if } f_{j+1} \neq f_j, \\ f_{j+\frac{1}{2}} = f_{j+1} & \text{else.} \end{cases} \quad (6.20)$$

Remark 20. This entropic average has already been considered for an isotropic distribution function by Dellacherie ([14]) to build an entropic scheme. Next, in the previous chapter, this approach has been generalized to a system for the M_1 model. Here we propose to use this entropic average to preserve the collision properties and to simplify demonstrations.

Remark 21. For the sake of simplicity operator $Q_{ee,j}$ defined by $G_{j+\frac{1}{2}}^i$ (6.52) can be rewritten by using $G_{j+\frac{1}{2}}^i$ as

$$G_{j+\frac{1}{2}}^i = A_{j+\frac{1}{2}} \frac{f_{j+1}^i - f_j^i}{\Delta \zeta_{j+\frac{1}{2}}} + B_{j+\frac{1}{2}} f_{j+\frac{1}{2}}^i$$

where

$$A_{j+\frac{1}{2}} = \zeta_{j+\frac{1}{2}}^2 \sum_{k=0}^m \min \left(\frac{1}{\zeta_{j+\frac{1}{2}}^3}, \frac{1}{\zeta_{k+\frac{1}{2}}^3} \right) F_{k+\frac{1}{2}}^0 \zeta_{k+\frac{1}{2}}^4 \Delta \zeta_{k+\frac{1}{2}}, \quad (6.21)$$

$$B_{j+\frac{1}{2}} = -\zeta_{j+\frac{1}{2}}^3 \sum_{k=0}^m \min \left(\frac{1}{\zeta_{j+\frac{1}{2}}^3}, \frac{1}{\zeta_{k+\frac{1}{2}}^3} \right) \zeta_{k+\frac{1}{2}}^3 (F_{k+1}^0 - F_k^0). \quad (6.22)$$

6.3.2 Properties of the semi-discretized scheme

Proposition 11. Q_j satisfies the following fundamental properties.

1). The operator Q_j satisfies mass and energy conservations property i.e.

$$\langle \sum_{j=0}^m \zeta_j^2 Q_j \left(\frac{1}{\zeta^2} \right) \Delta \zeta_j \rangle = 0 .$$

2) By defining $f_{j+\frac{1}{2}}$ through the entropic average given by (6.20), the operator Q_j satisfies the entropy dissipation property

$$\langle \sum_{j=0}^m \zeta_j^2 Q_j \log f_j \Delta \zeta_j \rangle \leq 0 .$$

Proof. First we aim to prove the mass conservation. A simple computation gives

$$\langle \sum_{j=0}^m \zeta_j^2 Q_j \Delta \zeta_j \rangle = G_{m+\frac{1}{2}} - G_{-\frac{1}{2}} .$$

We suppose G as a compact support function so $G_{m+\frac{1}{2}} = 0$. Besides $G_{-\frac{1}{2}} = 0$ because $\zeta_{-\frac{1}{2}} = 0$. Therefore we get

$$\langle \sum_{j=0}^m \zeta_j^2 Q_j \Delta \zeta_j \rangle = 0 .$$

Then we will show the conservation of energy which can be rewritten into

$$\langle \sum_{j=0}^m \zeta_j^2 Q_j \zeta_j^2 \Delta \zeta_j \rangle = \langle - \sum_{j=0}^{m-1} G_{j+\frac{1}{2}} (\zeta_{j+1}^2 - \zeta_j^2) \rangle . \quad (6.23)$$

By using the expression of $G_{j+\frac{1}{2}}$ given in (6.18), equation (6.23) can be simplified into

$$\begin{aligned} \langle \sum_{j=0}^m \zeta_j^2 Q_j \zeta_j^2 \Delta \zeta_j \rangle = & \langle - \sum_{j=0}^{m-1} \zeta_{j+\frac{1}{2}} \sum_{k=0}^m \tilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) \left[F_{k+\frac{1}{2}}^0 \frac{1}{\zeta_{j+\frac{1}{2}}} \frac{f_{j+1} - f_j}{\Delta \zeta_{j+\frac{1}{2}}} \right. \\ & \left. - \frac{1}{\zeta_{k+\frac{1}{2}}} f_{j+\frac{1}{2}} \frac{F_{k+1}^0 - F_k^0}{\Delta \zeta_{k+\frac{1}{2}}} \right] \zeta_{k+\frac{1}{2}}^2 \Delta \zeta_{k+\frac{1}{2}} (\zeta_{j+1}^2 - \zeta_j^2) \rangle . \end{aligned} \quad (6.24)$$

Since we have $\zeta_{j+1}^2 - \zeta_j^2 = 2\zeta_{j+\frac{1}{2}}\Delta\zeta_{j+\frac{1}{2}}$, equation (6.24) can be rewritten into

$$\begin{aligned} < \sum_{j=0}^m \zeta_j^2 Q_j \zeta_j^2 \Delta\zeta_j > = - \int_{-1}^1 \int_{-1}^1 \sum_{j,k=0}^m \zeta_{j+\frac{1}{2}}^2 \zeta_{k+\frac{1}{2}}^2 \Delta\zeta_{j+\frac{1}{2}} \Delta\zeta_{k+\frac{1}{2}} \tilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) f_{k+\frac{1}{2}} \frac{1}{\zeta_{j+\frac{1}{2}}} \\ & \quad \frac{f_{j+1} - f_j}{\Delta\zeta_{j+\frac{1}{2}}} d\mu d\mu' \\ & \quad + \int_{-1}^1 \int_{-1}^1 \sum_{j,k=0}^m \zeta_{j+\frac{1}{2}}^2 \zeta_{k+\frac{1}{2}}^2 \Delta\zeta_{j+\frac{1}{2}} \Delta\zeta_{k+\frac{1}{2}} \tilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) f_{j+\frac{1}{2}} \frac{1}{\zeta_{k+\frac{1}{2}}} \\ & \quad \frac{f_{k+1} - f_k}{\Delta\zeta_{k+\frac{1}{2}}} d\mu d\mu' , \end{aligned}$$

and the conservation of energy follows by exchanging indexes k and j . Next, we prove that the numerical operator Q_j is entropic. Firstly,

$$< \sum_{j=0}^m \zeta_j^2 Q_j \log f_j \Delta\zeta_j > = < - \sum_{j=0}^{m-1} \log \left(\frac{f_{j+1}}{f_j} \right) G_{j+\frac{1}{2}} > . \quad (6.25)$$

From the definition of $G_{j+\frac{1}{2}}$, equation (6.25) can be written as

$$\begin{aligned} < \sum_{j=0}^m \zeta_j^2 Q_j \log f_j \Delta\zeta_j > = - \int_{-1}^1 \int_{-1}^1 \sum_{j,k=0}^m \zeta_{j+\frac{1}{2}}^2 \zeta_{k+\frac{1}{2}}^2 \Delta\zeta_{j+\frac{1}{2}} \Delta\zeta_{k+\frac{1}{2}} \tilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) f_{k+\frac{1}{2}} f_{j+\frac{1}{2}} \\ & \quad \left[\frac{1}{\zeta_{j+\frac{1}{2}}} \frac{f_{j+1} - f_j}{f_{j+\frac{1}{2}} \Delta\zeta_{j+\frac{1}{2}}} - \frac{1}{\zeta_{k+\frac{1}{2}}} \frac{f_{k+1} - f_k}{f_{k+\frac{1}{2}} \Delta\zeta_{k+\frac{1}{2}}} \right] \frac{\log f_{j+1} - \log f_j}{\zeta_{j+\frac{1}{2}} \Delta\zeta_{j+\frac{1}{2}}} d\mu d\mu' . \end{aligned} \quad (6.26)$$

By using the change of variables $(\mu, \mu') \mapsto (\mu', \mu)$ and the change of index $(j, k) \mapsto (k, j)$ together to the entropic average (6.20) defining $f_{j+\frac{1}{2}}$, equation (6.26) can be rewritten into

$$\begin{aligned} < \sum_{j=0}^m \zeta_j^2 Q_j \log f_j \Delta\zeta_j > = - \frac{1}{2} \int_{-1}^1 \int_{-1}^1 \sum_{j,k=0}^m \zeta_{j+\frac{1}{2}}^2 \zeta_{k+\frac{1}{2}}^2 \Delta\zeta_{j+\frac{1}{2}} \Delta\zeta_{k+\frac{1}{2}} \tilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) f_{k+\frac{1}{2}} f_{j+\frac{1}{2}} \\ & \quad \left[\frac{1}{\zeta_{j+\frac{1}{2}}} \frac{\log f_{j+1} - \log f_j}{\Delta\zeta_{j+\frac{1}{2}}} - \frac{1}{\zeta_{k+\frac{1}{2}}} \frac{\log f_{k+1} - \log f_k}{\Delta\zeta_{k+\frac{1}{2}}} \right]^2 d\mu d\mu' , \end{aligned}$$

and the entropic dissipation property follows. $\square$

6.4 Continuous moment model and entropy

In this section, we explain the construction of the M_N model obtained from the kinetic one described in section 6.2. Besides collision operator properties are shown to be still preserved by the moments extraction.

6.4.1 N-moment model for the new kinetic equation

Firstly the M_N model constructed from the previous kinetic equation (6.10) is presented. This system is established by using a minimum entropy principle for the angle variable by keeping the energy of particles as a kinetic variable.

By extracting moment of (6.10) with respect to μ^i , we obtain

$$\partial_t f^i + \zeta \partial_x f^{i+1} = Q^i, \quad i \in \{0, N\}, \quad (6.27)$$

with

$$Q^i = Q_{ee}^i + Q_{ei}^i.$$

The expression Q_{ee}^i and Q_{ei}^i are computed in the following property.

Property 22. *The moments of Q_{ee} are given by*

$$Q_{ee}^i = \partial_\zeta \left(\zeta \int_0^\infty \tilde{J}(\zeta, \zeta') \left[F^0(\zeta') \frac{1}{\zeta} \partial_\zeta F^i(\zeta) - F^i(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} F^0(\zeta') \right] \zeta'^2 d\zeta' \right),$$

whereas the moments of Q_{ei} are expressed as

$$Q_{ei}^i = \alpha_{ei} \frac{i}{\zeta^3} \left((i-1)f^{i-2} - (i+1)f^i \right).$$

The expression of the electron-electron collision Q_{ee}^i is simple whereas the moment extraction Q_{ei}^i for the electron-ion collision is computed in Appendix 6.9.

Then the minimum entropy principle [23, 27] under moments constraint implies that the distribution function f should be taken under the following form

$$f(\zeta, \mu) = \exp(\bar{\alpha}(\zeta, t, x) \cdot \bar{\mu}), \quad (6.28)$$

with $\bar{\alpha} = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_N \end{pmatrix}$ and $\bar{\mu} = \begin{pmatrix} \mu^0 \\ \vdots \\ \mu^N \end{pmatrix}$.

(6.28) implies that $\bar{\alpha}$ and f are in bijection in the realizability domain (6.8). Therefore f^{N+1} can be expressed as

$$f^{N+1} = \zeta^2 < f \mu^{N+1} >.$$

By setting

$$\bar{f} = \begin{pmatrix} f^0 \\ \vdots \\ f^N \end{pmatrix}, \quad \tilde{f} = \begin{pmatrix} f^1 \\ \vdots \\ f^{N+1} \end{pmatrix} \quad \text{and} \quad \bar{Q} = \begin{pmatrix} Q^0 \\ \vdots \\ Q^N \end{pmatrix}, \quad (6.29)$$

it holds that $\bar{f}$ satisfies (6.27) and we obtain the system written in a concise form

$$\partial_t \bar{f} + \zeta \partial_x \tilde{f} = \bar{Q}. \quad (6.30)$$

6.4.2 Realizability domain

The continuous problem defined by equation (6.30) satisfy the following property.

Proposition 12. *By using the continuous problem defined by equation (6.30) and under the assumption $\bar{f}(t=0, \zeta) \in \mathcal{A}$, the realizability domain is conserved, i.e. $\bar{f}(t, \zeta) \in \mathcal{A}$ for any $t \geq 0$.*

The proof of the Proposition 12 is shown by using the following lemma.

Lemma 23. *Let g be solution to*

$$\frac{\partial g}{\partial t} = Q_{ee}(g) + Q_{ei}(g), \quad (6.31)$$

with

$$Q_{ee}(g) = \frac{2\alpha_{ee}}{3\zeta^2} \partial_\zeta \left(\tilde{A}(F^{0+}, \zeta) \partial_\zeta g + \tilde{B}(F^{0+}, \zeta) g \right), \quad (6.32)$$

$$Q_{ei}(g) = \frac{\alpha_{ei}}{\zeta^3} \frac{\partial}{\partial \mu} \left((1 - \mu^2) \frac{\partial f}{\partial \mu} \right). \quad (6.33)$$

Here $\tilde{A}$ and $\tilde{B}$ are given by

$$\tilde{A}(F^{0+}, \zeta) = \zeta^2 \int_0^\infty \min\left(\frac{1}{\zeta^3}, \frac{1}{w^3}\right) w^4 F^0(w)^+ dw, \quad (6.34)$$

$$\tilde{B}(F^{0+}, \zeta) = -\zeta^3 \int_0^\infty \min\left(\frac{1}{\zeta^3}, \frac{1}{w^3}\right) w^3 \partial_w \left(\frac{f^0(w)^+}{w^2} \right) dw = 3 \int_0^\zeta w^2 F^0(w)^+ dw. \quad (6.35)$$

Therefore by using the continuous problem defined by equation 6.31 and under the assumption $g(0, \zeta) \in \mathcal{A}$, the realizability domain $\mathcal{A}$ is conserved, i.e. $g(t, \zeta) \in \mathcal{A}$ for any $t \geq 0$.

Notice that definition (6.32) is a simplification of definition (6.8). Moreover, recall that $F^{0+} = \max(0, F^0)$.

Proof. Let H be a convex positive function such as $H' \leq 0$ and defined by

$$H(x) = \begin{cases} -Cx & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}, \quad C > 0.$$

By multiplying equation (6.31) by $\zeta^2 H'(g)$ and integrating over Ω , we get

$$\int_\Omega \zeta^2 \partial_t H(g) d\Omega = \int_\Omega \zeta^2 Q_{ee}(g) H'(g) d\Omega + \int_\Omega \zeta^2 Q_{ei}(g) H'(g) d\Omega. \quad (6.36)$$

We show firstly that $\int_{\Omega} \zeta^2 Q_{ee}(g) H'(g) d\Omega \leq 0$. The first right-hand side term of (6.36) writes

$$4\pi \int_0^{\infty} \zeta^2 Q_{ee}(g) H'(g) d\zeta = 4\pi \left(\int_0^{\infty} \frac{2\alpha_{ee}}{3} \partial_{\zeta} \left(\tilde{A}(F^{0+}, \zeta) \partial_{\zeta} g \right) H'(g) d\zeta \right. \\ \left. + \int_0^{\infty} \frac{2\alpha_{ee}}{3} \partial_{\zeta} \left(\tilde{B}(F^{0+}, \zeta) g \right) H'(g) d\zeta \right). \quad (6.37)$$

By using integration by parts on the first right-hand side term of (6.37) together with the convexity of H implies the inequality

$$\int_0^{\infty} \frac{2\alpha_{ee}}{3} \partial_{\zeta} \left(\tilde{A}(F^{0+}, \zeta) \partial_{\zeta} g \right) H'(g) d\zeta \leq 0.$$

Moreover the second term of (6.37) can be simplified as

$$\int_0^{\infty} \frac{2\alpha_{ee}}{3} \partial_{\zeta} \left(\tilde{B}(F^{0+}, \zeta) g \right) H'(g) d\zeta = \int_0^{\infty} 2\alpha_{ee} H'(g) \partial_{\zeta} g \int_0^{\zeta} w^2 F^{0+}(w) dw d\zeta \\ + \int_0^{\infty} 2\alpha_{ee} H'(g) \zeta^2 g F^{0+} d\zeta.$$

By using an integration by part, we get

$$\int_0^{\infty} \frac{2\alpha_{ee}}{3} \partial_{\zeta} \left(\tilde{B}(F^{0+}, \zeta) g \right) H'(g) d\zeta = \int_0^{\infty} 2\alpha_{ee} F^{0+} (-H(g) + H'(g)g) d\zeta.$$

Moreover, by construction of H , $\forall x$, $(-H(x) + H'(x)x) x = 0$. Therefore, (6.37) gives

$$\int_{\Omega} \zeta^2 Q_{ee}(g) H'(g) d\Omega \leq 0. \quad (6.38)$$

Then we show that $\int_0^{\infty} \zeta^2 Q_{ei}(g) H'(g) d\zeta \leq 0$. The second right-hand side term of (6.36) writes

$$\int_{\Omega} \zeta^2 Q_{ei}(g) H'(g) d\Omega = \int_0^{\infty} \int_{-1}^1 \zeta^2 \frac{\partial}{\partial \mu} \left((1 - \mu^2) \frac{\partial g}{\partial \mu} \right) H'(g) d\mu d\zeta. \quad (6.39)$$

By using an integration by parts, equation (6.39) writes

$$\int_{\Omega} \zeta^2 Q_{ei}(g) H'(g) d\Omega = - \int_0^{\infty} \int_{-1}^1 \zeta^2 (1 - \mu^2) \left(\frac{\partial g}{\partial \mu} \right)^2 H''(g) d\mu d\zeta.$$

Function H can be taken more regular by assuming that $H'(0) = H''(0) = 0$. In this way, the second derivative can be considered for the previous equation. By convexity of H , we get

$$\int_{\Omega} \zeta^2 Q_{ei}(g) H'(g) d\Omega \leq 0. \quad (6.40)$$

From equations (6.38) and (6.40), we obtain

$$\int_{\Omega} \zeta^2 \partial_t H(g) d\Omega \leq 0 .$$

After integrating the previous inequality, we get

$$\int_{\Omega} \zeta^2 H(g(t, \Omega)) d\Omega \leq \int_{\Omega} \zeta^2 H(g(0, \Omega)) d\Omega .$$

By using the definition of H , $\int_{\Omega} \zeta^2 H(g) d\Omega = 0$. Then $H(g, \Omega) = 0$ a.e. and the positivity of g is preserved, i.e. $g \geq 0$. $\square$

Let us now finish the proof of Proposition 12:

Proof. Let us choose $\bar{f}(0, \zeta) \in \mathcal{A}$ therefore

$$\bar{f}(0, \zeta) = \int_{-1}^1 f(0, \zeta) \bar{\mu} d\mu \quad (6.41)$$

with $f(0, \zeta) \geq 0$. The distribution function $f(0, \zeta) \geq 0$ is solution of equation

$$\frac{\partial f}{\partial t} = Q_{ee}(f) + Q_{ei}(f) , \quad (6.42)$$

whose collision operators Q_{ee} and Q_{ei} defined respectively by formulations (6.32) and (6.33). Extracting moments of the equation (6.42) and setting $\bar{g}(t, \zeta) = \int_{-1}^1 f(t, \zeta) \bar{\mu} d\mu$ leads to

$$\frac{\partial}{\partial t} \bar{g}(t, \zeta) = Q_{ee}(\bar{g}(t, \zeta)) + Q_{ei}(\bar{g}(t, \zeta)) . \quad (6.43)$$

From lemma 23, we deduce that $g(t, \zeta, \mu) \geq 0$. By uniqueness of the solution we get $\bar{g}(t, \zeta, \mu) = \bar{f}(t, \zeta, \mu)$ and the solution of the continuous problem defined by equation (6.30) belongs to the realizability domain.

Then by taking $g(t, \zeta) = f^0(\zeta)$, we deduce that $f^0(t, \zeta) \geq 0$ which implies that $f^{0+}(\zeta) = f^0(\zeta)$ and the proposition 12 follows. $\square$

6.4.3 H-theorem

To solve the minimization problem, let remember the realizability domain $\mathcal{A}$ defined in 5.33. The following lemma shows the existence of a solution for the minimization problem

$$\left(\min_{\bar{\alpha} \in \mathbb{R}^N} \left\{ \langle \exp(\bar{\alpha} \cdot \bar{\mu}) \rangle - \bar{\alpha} \cdot \bar{F} \right\} \right) (P) ,$$

where $\bar{\alpha} = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_N \end{pmatrix}$ and $\bar{F} = \begin{pmatrix} F^0 \\ \vdots \\ F^N \end{pmatrix}$.

Lemma 24. *The value F^{N+1} is defined by values of $(F^i)_{i \in \{0, N\}}$. Indeed, by taking f under the form*

$$f(\alpha) = \exp \left(\sum_{i=1}^N \alpha_i \mu^i \right) ,$$

and by choosing $F^i(\zeta) \in \mathcal{A}$, the minimization problem (P) subject to the realisability constraints $\mathcal{A}$ has a unique solution.

Proof. Let us introduce

$$J(\bar{\alpha}) = \min_{\bar{\alpha} \in \mathbb{R}^N} \left\{ \langle \exp(\bar{\alpha} \cdot \bar{\mu}) \rangle - \bar{\alpha} \cdot \bar{F} \right\} .$$

First we aim to prove the existence of solutions for the problem (P). By coercivity and convexity of J ([20, 21, 28]), its unique minimum satisfies

$$J'(\bar{\alpha}) = 0 . \tag{6.44}$$

That is

$$F^i = \langle f(\alpha) \mu^i \rangle ,$$

i.e. the unique minimum of the function J realizes moments as desired. □

One of the main result of this paper is the following theorem which proves that the system (6.29, 6.30) is entropic.

Theoreme 25. *$E = \langle f \ln f - f \rangle \zeta^2$ is an entropy for the system (6.29, 6.30). More precisely, we have*

$$\partial_t E + \nabla \cdot F \leq 0 ,$$

where F is the entropic flux given by $F = \zeta \langle (f \ln f - f) \mu \rangle \zeta^2$.

Proof. Equation (6.30) multiplied by $\bar{\alpha}$ can be rewritten into

$$\bar{\alpha} \cdot \partial_t \langle f \bar{\mu} \rangle \zeta^2 + \zeta \bar{\alpha} \cdot \partial_x \langle f \bar{\mu} \rangle \zeta^2 = \langle \zeta^2 \bar{\alpha} \cdot \bar{\mu} Q(f) \rangle . \quad (6.45)$$

The first term of the left hand side of (6.45) can be simplified into

$$\bar{\alpha} \cdot \partial_t \langle f \bar{\mu} \rangle = \partial_t \langle (\bar{\mu} \cdot \bar{\alpha}) f \rangle - \langle \partial_t (\bar{\mu} \cdot \bar{\alpha}) f \rangle .$$

By using the definition (6.28) of f , we get

$$\bar{\alpha} \cdot \partial_t \langle f \bar{\mu} \rangle = \partial_t \langle \log f f \rangle - \langle \partial_t f \rangle . \quad (6.46)$$

In the same way we have

$$\zeta \bar{\alpha} \cdot \partial_x \langle f \bar{\mu} \rangle = \zeta \partial_x \langle (\bar{\alpha} \cdot \bar{\mu}) \mu f \rangle - \zeta \cdot \langle \partial_x (\bar{\alpha} \cdot \bar{\mu}) \mu f \rangle ,$$

and then

$$\zeta \bar{\alpha} \cdot \partial_x \langle f \bar{\mu} \rangle = \zeta \partial_x \langle \log f f \mu \rangle - \langle \zeta \mu \partial_x f \rangle . \quad (6.47)$$

Then, from equations (6.46) and (6.47), we obtain

$$\partial_t \int_0^\infty \langle f \log f - f \rangle \zeta^2 d\zeta + \partial_x \int_0^\infty \zeta \langle (f \log f - f) \mu \rangle \zeta^2 d\zeta = \int_0^\infty \langle \log f Q(f) \zeta^2 \rangle d\zeta ,$$

and Theorem 25 follows from Proposition 10. $\square$

6.5 Semi-discretization of the problem

In this section, the moments procedure with respect to the angle variable is applied on the semi-discrete scheme proposed in section 6.3. This allows us to derive an entropic scheme for the M_N model derived in section 6.4.

6.5.1 N-moment model for the semi-discretized kinetic equation

Firstly, moments of f_j are given by

$$f_j^i = \zeta^2 \langle \mu^i f_j(\alpha) \rangle , \quad \forall i \in \{0, N\} , \quad \forall j \in \{1, m\} . \quad (6.48)$$

Then the moments of f_j satisfy for any $i \in \{0, \dots, N\}$

$$\partial_t f_j^i + \zeta_j \partial_x f_j^{i+1} = Q_j^i ,$$

where $f_j^i = \langle f_j \mu^i \rangle \zeta_j^2$ and $Q_j^i = \langle Q_j \mu^i \rangle \zeta_j^2$.

To solve this problem, we introduce the discrete realizability domain $\mathcal{A}_d$ defined by

$$\mathcal{A}_d = \left\{ \overline{g}_j = \begin{pmatrix} g_j^0 \\ \vdots \\ g_j^N \end{pmatrix} \in \mathbb{R}^N, \text{ such that } \exists g_j \geq 0 \text{ and } \zeta^2 < \mu^i f_j(\alpha) \Rightarrow f_j^i \right. \\ \left. \forall i \in \{0, N\}, \forall j \in \{1, m\} \right\}. \quad (6.49)$$

By extracting moments on equation (6.15), we obtain the N-moment discretized system as

$$\partial_t \overline{f}_j + \zeta_j \partial_x \widetilde{f}_j = \overline{Q}_j, \quad (6.50)$$

where

$$\overline{f}_j = \begin{pmatrix} f_j^0 \\ \vdots \\ f_j^m \end{pmatrix}, \widetilde{f}_j = \begin{pmatrix} f_j^1 \\ \vdots \\ f_j^{m+1} \end{pmatrix}, \overline{Q}_j = \begin{pmatrix} Q_j^0 \\ \vdots \\ Q_j^n \end{pmatrix} = \langle \overline{\mu} Q_j \rangle.$$

Definition 4. The discrete collision operators $Q_{ee,j}^i$ and $Q_{ei,j}^i$ involved in (6.50) are respectively given by

$$\begin{cases} Q_{ee,j}^i = \frac{G_{j+\frac{1}{2}}^i - G_{j-\frac{1}{2}}^i}{\Delta \zeta_j}, \\ Q_{ei,j}^i = \alpha_{ei} \frac{i}{\zeta_j^2} ((i-1)f_j^{i-2} - (i+1)f_j^i), \end{cases} \quad (6.51)$$

where

$$G_{j+\frac{1}{2}}^i = \zeta_{j+\frac{1}{2}} \sum_{k=0}^m \widetilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) \left[F_{k+\frac{1}{2}}^0 \frac{1}{\zeta_{j+\frac{1}{2}}} \frac{F_{j+1}^i - F_j^i}{\Delta \zeta_{j+\frac{1}{2}}} - \frac{1}{\zeta_{k+\frac{1}{2}}} F_{j+\frac{1}{2}}^i \frac{F_{k+1}^0 - F_k^0}{\Delta \zeta_{k+\frac{1}{2}}} \right] \\ \zeta_{k+\frac{1}{2}}^2 \Delta \zeta_{k+\frac{1}{2}}, \quad (6.52)$$

with $\widetilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}})$ is given by (6.19) and

$$F_{j+\frac{1}{2}}^i = \int_0^1 f_{j+\frac{1}{2}} \mu^i d\mu. \quad (6.53)$$

Recall that $f_{j+\frac{1}{2}}$ is defined in (6.20).

Remark 26. Remember that chapter 5 shows that there exists an intermediate state $(f_{j+\frac{1}{2}}^i)_{i \in \{0, N\}} \in \mathcal{A}_d$ related to $(f^i)_{i \in \{0, N\}}$ expressed by

$$f_{j+1}^i - f_j^i = \left(f_{j+\frac{1}{2}}^{k+i} \right)_{k,i=0,N} (\overline{\alpha_{j+1}} - \overline{\alpha_j}),$$

where

$$f_{j+\frac{1}{2}}^i = \int_0^1 f_{j+\frac{1}{2}} \mu^i d\mu,$$

and $\overline{\alpha_j}$ an approximation of $\overline{\alpha(\zeta_j)}$. For $\theta \in [0, 1]$, $f_{j+\frac{1}{2}}$ is defined by

$$f_{j+\frac{1}{2}} = \int_0^1 \exp(\overline{\alpha_j} \theta + (1 - \theta) \overline{\alpha_{j+1}}) \cdot \overline{\mu} d\theta,$$

which can be simplified after some computations and by continuity into

$$\begin{cases} f_{j+\frac{1}{2}} = \frac{\exp(\overline{\alpha_{j+1}} \cdot \overline{\mu}) - \exp(\overline{\alpha_j} \cdot \overline{\mu})}{\overline{\alpha_{j+1}} \cdot \overline{\mu} - \overline{\alpha_j} \cdot \overline{\mu}} & \text{if } \overline{\alpha_{j+1}} \cdot \overline{\mu} \neq \overline{\alpha_j} \cdot \overline{\mu}, \\ f_{j+\frac{1}{2}} = \exp(\overline{\alpha_{j+1}} \cdot \overline{\mu}) & \text{else.} \end{cases} \quad (6.54)$$

By using definition (6.13) of $f_{j+\frac{1}{2}}^i$, equation (6.54) can be rewritten as

$$\begin{cases} f_{j+\frac{1}{2}} = \frac{f_{j+1} - f_j}{\log(f_{j+1}) - \log(f_j)} & \text{if } f_{j+1} \neq f_j, \\ f_{j+\frac{1}{2}} = f_{j+1} & \text{else.} \end{cases} \quad (6.55)$$

Therefore the intermediate state $f_{j+\frac{1}{2}}$ defined in [25] is equivalent to the entropic average (6.53).

Remark 27. From a computational point of view, an approximation of (6.53) can be obtained through a usual quadrature formula.

6.5.2 Realizability domain

The semi-discretized problem defined by equations (6.50, 6.51) satisfy the following property.

Proposition 13. By using the semi-discretized problem (6.15, 6.17) and under assumption that $\underline{f}_j(t = 0) \in \mathcal{A}_d$, the realizability domain is conserved, i.e. $\underline{f}_j(t) \in \mathcal{A}_d$ for any $t \geq 0$.

Lemma 28. Let g_j solution to

$$\frac{\partial g_j}{\partial t} = \frac{G_{j+\frac{1}{2}} - G_{j-\frac{1}{2}}}{\zeta_j^2 \Delta \zeta_j} + Q_{ei,j}, \quad (6.56)$$

where

$$\begin{aligned} G_{j+\frac{1}{2}} &= \widetilde{A}_{j+\frac{1}{2}} \frac{g_{j+1} - g_j}{\Delta \zeta_{j+\frac{1}{2}}} - \widetilde{B}_{j+\frac{1}{2}} g_{j+1}, \\ \widetilde{A}_{j+\frac{1}{2}} &= \zeta_{j+\frac{1}{2}}^2 \sum_{k=0}^m \min \left(\frac{1}{\zeta_{j+\frac{1}{2}}^3}, \frac{1}{\zeta_{k+\frac{1}{2}}^3} \right) \left(F_{k+\frac{1}{2}}^0 \right)^+ \zeta_{k+\frac{1}{2}}^4 \Delta \zeta_{k+\frac{1}{2}}, \end{aligned} \quad (6.57)$$

$$\widetilde{B}_{j+\frac{1}{2}} = \left(B_{j+\frac{1}{2}} \right)^+ \frac{g_{j+\frac{1}{2}}^+}{g_{j+1}^+}. \quad (6.58)$$

$B_{j+\frac{1}{2}}$ is given in remark 21.

Therefore by using the continuous problem defined by equation 6.31 and under the assumption $g_j(0) \in \mathcal{A}_d$, the realizability domain $\mathcal{A}_d$ is conserved, i.e. $g_j(t)$ belongs to the realizability domain $\mathcal{A}_d$ for any $t \geq 0$.

Proof. Let H be a convex positive function such as $H' \leq 0$ and defined by

$$H(x) = \begin{cases} -Cx & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}, \quad C > 0.$$

By multiplying equation (6.56) by $\zeta_{j+\frac{1}{2}}^2 H'(g_j)$ and integrating over ζ and μ , we get

$$\begin{aligned} \int_{-1}^1 \sum_j \zeta_{j+\frac{1}{2}}^2 \frac{\partial H(g_j)}{\partial t} \Delta \zeta_{j+\frac{1}{2}} d\mu &= \int_{-1}^1 \sum_j (G_{j+\frac{1}{2}} - G_{j-\frac{1}{2}}) H'(g_j) d\mu + \\ &\quad \int_{-1}^1 \sum_j \zeta_{j+\frac{1}{2}}^2 Q_{ei,j} \Delta \zeta_{j+\frac{1}{2}} d\mu. \end{aligned} \quad (6.59)$$

From expression (6.21) together to a shift of index, it comes that the first term of the right-hand side of equation (6.59) can be simplified into

$$\begin{aligned} \int_{-1}^1 \sum_j \zeta_{j+\frac{1}{2}}^2 Q_{ee}(g_j) H'(g_j) \Delta \zeta_{j+\frac{1}{2}} d\mu &= \int_{-1}^1 \sum_j \tilde{A}_{j+\frac{1}{2}} \frac{g_{j+1} - g_j}{\Delta \zeta_{j+\frac{1}{2}}} (H'(g_{j+1}) - H'(g_j)) \\ &\quad + \int_{-1}^1 \sum_j (\tilde{B}_{j+\frac{1}{2}} g_{j+1} - \tilde{B}_{j-\frac{1}{2}} g_j) H'(g_j) d\mu. \end{aligned} \quad (6.60)$$

The second term of the right hand side of (6.60) can be rewritten as

$$\begin{aligned} \sum_j (\tilde{B}_{j+\frac{1}{2}} g_{j+1} - \tilde{B}_{j-\frac{1}{2}} g_j) H'(g_j) &= \sum_j \left[(\tilde{B}_{j+\frac{1}{2}} - \tilde{B}_{j-\frac{1}{2}}) g_j H'(g_j) + \right. \\ &\quad \left. \tilde{B}_{j+\frac{1}{2}} (g_{j+1} - g_j) H'(g_j) \right]. \end{aligned} \quad (6.61)$$

Moreover by convexity H , H satisfies

$$H(g_{j+1}) \geq H(g_j) + (g_{j+1} - g_j) H'(g_j) + \frac{1}{2} (g_{j+1} - g_j)^2 H''(g_j). \quad (6.62)$$

Then by non-negativity of $\tilde{B}_{j+\frac{1}{2}}$, equation (6.61) becomes

$$\begin{aligned} \sum_j (\tilde{B}_{j+\frac{1}{2}} g_{j+1} - \tilde{B}_{j-\frac{1}{2}} g_j) H'(g_j) &\leq \sum_j \left[(\tilde{B}_{j+\frac{1}{2}} - \tilde{B}_{j-\frac{1}{2}}) g_j H'(g_j) \right. \\ &\quad \left. + \tilde{B}_{j+\frac{1}{2}} (H(g_{j+1}) - H(g_j)) \right]. \end{aligned} \quad (6.63)$$

By using a shift of index, we get

$$\sum_j (\widetilde{B}_{j+\frac{1}{2}} g_{j+\frac{1}{2}} - \widetilde{B}_{j-\frac{1}{2}} g_j) H'(g_j) \leq \sum_j (\widetilde{B}_{j+\frac{1}{2}} - \widetilde{B}_{j-\frac{1}{2}}) (-H(g_j) + g_j H'(g_j)) . \quad (6.64)$$

By construction of H , we get $(-H(g_j) + g_j H'(g_j)) = 0$ and thus

$$\sum_j (\widetilde{B}_{j+\frac{1}{2}} g_{j+\frac{1}{2}} - \widetilde{B}_{j-\frac{1}{2}} g_j) H'(g_j) \leq 0 . \quad (6.65)$$

Moreover since H' is a non-decreasing function, the first right hand term side of (6.60) is negative. Therefore, from (6.65), we obtain the following inequality

$$\int_{-1}^1 \sum_j \zeta_{j+\frac{1}{2}}^2 Q_{ee}(g_j) H'(g_j) \Delta \zeta_{j+\frac{1}{2}} d\mu \leq 0 .$$

Besides by arguing like for proof of lemma 23, the second term of the right-hand side of equation (6.59) is negative. Indeed, integration by parts are effective in μ which is a continuous variable.

Then by using definition of H , lemma 28 follows. $\square$

Let us now finish the proof of Proposition 13:

Proof. By using lemma 28 with $g_j = f_j^0$, we deduce that $f_j^0 \geq 0$. Then by arguing in the same way, we get $f_j \geq 0$. Therefore, it remains to solve

$$\frac{\partial f_j}{\partial t} = \frac{G_{j+\frac{1}{2}} - G_{j-\frac{1}{2}}}{\zeta_j^2 \Delta \zeta_j} , \quad (6.66)$$

with

$$G_{j+\frac{1}{2}} = A_{j+\frac{1}{2}} \frac{f_{j+1} - f_j}{\Delta \zeta_{j+\frac{1}{2}}} - (B_{j+\frac{1}{2}})^+ g_{j+1} ,$$

where $A_{j+\frac{1}{2}}$ and $B_{j+\frac{1}{2}}$ are respectively given by (6.21) and (6.22). By arguing like for the proof of the lemma but on equation (6.66), Proposition 13 follows. $\square$

6.5.3 Entropic property of the scheme

Finally, this subsection is devoted to demonstrate the entropic property of the scheme. We recall that this scheme is based on an entropic average given in (6.53).

Theoreme 29. $E = \sum_{j=0}^m \langle f_j \ln f_j - f_j \rangle \zeta_j^2$ is an entropy for the system (6.50). More precisely, we have

$$\partial_t E + \nabla \cdot F \leq 0 ,$$

where $F = \sum_{j=0}^m \zeta_j \langle (f_j \ln f_j - f_j) \mu \rangle \zeta_j^2$ is the entropic flux.

Proof. First, after multiplying equation (6.50) by $\bar{\alpha}$, we get

$$\bar{\alpha}_j \cdot \partial_t \bar{f}_j + \zeta_j \bar{\alpha}_j \cdot \partial_x \bar{f}_j = \bar{\alpha}_j \cdot \bar{Q}_j .$$

Hence, by using the expression f_j ,

$$\bar{\alpha}_j \cdot \partial_t \langle \bar{\mu} f_j \rangle \zeta_j^2 + \zeta_j \bar{\alpha}_j \cdot \partial_x \langle \bar{\mu} f_j \rangle \zeta_j^2 = \langle \bar{\alpha}_j \cdot \bar{\mu} Q_j \zeta_j^2 \rangle . \quad (6.67)$$

The first left hand side term of (6.67) can be expressed such as

$$\bar{\alpha}_j \cdot \partial_t \langle \bar{\mu} f_j \rangle = \partial_t \langle \bar{\alpha}_j \cdot \bar{\mu} f_j \rangle - \partial_t (\bar{\alpha}_j \cdot \bar{\mu}) f_j ,$$

which can be simplified into

$$\bar{\alpha}_j \cdot \partial_t \langle \bar{\mu} f_j \rangle = \partial_t \langle \log f_j f_j \rangle - \langle \partial_t f_j \rangle . \quad (6.68)$$

In the same way, the second term of the right hand side can be rewritten into

$$\zeta_j \bar{\alpha}_j \cdot \partial_x \langle \bar{\mu} f_j \rangle = \zeta_j \partial_x \left(\langle \log f_j f_j \mu \rangle \right) - \langle \zeta_j \mu \partial_x f_j \rangle . \quad (6.69)$$

Then, from equations (6.68) and (6.69) and the definition of f_j , it comes

$$\partial_t \sum_{j=0}^m \langle f_j \ln f_j - f_j \rangle \zeta_j^2 \Delta \zeta_j + \partial_x \sum_{j=0}^m \zeta_j \langle (f_j \ln f_j - f_j) \mu \rangle \zeta_j^2 \Delta \zeta_j = \langle \log f_j Q_j \zeta_j^2 \rangle .$$

From the dissipative property of Q_j given in Proposition 11, Theorem 29 follows. $\square$

6.6 Conclusion

We have proposed firstly a model for plasma electron transport with a new consistent collision operator for the electron-electron interactions. Its definition allows mainly to conserve fundamental properties such as the mass and energy conservation together with the entropy dissipation. Next we have motivated the development of a semi-discretized scheme in the energy variable for the new continuous model. This model still preserves fundamental properties by using a specify entropic average. Moreover the M_N model derived from the new continuous model is proved to be entropic. This M_N model is based on an entropy minimization principle and an integration of the kinetic model w.r.t. the angle variable. Finally the moment extraction on the semi-discretization model is proved to be entropic.

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6.7 Appendix A: Moment closure for the homogeneous system for isotropic distribution function

To establish this closure we consider only the homogeneous case with the collision operator C_{ee} ,

$$\partial_t f = C_{ee}(f, f) = \alpha_{ee} \nabla_v \cdot \int_{v' \in \mathbb{R}^3} \Phi(u) [f(v') \nabla_v f(v) - f(v) \nabla_{v'} f(v')] dv', \quad (6.70)$$

where $\Phi(u)$ is given in (2.8) and $u = v' - v$. To simplify the moment extraction of this operator, we will assume that the main contribution from this operator comes from the isotropic part of f . In the remainder of this subsection, we will assume that the distribution function f is isotropic, i.e. $f = f(|v|) = f(\zeta)$.

We need to introduce the notation $\Gamma(v)$ which is the flux term given by divergence in v ,

$$\Gamma(v) = \int_{v' \in \mathbb{R}^3} \Phi(u) [f(v') \nabla_v f(v) - f(v) \nabla_{v'} f(v')] dv'. \quad (6.71)$$

The $\Gamma(v)$ term satisfies $C_{ee}(f, f) = \alpha_{ee} \nabla_v \cdot \Gamma(v)$. Moreover if the distribution function f is also isotropic, $\Gamma(v)v$ is isotropic. In fact, in this case $\Gamma(v)v = g(|v|)$ does not depend on the direction of vector v . If we set $\zeta = |v|$, and we denote by φ some regular test function, it comes that

$$\begin{aligned} 0 &= \int_{v \in \mathbb{R}^3} (\partial_t f - \alpha_{ee} \nabla_v \cdot \Gamma(v)) \varphi(\zeta) dv, \\ &= \int_{v \in \mathbb{R}^3} \partial_t f \varphi(\zeta) dv + \alpha_{ee} \int_{v \in \mathbb{R}^3} \nabla_v \varphi(\zeta) \cdot \Gamma(v) dv, \\ &= \int_0^\infty \partial_t \left(\zeta^2 \int_{S_2} f d\mu \right) \varphi(\zeta) d\zeta + \alpha_{ee} \int_0^\infty \frac{1}{\zeta} \varphi'(\zeta) \int_{S_2} \Gamma(v) \cdot v d\mu \zeta^2 d\zeta, \\ &= \int_0^\infty \partial_t f^0 \varphi(\zeta) d\zeta + \alpha_{ee} \int_0^\infty \frac{1}{\zeta} \varphi'(\zeta) 4\pi \Gamma(v) \cdot v \zeta^2 d\zeta, \text{ as } \Gamma(v) \cdot v \text{ is isotropic,} \\ &= \int_0^\infty \left(\partial_t f^0 - 4\pi \alpha_{ee} \partial_\zeta (\zeta \Gamma(v) \cdot v) \right) \varphi(\zeta) d\zeta, \end{aligned}$$

which imply that

$$\partial_t f^0 = 4\pi \alpha_{ee} \partial_\zeta (\zeta \Gamma(v) \cdot v). \quad (6.72)$$

Now to compute $\Gamma(v) \cdot v$, let us introduce the notation $\zeta' = |v'|$ i.e. $v' = \zeta' \Omega'$. From (6.71), the term

$$[f(v') \nabla_v f(v) - f(v) \nabla_{v'} f(v')] dv'$$

can be rewritten in spherical coordinates as

$$\left[f(\zeta') \frac{v}{\zeta} \partial_\zeta f(\zeta) - f(\zeta) \Omega' \partial_{\zeta'} f(\zeta') \right] d\Omega' \zeta'^2 d\zeta'. \quad (6.73)$$

By symmetry of $\Phi(u)$, $\Gamma(v) \cdot v$ writes

$$\Gamma(v) \cdot v = \int_{v' \in \mathbb{R}^3} \Phi(u) v \cdot [f(v') \nabla_v f(v) - f(v) \nabla_{v'} f(v')] dv'. \quad (6.74)$$

From (6.73), $\Gamma(v) \cdot v$ term becomes in spherical coordinates

$$\Gamma(v) \cdot v = \int_0^\infty \int_{\Omega' \in \mathcal{S}_2} \Phi(u) v \cdot \left[f(\zeta') \frac{v}{\zeta} \partial_\zeta f(\zeta) - f(\zeta) \Omega' \partial_{\zeta'} f(\zeta') \right] d\Omega' \zeta'^2 d\zeta'. \quad (6.75)$$

By setting some scalar product as $A_v = \Phi(u) v \cdot v$ and $A_{\Omega'} = \Phi(u) v \cdot \Omega'$, (6.75) can be expressed into

$$\Gamma(v) \cdot v = \int_0^\infty \int_{\Omega' \in \mathcal{S}_2} \left[f(\zeta') \frac{1}{\zeta} \partial_\zeta f(\zeta) A_v - f(\zeta) \partial_{\zeta'} f(\zeta') A_{\Omega'} \right] d\Omega' \zeta'^2 d\zeta'. \quad (6.76)$$

Since $u = v - v' = v - \Omega' \zeta'$, the Landau tensor ϕ can be developed into

$$\Phi(u) = \frac{(\zeta^2 + \zeta'^2 - 2\zeta' v \cdot \Omega') I - (v \otimes v + \zeta'^2 \Omega' \otimes \Omega' - \zeta'(\Omega' \otimes v + v \otimes \Omega'))}{(\zeta^2 + \zeta'^2 - 2\zeta' v \cdot \Omega')^{\frac{3}{2}}}. \quad (6.77)$$

From (6.77), we obtain

$$\Phi(u) v = \frac{(\zeta'^2 - \zeta' v \cdot \Omega') v - (\zeta'^2 (\Omega' \cdot v) - \zeta^2 \zeta') \Omega'}{(\zeta^2 + \zeta'^2 - 2\zeta' v \cdot \Omega')^{\frac{3}{2}}}. \quad (6.78)$$

By using equation (6.78), the first scalar product $A_v = \Phi(u) v \cdot v$ writes

$$A_v = \frac{(\zeta'^2 - \zeta' v \cdot \Omega') \zeta^2 + \zeta' (\zeta^2 - \zeta' (\Omega' \cdot v)) (\Omega' \cdot v)}{(\zeta^2 + \zeta'^2 - 2\zeta' v \cdot \Omega')^{\frac{3}{2}}}.$$

Recall that μ represents the cosine of that angle between the vector Ω and v , such that $\Omega \cdot v = \zeta \mu$. By using μ variable, A_v can be simplified into

$$A_v = \frac{\zeta^2 \zeta'^2 (1 - \mu^2)}{(\zeta^2 + \zeta'^2 - 2\zeta \zeta' \mu)^{\frac{3}{2}}}. \quad (6.79)$$

In the same way, we get

$$A_{\Omega'} = \frac{\zeta^2 \zeta' (1 - \mu^2)}{(\zeta^2 + \zeta'^2 - 2\zeta\zeta'\mu)^{\frac{3}{2}}} . \quad (6.80)$$

The computation of the $\Gamma(v) \cdot v$ term can now be completed. From (6.79) and (6.80), expression (6.76) of the term $\Gamma(v) \cdot v$ can be simplified into

$$\Gamma(v) \cdot v = \int_0^\infty \int_{-1}^{+1} \frac{\zeta^2 \zeta'^2 (1 - \mu^2)}{(\zeta^2 + \zeta'^2 - 2\zeta\zeta'\mu)^{\frac{3}{2}}} \left[f(\zeta') \frac{1}{\zeta} \partial_\zeta f(\zeta) - f(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} f(\zeta') \right] 2\pi d\mu \zeta'^2 d\zeta' , \quad (6.81)$$

with $d\Omega' = 2\pi d\mu$. As,

$$\int_{-1}^{+1} \frac{\zeta^2 \zeta'^2 (1 - \mu^2)}{(\zeta^2 + \zeta'^2 - 2\zeta\zeta'\mu)^{\frac{3}{2}}} d\mu = \frac{4}{3} \inf\left(\frac{1}{\zeta^3}, \frac{1}{\zeta'^3}\right) ,$$

$\Gamma(v) \cdot v$ can be rewritten as

$$\Gamma(v) \cdot v = \int_0^\infty \frac{8\pi}{3} \inf\left(\frac{1}{\zeta^3}, \frac{1}{\zeta'^3}\right) \zeta^2 \zeta'^2 \left[f(\zeta') \frac{1}{\zeta} \partial_\zeta f(\zeta) - f(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} f(\zeta') \right] \zeta'^2 d\zeta' . \quad (6.82)$$

In this case, as we have assumed that f is isotropic, $f^0(\zeta) = \zeta^2 \int_{S_2} f(\zeta) d\Omega = 4\pi \zeta^2 f(\zeta)$. So from (6.72) and (6.82) it comes that

$$\partial_t f^0 = \alpha_{ee} \partial_\zeta \left(\zeta \int_0^\infty \frac{2}{3} \inf\left(\frac{1}{\zeta^3}, \frac{1}{\zeta'^3}\right) \zeta^2 \zeta'^2 \left[\frac{f^0(\zeta')}{\zeta'^2} \frac{1}{\zeta} \partial_\zeta \left(\frac{f^0(\zeta)}{\zeta^2} \right) - \frac{f^0(\zeta)}{\zeta^2} \frac{1}{\zeta'} \partial_{\zeta'} \left(\frac{f^0(\zeta')}{\zeta'^2} \right) \right] \zeta'^2 d\zeta' \right) .$$

We retrieve exactly the formula for the collision operator involved by the equation on f^0 . We neglect the operator Q_{ee} for the equation on f^1 because we retain only the isotropic part of this operator and f^1 represents the anisotropic part of the distribution function.

6.8 Appendix B: Expression of Q_{ei} in spherical coordinate (ζ, μ, φ)

To establish the expression of Q_{ei} in spherical coordinate (ζ, μ, φ) defined by (6.89), we consider only the homogeneous case with the collision operator C_{ei} ,

$$\partial_t f = \alpha_{ei} \nabla_v \cdot (\Phi(u) \nabla_v(f)) , \quad (6.83)$$

where $\Phi(u)$ is an orthogonal projection operator issued from the coulomb potential, acting on the relative velocity u defined by

$$\Phi(u) = \frac{1}{|u|^3} (|u|^2 Id - u \otimes u) .$$

We express equation (6.83) along the coordinate system (ζ, μ, φ) , which describes the direction of propagation Ω as

$$\Omega = \begin{pmatrix} \mu \\ \sqrt{1-\mu^2} \cos \varphi \\ \sqrt{1-\mu^2} \sin \varphi \end{pmatrix}.$$

In this frame, the velocity v is defined as

$$v = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \zeta \Omega = \begin{pmatrix} \zeta \mu \\ \zeta \sqrt{1-\mu^2} \cos \varphi \\ \zeta \sqrt{1-\mu^2} \sin \varphi \end{pmatrix}.$$

The gradient $\nabla_v f$ can be rewritten as

$$\nabla_v f = \frac{\partial f}{\partial \zeta} \nabla_v \zeta + \frac{\partial f}{\partial \mu} \nabla_v \mu + \frac{\partial f}{\partial \varphi} \nabla_v \varphi. \quad (6.84)$$

Since we have

$$\begin{pmatrix} \nabla_v \zeta \\ \nabla_v \mu \\ \nabla_v \varphi \end{pmatrix} = \frac{D(\zeta, \mu, \varphi)}{D(v_x, v_y, v_z)} = \left(\frac{D(v_x, v_y, v_z)}{D(\zeta, \mu, \varphi)} \right)^{-1} = \begin{pmatrix} \mu & \sqrt{1-\mu^2} \cos \varphi & \sqrt{1-\mu^2} \sin \varphi \\ \frac{1-\mu^2}{\zeta} & \frac{-\mu \sqrt{1-\mu^2} \cos \varphi}{\zeta} & \frac{-\mu \sqrt{1-\mu^2} \sin \varphi}{\zeta} \\ 0 & \frac{-\sin \varphi}{\sqrt{1-\mu^2} \zeta} & \frac{\cos \varphi}{\sqrt{1-\mu^2} \zeta} \end{pmatrix},$$

the gradient vector can be simplified as

$$\nabla_v \zeta = \vec{\Omega}, \quad \nabla_v \mu = \frac{\sqrt{1-\mu^2}}{\zeta} \vec{e}_\mu \text{ and } \nabla_v \varphi = \frac{1}{\sqrt{1-\mu^2} \zeta} \vec{e}_\varphi. \quad (6.85)$$

By using (6.85), (6.84) writes

$$\nabla_v f = \frac{\partial f}{\partial \zeta} \vec{\Omega} + \frac{\partial f}{\partial \mu} \frac{\sqrt{1-\mu^2}}{\zeta} \vec{e}_\mu + \frac{\partial f}{\partial \varphi} \frac{1}{\sqrt{1-\mu^2} \zeta} \vec{e}_\varphi. \quad (6.86)$$

We denote by Ψ some test function and we obtain

$$\int_{\mathbb{R}^3} \partial_t f \Psi(v) dv = \int_{\mathbb{R}^3} \nabla_v \cdot (\Phi(v) \nabla_v f) \Psi(v) dv.$$

Since

$$\int_{\mathbb{R}^3} \nabla_v \cdot (\Phi(v) \nabla_v f) \Psi(v) dv = - \int_{\mathbb{R}^3} \Phi(v) \nabla_v f \cdot \nabla_v \Psi(v) dv,$$

and by using the expression of the divergence term (6.86), we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla_{\mathbf{v}} \cdot (\Phi(\mathbf{v}) \nabla_{\mathbf{v}} f) \Psi(\mathbf{v}) d\mathbf{v} = & - \int_{\mathbb{R}^3} \Phi(\mathbf{v}) \left(\frac{\partial f}{\partial \zeta} \vec{\Omega} + \frac{\partial f}{\partial \mu} \frac{\sqrt{1-\mu^2}}{\zeta} \vec{e}_\mu + \frac{\partial f}{\partial \varphi} \frac{1}{\sqrt{1-\mu^2}\zeta} \vec{e}_\varphi \right) \\ & \left(\frac{\partial \Psi}{\partial \zeta} \vec{\Omega} + \frac{\partial \Psi}{\partial \mu} \frac{\sqrt{1-\mu^2}}{\zeta} \vec{e}_\mu + \frac{\partial \Psi}{\partial \varphi} \frac{1}{\sqrt{1-\mu^2}\zeta} \vec{e}_\varphi \right) d\mathbf{v}, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla_{\mathbf{v}} \cdot (\Phi(\mathbf{v}) \nabla_{\mathbf{v}} f) \Psi(\mathbf{v}) d\mathbf{v} = & - \int_{\mathbb{R}^3} \left(\frac{\partial f}{\partial \mu} \frac{\sqrt{1-\mu^2}}{\zeta^2} \vec{e}_\mu + \frac{\partial f}{\partial \varphi} \frac{1}{\sqrt{1-\mu^2}\zeta^2} \vec{e}_\varphi \right) \\ & \left(\frac{\partial \Psi}{\partial \mu} \frac{\sqrt{1-\mu^2}}{\zeta^2} \vec{e}_\mu + \frac{\partial \Psi}{\partial \varphi} \frac{1}{\sqrt{1-\mu^2}\zeta^2} \vec{e}_\varphi \right) d\mathbf{v}, \end{aligned} \quad (6.87)$$

because Φ is an orthogonal projection on $\vec{\Omega}$ ($\Phi(\mathbf{v})\vec{\Omega} = 0$, $\Phi(\mathbf{v})\vec{e}_\mu = \frac{1}{\zeta}\vec{e}_\mu$ and $\vec{e}_\varphi = \frac{1}{\zeta}\vec{e}_\varphi$). Besides $(\vec{e}_\zeta, \vec{e}_\mu, \vec{e}_\varphi)$ is an orthonormal basis, so (6.87) writes

$$\int_{\mathbb{R}^3} \nabla_{\mathbf{v}} \cdot (\Phi(\mathbf{v}) \nabla_{\mathbf{v}} f) \Psi(\mathbf{v}) d\mathbf{v} = \int_{\mathbb{R}^3} \left(\frac{\partial f}{\partial \mu} \frac{\partial \Psi}{\partial \mu} \frac{1-\mu^2}{\zeta^3} + \frac{\partial f}{\partial \varphi} \frac{\partial \Psi}{\partial \varphi} \frac{1}{(1-\mu^2)\zeta^3} \right) d\mathbf{v}. \quad (6.88)$$

As $d\mathbf{v} = \zeta^2 d\zeta d\mu d\varphi$, (6.88) writes

$$\int_{\mathbb{R}^3} \nabla_{\mathbf{v}} \cdot (\Phi(\mathbf{v}) \nabla_{\mathbf{v}} f) \Psi(\mathbf{v}) d\mathbf{v} = \int_{\Omega} \int_0^\infty \frac{1}{\zeta} \left(\frac{\partial f}{\partial \mu} \frac{\partial \Psi}{\partial \mu} (1-\mu^2) + \frac{\partial f}{\partial \varphi} \frac{\partial \Psi}{\partial \varphi} \frac{1}{(1-\mu^2)} \right) d\zeta d\mu d\varphi.$$

Then by using integration by parts we obtain

$$\int_{\mathbb{R}^3} \nabla_{\mathbf{v}} \cdot (\Phi(\mathbf{v}) \nabla_{\mathbf{v}} f) \Psi(\mathbf{v}) d\mathbf{v} = \int_{\mathbb{R}^3} \frac{1}{\zeta^3} \left[\frac{\partial}{\partial \mu} \left((1-\mu^2) \frac{\partial f}{\partial \mu} \right) + \frac{\partial}{\partial \varphi} \left(\frac{1}{1-\mu^2} \frac{\partial f}{\partial \varphi} \right) \right] \Psi(\mathbf{v}) d\mathbf{v}.$$

Therefore we obtain the moment closure for the electron-ion collision (6.9)

$$Q_{ei} = \frac{\alpha_{ei}}{\zeta^3} \left[\frac{\partial}{\partial \mu} \left((1-\mu^2) \frac{\partial f}{\partial \mu} \right) + \frac{\partial}{\partial \varphi} \left(\frac{1}{1-\mu^2} \frac{\partial f}{\partial \varphi} \right) \right]. \quad (6.89)$$

6.9 Appendix C: Moment extraction for the electron-ion collision

In this appendix, we will show that the moment extraction for the electron-ion collision Q_{ei} can be written as

$$Q_{ei}^i(f) = \alpha_{ei} \frac{i}{|v|^3} \left((i-1)f^{i-2} - (i+1)f^i \right). \quad (6.90)$$

We denote by ψ some test function and we will establish the weak formulation associated with the electron-ion collision operator Q_{ei}

$$\int_0^\infty \langle Q_{ei}(f) \mu^i \zeta^2 \rangle \psi(\zeta) d\zeta = \alpha_{ei} \int_0^\infty \int_{-1}^1 \frac{1}{\zeta^3} \left[\frac{\partial}{\partial \mu} \left((1 - \mu^2) \frac{\partial f}{\partial \mu} \right) + \frac{\partial}{\partial \varphi} \left(\frac{1}{1 - \mu^2} \frac{\partial f}{\partial \varphi} \right) \right] \mu^i \zeta^2 \psi(\zeta) d\mu d\zeta .$$

We consider here the isotropic case. So $\frac{\partial}{\partial \varphi} f = 0$ and the weak formulation can be rewritten by using Green formula

$$\int_0^\infty \langle Q_{ei}(f) \mu^i \zeta^2 \rangle \psi(\zeta) d\zeta = -\alpha_{ei} \int_0^\infty \int_{-1}^1 \frac{1}{\zeta^3} (1 - \mu^2) \frac{\partial f}{\partial \mu} i \mu^{i-1} \psi(\zeta) d\mu d\zeta .$$

By using again the green formula we get

$$\int_0^\infty \langle Q_{ei}(f) \mu^i \zeta^2 \rangle \psi(\zeta) d\zeta = \alpha_{ei} \int_0^\infty \frac{i}{\zeta^3} \left[(i-1) \zeta^2 \int_{-1}^1 f \mu^{i-2} d\mu - (i+1) \zeta^2 \int_{-1}^1 f \mu^i d\mu \right] \psi(\zeta) d\zeta .$$

So

$$Q_{ei}^i(f) = \alpha_{ei} \frac{i}{\zeta^3} \left((i-1) f^{i-2} - (i+1) f^i \right) .$$

Remark 30. We can find these results back by taking moments of the definition (2.10) of the electron-ion collision operator. We denote by ψ some test function and we will establish the weak formulation associated with the electron-ion collision operator C_{ei}

$$\int_0^\infty \langle C_{ei}(f) \mu^i \zeta^2 \rangle \psi(\zeta) d\zeta = 2\pi \alpha_{ei} \int_0^\infty \int_{-1}^1 \nabla_v \cdot [\Phi(v) \nabla_v f(v)] \psi(\zeta) \mu^i \zeta^2 d\mu d\zeta .$$

In the Cartesian coordinate system, we get

$$\int_0^\infty \langle C_{ei}(f) \mu^i \zeta^2 \rangle \psi(\zeta) d\zeta = \alpha_{ei} \int_{\mathbb{R}_v^3} \nabla_v \cdot [\Phi(v) \nabla_v f(v)] \psi(|v|) \mu^i dv .$$

So

$$\int_0^\infty \langle C_{ei}(f) \mu^i \zeta^2 \rangle \psi(\zeta) d\zeta = -\alpha_{ei} \int_{\mathbb{R}_v^3} \Phi(v) \nabla_v f(v) \nabla_v \cdot [\mu^i \psi(|v|)] dv . \quad (6.91)$$

$\nabla_v \cdot [\mu^i \psi(|v|)]$ can be written as

$$\nabla_v \cdot [\mu^i \psi(|v|)] = \nabla_v \mu^i \psi(|v|) + \mu^i \nabla_v \psi(|v|) . \quad (6.92)$$

Since

$$\begin{aligned} \nabla_v \Omega &= \nabla_v \left(\frac{1}{|v|} v \right) = \nabla_v \left(\frac{1}{|v|} \right) \otimes v + \frac{1}{|v|} v \nabla_v v = \frac{1}{|v|^2} \nabla_v |v| \otimes v + \frac{1}{|v|} Id \\ &= \frac{1}{|v|} (Id - \Omega \otimes \Omega) = \frac{1}{|v|^3} (|v|^2 Id - v \otimes v) = \Phi(v) \end{aligned} \quad (6.93)$$

By using (6.93), we get

$$\nabla_v \mu = \nabla_v \Omega_x = \Phi(v) \vec{e}_x,$$

and so

$$\nabla_v \mu^i = i \mu^{i-1} \Phi(v) \vec{e}_x.$$

Therefore (6.92) can be rewritten as

$$\nabla_v \cdot [\mu^i \psi(|v|)] = i \mu^{i-1} \Phi(v) \vec{e}_x \psi(|v|) + \mu^i \nabla_v \psi(|v|),$$

and equation (6.91) can be reformulated into

$$\begin{aligned} \int_0^\infty \langle C_{ei}(f) \mu^i \zeta^2 \rangle \psi(\zeta) d\zeta &= -\alpha_{ei} \int_{\mathbb{R}_v^3} \left[i \mu^{i-1} (\nabla_v f(v) \cdot \Phi(v) \Phi(v) \vec{e}_x) \psi(|v|) \right. \\ &\quad \left. + \frac{1}{v} \mu^i \psi'(|v|) \nabla_v f(v) \cdot (\Phi(v) \Omega) \right] dv. \end{aligned} \quad (6.94)$$

Since $\Phi(v)v = 0$, we get

$$-\frac{\alpha_{ei}}{v} \int_{\mathbb{R}_v^3} \mu^i \psi'(|v|) \nabla_v f(v) \cdot (\Phi(v) \Omega) dv = 0.$$

Hence (6.94) gives

$$\int_0^\infty \langle C_{ei}(f) \mu^i \zeta^2 \rangle \psi(\zeta) d\zeta = -\alpha_{ei} \int_{\mathbb{R}_v^3} \left[i \mu^{i-1} (\nabla_v f(v) \cdot \Phi(v) \Phi(v) \vec{e}_x) \psi(|v|) \right] dv,$$

which can be simplified as

$$\int_0^\infty \langle C_{ei}(f) \mu^i \zeta^2 \rangle \psi(\zeta) d\zeta = -\alpha_{ei} \int_{\mathbb{R}_v^3} \nabla_v \cdot \left[\frac{i \mu^{i-1}}{|v|} \psi(|v|) \Phi(v) \right] f dv.$$

The development of the divergence term gives

$$\begin{aligned} \int_0^\infty \langle C_{ei}(f) \mu^i \zeta^2 \rangle \psi(\zeta) d\zeta &= \alpha_{ei} \int_{\mathbb{R}_v^3} \mu^{i-1} S(v) \vec{e}_x \cdot \nabla_v \left(\frac{1}{|v|^4} \psi(|v|) \right) f dv \\ &\quad + \alpha_{ei} \int_{\mathbb{R}_v^3} \nabla_v (\mu^{i-1} S(v)) \cdot \vec{e}_x \frac{i}{|v|^4} \psi(|v|) f dv, \end{aligned} \quad (6.95)$$

with $S(v) = v^2 Id - v \otimes v$. (6.95) can be simplified into

$$\begin{aligned} \int_0^\infty \langle C_{ei}(f) \mu^i \zeta^2 \rangle \psi(\zeta) d\zeta &= \alpha_{ei} \int_{\mathbb{R}_v^3} \vec{e}_x \cdot \mu^{i-1} S(v) \frac{\partial}{\partial |v|} \left(\frac{1}{|v|^4} \psi(|v|) \right) \Omega f dv \\ &\quad - \alpha_{ei} \int_{\mathbb{R}_v^3} 2v \cdot \vec{e}_x \frac{i \mu^{i-1}}{|v|^4} \psi(|v|) f dv \\ &\quad + \alpha_{ei} \int_{\mathbb{R}_v^3} \mu^{i-2} (i-1) S(v) \Phi(v) \vec{e}_x \cdot \vec{e}_x \frac{i}{|v|^4} \psi(|v|) f dv. \end{aligned} \quad (6.96)$$

Since

$$\frac{\Phi(v)S(v)}{|v|^4} = \frac{\Phi(v)\Phi(v)}{|v|} = \frac{1}{|v|^2}\Phi(v) ,$$

we get

$$\frac{\Phi(v)S(v)}{|v|^4}\vec{e}_x.\vec{e}_x = \frac{1}{|v|^2}\Phi(v)\vec{e}_x.\vec{e}_x = \frac{1}{|v|^5}(|v|^2 - v_x^2) = \frac{1}{|v|^5}(|v|^2 - \mu^2 v^2) .$$

Therefore (6.96) can be rewritten as

$$\int_0^\infty \langle C_{ei}(f)\mu^i \zeta^2 \rangle \psi(\zeta) d\zeta = -\alpha_{ei} \int_{\mathbb{R}_v^3} \frac{2i\mu^i}{|v|^3} \psi(|v|) f dv + \alpha_{ei} \int_{\mathbb{R}_v^3} \frac{\mu^{i-2}}{|v|^3} i(i-1)(1-\mu^2) \psi(|v|) f dv ,$$

which can be expressed in spherical coordinate by

$$\int_0^\infty \langle C_{ei}(f)\mu^i \zeta^2 \rangle \psi(\zeta) d\zeta = \alpha_{ei} \int_0^\infty \frac{1}{|v|^3} \left[-i(i+1) \langle \mu^i f \zeta^2 \rangle + i(i-1) \langle \mu^{i-2} f \zeta^2 \rangle \right] \psi(\zeta) d\zeta .$$

and the expression (6.90) of Q_{ei}^i follows.

CHAPTER 7

CONCLUSION AND PERSPECTIVES

In this thesis, we have presented different mathematical approaches to modeling some important processes in Inertial Confinement Fusion.

These processes have been first described by a kinetic scheme. In this way, the time evolution of the distribution function was given by the Maxwell-Fokker-Planck-Landau system. To solve this system, a numerical solver developed by R. Ducloux called *KETS* has been improved. Indeed, different high order schemes have been proposed. The third and fourth order schemes are based on MUSCL schemes. Their construction and their different conservation properties have been shown. Moreover some work on boundary conditions have been proposed for *KETS*.

Then, the efficiency of each scheme has been illustrated through four different crucial processes of ICF. First the study of the two streams instability has shown the good behavior of each scheme and thus validates each one. Some comparison with *PIC* code and theoretical solution for linear part have confirmed the accuracy of results. Moreover, comparisons between simulation results have shown that the third order scheme seems to be the best compromise between time executed by the simulation and accuracy of the numerical solution. Then the test based on laser-plasma interaction showed the relevance of the use of such solver in well known difficult situations. Moreover the relaxation of the hot spot has presented the good behavior of the code *KETS* in weakly collisional regime. The good behavior of the scheme has also been confirmed by some simulation presented by O.V. Batishchev. Besides in highly collisional regime, this scheme has been shown to be well adapted and very close to physical results by using some comparison with Braginskii theory. Nevertheless, in the four configurations of ICF, the computational time for each simulation was too much important in spite of the massive parallelization.

To reduce this computational time, another approach between the kinetic and the hydrodynamic limits has been proposed. This model called the M_1 model was written by considering the moment system with respect to the angular variable. The choice of the entropy minimization principle for the angular closure enables us to guarantee physical properties such as the positivity of the distribution function. The M_1 model under which the realizable set is invariant has been proved to be entropic. Then a semi-discrete scheme in energy based on an entropic average has been derived.

The numerical results obtained with this approach are in a very good agreement with physical properties. Indeed relaxation of a Maxwellian or the sum of Maxwellian have shown that the entropy is conserved. Moreover these test cases highlight the good relaxation to equilibrium states of collisional operator. Moreover a comparison with the kinetic code *KETS* has shown the accuracy of M_1 . The two stream instability shows that M_1 code and kinetic code have the same accuracy for small perturbation of beams. Moreover M_1 results obtained with the propagation of laser beam in vacuum are very close to those obtained with the kinetic code.

Finally the previous intermediate approach M_1 has been generalized to a model which uses N moments. The M_N continuous approach also conserves fundamental physical properties. In particular, the model has been proved to be entropic by the use of an entropic average. Next a semi-discretized scheme in energy has been developed and also proved to be entropic.

To extend this model, several applications of this work can be considered. Indeed it will be interesting to see how use the model with mobile ions. The idea is to add the ion movement equation to the Fokker-Planck-Landau equation. Of course, the ion consideration in kinetic code will involve a huge computational time. So it will be interesting to consider M_1 model with mobile ion for a more efficient analysis. Then the forthcoming work should be to extend this work for a plasma in the relativistic regime. There exists many high intensity laser installations that can achieve highly relativistic intensities. Moreover these installation generate extremely high currents of relativistic electrons in interaction with solid targets. This is why it will be interesting to repeat the work of this thesis with relativistic electrons. The difference with the usual Fokker-Planck-Landau equation consists in considering the momentum instead of velocity of electrons. In ICF, the possible perspective that emerges from this work is to apply the M_1 model for one of the new approach in fusion called shock ignition. The shock ignition concept of ICF relies on the ability to launch a strong converging shock into a pre-assembled fuel. This thesis already presents some useful tools for this application such as the Bremsstrahlung Inverse effect or the Batishchev test case. This work could naturally be extended toward explaining in more depth energetic electron transport for shock ignition.