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LIE-ADMISSIBLE STRUCTURES ON WITT TYPE ALGEBRAS AND AUTOMORPHIC ALGEBRAS

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Contents

Introduction

The Witt algebra and its universal central extension, the Virasoro algebra, have been intensively studied. They arised in many research fields in Mathematics and in Theoretical Physics and have numerous applications (e.g. see [11]).

From a pure algebraic point of view, the construction of the Witt algebra is the following. Let $W := \bigoplus_{n\in\mathbb{Z}} W_n$ be a $\mathbb{Z}$-graded vector space with one dimensional homogeneous spaces $W_n = \mathrm{vect}(L_n)$. We define on W the Lie bracket $[L_n, L_m] := (m-n)L_{n+m}$. The algebra $(W, [,])$ is called the Witt algebra.

Thus the Witt algebra is an infinite-dimensional Lie algebra and is $\mathbb{Z}$-graded.

The Witt algebra is not just an abstract algebraic construction but occurs in many mathematical fields. The Witt algebra W can be geometrically constructed as the Lie algebra of meromorphic vector fields defined on the Riemann sphere that are holomorphic except at two fixed points. Indeed, let the Riemann sphere be viewed as $\mathbb{C} \cup \{\infty\}$, i.e. the compactification of the complex plane and let the two fixed points be 0 and ∞. Then a basis of W is given by $\{L_n := z^{n+1}\frac{\partial}{\partial z} \mid n \in \mathbb{Z}\}$ and $[L_m, L_n] = (n-m)L_{m+n}$.

The Witt algebra admits an universal (one-dimensional) central extension. A Lie algebra $\hat{L}$ is a (one-dimensional) central extension of a Lie algebra L over a field $\mathbb{K}$ if there exists a Lie algebra exact sequence $0 \to \mathbb{K}c \to \hat{L} \to L$ where $\mathbb{K}c$ is the one-dimensional trivial Lie algebra and the image of $\mathbb{K}c$ is contained in the center of $\hat{L}$. It is well-known this is equivalent to $\hat{L} = L \oplus \mathbb{K}c$ with the Lie bracket $[x, y]_{\hat{L}} = [x, y]_L + \varphi(x, y) \cdot c$, where φ is a 2-cocycle of L. The universal central extension of the Witt algebra is called the Virasoro algebra and its usual normalization is given by the normalized 2-cocycle $\varphi(L_n, L_m) = \frac{1}{12}(n^3 - n)\delta^0_{n+m}$.

In mathematics we find many generalizations of the Witt algebras. For example there are the generalized Witt algebras [15], which are graded Lie algebras over an abelian group and whose homogeneous spaces are not necessary finite-dimensional. We also find generalization to Lie super-algebras: the super-Witt and super-Virasoro algebras [2].

In this thesis we are interested in two particular generalizations of the Witt algebra.

The first are the Witt type algebras. Starting from the algebraic construction of the Witt algebra, the following question occurs: let Γ be an abelian group, and let $V := \bigoplus_{\gamma\in\Gamma} V_\gamma$

be a Γ-graded $\mathbb{K}$-vector space with one-dimensional homogeneous spaces $V_\gamma := \mathbb{K}e_\alpha$; what are all Γ-graded Lie algebra structures on V?
This question was posed by Kirillov at the European School on Groups of Luminy in 1991. A partial solution to this question are the Witt type algebras. They have been introduced by R. Yu [38] in 1997. Given a map $f : \Gamma \to \mathbb{K}$, Rupert Yu considers the Γ-graded algebra V with the bracket $[e_\alpha, e_\beta] = (f(\beta) - f(\alpha))e_{\alpha+\beta}$. Under some conditions on f, V becomes a Lie algebra called Witt type algebra. For example, any additive function f gives a Lie algebra structure. If Γ is a free abelian group and f is an injective additive function, a Witt type algebra admits an universal central extension very close to the Virasoro case.
The second generalization which is interesting to us are the Krichever-Novikov algebras. This kind of generalization of the Witt and Virasoro algebra arose in the study of conformal field theory and was given by Krichever and Novikov in 1987 [18, 19] for Riemann surfaces of higher genus. Let X be a fixed compact Riemann surface of genus g. We choose two 'generic' points P_+ and P_- and consider the Lie algebra of meromorphic vector fields on X which are holomorphic on $X \setminus \{P_+, P_-\}$. For $g = 0$, i.e. if X is the Riemann sphere, this algebra is exactly the Witt algebra and its central extension is the Virasoro algebra as described before. For higher genus g, this algebra is not graded anymore but a weaker structure is still present: the almost-graded structure (see its definition below). Krichever and Novikov showed that this algebra admits a central extension respecting the almost-graded structure. Note that, in contrast to the Witt algebra, there are many non-equivalent central extensions but only one which is compatible with the almost-grading (see [33]). This algebra (with or without central extension) is now usually called the Krichever-Novikov algebra (short: KN-algebra). Martin Schlichenmaier [34–36] give extension of the KN-algebras to the multi-points case and give moreover explicit generators of the KN-algebras which gives an almost-graded structure for the KN-algebras.

This thesis is split in two quite different parts but they are related to the Witt algebra. In fact in the first part we study the Witt type algebras and in the second part, automorphic algebras are linked to the theory of KN-algebras. Let us described each part more precisely.

The first part is about Lie-admissible structures on Witt type algebras. For any algebra $(A, *)$ over a field of characteristic different from 2, we define the algebra $A^- := (A, [,])$ and the algebra $A^+ := (A, \circ)$ where $[x, y] = x * y - y * x$ and $x \circ y = \frac{1}{2}(x * y + y * x)$. If A^- is a Lie algebra, then A is called a Lie-admissible algebra. Lie-admissible algebras were introduced by A.A. Albert in 1948 [1]. Much of the structure theory of Lie-admissible algebras has been initially carried out under additional conditions such as the flexible identity $(x * y) * x = x * (y * x)$ or the power-associativity (i.e. every element generates an associative sub algebra).

Finding flexible or third power-associative Lie-admissible algebras with prescribed algebra A^- has been a main problem in algebra. Benkart and Osborn [6] and Myung and Okubo [22] gave in 1981 all flexible Lie-admissible algebras when A^- is a finite-dimensional simple Lie algebra. Benkart extended this result in 1990 [5] and gave all third power-associative Lie-admissible algebra when A^- is semi-simple and finite-dimensional. For infinite-dimensional algebras the problem has been solved in the following cases: simple generalized Witt algebras [15], Witt and Virasoro algebras [23] and Kac-Moody algebras [14]. Lie-admissible algebras are also related with many problems in physics (see [25–29]).

Another class of Lie-admissible algebras are the left-symmetric algebras. An algebra $(A, *)$ is left-symmetric if $(x, y, z) = (y, x, z)$ where $(x, y, z) = (x * y) * z - x * (y * z)$ is the associator. We can show that such an algebra is Lie-admissible. Left-symmetric algebras arise in many areas of mathematics and physics. They have already been introduced by A. Cayley in 1896, in the context of rooted tree algebras; see [7]. Then they were forgotten for a long time until Vinberg [37] in 1960 and Koszul [17] in 1961 introduced them in the context of convex homogeneous cones and affinely flat manifolds. They appear now in many mathematical theories like the theory of vector fields, theory of operads, or affine structures on Lie groups. The graded left-symmetric structure on the Witt and Virasoro algebras have been classified (see [3]).

In the first part of this thesis, we study the problem of finding Lie-admissible structures for the case of Witt type algebras.

In the first section we give all preliminary definitions and properties. We begin by introducing the definitions of the Lie-admissibility, flexibility and third power-associativity. We then define the Poisson structures and we explain their link with the flexible Lie-admissible algebras, we also briefly remind the theory of central extension. Finally we treat in details the Witt type algebras. In particular we show which among them are simple or graded-simple algebras.

In the second section we determine all third-power associative Lie-admissible structures and flexible Lie-admissible structures on Witt type algebras. We obtain a general result for any Witt type algebra. For simple Witt type algebras, the third-power-associative or flexible structures are of the same form as for the simple algebras studied before [5, 15, 23]. But our results are true even for non-simple Witt type algebras and then we see how the simplicity acts on the third-power associative and flexible structures. Since Witt type algebras are generalizations of the Witt algebra, our results of course coincide with those given by H. Myung [23] in the case of the natural Witt algebra. Finally, we search for Poisson structures on Witt type algebras. In fact for a flexible Lie-admissible algebra $(A, *)$, finding Poisson structures on A is equivalent to requiring the associativity of the commutator law $\circ$. Thus, since there exists non trivial flexible Lie-admissible structures on some Witt type algebras, looking for Poisson structures is rather a natural question. We give a condition for their existence.

Next we deal with central extensions of Witt type algebras. We compute their second Lie algebra cohomology with values in the trivial module. It is interesting to see that this group depends on the gradation by the group Γ. Rupert Yu already studied central extensions for a class of Witt type algebras which are very close to the Witt algebra. Using his result, we generalize Myung's paper on the Virasoro algebra [23]. More precisely we find third power-associative Lie-admissible structures and flexible Lie-admissible structures on the central extension of some Witt type algebras.

The computation of the 2-cocycles of Witt type algebras leads us to the problem of finding symplectic structures. In fact a symplectic structure on a Lie algebra is given by a non-degenerate 2-cocycle. It is well known that in the finite-dimensional case there is no symplectic simple Lie algebras. We prove that the situation is different in the infinite-dimensional case. Indeed we find symplectic structures on some of the simple Witt type algebras. For a finite-dimensional Lie algebra A, any symplectic form induce a left-symmetric product such that $A^- = (A, [,])$. A natural question is, whether this is also true for infinite-dimensional Lie algebras. The answer to this question seems to be very difficult. Nevertheless, for some specific Witt type algebras we can give explicitly all symplectic forms which induce a graded left-symmetric product. In particular, we study the case of the classical Witt algebra and, as the classification of its graded left-symmetric structures is known [3], we determine which ones are induced by a symplectic form.

All these results are published in the Journal of Geometry and Physics [4].

In the second part of the thesis we study the automorphic algebras.
Starting from arbitrary compact Riemann surfaces we consider the action of finite subgroups of the automorphism group of the surface on certain geometrically defined Lie algebras.

These algebras are the algebras of all meromorphic functions, the algebras of meromorphic functions with poles only on a finite subset of points, and their induced current algebras, related to finite-dimensional Lie algebras. In the case of finitely many points, where poles are allowed, these algebras are algebras of Krichever-Novikov type and they allow an almost-grading.

More precisely, for a finite subgroup G of automorphisms acting on the Riemann surface, we relate the invariance subalgebras living on the surface to the algebras on the quotient surface under the group action. The almost-graded Krichever-Novikov algebras structure on the quotient gives in this way a subalgebra of a certain Krichever-Novikov algebra (with almost-grading) on the original Riemann surface.
Specially discussed is the situation where the finite subgroup of the automorphism group has also a faithful representation on the finite dimensional Lie algebra used to construct the current algebra.
According to the difference on the automorphism groups, the situation is divided in three cases: genus $g = 0$ (Riemann sphere), $g = 1$ (the complex torus), and g greater or equal 2. In this thesis we study in details the cases of the Riemann sphere and the torus.

The second chapter is organized as follows. In the first section we remind all the material needed about Riemann Surfaces. We give the basic definitions and properties on Riemann surfaces, holomorphic maps and function and meromorphic functions. In particular the situation of compact Riemann surfaces is described. Moreover, explicit descriptions of the Riemann sphere and of the tori are given.

In the second section we are interested in the meromorphic functions with prescribed poles on a compact Riemann surface X. For a finite set Γ containing at least two points of X, we consider the algebra $\mathcal{M}(X, \Gamma)$ of the global meromorphic functions which are holomorphic in $X \setminus \Gamma$. This algebra is a Krichever-Novikov algebra and is almost-graded. We give a description of the general situation of the Krichever-Novikov algebras and of their almost-graded structure. The almost-graded structure of the Krichever-Novikov algebras depends on a splitting of the set Γ into two non-empty subsets. We treat in more details the situation of the algebra $\mathcal{M}(X, \Gamma)$ and we give explicit examples for a two points set Γ in the case of the Riemann sphere and of the tori. Moreover we explain how we can find an almost-graded structure in the case when Γ is a single point.

The third section is dedicated to the finite groups acting on a Riemann surface. In particular we are interested in the finite groups G acting holomorphically and effectively on a Riemann surface X. Then we explain how the quotient space X/G can be equipped with a Riemann surface structure such that the natural projection $\pi : X \to X/G$ is holomorphic. There is a close link between stabilizer subgroups of the action and the ramification points of the projection π. In the case of compact Riemann surfaces, using the Hurwitz Formula we are able to describe explicitly the possible ramifications (and the stabilizer subgroups) of the map π. In the particular case of the Riemann sphere we give explicitly the groups which realize these ramifications and we give the complete classification of the finite automorphism groups. For the complex tori T, the finite groups acting on T contains two types of automorphisms; the translations, which are not fixing any point of T, and the rotations, which are fixing points. Using the possible ramifications of the map π we can conclude on the genus of the quotient Riemann surface T/G. A quick description of the higher genus situation is also given.

In the last section, we study the automorphic algebras. Automorphic algebras are algebras which are invariant under the action of a finite automorphism group G of a Riemann surface X. First we consider the associative algebra $\mathcal{M}(X)$ of the meromorphic functions of X and we show that the automorphic algebra $\mathcal{M}_G(X)$ of G-invariant meromorphic functions is isomorphic to the algebra $\mathcal{M}(X/G)$. Most interesting is to consider the Krichever-Novikov algebras $\mathcal{M}(X, \Gamma)$ where Γ is a finite set of points of X. We prove that the automorphic algebra $\mathcal{M}_G(X, \Gamma)$ is also a Krichever-Novikov algebra and then admits an almost-graded structure. Next we extend our study to the current Lie algebra which is typically constructed by taking the tensor product of a finite-dimensional Lie algebra $\mathcal{L}$ and the associative algebra $\mathbb{C}[z, z^{-1}]$. In fact as for the Krichever-Novikov algebras $\mathcal{M}(X, \Gamma)$ a natural extension is to consider the Lie algebras $\mathcal{L}(X, \Gamma) := \mathcal{M}(X, \Gamma) \otimes \mathcal{L}$. We consider the action of special finite subgroups of $Aut(\mathcal{M}(X) \otimes \mathcal{L})$ that we obtain by using a faithful representation of a finite automorphism group G of the Riemann surface X on the finite dimensional Lie algebra $\mathcal{L}$. In fact we show that considering these special groups which are groups acting simultaneously on the both algebras $\mathcal{M}(X)$ and $\mathcal{L}$, is an important situation and we study the automorphic algebra $\mathcal{L}_G(X, \Gamma)$.
Finally, since the Lie algebra $\mathcal{L}(X, \Gamma)$ admits a natural almost-graded structure, we try to describe the situation for the automorphic algebra $\mathcal{L}_G(X, \Gamma)$ in the special case of the Riemann sphere and of complex tori.

Chapter 1

Lie-admissible structures on Witt type algebras.

1.1 Preliminaries

1.1.1 Flexibility - 3rd power-associativity

In the following we give the definitions of 3^{rd} power-associative and flexible algebras. Also we define the notion of Poisson structures.

Let $\mathbb{K}$ be a field of characteristic different from 2. For any algebra $(A, *)$, we define two products, $[x, y] := x * y - y * x$ and $x \circ y := \frac{1}{2}(x * y + y * x)$. We denote by A^- the algebra $(A, [\ ,\ ])$ and by A^+ the algebra $(A, \circ)$.

Definition 1.1. *An algebra $(A, *)$ is said Lie-admissible if the algebra A^- is a Lie algebra.*

Remark: Since the bracket $[\ ,\ ]$ is skew-symmetric, an algebra $(A, *)$ is Lie-admissible if and only if the bracket $[\ ,\ ]$ verifies the Jacobi identity.
Any associative algebra is Lie-admissible. Therefore we are mainly interested in non-associative Lie-admissible algebra.

Definition 1.2. *For a given Lie algebra $(L, [\ ,\ ])$, a Lie-admissible product $*$ on L is said compatible with $(L, [\ ,\ ])$ or just compatible, if $[x, y] = x * y - y * x,\ \forall x, y \in L$.*

Definition 1.3. *Let $(A, *)$ be an algebra. For $x, y, z \in A$ we define the associator $(x, y, z) := (x * y) * z - x * (y * z)$. Then:*

1. *The product $*$ or equivalently the algebra $(A, *)$ is said 3^{rd} power-associative if*
$$(x, x, x) = 0, \ \forall x \in A.$$
2. *The product $*$ or equivalently the algebra $(A, *)$ is said flexible if*
$$(x, y, x) = 0, \ \forall x, y \in A.$$

We give equivalent formulations of these definitions, in terms of [,] and $\circ$:

Proposition 1.4. 1. *The following properties are equivalent:*

i) $(A, *)$ *is 3^{rd} power-associative,*

ii) $[x, x \circ x] = 0, \ \forall x \in A,$

iii) $2[x, x \circ y] + [y, x \circ x] = 0, \ \forall x, y \in A,$

iv) $[x, y \circ z] + [y, z \circ x] + [z, y \circ x] = 0, \ \forall x, y, z \in A.$

2. *The following properties are equivalent:*

i) $(A, *)$ *is flexible,*

ii) $[x, y] \circ x = [x, y \circ x], \ \forall x, y \in A,$

iii) $[x, y \circ z] = [x, y] \circ z + y \circ [x, z] \quad \forall x, y, z \in A,$
Hence A is flexible if and only if $ad(x) := [x, .] \in Der(A, \circ), \ \forall x \in A.$

Proof. We suppose that $*$ is 3^{rd} power-associative. Remark that $x * x = x \circ x$. Then:

$$\begin{aligned}(x, x, x) &= (x * x) * x - x * (x * x) \\ &= (x \circ x) * x - x * (x \circ x) \\ &= [x, x \circ x].\end{aligned}$$

This proves the equivalence between i) and ii)

Now we use the identity $[x, x \circ x] = 0,\ \forall x \in A$ with $x + y$:

$$\begin{aligned}
& [x+y,(x+y)\circ(x+y)] = 0 \\
\Longleftrightarrow\ & [x, x\circ x] + [x, 2x\circ y] + [x, y\circ y] + [y, x\circ x] + [y, 2x\circ y] + [y, y\circ y] = 0 \\
\Longleftrightarrow\ & 2[x, x\circ y] + 2[y, x\circ y] + [x, y\circ y] + [y, x\circ x] = 0. \qquad (1.1)
\end{aligned}$$

We use now $[x, x \circ x] = 0,\ \forall x \in A$ with $x - y$:

$$\begin{aligned}
& [x-y,(x-y)\circ(x-y)] = 0 \\
\Longleftrightarrow\ & [x, x\circ x] - [x, 2x\circ y] + [x, y\circ y] - [y, x\circ x] + [y, 2x\circ y] - [y, y\circ y] = 0 \\
\Longleftrightarrow\ & -2[x, x\circ y] + 2[y, x\circ y] + [x, y\circ y] - [y, x\circ x] = 0. \qquad (1.2)
\end{aligned}$$

Finally (1.1)-(1.2) gives

$$2[x, x\circ y] + [y, x\circ x] = 0. \qquad (1.3)$$

This proves $ii) \longrightarrow iii)$.

To get the last identity, we use (1.3) with $x + y$ and z. That gives

$$\begin{aligned}
& 2[x+y,(x+y)\circ z] + [z,(x+y)\circ(x+y)] = 0 \\
\Longleftrightarrow\ & 2[x, x\circ z] + 2[x, y\circ z] + 2[y, x\circ z] \\
& \quad + 2[y, y\circ z] + [z, x\circ x] + 2[z, x\circ y] + [z, y\circ y] = 0.
\end{aligned}$$

From (1.3) we have $2[x, x\circ z] + [z, x\circ x] = 0$ and $2[y, y\circ z] + [z, y\circ y] = 0$. Hence we get:

$$\begin{aligned}
& 2[x, y\circ z] + 2[y, x\circ z] + 2[z, x\circ y] = 0 \\
\Longleftrightarrow\ & [x, y\circ z] + [y, z\circ x] + [z, x\circ y] = 0. \qquad (1.4)
\end{aligned}$$

This proves $iii) \longrightarrow iv)$.

To conclude notice that iv) trivially implies ii)

We suppose now that $*$ is flexible. That means that $(x, y, x) = 0$, $\forall x, y \in A$. Hence we have:

$$\begin{aligned}
&(x * y) * x - x * (y * x) = 0 \\
\Longleftrightarrow\ & \frac{1}{2}[x * y, x] + (x * y) \circ x - \frac{1}{2}[x, y * x] - x \circ (y * x) = 0 \\
\Longleftrightarrow\ & \frac{1}{2}\left[\frac{1}{2}[x, y] + x \circ y, x\right] + \left(\frac{1}{2}[x, y] + x \circ y\right) \circ x \\
& - \frac{1}{2}\left[x, \frac{1}{2}[y, x] + y \circ x\right] - x \circ \left(\frac{1}{2}[y, x] + y \circ x\right) = 0 \\
\Longleftrightarrow\ & \frac{1}{4}\left[[x, y], x\right] + \frac{1}{2}[x \circ y, x] + \frac{1}{2}[x, y] \circ x + (x \circ y) \circ x \\
& - \frac{1}{4}\left[x, [y, x]\right] - \frac{1}{2}[x, y \circ x] - \frac{1}{2}x \circ [y, x] - x \circ (y \circ x) = 0 \\
\Longleftrightarrow\ & [x \circ y, x] + [x, y] \circ x = 0 \\
\Longleftrightarrow\ & [x, y] \circ x = [x, y \circ x].
\end{aligned} \tag{1.5}$$

We have to prove the last equivalence. We remark that:

$$[x, y \circ z] = [x, y] \circ z + y \circ [x, z], \quad \forall x, y, z \in A \tag{1.6}$$

$$\begin{aligned}
\Longleftrightarrow\ & \left[x, \frac{1}{2}(y * z + z * y)\right] = \frac{1}{2}([x, y] * z + z * [x, y]) + \frac{1}{2}(y * [x, z] + [x, z] * y) \\
\Longleftrightarrow\ & [x, y * z] + [x, z * y] = [x, y] * z + z * [x, y] + y * [x, z] + [x, z] * y \\
\Longleftrightarrow\ & x * (y * z) - (y * z) * x + x * (z * y) - (z * y) * x \\
& \quad = (x * y) * z - (y * x) * z + z * (x * y) - z * (y * x) \\
& \quad\quad + y * (x * z) - y * (z * x) + (x * z) * y - (z * x) * y \\
\Longleftrightarrow\ & (x, y, z) + (z, y, x) + (y, z, x) + (x, z, y) = (y, x, z) + (z, x, y).
\end{aligned} \tag{1.7}$$

But $*$ is flexible if and only if

$$(x, z, y) + (y, z, x) = 0. \tag{1.8}$$

In fact by polarizing $(x, y, x) = 0$ we get:

$$\begin{aligned}
&(x+y, z, x+y) = 0 \\
\Longleftrightarrow &((x+y)*z)*(x+y) - (x+y)*(z*(x+y)) = 0 \\
\Longleftrightarrow &(x*z)*x + (x*z)*y + (y*z)*x + (y*z)*y \\
&\qquad - x*(z*x) - x*(z*y) - y*(z*x) - y*(z*y) = 0 \\
\Longleftrightarrow &(x*z)*y + (y*z)*x - x*(z*y) - y*(z*x) = 0 \\
\Longleftrightarrow &(x, z, y) + (y, z, x) = 0.
\end{aligned}$$

Finally, the flexibility of $*$ gives (1.8). But if (1.8) is true, then (1.7) is true and then (1.6) too. Conversely, the identity (1.7) with $z = x$ gives clearly the flexibility of $*$. □

1.1.2 Poisson structures

Definition 1.5.

1. *Let A be a $\mathbb{K}$-vector space with two products $\{\ ,\ \}$ et $\cdot$. $(A, \{\ ,\ \}, \cdot)$ is a Poisson algebra if $(A, \{\ ,\ \})$is a Lie algebra and if $(A, \cdot)$ is an associative and commutative algebra such that:*

$$\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\}, \quad \forall x, y, z \in A. \tag{1.9}$$

2. *Let $(L, [\ ,\ ])$ be a Lie algebra. A Poisson structure on L is a product $\cdot$ such that $(L, [\ ,\ ], \cdot)$ is a Poisson algebra.*

3. *Let $(A, *)$ be an algebra. A is Poisson-admissible if $(A, [\ ,\ ], \circ)$ is a Poisson algebra. We Remind that $[\ ,\ ]$ and $\circ$ are defined by $[x, y] := x*y - y*x$ and $x \circ y := \frac{1}{2}(x*y + y*x)$.*

Proposition 1.6. *An algebra $(A, *)$ is Poisson-admissible if and only if A is flexible, Lie-admissible and for all $x, y, z \in A$ we have*

$$2(x, y, z) = (y*z)*x - (y*x)*z + x*(z*y) - z*(x*y).$$

Proof. We suppose that A is flexible, and Lie-admissible. By definition the product $\circ$ is a commutative product. Moreover by proposition 1.4 the flexibility of $*$ is equivalent to

$$[x, y \circ z] = [x, y] \circ z + y \circ [x, z] \quad \forall x, y, z \in A.$$

That is exactly the identity 1.9 applied to [,] and $\circ$.
Hence we just have to show that $\circ$ is an associative law if and only if

$$2(x, y, z) = (y * z) * x - (y * x) * z + x * (z * y) - z * (x * y).$$

By direct computation we get:

$$\begin{aligned}
&(x \circ y) \circ z = x \circ (y \circ z) \\
\Longleftrightarrow &(x \circ y) * z + z * (x \circ y) = x * (y \circ z) + (y \circ z) * x \\
\Longleftrightarrow &(x * y + y * x) * z + z * (x * y + y * x) \\
&\qquad = (y * z + z * y) * x + x * (y * z + z * y) \\
\Longleftrightarrow &(x * y) * z - x * (y * z) - (z * y) * x + z * (y * x) \\
&\qquad = (y * z) * x - (y * x) * z + x * (z * y) - z * (x * y) \\
\Longleftrightarrow &(x, y, z) - (z, y, x) \\
&\qquad = (y * z) * x - (y * x) * z + x * (z * y) - z * (x * y).
\end{aligned}$$

Then the law $\circ$ is associative if and only if

$$(x, y, z) - (z, y, x) = (y * z) * x - (y * x) * z + x * (z * y) - z * (x * y). \qquad (1.10)$$

But we computed before (1.8) that the flexibility of $*$ is equivalent to

$$(x, y, z) + (z, y, x) = 0.$$

Hence the identity (1.10) becomes

$$2(x, y, z) = (y * z) * x - (y * x) * z + x * (z * y) - z * (x * y).$$

That ends the proof. □

Proposition 1.7. *An algebra $(A, *)$ is Poisson-admissible if and only if A is flexible, Lie-admissible and for all $x, y, z \in \mathcal{A}$*

$$(x, y, z) = \frac{1}{4}[y, [z, x]].$$

Proof. As for the previous proposition we just have to show that the associativity of $\circ$ is equivalent to

$$(x, y, z) = \frac{1}{4}[y, [z, x]].$$

We proved in proposition 1.4 that $(A, *)$ is flexible if and only if for all $x \in A$, $ad(x)$ is a derivation of $(A, \circ)$. Moreover $(A, *)$ is Lie-admissible if and only if for all $x \in A$, $ad(x)$ is a derivation of $(A, [\ ,\ ])$.

Hence $(A, *)$ est flexible and Lie-admissible if and only if for all $x \in A$, $ad(x)$ is a derivation of $(A, *)$.

By writing $x \circ y = x * y - \frac{1}{2}[x, y]$, we get that:

$$\begin{aligned}
&(x \circ y) \circ z = x \circ (y \circ z)\\
\Longleftrightarrow &(x * y) * z - \frac{1}{2}[x * y, z] - \frac{1}{2}[x, y] * z + \frac{1}{4}[[x, y], z]\\
&\qquad\qquad = x * (y * z) - \frac{1}{2}[x, y * z] - \frac{1}{2}x * [y, z] + \frac{1}{4}[x, [y, z]]\\
\Longleftrightarrow &(x, y, z) - \frac{1}{2}[x * y, z] - \frac{1}{2}[x, y] * z + \frac{1}{2}[x, y * z] + \frac{1}{2}x * [y, z]\\
&\qquad\qquad = \frac{1}{4}[x, [y, z]] - \frac{1}{4}[[x, y], z]\\
\Longleftrightarrow &(x, y, z) + \frac{1}{2}[z, x * y] + \frac{1}{2}[x, y * z] - \frac{1}{2}[x, y] * z + \frac{1}{2}x * [y, z]\\
&\qquad\qquad = \frac{1}{4}[x, [y, z]] + \frac{1}{4}[z, [x, y]]\\
\Longleftrightarrow &(x, y, z) + \frac{1}{2}[z, x] * y + \frac{1}{2}x * [z, y] + \frac{1}{2}[x, y] * z + \frac{1}{2}y * [x, z]\\
&\qquad\qquad - \frac{1}{2}[x, y] * z + \frac{1}{2}x * [y, z] = -\frac{1}{4}[y, [z, x]]\\
\Longleftrightarrow &(x, y, z) + \frac{1}{2}[z, x] * y - \frac{1}{2}y * [z, x] = -\frac{1}{4}[y, [z, x]]\\
\Longleftrightarrow &(x, y, z) + \frac{1}{2}[[z, x], y] = -\frac{1}{4}[y, [z, x]]\\
\Longleftrightarrow &(x, y, z) = \frac{1}{4}[y, [z, x]].
\end{aligned}$$

$\square$

1.1.3 Central extensions

Definition 1.8. *Let $(L, [\ ,\ ])$ be a Lie algebra. A skew-symmetric bilinear form $\omega : L \times L \to \mathbb{K}$ is a 2-cocycle (scalar 2-cocycle) if:*

$$\omega([x, y], z) + \omega([y, z], x) + \omega([z, x], y) = 0, \ \forall x, y, z \in L.$$

For any linear form $h : L \to \mathbb{K}$, the bilinear form dh defined by $dh(x,y) := h([x,y]),\ \forall x,y \in L$, is a 2-cocycle.
A 2-cocycle ω is a 2-coboundary if there exists a linear form h of L such that $\omega = dh$.
The set of the 2-cocycles is denoted by $Z^2(L,\mathbb{K})$ and the set of the 2-coboundaries is noted by $B^2(L,\mathbb{K})$. The second (scalar) cohomology group of L is the quotient group $H^2(L,\mathbb{K}) := Z^2(L,\mathbb{K})/B^2(L,\mathbb{K})$.

Definition 1.9. *Let $(L,[\ ,\ ]_L)$ be a Lie algebra and ω a 2-cocycle of L. On the vector space $E := L \oplus \mathbb{K}c$ we define the bracket $[\ ,\ ]_E$ by:*

$$[x,y]_E := [x,y]_L + \omega(x,y)c,\ \forall x,y \in L,$$
$$[x,c]_E := 0,\ \forall x \in L.$$

The algebra $(E,[\ ,\ ]_E)$ is a Lie algebra called the central extension of $(L,[\ ,\ ]_L)$ by means of the 2-cocycle ω. The central extensions of L, E_1 by means of the 2-cocycle ω_1 and E_2 by means of the 2-cocycle ω_2 are equivalent if and only if $\omega_1 - \omega_2 \in B^2(L,\mathbb{K})$ (that means they are in the same class in $H^2(L,\mathbb{K})$).

Remark : Two equivalent central extensions are isomorphic Lie algebras.

Definition 1.10. *Let $(L,[\ ,\ ])$ be a Lie algebra. We say that (L,ω) is a symplectic Lie algebra if ω is a non-degenerate 2-cocycle of L. Note that in the finite dimensional case, L must be even-dimensional. The form ω is called a symplectic structure or a symplectic form on L.*

1.1.4 Witt type algebras

Now we summarize the definitions about Witt type algebras and we list important results on simplicity. Also we recall the classification of Witt type algebras. More details can be found in the paper of R. Yu [38].

1.1.4.1 Definitions and first properties

Let $\mathbb{K}$ be a commutative field.
Let Γ be an abelian group and $V := \bigoplus_{\alpha \in \Gamma} V_\alpha$ a Γ-graded $\mathbb{K}$-vector space such that $\dim V_\alpha = 1$ for all $\alpha \in \Gamma$. Let $\{e_\alpha\}_{\alpha \in \Gamma}$ be a basis of V such that $V_\alpha = \mathbb{K} e_\alpha$.

During the European School on Groups of Luminy in 1991, A.A. Kirillov posed the following problem: characterize all Lie algebra structures on V. Witt type algebras give a partial answer to this problem.

Definition 1.11. *Let $f : \Gamma \to \mathbb{K}$. We define on V the product $[\ ,\ ] : V \times V \to V$ given by*

$$[e_\alpha, e_\beta] = (f(\beta) - f(\alpha)) e_{\alpha+\beta}$$

The algebra $(V, [\ ,\ ])$ is denoted $V(f)$.
A Witt type algebra is an algebra $V(f)$ which is a Lie algebra. Since the bracket is skew-symmetric, $V(f)$ is a Lie algebra if and only if the Jacobi identity holds for $[\ ,\ ]$.

Remark: Replacing f with $f - f(0)$, that does not change the bracket and $V(f) = V(f - f(0))$. Hence, we can always consider functions f with $f(0) = 0$.

Definition 1.12. *Let $\mathcal{E}$ be the set of functions f such that:*

(E1) $f(0) = 0$,

(E2) $f(\alpha+\beta)(f(\alpha) - f(\beta)) = (f(\alpha) + f(\beta))(f(\alpha) - f(\beta)) \quad \forall \alpha, \beta \in \Gamma$.

In regard of the definition of Witt type algebras, they are completely characterized by the map f. We give more information about these maps.

Proposition 1.13. *A Lie algebra V is a Witt type algebra if and only if $V = V(f)$ with $f \in \mathcal{E}$.*

Proof. We suppose that V is a Witt type algebra. By definition, $V = V(f)$ for some map $f : \Gamma \to \mathbb{K}$ and V is a Lie algebra. As remarked above, we can replace f by $f - f(0)$ and

suppose that $f(0)=0$. That means that $(E1)$ holds.
As $V(f)$ is a Lie algebra, the Jacobi identity holds for its bracket and

$$\begin{aligned}
&\sum_{cyclic}[e_\alpha,[e_\beta,e_\gamma]]=0\\
\Longleftrightarrow&\sum_{cyclic}[e_\alpha,(f(\gamma)-f(\beta))e_{\beta+\gamma}]=0\\
\Longleftrightarrow&\sum_{cyclic}(f(\gamma)-f(\beta))[e_\alpha,e_{\beta+\gamma}]=0\\
\Longleftrightarrow&\sum_{cyclic}(f(\gamma)-f(\beta))(f(\beta+\gamma)-f(\alpha))e_{\alpha+\beta+\gamma}=0\\
\Longleftrightarrow&(f(\gamma)-f(\beta))(f(\beta+\gamma)-f(\alpha))+(f(\alpha)-f(\gamma))(f(\gamma+\alpha)-f(\beta))\\
&\qquad\qquad+(f(\beta)-f(\alpha))(f(\alpha+\beta)-f(\gamma))=0.
\end{aligned}$$

For $\gamma=0$ we get

$$\begin{aligned}
&-f(\beta)(f(\beta)-f(\alpha))+f(\alpha)(f(\alpha)-f(\beta))+(f(\beta)-f(\alpha))f(\alpha+\beta)=0\\
\Longleftrightarrow&f(\alpha+\beta)(f(\beta)-f(\alpha))=(f(\beta)+f(\alpha))(f(\beta)-f(\alpha)).
\end{aligned}$$

This gives the identity $(E2)$.

Conversely, we suppose that $V=V(f)$ with $f\in\mathcal{E}$. As $(E2)$ holds we have:

$$\begin{aligned}
&\sum_{cyclic}[e_\alpha,[e_\beta,e_\gamma]]\\
=&\sum_{cyclic}(f(\gamma)-f(\beta))(f(\beta+\gamma)-f(\alpha))e_{\alpha+\beta+\gamma}\\
=&\sum_{cyclic}[f(\beta+\gamma)(f(\gamma)-f(\beta))-f(\alpha)(f(\gamma)-f(\beta))]e_{\alpha+\beta+\gamma}.
\end{aligned}$$

But by using $(E2)$, the previous equality becomes then

$$\begin{aligned}
=&\sum_{cyclic}[(f(\gamma)+f(\beta))(f(\gamma)-f(\beta))-f(\alpha)(f(\gamma)-f(\beta))]e_{\alpha+\beta+\gamma}\\
=&\sum_{cyclic}[f(\gamma)^2-f(\beta)^2-f(\alpha)(f(\gamma)-f(\beta))]e_{\alpha+\beta+\gamma}\\
=&[f(\gamma)^2-f(\beta)^2-f(\alpha)(f(\gamma)-f(\beta))\\
&\quad+f(\alpha)^2-f(\gamma)^2-f(\beta)(f(\alpha)-f(\gamma))\\
&\qquad+f(\beta)^2-f(\alpha)^2-f(\gamma)(f(\beta)-f(\alpha))]e_{\alpha+\beta+\gamma}=0
\end{aligned}$$

That is, the Jacobi identity, holds and V is a Lie algebra. □

The following lemma gives some properties of maps in $\mathcal{E}$:

Lemma 1.14. *Let $f \in \mathcal{E}$ and let $\alpha, \beta \in \Gamma$.*

1. *If $f(\alpha) \neq f(\beta)$, hence $f(\alpha+\beta) = f(\alpha) + f(\beta)$.*
2. *$f(\alpha) = \pm f(-\alpha)$.*
3. *Set $\Gamma_0 := f^{-1}(0)$.*
   a) *The set Γ_0 is a subgroup of Γ.*
   b) *If $\alpha \sim \beta$ modulo Γ_0, hence $f(\alpha) = f(\beta)$.*

Proof. 1. If $f(\alpha) \neq f(\beta)$, then $f(\alpha) - f(\beta) \neq 0$. By $(E2)$ we have

$$f(\alpha+\beta) = f(\alpha) + f(\beta).$$

2. In $(E2)$, we set $\beta = -\alpha$. We get

$$(f(\alpha) + f(-\alpha))(f(\alpha) - f(-\alpha)) = 0.$$

So $f(\alpha) = \pm f(-\alpha)$.

3. a) Let $\alpha, \beta \in \Gamma_0$. Apply $(E2)$ to $\alpha+\beta$ and $-\beta$ and note that $f(-\beta) = \pm f(\beta) = 0$. Hence we get

$$f(\alpha+\beta)^2 = 0.$$

Hence $\alpha+\beta \in \Gamma_0$. Moreover $0 \in \Gamma_0$, so Γ_0 is a subgroup of Γ.
b) We suppose that $\alpha \sim \beta$ modulo Γ_0. We can suppose $f(\alpha) \neq 0$ (otherwise $\alpha \in \Gamma_0$, $\beta \in \Gamma_0$ and $f(\alpha) = f(\beta) = 0$). Hence $f(\alpha) \neq f(\beta - \alpha)$ since $\beta - \alpha \in \Gamma_0$ and $f(\alpha - \beta) = 0$. By 1) we conclude that

$$f(\beta) = f(\alpha+\beta-\alpha) = f(\alpha) + f(\beta-\alpha) = f(\alpha).$$

□

Lemma 1.15. *Let $f \in \mathcal{E}$ and set $\tilde{\Gamma} := \{\alpha \in \Gamma /\ f(\alpha) = f(-\alpha)\}$.*
Then $\tilde{\Gamma} = \Gamma_0$ or $\tilde{\Gamma} = \Gamma$.

Proof. If $\mathrm{car}(\mathbb{K}) = 2$, we clearly have $\tilde{\Gamma} = \Gamma$ as $\forall \alpha \in \Gamma,\ f(-\alpha) = \pm f(\alpha)$.
So we suppose that $\mathrm{car}(\mathbb{K}) \neq 2$.
Suppose first that $\Gamma_0 = \{0\}$. We show that if $\tilde{\Gamma} \neq \{0\}$. Then $\tilde{\Gamma}$ is necessary equal to Γ.
Let $\alpha \in \tilde{\Gamma}$ which is not in Γ_0 i.e. α such that $f(\alpha) = f(-\alpha) \neq 0$ and let γ be any element of Γ.
First situation: $f(\alpha) = f(\gamma)$. If $f(\gamma) = -f(-\gamma)$, then $f(\alpha) \neq f(-\gamma)$ (otherwise $f(\alpha) = -f(-\alpha)$) and then $f(\alpha - \gamma) = f(\alpha) + f(-\gamma) = 0$. Therefore $\alpha - \gamma \in \Gamma_0$. Hence $\alpha = \gamma$. But that is absurd because $f(\gamma) = -f(-\gamma)$ and $f(\alpha) = f(-\alpha)$, we should get $f(\alpha) = 0$.
Hence $f(\gamma) = f(-\gamma)$ and $\pm\gamma \in \tilde{\Gamma}$. As well, if $f(-\gamma) = f(\alpha)$ we show that $\pm\gamma \in \tilde{\Gamma}$.
Second situation: $f(\alpha) \neq f(\gamma)$ and $f(\alpha) \neq f(-\gamma)$.
Suppose that $\gamma \notin \tilde{\Gamma}$ i.e. that $f(\gamma) = -f(-\gamma)$. As $f(\alpha) \neq f(\gamma)$ and as $f(-\alpha) = f(\alpha) \neq f(-\gamma)$ we have:

$$f(\alpha + \gamma) = f(\alpha) + f(\gamma)$$

and

$$f(-\alpha - \gamma) = f(-\alpha) + f(-\gamma) = f(\alpha) - f(\gamma).$$

But $f(-\alpha - \gamma) = \pm f(\alpha + \gamma)$ so $2f(\alpha) = 0$ or $2f(\gamma) = 0$. This is absurd. So $\gamma \in \tilde{\Gamma}$.
Finally, if $\Gamma_0 = \{0\}$ and if $\tilde{\Gamma} \neq \Gamma_0$ then $\tilde{\Gamma} = \Gamma$. That is the expected result in the situation $\Gamma_0 = \{0\}$.
Suppose now that $\Gamma_0 \neq \{0\}$. From the lemma 1.14 the map f is well-defined on the quotient $G := \Gamma/\Gamma_0$. Consider then the induced map $\tilde{f} : G := \Gamma/\Gamma_0 \to \mathbb{K}$ which belong again to $\mathcal{E}$. For this new map we have $G_0 = \{0\}$ and we can apply the previous computation: $\tilde{G} = G_0$ or $\tilde{G} = G$. But $\tilde{G} = \tilde{\Gamma}/\Gamma_0$. Hence $\tilde{G} = G_0 = \{0\} \iff \tilde{\Gamma} = \Gamma_0$ and $\tilde{G} = G \iff \tilde{\Gamma} = \Gamma$ because $\Gamma_0 \subset \tilde{\Gamma}$. □

Proposition 1.16. *Suppose that Γ is 2-torsion free and 3-torsion free. Let $f \in \mathcal{E}$ be an injective function, then f is additive.*

Proof. From part 1) of the Lemma 1.14 and the assumption that f is injective, we just have to show that for $\alpha \neq 0$, we have $f(2\alpha) = 2f(\alpha)$.
By (E2) we have

$$f(2\alpha - \alpha)(f(2\alpha) - f(-\alpha)) = (f(2\alpha) + f(-\alpha))(f(2\alpha) - f(-\alpha)). \tag{1.11}$$

Moreover since Γ is 2-torsion free, $\alpha \neq -\alpha$. As f is injective, $f(-\alpha) \neq f(\alpha)$. Hence $f(-\alpha) = -f(\alpha)$.

Similarly, since Γ is 3-torsion free, $2\alpha \neq -\alpha$. As f is injective, $f(2\alpha) \neq f(-\alpha)$. Thus in the identity 1.11 we can divide by $f(2\alpha) - f(-\alpha)$ and we get $f(\alpha) = f(2\alpha) + f(-\alpha)$ i.e.

$$2f(\alpha) = f(2\alpha).$$

$\square$

1.1.4.2 Simplicity of Witt type algebras

The Witt type algebras generalize the Witt algebra. So it is natural to study when these algebras are simple.

Definition 1.17. *Let $\mathcal{A}$ be a Γ-graded algebra. A graded ideal of $\mathcal{A}$ is an ideal which is a graded sub-vector space of $\mathcal{A}$.*
The algebra $\mathcal{A}$ is said graded-simple if there is no proper graded ideal in $\mathcal{A}$.
A Lie algebra L is said perfect if $[L, L] = L$. In particular simple and graded-simple Lie algebras are perfect.

Theorem 1.18. *Let f be a non-zero function of $\mathcal{E}$. $V(f)$ is graded-simple if and only if $\tilde{\Gamma} = \Gamma_0$.*
In particular if car$(\mathbb{K}) = 2$, $V(f)$ is never graded-simple.

Proof. $(\Longrightarrow)$
We suppose that $\tilde{\Gamma} \neq \Gamma_0$. By Lemma 1.15, $\tilde{\Gamma} = \Gamma$. Hence $f(\alpha) = f(-\alpha)$ for all $\alpha \in \Gamma$.
In these situation, as $[e_\alpha, e_\beta] = (f(\beta) - f(\alpha))e_{\alpha+\beta}$ it is obvious that $e_0 \notin [V(f), V(f)]$.
So $V(f)$ is not a perfect Lie algebra and cannot be graded-simple.
$(\Longleftarrow)$
We suppose that f is non-zero and $\tilde{\Gamma} = \Gamma_0$. Hence car$(\mathbb{K}) \neq 2$ (otherwise $\tilde{\Gamma} = \Gamma$). Let I be a non-trivial graded ideal of $V(f)$. We define $\Gamma' := \{\gamma \in \Gamma / e_\gamma \in I\}$. Since I is not reduced to zero, Γ' is not empty. We want to show that $\alpha + \beta \in \Gamma'$ for all $\alpha \in \Gamma'$ and $\beta \in \Gamma$. In this case we have $\Gamma' = \Gamma$. In fact, let γ be in Γ' since $-\gamma \in \Gamma$ we have $0 = \gamma - \gamma \in \Gamma'$. Hence for all $\beta \in \Gamma$ we have $\beta = 0 + \beta \in \Gamma'$ and then $\Gamma' = \Gamma$. As a consequence, $I = V(f)$. Let $\alpha \in \Gamma'$ and $\beta \in \Gamma$. There is three situations :

1. If $f(\alpha) \neq f(\beta)$ so $0 \neq [e_\alpha, e_\beta] \in \mathbb{K}e_{\alpha+\beta}$. As I is an ideal, $e_{\alpha+\beta} \in I$ and hence $\alpha+\beta \in \Gamma'$.

2. If $f(\alpha) = f(\beta) \neq 0$ so $\beta \notin \Gamma_0 = \tilde{\Gamma}$. Thus $f(-\beta) = -f(\beta)$. From (E2) we get

$$\begin{aligned} & f(2\beta-\beta)(f(2\beta)+f(\beta)) = (f(2\beta)-f(\beta))(f(2\beta)+f(\beta)) \\ \Longleftrightarrow & (f(2\beta)+f(\beta))(2f(\beta)-f(2\beta)) = 0 \\ \Longleftrightarrow & f(2\beta) = -f(\beta) \text{ or } f(2\beta) = 2f(\beta). \end{aligned}$$

So we cannot have $f(\alpha) = f(\beta) = f(2\beta)$, because in this case $f(\beta) = 2f(\beta)$ or $f(\beta) = -f(\beta)$. If $f(\beta) = 2f(\beta)$ then $\beta \in \Gamma_0$, against our assumption. If $f(\beta) = -f(\beta)$ then $f(\beta) = 0$ since $\mathrm{car}(\mathbb{K}) \neq 2$. Hence $\beta \in \Gamma_0$ is this case too. Thus $f(\alpha) \neq f(2\beta)$.
Moreover we cannot have $f(-\beta) = f(\alpha+2\beta)$. In this case, as $f(\alpha) \neq f(2\beta)$, we should have

$$f(-\beta) = f(\alpha)+f(2\beta) \text{ i.e. } -f(\beta) = f(\alpha)+f(2\beta),$$

but $f(2\beta) = -f(\beta)$ or $f(2\beta) = 2f(\beta)$ and:
If $f(2\beta) = -f(\beta)$ so $f(\alpha) = 0$ which is supposed to be wrong.
If $f(2\beta) = 2f(\beta)$ so $f(\alpha) = -3f(\beta) \Longleftrightarrow f(\alpha) = 0$ because $\mathrm{car}(\mathbb{K}) \neq 2$.

In conclusion we have proven that $f(-\beta) \neq f(\alpha+2\beta)$ and $f(\alpha) \neq f(2\beta)$. By 1) we have $\alpha+2\beta \in \Gamma'$ and $\alpha+\beta = \alpha+2\beta-\beta \in \Gamma'$.

3. If $f(\alpha) = f(\beta) = 0$, since f is non-zero, there exists $\gamma \in \Gamma$ such that $f(\gamma) \neq 0$. As $\tilde{\Gamma} = \Gamma_0$, $f(-\gamma) = -f(\gamma)$. By 1) (since $0 = f(\alpha) \neq f(\gamma)$) we get $\alpha+\gamma \in \Gamma'$. As well since $f(\alpha+\gamma) = f(\alpha)+f(\gamma) = f(\gamma) \neq f(-\gamma) = f(\beta-\gamma)$, we get by 1) $(\alpha+\gamma)+(\beta-\gamma) \in \Gamma'$.
Thus $\alpha+\beta \in \Gamma'$, $\forall \alpha \in \Gamma'$ and $\forall \beta \in \Gamma$.

□

Theorem 1.19. *Let $f \in \mathcal{E}$. The algebra $V(f)$ is simple if and only if the following statements hold:*

a) f is non-zero and injective,

b) Γ is 2-torsion free.

Proof. See [38]. □

The following theorem gives in some cases a condition for two Witt type algebras to be isomorphic.

Theorem 1.20. *Let Γ be a free abelian group and let $f, g \in \mathcal{E}$ be two injective functions. The algebras $V(f)$ and $V(g)$ are isomorphic (as Lie algebras) if and only if there exists $\lambda \in \mathbb{K}^*$ and $\sigma \in Aut(\Gamma)$ such that $\lambda g = f \circ \sigma$.*

In the following we give a classification of Witt type algebras.

Definition 1.21. *Let $\mathcal{A}$ be the set of additive functions from Γ to $\mathbb{K}$.*
Let $\mathcal{P}$ be the set of functions from Γ to $\mathbb{K}$ with the following property: there exists a surjective group morphism $\tau : \Gamma \to \mathbb{Z}/3\mathbb{Z}$ and $\lambda \in \mathbb{K}^*$ such that

$$f(\alpha) = \begin{cases} 0 & if \quad \tau(\alpha) = 0, \\ \lambda & if \quad \tau(\alpha) = 1, \\ -\lambda & if \quad \tau(\alpha) = -1. \end{cases}$$

Let $\mathcal{C}$ be the set of functions from Γ to $\mathbb{K}$ with the following property: there exists a subgroup Γ_0 of Γ, $\Gamma_0 \neq \Gamma$ and $\lambda \in \mathbb{K}^*$ such that

$$f(\alpha) = \begin{cases} 0 \ if \, \alpha \in \Gamma_0, \\ \lambda \ otherwise. \end{cases}$$

Theorem 1.22. *(Yu [38])*
The set $\mathcal{E}$ is the union of the sets $\mathcal{A}$, $\mathcal{P}$ and $\mathcal{C}$. If $car(\mathbb{K}) \notin \{2, 3\}$, the union is a disjoint.

Proof.

1. Card$f(\Gamma) = 1$ if and only if $f = 0$:
   Since $f(0) = 0$, if Card$f(\Gamma) = 1$ so it is obvious that $f = 0$.

2. Card$f(\Gamma) = 2$ if and only if $f \in \mathcal{C}$:
   Γ_0 is a subgroup of Γ and $f(\Gamma_0) = \{0\}$. If Card$f(\Gamma) = 2$ so $f(\Gamma \backslash \Gamma_0) := \lambda \in \mathbb{K}^*$. It is easy to check that such a function is in $\mathcal{E}$.

3. Card$f(\Gamma) = 3$ if and only if car$(\mathbb{K}) \neq 2$ and $f \in \mathcal{P}$:
   If car$(\mathbb{K}) \neq 2$ and if $f \in \mathcal{P}$ so it is obvious that Card$f(\Gamma) = 3$.
   Assume now that $f(\Gamma) = \{0, \mu, \lambda\}$ with $\lambda \neq \mu$ non-zero. Since $\lambda \neq \mu$, by the lemma 1.14 we have $\lambda + \mu \in f(\Gamma)$ and then $\lambda + \mu = 0$. Hence $\mu = -\lambda$ and as $\lambda \neq \mu$, it

is not possible if $\mathrm{car}(\mathbb{K}) = 2$. So $\mathrm{car}(\mathbb{K}) \neq 2$. We have to show that there exists a surjective morphism $\tau : \Gamma \to \mathbb{Z}/3\mathbb{Z}$ satisfying the conditions of the definition of $\mathcal{P}$. Assume that $\tilde{\Gamma} = \Gamma$. Let $\alpha, \beta \in \Gamma$ such that $f(\alpha) = \lambda$ and $f(\beta) = -\lambda$. Hence, since $f(\alpha) \neq f(\beta)$ we have $f(\alpha+\beta) = f(\alpha) + f(\beta) = 0$. As well, $0 = f(\alpha+\beta) \neq f(-\beta) = -\lambda$. Thus:

$$\lambda = f(\alpha) = f(\alpha+\beta-\beta) = f(\alpha+\beta) + f(-\beta) = -\lambda.$$

That is not possible because $\lambda \neq 0$. hence $\tilde{\Gamma} = \Gamma_0$.
We define τ as follow:

$$\tau : \Gamma \to \mathbb{Z}/3\mathbb{Z}; \tau(\alpha) = \begin{cases} 0 & \text{si} \quad f(\alpha) = 0, \\ 1 & \text{si} \quad f(\alpha) = \lambda, \\ -1 & \text{si} \quad f(\alpha) = -\lambda, \end{cases}$$

It is quite easy to verify that τ is a morphism and that f belong to $\mathcal{P}$.

4. If $\mathrm{Card} f(\Gamma) \geq 4$ then f is additive and non-zero:
   Let β and γ be two elements of Γ. As $\mathrm{Card}(f(\Gamma)) \geq 4$ we can find $\alpha \in \Gamma$ such that $f(\alpha) \neq f(\gamma)$, $f(\alpha) \neq f(\beta+\gamma)$ and $f(\alpha) \neq f(\beta) - f(\gamma)$.
   Hence $f(\beta) \neq f(\alpha) + f(\gamma)$ and as $f(\alpha) \neq f(\gamma)$, we have $f(\alpha+\gamma) = f(\alpha) + f(\gamma)$. Thus $f(\beta) \neq f(\alpha+\gamma)$. Finally:

$$f(\beta+\gamma) + f(\alpha) = f(\beta + (\gamma+\alpha)) = f(\beta) + f(\gamma+\alpha) = f(\beta) + f(\gamma) + f(\alpha).$$

   This proves that $f(\beta+\gamma) = f(\beta) + f(\gamma)$.

□

Remark: If $\mathrm{car}\mathbb{K} \in \{2,3\}$, the union is not disjoint. In fact, if $\mathrm{car}(\mathbb{K}) = 2$ then $\mathcal{P} \subset \mathcal{C}$ and if $\mathrm{car}(\mathbb{K}) = 3$, $\mathcal{P} \subset \mathcal{A}$.

We now specialize Theorem 1.19 by distinguishing whether f belongs to $\mathcal{A}, \mathcal{P}$ or $\mathcal{C}$. This is not in the paper of Yu [38] but is quite obvious in regards of Theorem 1.19.

Proposition 1.23. *Let f be in $\mathcal{A}$, then $V(f)$ is simple if and only if f is injective. As the functions of $\mathcal{A}$ are additive, $V(f)$ is simple if and only if $\Gamma_0 = f^{-1}(0) = \{0\}$.*

Proof. If $\mathrm{car}(\mathbb{K}) = 2$, no Witt type algebra is graded-simple.
Hence we suppose that $\mathrm{car}\mathbb{K} \neq 2$. If Γ is not 2-torsion free there exists $\alpha \neq 0$ such that $2\alpha = 0$ and since $f(2\alpha) = 2f(\alpha) = 0$ we have $\alpha \in \Gamma_0$. So $\Gamma_0 \neq \{0\}$ and f is not injective. □

Proposition 1.24. *Witt type algebras $V(f)$ with $f \in \mathcal{C}$ are not simple.*

Proof. If $card(\Gamma) > 2$ then f can't be one-to-one since $f(\Gamma) = \{0, \lambda\}$. So $V(f)$ is not simple . If $card(\Gamma) = 2$ then $\Gamma = \mathbb{Z}/2\mathbb{Z}$, which is a 2-torsion group. Hence $V(f)$ is not simple. $\square$

Proposition 1.25. *Let $f \in \mathcal{P}$ then $V(f)$ is simple if and only if $\Gamma_0 = 0$.*

Proof. If $\mathrm{car}(\mathbb{K}) = 2$, there is no simple Witt type algebras. If $\mathrm{car}(\mathbb{K}) \neq 2$, f is one-to-one if and only if τ is an injective morphism. Moreover $\Gamma_0 = f^{-1}(0) = \tau^{-1}(0)$. Hence f is one-to-one if and only if $\Gamma_0 = 0$. On the other hand, as τ is a surjective morphism, f is injective if and only if τ is bijective. In this case $\Gamma \simeq \mathbb{Z}/3\mathbb{Z}$ which is 2-torsion free and so $V(f)$ is simple. $\square$

Corollary 1.26. *Let $f \in \mathcal{E}$. The algebra $V(f)$ is simple if and only if $f \in \mathcal{A} \cup \mathcal{P} \setminus \{0\}$ and $\Gamma_0 = \{0\}$.*

Proof. This follows directly from the previous propositions. $\square$

Proposition 1.27. *Suppose that $\mathrm{car}(\mathbb{K}) \neq 2$. Let f be in $\mathcal{E}$. Then $V(f)$ is graded-simple if and only if f is a non-zero function in $\mathcal{A} \cup \mathcal{P} \setminus \{0\}$.*
If $\mathrm{car}(\mathbb{K}) = 2$, there is no graded-simple Witt type algebras.

Proof. If $\mathrm{car}(\mathbb{K}) = 2$ then $f(-\alpha) = f(\alpha)$, $\forall \alpha \in \Gamma$. Hence $\tilde{\Gamma} = \Gamma$ and $V(f)$ is not simple-graded. Now suppose that $\mathrm{car}(\mathbb{K}) \neq 2$. For $f \in \mathcal{C}$, if $\alpha \notin \Gamma_0$ then $-\alpha \notin \Gamma_0$ and so $f(\alpha) = f(-\alpha)$. Hence $\tilde{\Gamma} = \Gamma$ and $V(f)$ is not simple-graded.
For $f \in \mathcal{A}$, $\tilde{\Gamma} = \Gamma_0$ since f is additive. In the same way for $f \in \mathcal{P}$, $\tilde{\Gamma} = \Gamma_0$ since τ is additive. So $V(f)$ is graded-simple for $f \in \mathcal{A} \cup \mathcal{P}$. $\square$

Any graded-simple Lie algebra L is perfect. That means that $\mathcal{D}(L) = L$ where $\mathcal{D}(L) := [L, L]$. A perfect algebra is not necessarily graded-simple but this is true for Witt type algebras. More precisely, the following assertions are equivalent:

1. $\mathcal{D}V(f) = V(f)$,

2. $\tilde{\Gamma} = \Gamma_0$,

3. $V(f)$ is graded-simple.

In fact if $\tilde{\Gamma} = \Gamma$ then $f(\alpha) = f(-\alpha), \forall \alpha \in \Gamma$. In this case it is clear that e_0 does not belong to $\mathcal{D}(V(f))$ since $[e_\alpha, e_{-\alpha}] = 0$. In characteristic different from 2, non-perfect Witt type algebras are $V(f)$ with $f \in \mathcal{C}$. In this case it is trivial to compute that

$$\mathcal{D}V(f) = V_{\Gamma \setminus \Gamma_0},$$

where $V_{\Gamma\setminus\Gamma_0} := \bigoplus_{\gamma \in \Gamma\setminus\Gamma_0} V_\gamma$.

1.2 Third power-associative, flexible and Poisson structures on Witt type algebras

In this section we determine all 3^{rd} power-associative structures and all flexible structures on the Witt type algebras. Moreover we investigate for Poisson structures on them.

We consider Witt type algebras $V(f)$ over a field of characteristic not 2 with $f \neq 0$. In fact if $f = 0$, any commutative product on $V(f)$ is Lie-admissible compatible and 3^{rd} power-associative.

If Λ is a subset of Γ we denote the vector space $\bigoplus_{\gamma\in\Lambda} V_\gamma$ by V_Λ. We search all third power-associative products $*$ compatible with the Lie algebra structure. As the bracket of $V(f)$ is known, finding $*$ is equivalent to finding the commutative product $\circ$ associated to $*$. Hence, according to the previous definitions and results we have to find a commutative product $\circ$ such that

$$[x, x \circ x] = 0, \ \forall x \in V(f). \tag{1.12}$$

Therefore the product $*$ is then given by $x * y = \frac{1}{2}[x, y] + x \circ y$.

Suppose now that there exists a commutative $\circ$ such that $[x, x \circ x] = 0, \ \forall x \in V(f)$. We introduce the following notation: for any $\alpha, \beta \in \Gamma$ we set $e_\alpha \circ e_\beta := \sum_{\gamma\in\Gamma} C_\gamma^{\alpha\beta} e_\gamma$ with $C_\gamma^{\alpha\beta} \neq 0$ for a finite number of $\gamma \in \Gamma$. For $\alpha = \beta$. We note $C_\gamma^{\alpha\alpha} := C_\gamma^\alpha$. Notice that $C_\gamma^{\alpha\beta} = C_\gamma^{\beta\alpha}$ since $\circ$ is commutative.

1.2.1 Case of Witt type algebras of the type $\mathcal{C}$

We consider in this section a Witt type algebra $V(f)$ with f in $\mathcal{C}$. This means that there exists a subgroup Γ_0 of Γ, $\Gamma_0 \neq \Gamma$ and $\lambda \in \mathbb{K}^*$ such that $f(\alpha) = 0$ if $\alpha \in \Gamma_0$ and $f(\alpha) = \lambda$ otherwise.

Let $\{e_\alpha\}$ be a basis of $V(f)$ such that $V_\alpha = \mathrm{vect}(e_\alpha)$, $\forall \alpha \in \Gamma$. For the elements of the basis, the bracket [,] of $V(f)$ is given by

$$[e_\alpha, e_\beta] = \begin{cases} 0 & \text{if } \alpha, \beta \in \Gamma_0 \text{ or } \alpha, \beta \in \Gamma \backslash \Gamma_0 \\ \lambda e_{\alpha+\beta} & \text{if } \beta \in \Gamma \backslash \Gamma_0 \text{ and if } \alpha \in \Gamma_0 \\ -\lambda e_{\alpha+\beta} & \text{if } \beta \in \Gamma_0 \text{ and if } \alpha \in \Gamma \backslash \Gamma_0 \end{cases}$$

We suppose that the product $*$ is third-power associative and Lie-admissible. In this case we have the following results:

Lemma 1.28. *Let $\alpha \in \Gamma_0$ and $\beta \in \Gamma \backslash \Gamma_0$. Then*

1. $e_\alpha \circ e_\alpha \in V_{\Gamma_0}$,
2. $e_\beta \circ e_\beta \in V_{\Gamma \backslash \Gamma_0}$.

Proof. We write the identity (1.12) with $x = e_\alpha$:

$$\begin{aligned} & [e_\alpha, e_\alpha \circ e_\alpha] = 0 \\ \Longleftrightarrow & \left[e_\alpha, \sum_{\gamma \in \Gamma} C_\gamma^\alpha e_\gamma \right] = 0 \\ \Longleftrightarrow & \sum_{\gamma \in \Gamma} C_\gamma^\alpha [e_\alpha, e_\gamma] = 0 \\ \Longleftrightarrow & \sum_{\gamma \in \Gamma} C_\gamma^\alpha (f(\gamma) - f(\alpha)) e_{\alpha+\gamma} = 0 \\ \Longleftrightarrow & \sum_{\gamma \in \Gamma} C_\gamma^\alpha f(\gamma) e_{\alpha+\gamma} = 0, \text{ because } \alpha \in \Gamma_0 \\ \Longleftrightarrow & \ C_\gamma^\alpha f(\gamma) = 0, \ \forall \gamma \in \Gamma. \end{aligned}$$

Hence $C_\gamma^\alpha = 0,\ \forall \gamma \in \Gamma \backslash \Gamma_0$, and

$$e_\alpha \circ e_\alpha := \sum_{\gamma \in \Gamma_0} C_\gamma^\alpha e_\gamma.$$

This proves the first assertion. A similar calculation gives the second one. $\square$

Lemma 1.29.

1. $e_\beta \circ e_\beta \in V_{\beta+\Gamma_0},\ \forall \beta \in \Gamma$.
2. $e_\alpha \circ e_\beta \in V_{\Gamma_0} \oplus V_{\beta+\Gamma_0}$ *for* $\alpha \in \Gamma_0$ *and* $\beta \in \Gamma \backslash \Gamma_0$. *Moreover :*

$$C_\gamma^{\alpha\beta} = \begin{cases} \frac{1}{2} C^\beta_{\gamma+\beta-\alpha} & \forall \gamma \in \Gamma_0, \\ \frac{1}{2} C^\alpha_{\gamma+\alpha-\beta} & \forall \gamma \in \beta + \Gamma_0, \\ 0 & \textit{otherwise.} \end{cases}$$

3. $e_\alpha \circ e_\beta \in V_{\Gamma_0}$ *for* $\alpha, \beta \in \Gamma_0$ *and* $e_\alpha \circ e_\beta \in V_{\Gamma \backslash \Gamma_0}$ *for* $\alpha, \beta \in \Gamma \backslash \Gamma_0$.

Proof. We use the polarized form of Equation (1.12):

$$2[x, x \circ y] + [y, x \circ x] = 0. \tag{1.13}$$

For $x = e_\beta$ and $y = e_\alpha$. Let $\alpha \in \Gamma_0$ and $\beta \in \Gamma \backslash \Gamma_0$. We have

$$\begin{aligned}
& 2[e_\beta, e_\beta \circ e_\alpha] + [e_\alpha, e_\beta \circ e_\beta] = 0 \\
\Longleftrightarrow\quad & 2\left[e_\beta, \sum_{\gamma \in \Gamma} C_\gamma^{\alpha\beta} e_\gamma\right] + \left[e_\alpha, \sum_{\gamma \in \Gamma \backslash \Gamma_0} C_\gamma^\beta e_\gamma\right] = 0 \\
\Longleftrightarrow\quad & 2\sum_{\gamma \in \Gamma} C_\gamma^{\alpha\beta} [e_\beta, e_\gamma] + \sum_{\gamma \in \Gamma \backslash \Gamma_0} C_\gamma^\beta [e_\alpha, e_\gamma] = 0 \\
\Longleftrightarrow\quad & 2\sum_{\gamma \in \Gamma} C_\gamma^{\alpha\beta} \left(f(\gamma) - f(\beta)\right) e_{\beta+\gamma} + \sum_{\gamma \in \Gamma \backslash \Gamma_0} C_\gamma^\beta (f(\gamma) - f(\alpha)) e_{\alpha+\gamma} = 0 \\
\Longleftrightarrow\quad & 2\sum_{\gamma \in \Gamma_0} C_\gamma^{\alpha\beta} \left(-\lambda\right) e_{\beta+\gamma} + \sum_{\gamma \in \Gamma \backslash \Gamma_0} C_\gamma^\beta \lambda e_{\alpha+\gamma} = 0.
\end{aligned}$$

We do the following changes of variables: $\gamma \to \gamma - \beta$ in the first sum and $\gamma \to \gamma - \alpha$ in the second one. Then we get

$$
\begin{aligned}
&\Longleftrightarrow \quad 2 \sum_{\gamma \in \beta + \Gamma_0} C^{\alpha\beta}_{\gamma-\beta}\,(-\lambda)\, e_\gamma + \sum_{\gamma \in \Gamma \setminus \Gamma_0} C^{\beta}_{\gamma-\alpha} \lambda e_\gamma = 0 \\
&\Longleftrightarrow \quad 2 \sum_{\gamma \in \beta + \Gamma_0} C^{\alpha\beta}_{\gamma-\beta}\,(-\lambda)\, e_\gamma + \sum_{\gamma \in \beta + \Gamma_0} C^{\beta}_{\gamma-\alpha} \lambda e_\gamma + \sum_{\gamma \in \Gamma \setminus \{\Gamma_0 \cup \beta + \Gamma_0\}} C^{\beta}_{\gamma-\alpha} \lambda e_\gamma = 0. \quad (1.14)
\end{aligned}
$$

The last equation gives:

$$C^{\beta}_{\gamma-\alpha} = 0,\ \forall \gamma \in \Gamma \setminus \{\Gamma_0 \cup \beta + \Gamma_0\}.$$

Since $\alpha \in \Gamma_0$, this is equivalent to

$$C^{\beta}_{\gamma} = 0,\ \forall \gamma \in \Gamma \setminus \{\Gamma_0 \cup \beta + \Gamma_0\}.$$

This means that

$$e_\beta \circ e_\beta \in V_{\beta + \Gamma_0}.$$

Combined with the first assertion of the previous lemma, this proves the first part of 2). In addition, equation (1.14) gives:

$$C^{\alpha\beta}_{\gamma-\beta} = \frac{1}{2} C^{\beta}_{\gamma-\alpha},\ \forall \gamma \in \beta + \Gamma_0,$$

which is equivalent to

$$C^{\alpha\beta}_{\gamma} = \frac{1}{2} C^{\beta}_{\gamma+\beta-\alpha},\ \forall \gamma \in \Gamma_0.$$

We use again equation (1.13) but with $x = e_\alpha$ and $y = e_\beta$ when $\alpha \in \Gamma_0$ and $\beta \in \Gamma \setminus \Gamma_0$. As before we get:

$$
\begin{aligned}
& \quad 2[e_\alpha, e_\alpha \circ e_\beta] + [e_\beta, e_\alpha \circ e_\alpha] = 0 \\
&\Longleftrightarrow \quad 2 \left[e_\alpha, \sum_{\gamma \in \Gamma} C^{\alpha\beta}_{\gamma} e_\gamma \right] + \left[e_\beta, \sum_{\gamma \in \Gamma_0} C^{\alpha}_{\gamma} e_\gamma \right] = 0 \\
&\Longleftrightarrow \quad 2 \sum_{\gamma \in \Gamma} C^{\alpha\beta}_{\gamma} [e_\alpha, e_\gamma] + \sum_{\gamma \in \Gamma_0} C^{\alpha}_{\gamma} [e_\beta, e_\gamma] = 0 \\
&\Longleftrightarrow \quad 2 \sum_{\gamma \in \Gamma} C^{\alpha\beta}_{\gamma} \left(f(\gamma) - f(\alpha) \right) e_{\alpha+\gamma} + \sum_{\gamma \in \Gamma_0} C^{\alpha}_{\gamma} (f(\gamma) - f(\beta)) e_{\beta+\gamma} = 0 \\
&\Longleftrightarrow \quad 2 \sum_{\gamma \in \Gamma \setminus \Gamma_0} C^{\alpha\beta}_{\gamma} \lambda e_{\alpha+\gamma} + \sum_{\gamma \in \Gamma_0} C^{\alpha}_{\gamma} (-\lambda) e_{\beta+\gamma} = 0.
\end{aligned}
$$

We do the following changes of variables: $\gamma \to \gamma - \alpha$ in the first sum et $\gamma \to \gamma - \beta$ in the second one,

$$\begin{aligned}
&\iff \quad 2\sum_{\gamma\in\Gamma\backslash\Gamma_0} C^{\alpha\beta}_{\gamma-\alpha}(-\lambda)e_\gamma + \sum_{\gamma\in\beta+\Gamma_0} C^{\alpha}_{\gamma-\beta}\lambda e_\gamma = 0\\
&\iff \quad 2\sum_{\gamma\in\beta+\Gamma_0} C^{\alpha\beta}_{\gamma-\alpha}(-\lambda)e_\gamma + 2\sum_{\gamma\in\Gamma\backslash\{\Gamma_0\cup\beta+\Gamma_0\}} C^{\alpha\beta}_{\gamma-\alpha}(-\lambda)e_\gamma + \sum_{\gamma\in\beta+\Gamma_0} C^{\alpha}_{\gamma-\beta}\lambda e_\gamma = 0.
\end{aligned} \tag{1.15}$$

Equation (1.15) gives: $C^{\alpha\beta}_{\gamma-\alpha} = 0,\ \forall\gamma \in \Gamma\backslash\{\Gamma_0 \cup \beta + \Gamma_0\}$ which is equivalent to

$$C^{\alpha\beta}_{\gamma} = 0,\ \forall\gamma \in \Gamma\backslash\{\Gamma_0 \cup \beta + \Gamma_0\} \text{ since } \alpha \in \Gamma_0.$$

Moreover equation (1.15) gives: $C^{\alpha\beta}_{\gamma-\alpha} = \frac{1}{2}C^{\alpha}_{\gamma-\beta},\ \forall\gamma \in \beta + \Gamma_0$ which is equivalent to

$$C^{\alpha\beta}_{\gamma} = \frac{1}{2}C^{\alpha}_{\gamma+\alpha-\beta},\ \forall\gamma \in \beta + \Gamma_0 \text{ since } \alpha \in \Gamma_0.$$

Thus the second assertion is proved.
To prove the last assertion of the lemma, just write Equation (1.13) for $x = e_\alpha$ and $y = e_\beta$ with $\alpha, \beta \in \Gamma_0$ and $\alpha, \beta \in \Gamma\backslash\Gamma_0$. □

Lemma 1.30. *Let α, β be in Γ then:*

1. *If $\alpha - \beta \in \Gamma_0$ then $e_\alpha \circ e_\beta \in V_{\alpha+\Gamma_0}$ and*

$$C^{\alpha\beta}_{\gamma} = \begin{cases} \frac{1}{2}C^{\beta}_{\gamma+\beta-\alpha} + \frac{1}{2}C^{\alpha}_{\gamma+\alpha-\beta}, & \forall\gamma \in \alpha + \Gamma_0,\\ 0, & \textit{otherwise}.\end{cases}$$

2. *If $\alpha - \beta \notin \Gamma_0$ then $e_\alpha \circ e_\beta \in V_{\alpha+\Gamma_0} \oplus V_{\beta+\Gamma_0}$ and*

$$C^{\alpha\beta}_{\gamma} = \begin{cases} \frac{1}{2}C^{\beta}_{\gamma+\beta-\alpha}, & \forall\gamma \in \alpha + \Gamma_0,\\ \frac{1}{2}C^{\alpha}_{\gamma+\alpha-\beta}, & \forall\gamma \in \beta + \Gamma_0,\\ 0, & \textit{otherwise}.\end{cases}$$

Proof. We use now the second polarized form of equation (1.12):

$$[x, y \circ z] + [y, z \circ x] + [z, x \circ y] = 0. \tag{1.16}$$

For $x = e_\alpha, y = e_\beta$ and $z = e_\mu$ with $\alpha, \beta \in \Gamma_0$ and $\mu \in \Gamma \backslash \Gamma_0$, we get

$$
\begin{aligned}
& [e_\alpha, e_\beta \circ e_\mu] + [e_\beta, e_\mu \circ e_\alpha] + [e_\mu, e_\alpha \circ e_\beta] = 0 \\
\Longleftrightarrow & \left[e_\alpha, \sum_{\gamma \in \Gamma_0} \frac{1}{2} C^\mu_{\gamma+\mu-\beta} e_\gamma + \sum_{\gamma \in \mu+\Gamma_0} \frac{1}{2} C^\beta_{\gamma+\beta-\mu} e_\gamma \right] \\
& + \left[e_\beta, \sum_{\gamma \in \Gamma_0} \frac{1}{2} C^\mu_{\gamma+\mu-\alpha} e_\gamma + \sum_{\gamma \in \mu+\Gamma_0} \frac{1}{2} C^\alpha_{\gamma+\alpha-\mu} e_\gamma \right] \\
& + \left[e_\mu, \sum_{\gamma \in \Gamma_0} C^{\alpha\beta}_\gamma e_\gamma \right] = 0 \\
\Longleftrightarrow & \sum_{\gamma \in \Gamma_0} \frac{1}{2} C^\mu_{\gamma+\mu-\beta} (f(\gamma) - f(\alpha)) e_{\alpha+\gamma} + \sum_{\gamma \in \mu+\Gamma_0} \frac{1}{2} C^\beta_{\gamma+\beta-\mu} (f(\gamma) - f(\alpha)) e_{\alpha+\gamma} \\
& + \sum_{\gamma \in \Gamma_0} \frac{1}{2} C^\mu_{\gamma+\mu-\alpha} (f(\gamma) - f(\beta)) e_{\gamma+\beta} + \sum_{\gamma \in \mu+\Gamma_0} \frac{1}{2} C^\alpha_{\gamma+\alpha-\mu} (f(\gamma) - f(\beta)) e_{\gamma+\beta} \\
& + \sum_{\gamma \in \Gamma_0} C^{\alpha\beta}_\gamma (f(\gamma) - f(\mu)) e_{\mu+\gamma} = 0 \\
\Longleftrightarrow & \sum_{\gamma \in \mu+\Gamma_0} \frac{1}{2} C^\beta_{\gamma+\beta-\mu} \lambda e_{\alpha+\gamma} + \sum_{\gamma \in \mu+\Gamma_0} \frac{1}{2} C^\alpha_{\gamma+\alpha-\mu} \lambda e_{\gamma+\beta} + \sum_{\gamma \in \Gamma_0} C^{\alpha\beta}_\gamma (-\lambda) e_{\mu+\gamma} = 0.
\end{aligned}
$$

We do a change of variables in the first two sums: $\gamma \to \gamma + \mu - \alpha$ and $\gamma \to \gamma + \mu - \beta$. Thus:

$$
\sum_{\gamma \in \Gamma_0} \frac{1}{2} C^\beta_{\gamma+\beta-\alpha} \lambda e_{\gamma+\mu} + \sum_{\gamma \in \Gamma_0} \frac{1}{2} C^\alpha_{\gamma+\alpha-\beta} \lambda e_{\gamma+\mu} + \sum_{\gamma \in \Gamma_0} C^{\alpha\beta}_\gamma (-\lambda) e_{\mu+\gamma} = 0. \tag{1.17}
$$

Hence, Equation (1.17) gives for $\alpha, \beta \in \Gamma_0$:

$$
C^{\alpha\beta}_\gamma = \frac{1}{2} C^\beta_{\gamma+\beta-\alpha} + \frac{1}{2} C^\alpha_{\gamma+\alpha-\beta}, \ \forall \gamma \in \Gamma_0. \tag{1.18}
$$

We use again the equation (1.16) for $x = e_\alpha, y = e_\beta$ and $z = e_\mu$ with $\alpha \in \Gamma_0$ and $\beta, \mu \in \Gamma \backslash \Gamma_0$. So we get

$$
\begin{aligned}
& [e_\alpha, e_\beta \circ e_\mu] + [e_\beta, e_\mu \circ e_\alpha] + [e_\mu, e_\alpha \circ e_\beta] = 0 \\
\Longleftrightarrow & \left[e_\alpha, \sum_{\gamma \in \Gamma \backslash \Gamma_0} C^{\beta\mu}_\gamma e_\gamma \right] + \left[e_\beta, \sum_{\gamma \in \Gamma_0} \frac{1}{2} C^\mu_{\gamma+\mu-\alpha} e_\gamma + \sum_{\gamma \in \mu+\Gamma_0} \frac{1}{2} C^\alpha_{\gamma+\alpha-\mu} e_\gamma \right] \\
& + \left[e_\mu, \sum_{\gamma \in \Gamma_0} \frac{1}{2} C^\beta_{\gamma+\beta-\alpha} e_\gamma + \sum_{\gamma \in \beta+\Gamma_0} \frac{1}{2} C^\alpha_{\gamma+\alpha-\beta} e_\gamma \right] = 0 \\
\Longleftrightarrow & \sum_{\gamma \in \Gamma \backslash \Gamma_0} C^{\beta\mu}_\gamma \lambda e_{\alpha+\gamma} + \sum_{\gamma \in \Gamma_0} \frac{1}{2} C^\mu_{\gamma+\mu-\alpha} (-\lambda) e_{\gamma+\beta} + \sum_{\gamma \in \Gamma_0} \frac{1}{2} C^\beta_{\gamma+\beta-\alpha} (-\lambda) e_{\mu+\gamma} = 0.
\end{aligned}
$$

We do a change of variables in the last two sums: $\gamma \to \gamma + \alpha - \beta$ and $\gamma \to \gamma + \alpha - \mu$. We get

$$\sum_{\gamma \in \Gamma \backslash \Gamma_0} C_\gamma^{\beta\mu} \lambda e_{\alpha+\gamma} + \sum_{\gamma \in \beta+\Gamma_0} \frac{1}{2} C^\mu_{\gamma+\mu-\beta}(-\lambda) e_{\gamma+\alpha}$$
$$+ \sum_{\gamma \in \mu+\Gamma_0} \frac{1}{2} C^\beta_{\gamma+\beta-\mu}(-\lambda) e_{\gamma+\alpha} = 0. \tag{1.19}$$

Equation (1.19) gives for $\beta, \mu \in \Gamma \backslash \Gamma_0$:
If $\beta + \Gamma_0 = \mu + \Gamma_0$, then

$$C_\gamma^{\mu\beta} = \begin{cases} \frac{1}{2} C^\beta_{\gamma+\beta-\mu} + \frac{1}{2} C^\mu_{\gamma+\mu-\beta} & \forall \gamma \in \mu + \Gamma_0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.20}$$

If $\beta + \Gamma_0 \neq \mu + \Gamma_0$, then

$$C_\gamma^{\alpha\mu} = \begin{cases} \frac{1}{2} C^\beta_{\gamma+\beta-\mu} & \forall \gamma \in \mu + \Gamma_0, \\ \frac{1}{2} C^\mu_{\gamma+\mu-\beta} & \forall \gamma \in \beta + \Gamma_0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.21}$$

Finally, by combining equations (1.18), (1.20), (1.21) and the second part of the previous lemma, we get the stated result. □

1.2.2 Case of Witt type algebras of the type $\mathcal{A}$ or $\mathcal{P}$

We suppose now that $V(f)$ is a Witt type algebra with $f \in \mathcal{P} \cup \mathcal{A}$.
We recall that $f \in \mathcal{A}$ means that the function f is additive and that $f \in \mathcal{P}$ means that there exists a surjective group morphism $\tau : \Gamma \to \mathbb{Z}/3\mathbb{Z}$ and an element $\lambda \in \mathbb{K}^*$ such that

$$f(\alpha) = \begin{cases} 0 & si \quad \tau(\alpha) = 0, \\ \lambda & si \quad \tau(\alpha) = 1, \\ -\lambda & si \quad \tau(\alpha) = -1. \end{cases}$$

If f is in $\mathcal{A}$, it is additive and the following property holds:

$$f(\alpha) = f(\beta) \text{ if and only if } \alpha - \beta \in \Gamma_0. \tag{1.22}$$

If f is in $\mathcal{P}$ then

$$f(\alpha) = f(\beta) \Longleftrightarrow \tau(\alpha) = \tau(\beta) \Longleftrightarrow \alpha - \beta \in \Gamma_0.$$

We use this property in the proofs below.

We suppose that the product $*$ is third-power associative and Lie-admissible.

Lemma 1.31.

$$\forall \alpha \in \Gamma, \quad e_\alpha \circ e_\alpha \in V_{\alpha+\Gamma_0}.$$

Proof. As in the above proofs we use the identity 1.12:

$$[x, x \circ x] = 0, \forall x \in V(f).$$

For $x = e_\alpha$, we get:

$$\sum_{\gamma \in \Gamma} C_\gamma^\alpha (f(\gamma) - f(\alpha)) e_{\alpha+\gamma} = 0. \tag{1.23}$$

Using now the property 1.22, we get :

$$C_\gamma^\alpha = 0, \ \forall \gamma \text{ such that } \gamma - \alpha \notin \Gamma_0 \text{ i.e. } \gamma \notin \alpha + \Gamma_0.$$

This proves the lemma. □

Lemma 1.32.
If $\alpha - \beta \notin \Gamma_0$, then:

$$\begin{cases} C_\gamma^{\alpha\beta} = \frac{1}{2} C_{\gamma+\alpha-\beta}^\alpha, & \forall \gamma \in \beta + \Gamma_0, \\ C_\gamma^{\alpha\beta} = \frac{1}{2} C_{\gamma+\beta-\alpha}^\beta, & \forall \gamma \in \alpha + \Gamma_0, \\ C_\gamma^{\alpha\beta} = 0, & \forall \gamma \notin \alpha + \Gamma_0 \cup \beta + \Gamma_0. \end{cases}$$

If $\alpha - \beta \in \Gamma_0$, then:

$$C_\gamma^{\alpha\beta} = 0, \quad \forall \gamma \notin \alpha + \Gamma_0 = \beta + \Gamma_0.$$

Proof. We use the second polarized form of the identity 1.12:

$$\forall x, y \in V(f), \quad 2[x, x \circ y] + [y, x \circ x] = 0.$$

For $x = e_\alpha$ and $y = e_\beta$ we get:

$$\begin{aligned}
&2\sum_{\gamma\in\Gamma} C_\gamma^{\alpha\beta}(f(\gamma)-f(\alpha))e_{\alpha+\gamma} + \sum_{\gamma\in\alpha+\Gamma_0} C_\gamma^{\alpha}(f(\gamma)-f(\beta))e_{\beta+\gamma} = 0.\\
\Longleftrightarrow &2\sum_{\gamma\in\Gamma} C_\gamma^{\alpha\beta}(f(\gamma)-f(\alpha))e_{\alpha+\gamma} + \sum_{\gamma\in\beta+\Gamma_0} C_{\gamma-\beta+\alpha}^{\alpha}(f(\alpha)-f(\beta))e_{\alpha+\gamma} = 0
\end{aligned}$$

Moreover $f(\gamma) = f(\alpha),\ \forall\gamma \in \alpha+\Gamma_0$.
Hence we have

$$\begin{cases} \forall\gamma\in\beta+\Gamma_0,\ 2C_\gamma^{\alpha\beta}(f(\beta)-f(\alpha)) + C_{\gamma-\beta+\alpha}^{\alpha}(f(\alpha)-f(\beta)) = 0,\\ \forall\gamma\notin\beta+\Gamma_0,\ \ 2C_\gamma^{\alpha\beta}(f(\gamma)-f(\alpha)) = 0. \end{cases}$$

Therefore there are two situations:

1. If $\alpha-\beta\in\Gamma_0$ then $f(\alpha)-f(\beta) = 0$ and the first identity is null. The second one gives $C_\gamma^{\alpha\beta} = 0,\ \forall\gamma\notin\alpha+\Gamma_0 = \beta+\Gamma_0$.
2. If $\alpha-\beta\notin\Gamma_0$ then $f(\alpha)-f(\beta)\neq 0$ and the first identity gives:

   $$\forall\gamma\in\beta+\Gamma_0,\ C_\gamma^{\alpha\beta} = \frac{1}{2}C_{\gamma+\alpha-\beta}^{\alpha}$$

   By changing the roles of α and β and since $C_\gamma^{\alpha\beta} = C_\gamma^{\beta\alpha},\ \forall\gamma\in\Gamma$, we get as well:

   $$\forall\gamma\in\alpha+\Gamma_0,\ C_\gamma^{\alpha\beta} = \frac{1}{2}C_{\gamma+\beta-\alpha}^{\beta}.$$

   And the second identity gives :

   $$\forall\gamma\notin\alpha+\Gamma_0\cup\beta+\Gamma_0,\ C_\gamma^{\alpha\beta} = 0.$$

That ends the proof. □

Lemma 1.33. *For α,β in Γ such that $\alpha-\beta\in\Gamma_0$,*

$$C_\gamma^{\alpha\beta} = \frac{1}{2}C_{\gamma+\alpha-\beta}^{\alpha} + \frac{1}{2}C_{\gamma+\beta-\alpha}^{\beta},\quad \forall\gamma\in\alpha+\Gamma_0.$$

Proof. We use the second polarized form of the identity 1.12:

$$\forall x,y,z\in\ V(f),\quad [x,y\circ z]+[y,z\circ x]+[z,x\circ y] = 0.$$

For $x = e_\alpha, y = e_\beta$ and $z = e_\mu$ with $\alpha - \beta \in \Gamma_0$ and $\mu \notin \alpha + \Gamma_0$ we get:

$$\begin{aligned}
&\sum_{\gamma\in\beta+\Gamma_0} \frac{1}{2}C^{\mu}_{\gamma+\mu-\beta}(f(\gamma)-f(\alpha))e_{\gamma+\alpha} + \sum_{\gamma\in\mu+\Gamma_0} \frac{1}{2}C^{\beta}_{\gamma+\beta-\mu}(f(\gamma)-f(\alpha))e_{\gamma+\alpha}\\
&+ \sum_{\gamma\in\mu+\Gamma_0} \frac{1}{2}C^{\alpha}_{\gamma+\alpha-\mu}(f(\gamma)-f(\beta))e_{\gamma+\beta} + \sum_{\gamma\in\alpha+\Gamma_0} \frac{1}{2}C^{\mu}_{\gamma+\mu-\alpha}(f(\gamma)-f(\beta))e_{\gamma+\beta}\\
&+ \sum_{\gamma\in\alpha+\Gamma_0} C^{\alpha\beta}_{\gamma}(f(\gamma)-f(\mu))e_{\gamma+\mu} = 0.
\end{aligned}$$

Since $\alpha - \beta \in \Gamma_0$ we get:

$$\begin{aligned}
&\sum_{\gamma\in\mu+\Gamma_0} \frac{1}{2}C^{\beta}_{\gamma+\beta-\mu}(f(\mu)-f(\alpha))e_{\gamma+\alpha} + \sum_{\gamma\in\mu+\Gamma_0} \frac{1}{2}C^{\alpha}_{\gamma+\alpha-\mu}(f(\mu)-f(\alpha))e_{\gamma+\beta}\\
&+ \sum_{\gamma\in\alpha+\Gamma_0} C^{\alpha\beta}_{\gamma}(f(\alpha)-f(\mu))e_{\gamma+\mu} = 0.
\end{aligned}$$

We do a change of variables in the first two sums:

$$\begin{aligned}
&\sum_{\gamma\in\alpha+\Gamma_0} \frac{1}{2}C^{\beta}_{\gamma-\alpha+\beta}(f(\mu)-f(\alpha))e_{\gamma+\mu} + \sum_{\gamma\in\beta+\Gamma_0} \frac{1}{2}C^{\alpha}_{\gamma-\beta+\alpha}(f(\mu)-f(\alpha))e_{\gamma+\mu}\\
&+ \sum_{\gamma\in\alpha+\Gamma_0} C^{\alpha\beta}_{\gamma}(f(\alpha)-f(\mu))e_{\gamma+\mu} = 0.
\end{aligned}$$

As $\mu \notin \alpha + \Gamma_0$, $f(\mu) - f(\alpha) \neq 0$ and so:

$$\forall\gamma \in \alpha + \Gamma_0, \quad C^{\alpha\beta}_{\gamma} = \frac{1}{2}C^{\alpha}_{\gamma+\alpha-\beta} + \frac{1}{2}C^{\beta}_{\gamma+\beta-\alpha}. \tag{1.24}$$

□

1.2.3 The general case

We consider now general Witt type algebra $V(f)$.

The lemmas of the two previous sections can be resumed by the following proposition which is a general result:

Proposition 1.34. *For all $\alpha, \beta \in \Gamma$,*

$$e_\alpha \circ e_\beta \in V_{\alpha+\Gamma_0} \oplus V_{\beta+\Gamma_0},$$

and

$$e_\alpha \circ e_\beta = \sum_{\gamma\in\Gamma}\left(\frac{1}{2}C^{\alpha}_{\gamma+\alpha-\beta}\delta^{\gamma+\Gamma_0}_{\beta+\Gamma_0}+\frac{1}{2}C^{\beta}_{\gamma+\beta-\alpha}\delta^{\gamma+\Gamma_0}_{\alpha+\Gamma_0}\right)e_\gamma.$$

Theorem 1.35. *Third power-associative, Lie-admissible compatible products on a Witt type algebra $V(f)$ with $f \neq 0$ are:*

$$x * y = \frac{1}{2}[x,y] + \sum_{\gamma\in\Gamma_0} u_\gamma(x)t_\gamma(y) + u_\gamma(y)t_\gamma(x),$$

where t_γ are the $V(f)$-morphisms defined by $t_\gamma(e_\alpha) := e_{\alpha+\gamma}$, $\forall\alpha \in \Gamma$ and where $\{u_\gamma : V(f) \to \mathbb{K};\ \gamma \in \Gamma_0\}$ is a collection of linear form.

Proof. Let $*$ be a third power-associative, Lie-admissible product, and $\circ$ the associated commutative product. We proved in the previous proposition that :

$$e_\alpha \circ e_\beta = \sum_{\gamma\in\Gamma}\left(\frac{1}{2}C^{\alpha}_{\gamma+\alpha-\beta}\delta^{\gamma+\Gamma_0}_{\beta+\Gamma_0}+\frac{1}{2}C^{\beta}_{\gamma+\beta-\alpha}\delta^{\gamma+\Gamma_0}_{\alpha+\Gamma_0}\right)e_\gamma, \tag{1.25}$$

where C^α_γ are the constants given by $e_\alpha \circ e_\alpha := \sum_{\gamma\in\alpha+\Gamma_0} C^\alpha_\gamma e_\gamma$.
This equation is equivalent to

$$e_\alpha \circ e_\beta = \sum_{\gamma\in\beta+\Gamma_0}\frac{1}{2}C^{\alpha}_{\gamma+\alpha-\beta}e_\gamma + \sum_{\gamma\in\alpha+\Gamma_0}\frac{1}{2}C^{\beta}_{\gamma+\beta-\alpha}e_\gamma.$$

After the following change of variables: $\gamma \to \gamma+\beta$ for the first sum $\gamma \to \gamma+\alpha$ for the second one, we get

$$e_\alpha \circ e_\beta = \sum_{\gamma\in\Gamma_0}\frac{1}{2}C^{\alpha}_{\gamma+\alpha}e_{\beta+\gamma} + \sum_{\gamma\in\Gamma_0}\frac{1}{2}C^{\beta}_{\gamma+\beta}e_{\alpha+\gamma}. \tag{1.26}$$

We define for each γ in Γ_0, the linear form u_γ by :

$$u_\gamma(e_\alpha) = \frac{1}{2}C^\alpha_{\alpha+\gamma}.$$

Hence the equation (1.26) becomes:

$$e_\alpha \circ e_\beta = \sum_{\gamma\in\Gamma_0} u_\gamma(e_\alpha)t_\gamma(e_\beta) + u_\gamma(e_\beta)t_\gamma(e_\alpha).$$

By linearity of $\circ$, u_γ and t_γ we get

$$x \circ y = \sum_{\gamma\in\Gamma_0} u_\gamma(x)t_\gamma(y) + u_\gamma(y)t_\gamma(x),\ \forall x,y \in V(f).$$

Finally, since $x * y = \frac{1}{2}[x, y] + x \circ y$, the result is proved.

Conversely, suppose now that there exists a collection of linear forms $u_\gamma : V(f) \to \mathbb{K}$ with $\gamma \in \Gamma_0$ such that

$$x \circ y = \sum_{\gamma \in \Gamma_0} u_\gamma(x) t_\gamma(y) + u_\gamma(y) t_\gamma(x).$$

Then

$$\begin{aligned}[x, x \circ x] &= [x, \sum_{\gamma \in \Gamma_0} 2u_\gamma(x) t_\gamma(x)] \\ &= \sum_{\gamma \in \Gamma_0} 2u_\gamma(x)[x, t_\gamma(x)].\end{aligned}$$

In addition, it is easy to compute that $[x, t_\gamma(x)] = [x, x], \ \forall \gamma \in \Gamma_0$ then:

$$[x, x \circ x] = \sum_{\gamma \in \Gamma_0} 2u_\gamma(x)[x, x] = 0.$$

This proves that $*$ is third power-associative. □

Remark: Since $e_\alpha \circ e_\alpha := \sum_{\gamma \in \alpha + \Gamma_0} C^\alpha_\gamma e_\gamma$, just a finite number of C^α_γ are non-zero. Hence we get that

$$\forall x \in V(f), \ u_\gamma(x) \neq 0 \text{ for a finite number of } \gamma \in \Gamma_0.$$

Proposition 1.36. *$(V(f), *)$ with $f \neq 0$ is flexible Lie-admissible and compatible if and only if*

$$x * y = \frac{1}{2}[x, y] + \sum_{\gamma \in \Gamma_0} u_\gamma(x) t_\gamma(y) + u_\gamma(y) t_\gamma(x)$$

with the same conditions as in the previous theorem and in addition $\mathcal{D}V(f) \subset \ker(u_\gamma)$, $\forall \gamma \in \Gamma_0$.

Proof. Remind that $(V(f), *)$ is flexible if and only if:

$$[x, y] \circ x = [x, y \circ x]. \tag{1.27}$$

The flexibility implies the third power-associativity, hence the product we $\circ$ is of the form:

$$x \circ y = \sum_{\gamma \in \Gamma_0} u_\gamma(x) t_\gamma(y) + u_\gamma(y) t_\gamma(x).$$

Hence Equation (1.27) with $x = e_\alpha,\ y = e_\beta$ gives

$$\sum_{\gamma\in\Gamma_0} u_\gamma([e_\alpha, e_\beta])t_\gamma(e_\alpha) + u_\gamma(e_\alpha)t_\gamma([e_\alpha, e_\beta]) = \sum_{\gamma\in\Gamma_0} u_\gamma(e_\beta)[e_\alpha, e_{\alpha+\gamma}] + u_\gamma(e_\alpha)[e_\alpha, e_{\beta+\gamma}]. \tag{1.28}$$

Remark that for all $\gamma \in \Gamma_0$, we have $[e_\alpha, e_{\alpha+\gamma}] = 0$ and $[e_\alpha, e_{\beta+\gamma}] = t_\gamma([e_\alpha, e_\beta])$. So (1.28) is equivalent to

$$\sum_{\gamma\in\Gamma_0} u_\gamma([e_\alpha, e_\beta])t_\gamma(e_\alpha) = 0.$$

This means exactly that $\forall\gamma \in \Gamma_0,\ u_\gamma([e_\alpha, e_\beta]) = 0$ or $\forall\gamma \in \Gamma_0,\ u_\gamma(\mathcal{D}V(f)) = 0$. That proves the proposition. □

Theorem 1.37. *There are non-trivial flexible Lie-admissible structures only on Witt type algebras with $f \in \mathcal{C}$.*

Proof. It is obvious by the previous proposition and since in characteristic different from 2, we have $\mathcal{D}V(f) \neq V(f)$ only for $f \in \mathcal{C}$. □

Remark. For $f \in \mathcal{C}$ we have $\mathcal{D}V(f) = V_{\Gamma\setminus\Gamma_0}$. Hence, flexible Lie-admissible products on $V(f)$ are given by

$$x \circ y = \sum_{\gamma\in\Gamma_0} u_\gamma(x)t_\gamma(y) + u_\gamma(y)t_\gamma(x),$$

with $u_{\gamma|V_{\Gamma\setminus\Gamma_0}} = 0$.

We can now search for Poisson structures on Witt type algebras. Remember that we just have to find flexible Lie-admissible products such that the associated commutative product $\circ$ is associative. So for Witt type algebras $V(f)$ with $f \in \mathcal{A}\cup\mathcal{P}$, Poisson structures are trivial structures. For $V(f)$ with $f \in \mathcal{C}$ we need to find for which collections of linear forms $\{u_\gamma : V(f) \to \mathbb{K}/\ u_{\gamma|V_{\Gamma\setminus\Gamma_0}} = 0;\ \gamma \in \Gamma_0)\}$, the product

$$x \circ y := \sum_{\gamma\in\Gamma_0} u_\gamma(x)t_\gamma(y) + u_\gamma(y)t_\gamma(x)$$

is associative.

We compute the associator of $\circ$:

$$
\begin{aligned}
(x \circ y) \circ z &= \left(\sum_{\gamma \in \Gamma_0} u_\gamma(x) t_\gamma(y) + u_\gamma(y) t_\gamma(x) \right) \circ z \\
&= \left(\sum_{\gamma \in \Gamma_0} u_\gamma(x) t_\gamma(y) \circ z + u_\gamma(y) t_\gamma(x) \circ z \right) \\
&= \left(\sum_{\gamma \in \Gamma_0} u_\gamma(x) \left(\sum_{\mu \in \Gamma_0} u_\mu(t_\gamma(y)) t_\mu(z) + u_\mu(z) t_\mu(t_\gamma(y)) \right) \right. \\
&\qquad \left. + u_\gamma(y) \left(\sum_{\mu \in \Gamma_0} u_\mu(t_\gamma(x)) t_\mu(z) + u_\mu(z) t_\mu(t_\gamma(x)) \right) \right) \\
&= \sum_{\gamma, \mu \in \Gamma_0} u_\gamma(x) u_\mu(t_\gamma(y)) t_\mu(z) + u_\gamma(x) u_\mu(z) t_{\mu+\gamma}(y) \\
&\qquad + u_\gamma(y) u_\mu(t_\gamma(x)) t_\mu(z) + u_\gamma(y) u_\mu(z) t_{\mu+\gamma}(x).
\end{aligned}
$$

As $\circ$ is commutative, we have $x \circ (y \circ z) = (y \circ z) \circ x$ and then:

$$
\begin{aligned}
x \circ (y \circ z) = \sum_{\gamma, \mu \in \Gamma_0} & u_\gamma(y) u_\mu(t_\gamma(z)) t_\mu(x) + u_\gamma(y) u_\mu(x) t_{\mu+\gamma}(z) \\
& + u_\gamma(z) u_\mu(t_\gamma(y)) t_\mu(x) + u_\gamma(z) u_\mu(x) t_{\mu+\gamma}(y).
\end{aligned}
$$

Finally we get

$$
\begin{aligned}
(x, y, z) = \sum_{\gamma, \mu \in \Gamma_0} & u_\gamma(x) u_\mu(t_\gamma(y)) t_\mu(z) + u_\gamma(x) u_\mu(z) t_{\mu+\gamma}(y) + u_\gamma(y) u_\mu(t_\gamma(x)) t_\mu(z) \\
& + u_\gamma(y) u_\mu(z) t_{\mu+\gamma}(x) \; - u_\gamma(y) u_\mu(t_\gamma(z)) t_\mu(x) - u_\gamma(y) u_\mu(x) t_{\mu+\gamma}(z) \\
& - u_\gamma(z) u_\mu(t_\gamma(y)) t_\mu(x) - u_\gamma(z) u_\mu(x) t_{\mu+\gamma}(y) \\
= \sum_{\gamma, \mu \in \Gamma_0} & [u_\gamma(x) u_\mu(t_\gamma(y)) + u_\gamma(y) u_\mu(t_\gamma(x))] t_\mu(z) \\
& - [u_\gamma(y) u_\mu(t_\gamma(z)) + u_\gamma(z) u_\mu(t_\gamma(y))] t_\mu(x) \\
& + u_\gamma(y) u_\mu(z) t_{\mu+\gamma}(x) - u_\gamma(y) u_\mu(x) t_{\mu+\gamma}(z) \\
& + [u_\gamma(x) u_\mu(z) - u_\gamma(z) u_\mu(x)] t_{\mu+\gamma}(y).
\end{aligned}
$$

Hence, as $u_{\gamma | V_{\Gamma \setminus \Gamma_0}} = 0$; for $x, y \in V_{\Gamma_0}$ and $z \in V_{\Gamma \setminus \Gamma_0}$ the identity $(x, y, z) = 0$ is equivalent to:

$$
\sum_{\gamma, \mu \in \Gamma_0} u_\gamma(x) u_\mu(t_\gamma(y)) t_\mu(z) + u_\gamma(y) u_\mu(t_\gamma(x)) t_\mu(z) - u_\gamma(y) u_\mu(x) t_{\mu+\gamma}(z) = 0. \quad (1.29)
$$

By writing the last equation (1.29) with $x = e_\alpha, y = e_\beta$ and $z = e_\sigma$, $\alpha, \beta \in \Gamma_0$ and $\sigma \in \Gamma'$ we get:

$$\sum_{\gamma,\mu\in\Gamma_0} [u_\gamma(e_\alpha)u_\mu(e_{\beta+\gamma}) + u_\gamma(e_\beta)u_\mu(e_{\alpha+\gamma})]e_{\sigma+\mu} - u_\gamma(e_\beta)u_\mu(e_\alpha)e_{\sigma+\mu+\gamma} = 0 \quad (1.30)$$

$$\iff \sum_{\gamma,\mu\in\Gamma_0} [u_\gamma(e_\alpha)u_\mu(e_{\beta+\gamma}) + u_\gamma(e_\beta)u_\mu(e_{\alpha+\gamma})]e_{\sigma+\mu} - \sum_{\gamma,\mu\in\Gamma_0} u_\gamma(e_\beta)u_\mu(e_\alpha)e_{\sigma+\mu+\gamma} = 0$$

$$\iff \sum_{\mu\in\Gamma_0}\left(\sum_{\gamma\in\Gamma_0}[u_\gamma(e_\alpha)u_\mu(e_{\beta+\gamma}) + u_\gamma(e_\beta)u_\mu(e_{\alpha+\gamma})]\right)e_{\sigma+\mu} - \sum_{\rho\in\Gamma_0}\sum_{\substack{\gamma+\mu=\rho\\ \gamma,\mu\in\Gamma_0}} u_\gamma(e_\beta)u_\mu(e_\alpha)e_{\sigma+\rho} = 0$$

$$\iff \sum_{\mu\in\Gamma_0}\left(\sum_{\gamma\in\Gamma_0}[u_\gamma(e_\alpha)u_\mu(e_{\beta+\gamma}) + u_\gamma(e_\beta)u_\mu(e_{\alpha+\gamma})]\right)e_{\sigma+\mu} - \sum_{\rho\in\Gamma_0}\sum_{\gamma\in\Gamma_0} u_\gamma(e_\beta)u_{\rho-\gamma}(e_\alpha)e_{\sigma+\rho} = 0$$

$$\iff \sum_{\mu\in\Gamma_0}\left(\sum_{\gamma\in\Gamma_0}[u_\gamma(e_\alpha)u_\mu(e_{\beta+\gamma}) + u_\gamma(e_\beta)u_\mu(e_{\alpha+\gamma})]\right)e_{\sigma+\mu} - \sum_{\mu\in\Gamma_0}\sum_{\gamma\in\Gamma_0} u_\gamma(e_\beta)u_{\mu-\gamma}(e_\alpha)e_{\sigma+\mu} = 0$$

$$\iff \sum_{\mu\in\Gamma_0}\left(\sum_{\gamma\in\Gamma_0}[u_\gamma(e_\alpha)u_\mu(e_{\beta+\gamma}) + u_\gamma(e_\beta)u_\mu(e_{\alpha+\gamma}) - u_\gamma(e_\beta)u_{\mu-\gamma}(e_\alpha)]\right)e_{\sigma+\mu} = 0$$

$$\iff \forall\mu\in\Gamma_0,\ \sum_{\gamma\in\Gamma_0}[u_\gamma(e_\alpha)u_\mu(e_{\beta+\gamma}) + u_\gamma(e_\beta)u_\mu(e_{\alpha+\gamma}) - u_\gamma(e_\beta)u_{\mu-\gamma}(e_\alpha)] = 0. \quad (1.31)$$

We remark that the last equation does not depend on σ. In fact if (1.31) is true, it is easy to prove that (1.30) is true for any $\sigma \in \Gamma$. So by linearity we get Equation (1.29) for any $x, y \in V_{\Gamma_0}$ and $z \in V(f)$. Moreover since $\forall \gamma \in \Gamma_0$, $u_{\gamma|_{V_{\Gamma'}}} = 0$, (1.29) is true for all $x, y, z \in V(f)$. Hence $(x, y, z) = 0$, $\forall x, y, z \in V(f)$ is clearly verified. Finally $\circ$ is associative if and only if for all $\alpha, \beta \in \Gamma_0$

$$\sum_{\gamma\in\Gamma_0}[u_\gamma(e_\alpha)u_\mu(e_{\beta+\gamma}) + u_\gamma(e_\beta)u_\mu(e_{\alpha+\gamma}) - u_\gamma(e_\beta)u_{\mu-\gamma}(e_\alpha)] = 0,\ \forall\mu\in\Gamma_0.$$

In particular if $\alpha = \beta$ we get

$$\sum_{\gamma\in\Gamma_0} u_\gamma(e_\alpha)[2u_\mu(e_{\alpha+\gamma}) - u_{\mu-\gamma}(e_\alpha)] = 0,\ \forall\mu\in\Gamma_0.$$

Theorem 1.38. *A flexible Lie-admissible product $*$ on a Witt type algebra $V(f)$ is a Poisson product if and only if $\forall\ \alpha, \beta, \mu \in \Gamma_0$*

$$\sum_{\gamma\in\Gamma_0}[u_\gamma(e_\alpha)u_\mu(e_{\beta+\gamma}) + u_\gamma(e_\beta)u_\mu(e_{\alpha+\gamma}) - u_\gamma(e_\beta)u_{\mu-\gamma}(e_\alpha)] = 0.$$

Open question: For easy choice of Γ_0 (like $\mathbb{Z}_2$ or $\mathbb{Z}_3$), we get $u_\gamma = 0$, $\forall\gamma \in \Gamma_0$. But we did not prove anything for the general case. Do examples of Witt type algebra with a non-trivial Poisson structure exist?

1.3 One-dimensional central extension of Witt type algebra

In his article [38], Rupert Yu he determines the one-dimensional central extension for some Witt type algebras which are very close to the Witt and Virasoro algebras:

Theorem 1.39 (Yu)**.** *Let Γ be a free abelian group and suppose that car$\mathbb{K} = 0$. If $f \in \mathcal{E}$ is injective, then the Witt type algebra $V(f)$ has a universal one-dimensional central extension given by the 2-coycle:*

$$\Phi(e_\alpha, e_\beta) = \delta_{\alpha,-\beta}(f(\alpha)^3 - f(\alpha)).$$

Remark. The Witt type algebras which are considered in this theorem are a subclass of the algebras $V(f)$ with $f \in \mathcal{A}$.

We are interested in one-dimensional central extensions for other classes of Witt type algebras.

1.3.1 Case of Witt type algebra of type $\mathcal{C}$.

Let $V(f)$ be a Witt type algebra with $f \in \mathcal{C}$. The set $\Gamma\backslash\Gamma_0$ is denoted by Γ'.

Proposition 1.40. *A bilinear form ω of $V(f)$ is a 2-cocycle if and only if there exists a skew-symetric bilinear form $\tilde{\omega}$ on V_{Γ_0} and a linear form h on $V_{\Gamma'}$ such that :*

1. $\omega_{|V_{\Gamma_0}\times V_{\Gamma_0}} = \tilde{\omega}$,
2. $\omega_{|V_{\Gamma_0}\times V_{\Gamma'}} = dh_{|V_{\Gamma_0}\times V_{\Gamma'}}$; $\omega_{|V_{\Gamma'}\times V_{\Gamma_0}} = dh_{|V_{\Gamma'}\times V_{\Gamma_0}}$,
3. $\omega_{|V_{\Gamma'}\times V_{\Gamma'}} = 0$.

Proof. Let ω be a 2-cocyle of $V(f)$. Then ω is skew-symmetric and

$$\omega([x, y], z) + \omega([y, z], x) + \omega([z, x], y) = 0, \qquad \forall x, y, z \in V. \tag{1.32}$$

For $x = e_\alpha,\ y = e_\beta$ and $z = e_\gamma$ with $\alpha, \beta \in \Gamma_0,\ \gamma \in \Gamma'$, we get:

$$\omega([e_\alpha, e_\beta], e_\gamma) + \omega([e_\beta, e_\gamma], e_\beta) + \omega([e_\gamma, e_\alpha], e_\beta) = 0 \tag{1.33}$$

$$\text{i.e. } \lambda\omega(e_{\beta+\gamma}, e_\alpha) - \lambda\omega(e_{\alpha+\gamma}, e_\beta) = 0. \tag{1.34}$$

Hence if $\alpha = 0$, we obtain

$$\omega(e_\gamma, e_\beta) = \omega(e_{\beta+\gamma}, e_0) = \frac{1}{\lambda}\omega([e_{\beta,} e_\gamma], e_0). \tag{1.35}$$

Let be $h : V_{\Gamma'} \to \mathbb{K}; y \mapsto \frac{1}{\lambda}\omega(y, e_0)$. Then h is a linear form of $V_{\Gamma'}$. Moreover for $\beta \in \Gamma',\ \gamma \in \Gamma_0$, by Equation (1.35) we have

$$\omega(e_\beta, e_\gamma) = h([e_\beta, e_\gamma]) = dh(e_\beta, e_\gamma).$$

This proves the second statement.
The first is clear. For the last part , we write Equation (1.32) for $x = e_\alpha,\ y = e_\beta$ and $z = e_\gamma$ with $\alpha, \beta \in \Gamma',\ \gamma \in \Gamma_0$. We easily get:

$$\omega(e_{\beta+\gamma}, e_\alpha) = \omega(e_{\alpha+\gamma}, e_\beta).$$

In particular if $\gamma = 0$,

$$\omega(e_\beta, e_\alpha) = \omega(e_\alpha, e_\beta).$$

Since ω is skew-symmetric we conclude that $\omega(e_\alpha, e_\beta) = 0,\ \forall \alpha, \beta \in \Gamma'$.

Conversely, one can easily verify that a skew-symmetric bilinear form verifying 1,2 and 3 also satisfies the 2-cocycle condition. □

Theorem 1.41. *Let f be in $\mathcal{C}$. The second cohomology space $H^2(V(f), \mathbb{K})$ is isomorphic to $C^2(V_{\Gamma_0}, \mathbb{K})$ the vector space of skew-symmetric bilinear form on V_{Γ_0}.*

Proof. A 2-cocycle ω of $V(f)$ is a 2-coboundary if there exists a linear form g such that $\omega = dg$. Since $[V_{\Gamma_0}, V_{\Gamma_0}] = \{0\}$ we have $\tilde{\omega} = 0$. Conversely if ω is such that $\tilde{\omega} = 0$, we choose a linear form g with $g_{|V_{\Gamma'}} = h$ and then $\omega = dg$. Therefore ω is a 2-coboundary if and only if $\tilde{\omega} = 0$. We consider the following map:

$$\Phi : Z^2(V(f), \mathbb{K}) \to C^2(V_{\Gamma_0}, \mathbb{K}); \omega \mapsto \tilde{\omega}.$$

The map Φ is onto. Moreover $\ker \Phi = \left\{\omega \in Z^2(V(f), \mathbb{K})/\tilde{\omega} = 0\right\} = B^2(V(f), \mathbb{K})$. Hence the vector space $Z^2(V(f), \mathbb{K})/B^2(V(f), \mathbb{K}) =: H^2(V(f), \mathbb{K})$ and the vector space $C^2(V_{\Gamma_0}, \mathbb{K})$ are isomorphic. $\square$

Since we know the 2-cocycles of $V(f)$, we are naturally interested by symplectic structures on $V(f)$. We give conditions on Γ and Γ_0 for the existence of symplectic structures.

Theorem 1.42. *Let f be in $\mathcal{C}$. If $|\Gamma|$ is finite, there exists a symplectic structure on the Witt type algebra $V(f)$ if and only if either $[\Gamma : \Gamma_0] = 2$ or $[\Gamma : \Gamma_0] = 1$ and $|\Gamma|$ is even.*
If $|\Gamma|$ is infinite and if $[\Gamma : \Gamma_0] \leq 2$, there exists a symplectic structure on the Witt type algebra $V(f)$.

Proof. The algebra $V(f)$ is symplectic if and only if there exists on $V(f)$ a non-degenerate 2-cocycle. Suppose that $[\Gamma : \Gamma_0] = 1$. Then $\Gamma_0 = \Gamma$ and any skew-symmetric bilinear form is a 2-cocycle. We just have to find a non-degenerate one. Since $|\Gamma| = \dim V(f)$, if $|\Gamma|$ is finite and even, it is possible. If $|\Gamma|$ is infinite, it is always possible. Indeed every infinite set I is the disjoint union of two equipotent subsets since I is equipotent to $I \times \{0\} \sqcup I \times \{1\}$. So $\Gamma = \Gamma_1 \cup \Gamma_2$ with a bijection ϕ between Γ_1 and Γ_2, then the skew-symmetric form defined by $\omega(e_\alpha, e_\beta) = \delta^{\beta}_{\phi(\alpha)}$ is non-degenerate.
We suppose now that $[\Gamma : \Gamma_0] = 2$. Then $\Gamma = \Gamma_0 \,\dot{\cup}\, (\alpha + \Gamma_0)$ with $\alpha \notin \Gamma_0$. Thus $V(f) = V_{\Gamma_0} \oplus V_{\alpha+\Gamma_0}$ and we define the following 2-cocycle on $V(f)$:

$$\tilde{\omega} = 0;\ h(e_\alpha) = 1 \text{ and } h(e_{\alpha+\gamma}) = 0,\ \forall \gamma \in \Gamma_0 \backslash \{0\}.$$

We verify that ω is non-degenerate: suppose that $\omega(x, y) = 0$ for all $y \in V(f)$. We can write $x = \sum_{\gamma \in \Gamma} a_\gamma e_\gamma$ and $y = \sum_{\mu \in \Gamma} b_\mu e_\mu$ with $a_\gamma,\ b_\gamma \neq 0$ for a finite number of $\gamma \in \Gamma$.

Hence

$$
\begin{aligned}
& \omega(\sum_{\gamma\in\Gamma} a_\gamma e_\gamma, \sum_{\gamma\in\Gamma} b_\gamma e_\gamma) = 0 \\
\iff & \omega(\sum_{\gamma\in\Gamma_0} a_\gamma e_\gamma, \sum_{\gamma\in\alpha+\Gamma_0} b_\gamma e_\gamma) + \omega(\sum_{\gamma\in\alpha+\Gamma_0} a_\gamma e_\gamma, \sum_{\gamma\in\Gamma_0} b_\gamma e_\gamma) = 0 \\
\iff & \sum_{\gamma,\mu\in\Gamma_0} a_\gamma b_{\alpha+\mu} h([e_\gamma, e_{\alpha+\mu}]) + \sum_{\gamma,\mu\in\Gamma_0} a_{\alpha+\gamma} b_\mu h([e_{\alpha+\gamma}, e_\mu]) = 0 \\
\iff & \lambda \left(\sum_{\gamma,\mu\in\Gamma_0} a_\gamma b_{\alpha+\mu} h(e_{\gamma+\alpha+\mu}) - \sum_{\gamma,\mu\in\Gamma_0} a_{\alpha+\gamma} b_\mu h(e_{\alpha+\gamma+\mu}) \right) = 0 \\
\iff & \lambda \left(\sum_{\substack{\gamma+\mu=0 \\ \gamma,\mu\in\Gamma_0}} a_\gamma b_{\alpha+\mu} - a_{\alpha+\gamma} b_\mu \right) = 0 \\
\iff & \sum_{\gamma\in\Gamma_0} a_\gamma b_{\alpha-\gamma} - a_{\alpha+\gamma} b_{-\gamma} = 0.
\end{aligned}
\tag{1.36}
$$

For $y = e_\mu$ with $\mu \in \Gamma_0$ (i.e $b_\mu = 1$ and $b_\gamma = 0,\ \forall\gamma \neq \mu$), Equation 1.36 gives $a_{\alpha-\mu} = 0$. Likewise for $y = e_{\alpha+\mu}$ with $\mu \in \Gamma_0$ we get $a_{-\mu} = 0$. So we get $a_\gamma = 0,\ \forall\gamma \in \Gamma$, this is to say $x = 0$. This proves that ω is non-degenerate.

Now suppose that $|\Gamma|$ is finite and that ω is a symplectic form. From Proposition 1.40 , a 2-cocycle ω verifies $\omega_{|V_{\Gamma'}\times V_{\Gamma'}} = 0$. Therefore if ω is non-degenerate, for each x in $V_{\Gamma'}$ there exists y in V_{Γ_0} such that $\omega(x, y) \neq 0$. This means that the following map is injective:

$$\overline{\overline{\omega}} : V_{\Gamma'} \to (V_{\Gamma_0})^*; x \mapsto \omega(x, .).$$

As Γ is finite, the vector spaces $V_{\Gamma'}$ and V_{Γ_0} are finite dimensional too and $\dim(V_{\Gamma'}) = |\Gamma'|$, $\dim(V_{\Gamma_0}) = |\Gamma_0|$. Hence $|\Gamma'| \leq |\Gamma_0|$ and then $|\Gamma| - |\Gamma_0| \leq |\Gamma_0|$. This is equivalent to $[\Gamma : \Gamma_0] \leq 2$. □

1.3.2 Case of Witt type algebras of type $\mathcal{P}$.

Now let f be in $\mathcal{P}$. We note $\Gamma_1 := f^{-1}(\{1\})$ and $\Gamma_{-1} := f^{-1}(\{-1\})$. Hence $V(f) = V_{\Gamma_{-1}} \oplus V_{\Gamma_0} \oplus V_{\Gamma_1}$.

A bilinear form ω is a 2-cocycle if and only if ω is skew-symmetric and if Identity (1.32) holds . Since the Identity (1.32) is linear, it just has to be verified for generators. Hence let $x = e_\alpha,\ y = e_\beta,\ z = e_\gamma$.

1. If $\alpha, \beta, \gamma \in \Gamma_i$ for $i \in \{-1, 0, 1\}$, Identity (1.32) holds.
2. If $\alpha, \beta \in \Gamma_0$ and $\gamma \in \Gamma_i,\ i \in \{-1, 1\}$, Identity (1.32) holds if and only if
$$\omega(e_\alpha, e_\gamma) = \omega(e_0, e_{\alpha+\gamma}),\ \forall\alpha \in \Gamma_0,\ \gamma \in \Gamma_i. \tag{1.37}$$
3. If $\alpha \in \Gamma_0$ and $\beta, \gamma \in \Gamma_i,\ i \in \{-1, 1\}$, Identity (1.32) holds if and only if
$$\omega_{|V_{\Gamma_i}\times V_{\Gamma_i}} = 0.$$
4. If $\alpha \in \Gamma_1,\ \beta \in \Gamma_{-1},\ \gamma \in \Gamma_0$, Identity (1.32) holds if and only if
$$\omega(e_{\alpha+\beta}, e_\gamma) = \frac{1}{2}\left(\omega(e_{\alpha+\gamma}, e_\beta) + \omega(e_{\beta+\gamma}, e_\alpha)\right).$$
5. If $\alpha, \beta \in \Gamma_i$ and $\gamma \in \Gamma_j$ with $i, j \in \{-1, 1\},\ i \neq j$, Identity (1.32) holds if and only if so does (1.37) .

Let be the linear form $h : V(f) \to \mathbb{K}$ defined by $h(e_\alpha) = \begin{cases} \frac{1}{\lambda}\omega(e_0, e_\alpha),\ \forall\alpha \in \Gamma_1 \\ -\frac{1}{\lambda}\omega(e_0, e_\alpha),\ \forall\alpha \in \Gamma_{-1} \end{cases}$ and taking any values on V_{Γ_0}.
Hence, we get:

Proposition 1.43. *On $V(f)$ with $f \in \mathcal{P}$, a bilinear form ω is a 2-cocycle if and only if there exists a linear form h of $V(f)$ such that ω is the skew-symmetric bilinear form defined by:*

1. $\omega_{|V_{\Gamma_i}\times V_{\Gamma_i}} = 0,\ i \in \{-1, 1\}$,
2. $\omega_{|V_{\Gamma_0}\times V_{\Gamma_i}} = d(h_{|V_{\Gamma_i}}),\ i \in \{-1, 1\}$,
3. $\omega(e_{\alpha+\beta}, e_\gamma) = \frac{1}{2}\left(\omega(e_{\alpha+\gamma}, e_\beta) + \omega(e_{\beta+\gamma}, e_\alpha)\right),\ \forall\alpha \in \Gamma_1,\ \beta \in \Gamma_{-1},\ \gamma \in \Gamma_0.$

Remark: The equation of the first part of Proposition 1.43 for $\gamma = 0$ gives $\omega(e_0, .)_{|V_{\Gamma_0}} = 0$. Therefore there is not homogeneous symplectic form on $V(f)$ for $f \in \mathcal{P}$.

Proposition 1.44. *On $V(f)$, a 2-cocycle ω is a 2-coboundary if and only if $\omega_{|V_{\Gamma_0}\times V_{\Gamma_0}} = 0$*

Proof. If ω is a 2-coboundary, then $\omega = dg$ with g a linear form on $V(f)$. Let $x, y \in V_{\Gamma_0}$ Then $\omega(x, y) = g([x, y]) = 0$.
Conversely suppose that $\omega_{|V_{\Gamma_0} \times V_{\Gamma_0}} = 0$. Thus, by the assertion 3 of the previous proposition we have:

$$\omega(e_{\alpha+\gamma}, e_\beta) + \omega(e_{\beta+\gamma}, e_\alpha) = 0, \ \forall \alpha \in \Gamma_1, \ \beta \in \Gamma_{-1}, \ \gamma \in \Gamma_0.$$

This proves that for $\alpha, \alpha' \in \Gamma_1$ and $\beta, \beta' \in \Gamma_{-1}$ such that $\alpha + \beta = \alpha' + \beta'$, we get

$$\omega(e_\alpha, e_\beta) = \omega(e_{\alpha'}, e_{\beta'}).$$

Hence for $\gamma \in \Gamma_0$, take $h(e_\gamma) = -\frac{1}{2\lambda}\omega(e_\alpha, e_\beta)$ for any α, β such that $\gamma = \alpha + \beta$. So $\omega = dh$ and ω is a 2-coboundary. □

Lemma 1.45. *An abelian group G such that for all $g \in G \backslash \{0\}$ the quotient group $G/\langle g \rangle$ is of finite order is a finite abelian group or is isomorphic to $\mathbb{Z}$.*

Proof. Obviously the property holds for every finite abelian group. So we suppose that G is an infinite abelian group. We show that G is torsion-free. In fact, if there exists a torsion element $g \in G$, then the subgroup $\langle g \rangle$ is of finite order. As the quotient group $G/\langle g \rangle$ is of finite order too, the group G should be of finite order. Hence G is torsion-free.
The quotient group $G/\langle g \rangle$ is of finite order. We denote by $\{q_1, ...q_n\}$ its elements. Hence G is generated by $\{q_1, ...q_n, g\}$, it is a finitely generated abelian group. But we said that G is torsion free. This means that G is a finitely generated free group, i.e. isomorphic to $\mathbb{Z}^l$ with $l > 0$. We suppose that $l > 1$. Hence the quotient group $G/\langle g \rangle$ with $g = (0, 1, ..., 1)$ is isomorphic to $\mathbb{Z}$, which is not possible. This proves that $l = 1$, i.e.that G is isomorphic to $\mathbb{Z}$. □

Proposition 1.46. *If Γ_0 is a finite abelian group or is isomorphic to $\mathbb{Z}$, all 2-cocycles of $V(f)$ are 2-coboundaries. In particular the second group of cohomology $H^2(V(f), \mathbb{K})$ is null.*

Proof. Let ω be a 2-cocycle of $V(f)$.
We said before that the identity 3 of the proposition 1.43 give us $\omega(e_0, .)_{|V_{\Gamma_0}} = 0$. Hence if we set $\beta = -\alpha \in \Gamma_{-1}$ in the identity 3, we get $\omega(e_{\alpha+\gamma}, e_{-\alpha}) = \omega(e_\alpha, e_{-\alpha+\gamma})$. Hence for all $\alpha \in \Gamma_1, \beta \in \Gamma_{-1}$, if $\gamma := \alpha + \beta \in \Gamma_0$, we have

$$\omega(e_\alpha, e_\beta) = \omega(e_{\alpha+\gamma}, e_{\beta-\gamma}).$$

Therefore if $\alpha' \in \Gamma_1,\ \beta' \in \Gamma_{-1}$ are such that $\gamma = \alpha + \beta = \alpha' + \beta'$ and $\alpha' - \alpha \in \langle \gamma \rangle$, we have

$$\omega(e_\alpha, e_\beta) = \omega(e_{\alpha'}, e_{\beta'}). \tag{1.38}$$

We consider the quotient group $\Gamma/\langle\gamma\rangle$ and the canonical projection $\pi : \Gamma \to \Gamma/\langle\gamma\rangle$. For a fixed $\gamma \in \Gamma_0$ we define the map $\Phi^\gamma : \pi(\Gamma_1) \to \mathbb{K}; \bar{\alpha} \mapsto \omega(e_\alpha, e_{\gamma-\alpha})$ which is well defined since Identity (1.38) holds. Moreover, we can remark that $\mathrm{card}(\pi(\Gamma_1)) = \mathrm{card}(\pi(\Gamma_0)) = \mathrm{card}(\Gamma_0/\langle\gamma\rangle)$.

Let γ be in Γ_0. For all $\alpha_1, \alpha_2 \in \Gamma_1$ and $\mu \in \Gamma_0$ we have

$$\omega(e_{\alpha_1-\alpha_1+\gamma-\mu}, e_\mu) = \omega(e_{\gamma-\mu}, e_\mu) = \omega(e_{\alpha_2-\alpha_2+\gamma-\mu}, e_\mu).$$

Hence by using again identity 3, we get

$$\omega(e_{\alpha_1+\mu}, e_{-\alpha_1+\gamma-\mu}) - \omega(e_{\alpha_1}, e_{-\alpha_1+\gamma-\mu}) = \omega(e_{\alpha_2+\mu}, e_{-\alpha_2+\gamma-\mu}) - \omega(e_{\alpha_2}, e_{-\alpha_2+\gamma-\mu}).$$

So, for all $\gamma, \mu \in \Gamma_0$ and $\alpha_1, \alpha_2 \in \Gamma_1$ we get

$$\Phi^\gamma(\bar{\alpha}_1 + \bar{\mu}) - \Phi^\gamma(\bar{\alpha}_1) = \Phi^\gamma(\bar{\alpha}_2 + \bar{\mu}) - \Phi^\gamma(\bar{\alpha}_2). \tag{1.39}$$

By the previous lemma and the condition on Γ_0, we know that for all $\gamma \in \Gamma_0$ the quotient group $\Gamma_0/\langle\gamma\rangle$ is of finite order. Since $\mathrm{card}(\pi(\Gamma_1)) = \mathrm{card}(\Gamma_0/\langle\gamma\rangle)$ the map Φ^γ has a finite number of value. For $\mu \in \Gamma_0$, let K be the order of μ. We have

$$\begin{aligned}\phi^\gamma(\bar{\alpha}) - \phi^\gamma(\bar{\alpha} + \bar{\mu}) = \phi^\gamma(\bar{\alpha} + K\bar{\mu}) - \phi^\gamma(\bar{\alpha} + (K-1)\bar{\mu}) + \phi^\gamma(\bar{\alpha} + (K-1)\bar{\mu})\\ -\phi^\gamma(\bar{\alpha} + (K-2)\bar{\mu}) + \ldots + \phi^\gamma(\bar{\alpha} + 2\bar{\mu}) - \phi^\gamma(\bar{\alpha} + \bar{\mu}).\end{aligned}$$

By (1.39) we have

$$\phi^\gamma(\bar{\alpha}) - \phi^\gamma(\bar{\alpha} + \bar{\mu}) = \phi^\gamma(\bar{\alpha} + (K-1)\bar{\mu}) - \phi^\gamma(\bar{\alpha} + K\bar{\mu}) = \ldots = \phi^\gamma(\bar{\alpha} + \bar{\mu}) - \phi^\gamma(\bar{\alpha} + 2\bar{\mu}).$$

Hence

$$\phi^\gamma(\bar{\alpha}) - \phi^\gamma(\bar{\alpha} + \bar{\mu}) = (1 - K)\left[\phi^\gamma(\bar{\alpha}) - \phi^\gamma(\bar{\alpha} + \bar{\mu})\right],$$

and then

$$\phi^\gamma(\bar{\alpha}) = \phi^\gamma(\bar{\alpha} + \bar{\mu}) = \phi^\gamma(\overline{\alpha + \mu}), \forall \alpha \in \Gamma_1,\ \gamma, \mu \in \Gamma_0.$$

That proves that each map $\Phi^{\gamma}, \gamma \in \Gamma$ take just one value. That proves that for $\alpha, \alpha' \in \Gamma_1$ and $\beta, \beta' \in \Gamma_1$ such that $\alpha + \beta = \alpha' + \beta'$ we have $\omega(e_\alpha, e_\beta) = \omega(e_{\alpha'}, e_{\beta'})$. In this case, identity (3) gives $\omega_{|V_{\Gamma_0} \times V_{\Gamma_0}} = 0$. Hence ω is a 2-coboundary. $\square$

1.4 Third power-associative structure on the central extensions of Witt type algebras.

In the following we study the relation between third power-associative structures on a Lie algebra L and third power-associative structures on a central extension E of L. Also we give the third power-associative structures on the central extension of some Witt type algebras.

Let $E := L \oplus \mathbb{K}c$ be a central extension of a Lie algebra L by means of a 2-cocycle ω. Let $*_E$ be a third power-associative product on E compatible with the Lie structure and $\circ_E$ the associated commutative product on E defined by $x \circ_E y = \frac{1}{2}(x *_E y + y *_E x)$. We remind that $*_E$ is third power-associative if and only if $[x, x \circ_E x]_E = 0,\ \forall x \in E$.

Since $\circ_E$ is commutative product, there exists:

1. $\circ_L$ a commutative product of L,
2. σ is a symmetric bilinear form of L,
3. A is an endomorphism of L,
4. λ is a linear form of L,
5. $\gamma \in L$ and $\mu \in \mathbb{K}$,

such that

$$\begin{aligned} x \circ_E y &= x \circ_L y + \sigma(x, y).c \quad \forall x, y \in L, \\ x \circ_E c = c \circ_E x &= A(x) + \lambda(x).c \quad \forall x \in L, \\ c \circ_E c &= \gamma + \mu.c. \end{aligned}$$

Moreover, since $[x, x \circ_E x]_E = 0,\ \forall x \in E$ we get:

1. $[x, x \circ_L x]_L = 0, \ \forall x \in L$ and then $\circ_L$is a third power-associative product of L,

2. $\omega(x, x \circ_L x) = 0, \ \forall x \in L$,

3. $\mathrm{ad}(x) \circ A = \mathrm{ad}(A(x)) \ \forall x \in L$ and A is ω-symmetric:
   We use the linearized form of $[x, x \circ_E x]_E = 0$: $2[x, x \circ_E y]_E + [y, x \circ_E x]_E = 0$; with $x \in L$ and $y = c$. We get

   $$2[x, x \circ_E c]_E = 0 \Longleftrightarrow [x, A(x)]_L + \omega(x, A(x)).c = 0.$$

   So $A(x) \in \ker \mathrm{ad}(x) \cap \ker \omega(x, .), \ \forall x \in L$ and that is equivalent to

   $$\omega(x, A(y)) = \omega(A(x), y), \ \forall x, y \in L$$

   and

   $$[x, A(y)]_L = [A(x), y]_L, \ \forall x, y \in L$$

   i.e. $\mathrm{ad}(x) \circ A = \mathrm{ad}(A(x)) \ \forall x \in L$ and A is ω-symmetric.

4. $\gamma \in Z(L) \cap \ker \omega$:
   We get this result by using the same equation with $x = c$ and $y \in L$.

Conversely suppose that there exists on L a third power-associative product $*_L$. We denote by $\circ_L$ the associated commutative product. We choose $\sigma, \ A, \ \lambda, \ \gamma, \ \mu$ verifying the four previous properties. Hence, we can define the product $\circ_E$ on E by

$$\begin{aligned} x \circ_E y &= x \circ_L y + \sigma(x, y).c \quad \forall x, y \in L, \\ x \circ_E c = c \circ_E x &= A(x) + \lambda(x).c \quad \forall x \in L, \\ c \circ_E c &= \gamma + \mu.c \quad . \end{aligned}$$

It is easy to check that if $\omega(x, x \circ_L x) = 0, \forall x \in E$, we have $[x, x \circ_E x]_E = 0, \ \forall x \in E$. Hence that defines a third power-associative product on E.
As a consequence and since $x *_E y = \frac{1}{2}[x, y] + x \circ_E y$ and $[E, c] = 0$, the following proposition holds.

Proposition 1.47. *Let L be a Lie algebra and $E := L \oplus \mathbb{K}.c$ be a central extension of L by means of the 2-cocycle ω.*
Any third power-associative structure $*_E$ on E induce a third power-associative structure on L by

$$x *_L y = p_L(x *_E y), \ \forall x, y \in L.$$

Moreover $*_L$ *verify* $\omega(x, x *_L x) = 0,\ \forall x \in L$ *(or equivalently* $\omega(x, x \circ_L x) = 0$ *)* .
Conversely any third power-associative structure $*_L$ *on* L *such that* $\omega(x, x{*_L}x) = 0,\ \forall x \in L$ *can be extended on* E *by*

$$\begin{aligned} x *_E y &= x *_L y + \sigma(x, y).c + \frac{1}{2}\omega(x, y).c \quad ,\\ x *_E c &= c *_E x = A(x) + \lambda(x).c \quad ,\\ c *_E c &= \gamma + \mu.c \quad , \end{aligned}$$

with σ *is a symmetric bilinear form of* L*;* A *an endomorphism of* L *such that* A *is* ω*-symmetric and* $ad(x) \circ A = ad(A(x))\ \forall x \in L$*;* λ *a linear form of* L*;* $\gamma \in Z(L) \cap \ker \omega$ *and* $\mu \in \mathbb{K}$.
Note that $*_E$ *is non-unique.*

We study now the particular case of Witt type algebra given in the theorem 1.39. That is an algebra $V(f)$ over a field $\mathbb{K}$ of characteristic zero with Γ a free abelian group and $f : \Gamma \to \mathbb{K}$ a injective function. By the theorem 1.39, $V(f)$ has a unique universal central extension given by the 2-cocycle

$$\Phi(e_\alpha, e_\beta) = \delta_{\alpha,-\beta}(f(\alpha)^3 - f(\alpha)).$$

Let W be such a Witt type algebra and $\mathcal{V}$ its central extension.

As f is injective we have $\Gamma_0 = \{0\}$, and then any third power associative structure on W is given by

$$x *_W y := \frac{1}{2}[x, y]_W + u(x)y + u(y)x,$$

where u is a linear form of W.

Proposition 1.48. *Third power-associative products on* $\mathcal{V}$ *compatible with the Lie algebra structure are:*

$$\begin{aligned} x *_{\mathcal{V}} y &= \frac{1}{2}[x, y]_W + u(x)y + u(y)x + \sigma(x, y).c \quad \forall x, y \in W,\\ x *_{\mathcal{V}} c = c \circ_{\mathcal{V}} x &= k.x + \lambda(x).c \quad \forall x \in W,\\ c *_{\mathcal{V}} c &= \mu.c, \end{aligned}$$

where $k, \mu \in \mathbb{K}$*,* σ *is a symmetric bilinear form of* W *and* λ, u *are linear forms of* W.

Proof. For all $x \in W$, $x \circ_W x = 2u(x)x$ and then, as Φ is skew-symmetric, we have $\Phi(x, x \circ_W x) = 0$, $\forall x \in W$. By the previous proposition that means that there is a third-power associative product on $\mathcal{V}$. Moreover in W we have $\ker \mathrm{ad}(x) \cap \ker \omega(x, .) = \mathbb{C} \cdot x$ and $Z(W) = \{0\}$. Hence the linear form A is necessary an homothecy (we note its ratio by k) and $\gamma = 0$. That gives the result, according to the previous proposition. □

Proposition 1.49. *The only flexible product on $\mathcal{V}$ which is compatible with the Lie algebra structure is the trivial one:*

$$x *_{\mathcal{V}} y = \frac{1}{2}[x, y]_W \quad \forall x, y \in W$$
$$x *_{\mathcal{V}} c = c \circ_{\mathcal{V}} x = 0 \quad \forall x \in W$$
$$c *_{\mathcal{V}} c = 0$$

Proof. Any compatible flexible product $*_{\mathcal{V}}$ on $\mathcal{V}$ is third power associative and then, is as describe in the previous proposition. The aim is to show that the flexibility implies that $\sigma = 0$, $\lambda = 0$, $u = \lambda = 0$ and $k = \mu = 0$.
Remind that the flexibility of $*_{\mathcal{V}}$ holds if and only if:

$$[u, v \circ_{\mathcal{V}} w]_{\mathcal{V}} = v \circ_{\mathcal{V}} [u, w]_{\mathcal{V}} + w \circ_{\mathcal{V}} [u, v]_{\mathcal{V}}, \ \forall u, v, w \in \mathcal{V}. \tag{1.40}$$

We set in 1.40 $u = x \in W$, $v = y \in W$ and $w = c$. Hence we get

$$[x, y \circ_{\mathcal{V}} c]_{\mathcal{V}} = c \circ_{\mathcal{V}} [x, y]_{\mathcal{V}}.$$

This is equivalent to

$$(k - \mu)\Phi(x, y) = \lambda([x, y]_W).$$

If $k - \mu \neq 0$, Φ should be a 2-coboundary but that is not true. So $k = \mu$ and $\lambda([x, y]_W) = 0$, $\forall x, y \in W$. Since $[W, W] = W$ we get that $\lambda = 0$.
Now we set in 1.40 $u = w = x$ and $v = y$. We get

$$[x, y \circ_{\mathcal{V}} x]_{\mathcal{V}} = [x, y]_{\mathcal{V}} \circ_{\mathcal{V}},$$

which is equivalent to

$$u(x)\Phi(x, y)c = \lambda(x)\Phi(x, y)c + \sigma([x, y]_W, x)c + k\Phi(x, y)x + u([x, y]_W)x, \ \forall x, y \in W.$$

So we have

$$u([x, y]_W) = -k\Phi(x, y) \ \forall x, y \in W, \tag{1.41}$$

and

$$u(x)\Phi(x,y) = \sigma([x,y]_W, x)\ \forall x, y \in W. \tag{1.42}$$

We suppose that $k \neq 0$. Hence according to 1.41, Φ should be a 2-coboundary. So $k = 0$ and as $k = \mu$, we have $k = \mu = 0$. Moreover if $k = 0$ we have $u([x,y])_W = 0$. As before that means that $u = 0$.
As $u = 0$, identity 1.42 becomes:

$$\sigma([x,y]_W, x) = 0\ \forall x, y \in W.$$

By polarization of this identity we get that σ is an invariant symmetric bilinear form on W. This means

$$\sigma([x,y],z) = \sigma(x,[y,z])\ \forall x,y,z \in W.$$

We can show that there is not non-zero invariant symmetric bilinear form on W. In fact, in the last identity we set $x = e_\alpha, y = e_0$ and $z = e_\beta$ with $\alpha, \beta \in \Gamma$. We get

$$\sigma([e_\alpha, e_0], e_\beta) = \sigma(e_\alpha, [e_0, e_\beta])\ \forall \alpha, \beta \in \Gamma.$$

This is equivalent to

$$f(\alpha+\beta)\sigma(e_\alpha, e_\beta) = 0\ \forall \alpha, \beta \in \Gamma.$$

Therefore for all $\alpha + \beta \neq 0$ (f is injective) we get

$$\sigma(e_\alpha, e_\beta) = 0.$$

By using again the invariance identity we get

$$\sigma([e_\alpha, e_{-\alpha}], e_0) = \sigma(e_\alpha, [e_{-\alpha}, e_0]),\ \ \forall \alpha \in \Gamma.$$

That is equivalent to:

$$-2f(\alpha)\sigma(e_0, e_0) = f(\alpha)\sigma(e_\alpha, e_{-\alpha}).$$

So for all $\alpha \neq 0$, $\sigma(e_\alpha, e_{-\alpha}) = -2\sigma(e_0, e_0)$. But for $\alpha + \beta \neq 0$ and $\beta \neq 0$ the invariance identity still gives $\sigma([e_\alpha, e_\beta], e_{-(\alpha+\beta)}) = \sigma(e_\alpha, [e_\beta, e_{-(\alpha+\beta)}])$. So

$$f(\beta-\alpha)\sigma(e_{\alpha+\beta}, e_{-(\alpha+\beta)}) = f(-\alpha-2\beta)\sigma(e_\alpha, e_{-\alpha}),$$

and since $\sigma(e_\alpha, e_{-\alpha}) = 2\sigma(e_0, e_0),\ \ \forall \alpha \in \Gamma$, we get

$$\sigma(e_0, e_0) = 0.$$

We conclude that $\sigma = 0$. □

Corollary 1.50. *Let $\mathfrak{W}$ be the Witt algebra and $\mathfrak{V}$ the Virasoro algebra. Third power-associative compatible products on the Virasoro algebra are:*

$$\begin{aligned} x *_{\mathfrak{V}} y &= \frac{1}{2}[x, y]_{\mathfrak{W}} + u(x)y + u(y)x + \sigma(x, y).c \quad \forall x, y \in \mathfrak{W} \\ x *_{\mathfrak{V}} c = c \circ_{\mathcal{V}} x &= k.x + \lambda(x).c \quad \forall x \in \mathfrak{W} \\ c *_{\mathfrak{V}} c &= \mu.c \end{aligned}$$

where $k, \mu \in \mathbb{K}$, σ is a symmetric bilinear form of $\mathfrak{W}$ and λ, u are linear forms of $\mathfrak{W}$. Moreover the only flexible compatible product on $\mathfrak{V}$ is the trivial one.

Remark. This corollary is a result of Myung [23]. Our proof for the flexible structure is quite different than the proof of Myung and gives an additional information: $\mu = 0$.

1.5 Left-symmetric structures induced by symplectic structures on Witt type algebras

This chapter is devoted to the study of some left-symmetric structures on certain Witt type algebras and their connection with symplectic structures.

An algebra $(A, *)$ is said left-symmetric if

$$(x, y, z) = (y, x, z), \quad \forall x, y, z \in A.$$

If $(A, *)$ is a left-symmetric algebra then A^- is a Lie algebra. For a given Lie algebra $(L, [\ ,\ ])$, a left-symmetric structure $*$ is said compatible with the Lie algebra structure if $[x, y] = x * y - y * x$.

Let $(L, [\ ,\ ], \omega)$ be a finite-dimensional symplectic Lie-algebra. Since ω is a non-degenerate bilinear form, there exists $\forall x, y \in L$ a unique element $x * y$ such that

$$\omega(x * y, z) = -\omega(y, [x, z]) \qquad \forall z \in \mathcal{G}.$$

In this way we define a left-symmetric product $*$ which is compatible with L.

J. Helmstetter has showed in [13] that if $(L, [\ ,\ ])$ is a finite-dimensional Lie algebra such that there exists a compatible left-symmetric structure then $[L, L] \neq L$. Consequently if L is a non-zero semi-simple Lie algebra then L is not symplectic.

Remark. We can prove that a semi-simple finite-dimensional Lie algebra L is not symplectic without using the Helmstatter's result. Indeed, if B is the Killing form of L and ω a 2-cocycle of L then there exists a derivation δ of L such that

$$\omega(x, y) = B(\delta(x), y), \quad \forall y \in L.$$

Since L is a finite-dimensional semi-simple Lie algebra, δ is an inner derivation. So there exists $t \in L$ such that $\delta = ad(t)$. If $t = 0$ then $\omega = 0$ and if $t \neq 0$, then

$$\omega(t, y) = B(ad(t)t, y) = 0, \ \forall y \in \mathcal{G}$$

This proves that ω is degenerate. We conclude that L is not symplectic.

The following examples proves that there exists an infinite-dimensional symplectic semi-simple Lie algebras. In fact if $k \in \mathbb{Z}$, then the following bilinear form defines a symplectic structure on the Witt algebra $\mathcal{W} := \bigoplus_{k\in\mathbb{Z}} < x_k >$:

$$\omega(x, y) = x^*_{2k+1}([x, y]), \ \forall x, y \in W, \ k \in \mathbb{Z}.$$

Definition 1.51. *Let $(L, [\ ,\ ], \omega)$ be a (infinite-dimensional) symplectic Lie algebra. If for each $x, y \in L$ there exists an element $x * y$ such that:*

$$\omega\,(x * y, z) = -\omega(y, [x, z]), \tag{1.43}$$

we say that ω is left-symmetric admissible. Note that $x * y$ is then unique since ω is non-degenerate.
If ω is left-symmetric admissible, then $*$ is a compatible left-symmetric product of L.

Let $\mathbb{K}$ be a field of characteristic zero. We consider a free abelian group Γ and an injective function $f : \Gamma \to \mathbb{K}$. In this case f belongs to $\mathcal{A}$ and the second cohomology group $H^2(V(f), \mathbb{C})$ is a one-dimensional vector space generated by the 2-cocycle

$$\Phi(e_\alpha, e_\beta) = \delta_{\alpha,-\beta}(f(\alpha)^3 - f(\alpha)).$$

Consequently any 2-cocycle ω of W is of the following form:

$$\omega = \lambda\Phi + dh, \quad \lambda \in \mathbb{K},\ h \in \mathcal{L}(W, \mathbb{K}).$$

Let $W = V(f) = \bigoplus_{\gamma\in\Gamma} < e_\gamma >$ be the Witt type algebra defined by f.

Definition 1.52. *A bilinear form ω of W is said homogeneous of degree $\rho \in \Gamma$ (or ρ-homogeneous) if*

$$\omega(e_\alpha, e_\beta) = 0 \quad \forall\alpha, \beta \text{ such that } \alpha + \beta + \rho \neq 0.$$

Proposition 1.53. *A homogeneous symplectic form on W is left-symmetric admissible and the induced left-symmetric structure is graded.*

Proof. Let ω be a ρ-homogeneous symplectic form on W. Let e_α, e_β be two homogeneous elements of W. We set $\gamma_0 := -\alpha - \beta - \rho$. We define the linear map h by

$$h : W \to \mathbb{K};\ e_\gamma \mapsto -\omega(e_\beta, [e_\alpha, e_\gamma]).$$

Since ω is ρ-homogeneous, we get :

$$h(e_\gamma) = 0 \quad \forall\gamma \in \Gamma\backslash\{\gamma_0\}.$$

Let g be the linear map defined by:

$$g : W \to \mathbb{K};\ e_\gamma \mapsto \omega(e_{\alpha+\beta}, e_\gamma).$$

Likewise, since ω is ρ-homogeneous and non-degenerate, we get

$$g(e_\gamma) = 0 \quad \forall\gamma \in \Gamma\backslash\{\gamma_0\},$$

and

$$g(e_{\gamma_0}) \neq 0.$$

Hence $\ker g \subset \ker h$ and there exists $\lambda_{\alpha\beta} \in \mathbb{K}$ such that $h = \lambda_{\alpha\beta} g$. Therefore $\omega(\lambda_{\alpha\beta} e_{\alpha+\beta}, e_\gamma) = -\omega(e_\beta, [e_\alpha, e_\gamma])$. Hence $e_\alpha * e_\beta := \lambda_{\alpha\beta} e_{\alpha+\beta}$. Then, by linearity, for all $x, y \in W$ there exists $x * y \in W$ such that $\omega(x * y, z) = -\omega(y, [x, z])$. We conclude that ω is left-symmetric admissible. Moreover it is clear that $*$ is graded. $\square$

Remark: Note that this proposition is even true for any Γ-graded Lie algebra with one-dimensional homogeneous spaces.

Proposition 1.54. *Let ω be a bilinear form on W. Then ω is a ρ-homogeneous symplectic form if and only if $\rho \notin 2\Gamma$ and $\omega = \mu d e^*_{-\rho}$ with $\mu \in \mathbb{K}^*$. Here $e^*_{-\rho}$ is the linear form defined by $e^*_{-\rho}(e_\gamma) = 0$ if $\gamma \neq -\rho$ and $e^*_{-\rho}(e_{-\rho}) = 1$.*

Proof. We suppose that ω is a ρ-homogeneous symplectic form. Hence there exists a linear form h on W such that

$$\omega = \lambda\Phi + dh.$$

As $\omega(e_0, e_0) = 0$ and ω is non-degenerate, then $\rho \neq 0$. Let $\alpha, \beta \in \Gamma$ such that $\alpha + \beta = 0$. Then

$$\begin{aligned} 0 = \omega(e_\alpha, e_\beta) &= \lambda(f(\alpha)^3 - f(\alpha)) + dh(e_\alpha, e_\beta) \\ &= \lambda(f(\alpha)^3 - f(\alpha)) + (f(\beta) - f(\alpha))h(e_0). \end{aligned}$$

Since f is injective and additive we get

$$h(e_0) = \frac{\lambda(f(\alpha)^3 - f(\alpha))}{2f(\alpha)} = \frac{\lambda}{2}(f(\alpha)^2 - 1), \quad \forall \alpha \neq 0.$$

Since Γ is infinite, there are two elements α_0 and β_0 such that $\beta_0 \neq \pm\alpha_0$. So $f(\alpha_0)^2 \neq f(\beta_0)^2$. Hence $\lambda = 0$, $h(e_0) = 0$ and $\omega = dh$.
Since $\omega = dh$ and ω is ρ-homogeneous we have

$$0 = \omega(e_0, e_\beta) = f(\beta)h(e_\beta), \ \forall \beta \neq -\rho.$$

Therefore $\{e_\alpha, \ \alpha \in \Gamma \backslash \{-\rho\}\} \subset \ker h$. So there exists $\mu \in \mathbb{K}$ such that $h = \mu e^*_{-\rho}$. Moreover $\mu \neq 0$ because $\omega \neq 0$. Finally if $\rho = 2\alpha$ where $\alpha \in \Gamma$, then $\omega(e_{-\alpha}, e_\beta) = h([e_{-\alpha}, e_\beta]) = 0$, $\forall \beta \in \Gamma$. Since ω is non-degenerate, we conclude that $\rho \notin 2\Gamma$.
Conversely the 2-cocycle $\mu d e^*_{-\rho}$ is clearly ρ-homogeneous and if $\rho \notin 2\Gamma$ then $\omega(e_\alpha, e_{-\rho-\alpha}) = \mu f(-\rho - 2\alpha) \neq 0$, $\forall \alpha \in \Gamma$. Hence ω is non-degenerate because $\omega(e_\alpha, e_\beta) \neq 0$ for all $\alpha, \beta \in \Gamma$ such that $\alpha + \beta + \rho = 0$. So ω is a ρ-homogeneous symplectic form. $\square$

Theorem 1.55. *Let ω be a symplectic form on W. If ω is left-symmetric admissible and if the induced left-symmetric structure is graded, then ω is homogeneous.*

Proof. Let $\alpha, \beta \in \Gamma$, then $e_\alpha * e_\beta$ is the unique element of W such that

$$\omega\left(e_\alpha * e_\beta, e_\gamma\right) = -\omega(e_\beta, [e_\alpha, e_\gamma]), \ \forall \gamma \in \Gamma. \tag{1.44}$$

Since $*$ is graded, there exists $\lambda_{\alpha\beta} \in \mathbb{K}$ such that $e_\alpha * e_\beta := \lambda_{\alpha\beta} e_{\alpha+\beta}$.
By the identity (1.44) we have:

$$\lambda_{\alpha\beta}\omega(e_{\alpha+\beta}, e_\gamma) = (f(\alpha) - f(\gamma))\omega(e_\beta, e_{\alpha+\gamma}). \tag{1.45}$$

In (1.45) we set $\alpha = 0$. Hence we get:

$$\lambda_{0\beta}\omega(e_\beta, e_\gamma) = -f(\gamma)\omega(e_\beta, e_\gamma),$$

which is equivalent to

$$(\lambda_{0\beta} + f(\gamma))\omega(e_\beta, e_\gamma) = 0. \tag{1.46}$$

For each $\beta \in \Gamma$, since ω is non-degenerate, there exists necessarily an element $\gamma_\beta \in \Gamma$ such that $\omega(e_\beta, e_{\gamma_\beta}) \neq 0$. By (1.46) we have

$$\lambda_{0\beta} = -f(\gamma_\beta).$$

Since $\lambda_{0\beta}$ is unique and f injective, γ_β is unique too. Thus $\omega(e_\beta, e_\gamma) \neq 0$ if and only if $\gamma = \gamma_\beta$. We set $\rho := -\gamma_0$.
In order to show that ω is ρ-homogeneous, we to prove that $\beta + \gamma_\beta$ does not depend on β and that we have $\beta + \gamma_\beta = \gamma_0 = -\rho, \ \forall \beta \in \Gamma$.
If we use again equation (1.45) and set $\gamma = 0$, we get

$$\lambda_{\alpha\beta}\omega(e_0, e_{\alpha+\beta}) = f(\alpha)\omega(e_\alpha, e_\beta). \tag{1.47}$$

We have $\forall \alpha \neq 0, \ f(\alpha)\omega(e_\alpha, e_\beta) \neq 0$ if and only if $\beta = \gamma_\alpha$ Hence $\forall \alpha \neq 0, \ \lambda_{\alpha\gamma_\alpha}\omega(e_0, e_{\alpha+\gamma_\alpha}) \neq 0$. In addition $\omega(e_0, e_\gamma) \neq 0$ if and only if $\gamma = \gamma_0 = -\rho$. Hence we have $\alpha + \gamma_\alpha = \gamma_0 = -\rho, \ \forall \alpha \neq 0$. Finally we proved that $\omega(e_\alpha, e_\beta) = 0$ for all $\alpha, \beta \in \Gamma$ such that $\alpha+\beta+\rho \neq 0$. That exactly means that ω is ρ-homogeneous. □

Corollary 1.56. *The left-admissible symplectic forms on W such that their induced left-symmetric structure is graded are:*

$$\omega_\rho := \mu d e^*_{-\rho},$$

with $\rho \notin 2\Gamma$ and $\mu \in \mathbb{K}^*$.

Moreover the left-symmetric product induced by ω_ρ is given by:

$$e_\alpha * e_\beta := \frac{f(\rho + 2\alpha + \beta) f(\rho + 2\beta)}{f(\rho + 2(\alpha + \beta))} e_{\alpha+\beta}, \ \forall \alpha, \beta \in \Gamma.$$

Proof. The first assertion results from Proposition 1.54 and Theorem 1.55 . We just have to give the induced left-symmetric product. By equation (1.45):

$$\lambda_{\alpha\beta}\omega(e_{\alpha+\beta}, e_\gamma) = f(\alpha - \gamma)\omega(e_\beta, e_{\alpha+\gamma}), \forall \alpha, \beta, \gamma \in \Gamma.$$

Let $\alpha, \beta, \gamma \in \Gamma$. Since $\omega = \omega_\rho := \mu d e^*_{-\rho}$ with $\rho \notin 2\Gamma$, we get:

$$\lambda_{\alpha\beta}\mu d e^*_{-\rho}(e_{\alpha+\beta}, e_\gamma) = f(\alpha - \gamma)\mu d e^*_{-\rho}(e_\beta, e_{\alpha+\gamma}),$$

which is equivalent to

$$\lambda_{\alpha\beta} f(\gamma - \alpha - \beta) e^*_{-\rho}(e_{\alpha+\beta+\gamma}) = f(\alpha - \gamma) f(\alpha + \gamma - \beta) e^*_{-\rho}(e_{\alpha+\beta+\gamma}).$$

For $\alpha + \beta + \gamma + \rho = 0$ we have

$$\lambda_{\alpha\beta} f(-\rho - 2(\alpha + \beta)) = f(2\alpha + \beta + \rho) f(-2\beta - \rho).$$

Remark that $f(-\rho - 2(\alpha + \beta)) \neq 0$ since $\rho \notin 2\Gamma$. Hence

$$\lambda_{\alpha\beta} = \frac{f(2\alpha + \beta + \rho) f(2\beta + \rho)}{f(\rho + 2(\alpha + \beta))}, \ \forall \alpha, \beta \in \Gamma.$$

$\square$

Remark. This result holds for a class of Witt type algebras which contains the Witt algebra. Graded left-symmetric structures on the Witt algebra are classified (see [3]). There are two classes of such structures denoted (in [3]) by $V_{\alpha,\epsilon}$, $\alpha, \epsilon \in \mathbb{C}$ statisfying $\epsilon = 0$ or $\epsilon^{-1} \notin \mathbb{Z}$ and $V^{\beta,k}$, $\beta \in \mathbb{C}$ and $k \in \mathbb{Z}$ satisfisfying $\beta \neq k$. Our graded left-symmetric structures induced by a symplectic form belong to the first class $V_{\alpha,\epsilon}$ with $\alpha = 1$ and $\epsilon = \frac{2}{\rho}$.

Chapter 2

Automorphic algebras.

2.1 Generalities on Riemann surfaces

In this part we remind definitions and general results about Riemann surfaces. We give more precisions for the situation of the Riemann sphere and for the tori.
Details of this part (and in particular the proofs) can be found in the book of Otto Forster [10] or in the book of Rick Miranda [21].

2.1.1 Definition and examples

Definition 2.1. *Let X be a Hausdorff topological space. A complex chart on X is an homeomorphism $\varphi : U \to V$ of a open subset $U \subset X$ to an open subset $V \subset \mathbb{C}$. Two complex charts $\varphi_1 : U_A \to V_1$ and $\varphi_2 : U_2 \to V_2$ are said to be compatible if the map*

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is biholomorphic.

A complex chart $\varphi : U \to V$ on X is said centered at the point p if $\varphi(p) = 0$.

A system $\mathcal{A} := \{\varphi_i : U_i \to V_i, i \in I\}$ of compatible charts such that $X = \bigcup_{i \in I} U_i$ is called an atlas. We define on the set of atlases an equivalence relation: two atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent if the set of charts $\mathcal{A}_1 \cup \mathcal{A}_2$ are again an atlas. This means that every chart of $\mathcal{A}_1$ is compatible with every chart of $\mathcal{A}_2$.

Definition 2.2. *A complex structure on a topological space X is an equivalence class Σ of atlases.*
A Riemann surface is a pair (X, Σ) of a connected Hausdorff topological space and a complex structure Σ.

Usually, to define a Riemann surface, we just give an altas $\mathcal{A}$ on X. Then the corresponding Riemann surface is the pair $(X, \bar{\mathcal{A}})$ where $\bar{\mathcal{A}}$ is the equivalence class of the atlas $\mathcal{A}$.

Let us give some examples of Riemann surfaces:

1. The complex plane $\mathbb{C}$ with the complex structure given by the one chart atlas $\{id_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}\}$.
2. The Riemann sphere $\mathbb{C}_\infty$. Let $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. The topology on $\mathbb{C}_\infty$ is given as follows: the open sets in $\mathbb{C}_\infty$ are the usual open sets of $\mathbb{C}$ together with the sets of the form $V \cup \{\infty\}$ where V is the complement of a compact subset $K \subset \mathbb{C}$. That makes $\mathbb{C}_\infty$ into an Hausdorff connected topological space. The complex structure is defined by the altlas $\{\varphi_i : U_i \to V_i, i = 1, 2\}$ with
$$U_1 = \mathbb{C} \qquad U_2 = \mathbb{C}^* \cup \{\infty\}$$
$$\varphi_1 = id_{\mathbb{C}} \qquad \varphi_2(z) := \begin{cases} 1/z & \text{for } z \in \mathbb{C}^* \\ 0 & \text{for } z = \infty. \end{cases}$$
Note that the Riemann sphere is a compact Riemann surface.
3. The complex tori: Let ω_1 and ω_2 be two complex numbers which are linearly independent over $\mathbb{R}$. We define the lattice L by
$$L := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{m_1\omega_1 + m_2\omega_2 \mid m_1, m_2 \in \mathbb{Z}\}.$$
The lattice L is a subgroup of $\mathbb{C}$. Let $X = \mathbb{C}/L$ be the quotient group. On X we put the usual quotient topology which makes the projection map π continuous. With this topology, $\mathbb{C}/L$ is an Hausdorff topological space and since $\mathbb{C}$ is connected, $\mathbb{C}/L$ is also connected. Moreover, this is a compact topological space since π is continuous. In fact $\mathbb{C}/L = \pi(P)$ where P is the compact parallelogram
$$P := \{a\omega_1 + b\omega_2 | a, b \in [0, 1]\}.$$

The complex structure of $\mathbb{C}/L$ is given as follow:
A chart is defined using the map π. Let $V \subset \mathbb{C}$ be an open subset such that no two points in V are equivalent under Γ. Then $U := \pi(V)$ is an open set and $\pi|V$ is a homeomorphism from V to U. Take the inverse $\varphi : U \to V$. This is a complex chart on $\mathbb{C}/L$. We have to check that the set $\mathcal{A}$ of charts obtained in this way is an atlas. It is clear that $\mathbb{C}/L$ is covered by the charts of $\mathcal{A}$. Then, we just have to check the compatibility of the charts. Let $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ be two charts of $\mathcal{A}$. Then consider the map

$$\psi := \varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2).$$

Let $z \in \varphi_1(U_1 \cap U_2)$. We have $\pi(\psi(z)) = \pi(z)$ and thus $\psi(z) - z \in \Gamma$. Since Γ is a discrete set, this proves that $\psi(z) - z$ is constant on every connected component of $\varphi_1(U_1 \cap U_2)$. Thus ψ is holomorphic and the same proof gives that ψ^{-1} is also holomorphic. That proves the compatibility of the charts of $\mathcal{A}$.

The complex structure on $\mathbb{C}/L$ is defined by the atlas $\mathcal{A}$.

Remark: For every compact Riemann surface X there exists a unique topological type given by the genus $g = g(X) \in \mathbb{N}$. Equivalently, $g(X)$ is the dimension of the space of global holomorphic differentials on X. In our examples, $g(\mathbb{C}_\infty) = 0$ and $g(\mathbb{C}/L) = 1$.

2.1.2 Holomorphic maps

Definition 2.3. *Suppose X and Y are two Riemann surfaces. A continuous map $f : X \to Y$ is called holomorphic if for every charts $\varphi_1 : U_1 \to V_1$ on X and $\varphi_2 : U_2 \to V_2$ on Y with $f(U_1) \subset U_2$, the map*

$$\varphi_2 \circ f \circ \varphi_1^{-1} : V_1 \to V_2$$

is holomorphic (in the sense of the theory of complex functions).
The map f is called an isomorphism if it is bijective and if the map f^{-1} is also holomorphic.
An isomorphism from X to X is called an automorphism. The set of all automorphisms of X is denoted by $Aut(X)$.
If there exists an isomorphism between two Riemann surfaces X and Y, then they are called

isomorphic.

Remark: Due to the local behaviour of holomorphic map (see below), if $F : X \to Y$ is a holomorphic and bijective map then F^{-1} is automatically holomorphic.
Note that $Aut(X)$ is a group with the usual composition law.

Theorem 2.4 (Local behaviour of holomorphic maps)**.** *Let $F : X \to Y$ be a non-constant holomorphic map between two Riemann surfaces X and Y. Fix $p \in X$. There is a unique integer $m \geq 1$ satisfying: for every chart $\Phi' : U' \to V'$ on Y centered at $F(p)$, there exists a chart $\Phi : U \to V$ on X centered at p such that $\Phi' \circ F \circ \Phi^{-1}(z) = z^m$.*

Corollary 2.5. *Let X and Y be Riemann surfaces and $f : X \to Y$ be a non-constant holomorphic map. Then f is open.*

Definition 2.6. *The multiplicity of a holomorphic map F at p is the unique integer m given in the previous theorem. The multiplicity of F at p is denoted by $mult_p(F)$.*

Definition 2.7. *Let $F : X \to Y$ be a nonconstant holomorphic map. A point $p \in X$ is a ramification point for F if $mult_p(F) \geq 2$. A point $y \in Y$ is a branch point if it is the image of a ramification point for F. Note that the set of ramification points as well as the set of branch points are discrete.*

Proposition 2.8. *Let $F : X \to Y$ be a non constant holomorphic map between compact Riemann surfaces. For each $y \in Y$ we define $d_y(F)$ as*

$$d_y(F) := \sum_{p \in F^{-1}(y)} mult_p(F).$$

Then $d_y(F)$ is (a finite) constant, independent of y.

Definition 2.9. *Let $F : X \to Y$ be a non constant holomorphic map between compact Riemann surfaces. The degree of F is the integer $d_y(F)$ for any $y \in Y$. It is denoted by $deg(F)$.*

Proposition 2.10. *A holomorphic map between compact Riemann surface is an isomorphism if and only if it has degree one.*

Theorem 2.11 (Hurwitz's formula)**.** *Let $F : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces. Then*

$$2g(X) - 2 = deg(F)(2g(Y) - 2) + \sum_{p \in X} [mult_p(F) - 1]$$

where $g(X)$ and $g(Y)$ are the genus of X and Y respectively.

Theorem 2.12 (Identity theorem)**.** *Suppose X and Y are Riemann surfaces and $f_1, f_2 : X \to Y$ are two holomorphic mappings which coincide on a set $A \subset X$ having a limit point $a \in X$. Then f_1 and f_2 are identically equal.*

2.1.3 Holomorphic functions

Definition 2.13. *Let X be a Riemann surface and Y an open subset of X. A function $f : Y \to \mathbb{C}$ is called holomorphic if for every chart $\varphi : U \to V$ on X, the function*

$$f \circ \varphi^{-1} : \varphi(U \cap Y) \to \mathbb{C}$$

is holomorphic.
This definition coincides with the definition of holomorphic maps between the Riemann surfaces X and $\mathbb{C}$.
The set of holomorphic functions of Y is denoted by $\mathcal{O}(Y)$

Theorem 2.14 (Riemann's Removable Singularities Theorem)**.** *Let U be an open subset of a Riemann surface and let $a \in U$. Suppose that a function $f \in \mathcal{O}(U \setminus \{a\})$ is bounded in some neighborhood of a. Then f can be extended uniquely to a function $\tilde{f} \in \mathcal{O}(U)$.*

Theorem 2.15 (Maximum principle)**.** *Suppose X is a Riemann surface and f a nonconstant holomorphic function of X. Then $|f|$ does not attain its maximum.*

Corollary 2.16. *Every holomolorphic function on a compact Riemann surface is constant.*

2.1.4 Meromorphic functions

Let X be a Riemann surface, let p be a point of X. By punctured neighborhood of p we mean a set of the form $U - \{p\}$ where U is a neighborhood of p. The concept of type of singularities for usual complex functions can be extended to functions on a Riemann surface.

Definition 2.17. *Let f be a holomorphic function in a punctured neighborhood of a $p \in X$.*

(a) We say f has a removable singularity at p if and only if there exists a chart $\phi : U \to V$ with $p \in U$, such that the composition $f \circ \phi^{-1}$ has a removable singularity at $\phi(p)$.

(b) We say f has a pole at p if and only if there exists a chart $\phi : U \to V$ with $p \in U$ such that the composition $f \circ \phi^{-1}$ has a pole at $\phi(p)$.

(c) We say f has an essential singularity at p if and only if there exists a chart $\phi : U \to V$ with $p \in U$ such that the composition $f \circ \phi^{-1}$ has an essential singularity at $\phi(p)$.

Note that this definition does not depend of the choice of the chart U.

If f is a holomorphic function on a punctured neighborhood of a point $p \in X$, the behaviour of $f(x)$ for x near p determine which kind of singularity f has at p.

(a) If $|f(x)|$ is bounded in a neighborhood of p, then f has a removable singularity at p. In this case, $\lim_{x \to p} f(x)$ exists and if we define $f(p)$ to be this limit, f is holomorphic at p.

(b) If $|f(x)|$ approaches ∞ as x approaches p, then f has a pole at p.

(c) If $|f(x)|$ has no limit as x approaches p, then f has an essential singularity at p.

Definition 2.18. *A function f on X is meromorphic at a point $p \in X$ if it is either holomorphic, has a removable singularity, or has a pole, at p. We say f is meromorphic on an open set W if it is meromorphic at every point of W.*

We can now give an equivalent definition for meromorphic functions

Definition 2.19. *Let X be a Riemann surface and Y an open subset of X. A meromorphic function $f : Y \to \mathbb{C}$ is a holomorphic function $f : Y' \to \mathbb{C}$ where Y' is an open subset of Y with:*

(i) $Y \setminus Y'$ is a set of isolated points.

(ii) For every point $p \in Y \setminus Y'$ we have

$$\lim_{x \to p} |f(x)| = \infty.$$

The points of $Y \setminus Y'$ are called the poles of f. The set of meromorphic functions of Y is denoted by $\mathcal{M}(Y)$.

Meromorphic functions are particular cases of holomorphic maps:

Theorem 2.20. *Let X be a Riemann surface and $f \in \mathcal{M}(X)$. If we define for each pole p of f, $f(p) := \infty$, then $f : X \to \mathbb{C}_\infty$ is a holomorphic map. Conversely, consider $f : X \to \mathbb{C}_\infty$ a holomorphic map. If $f^{-1}(\infty)$ consists of isolated points, then $f : X \to \mathbb{C}$ is a meromorphic function with poles in $f^{-1}(\infty)$. If $f^{-1}(\infty)$ does not consist of isolated points then f is identically equal to ∞ by the identity theorem.*

2.1.4.1 Laurent series and order

Let f be defined and holomorphic on a punctured neighborhood of $p \in X$. Let $\phi : U \to V$ be a chart on X with $p \in U$. We have that $f \circ \phi^{-1}$ is holomorphic in a neighborhood of $z_0 := \phi(p)$. Therefore we can expand $f \circ \phi^{-1}$ in a Laurent series about z_0:

$$f(\phi^{-1}(z)) = \sum_n c_n (z - z_0)^n$$

This is called the Laurent series for f about p with respect to ϕ. The Laurent series obviously depends of the choice of the chart ϕ. We can however use Laurent series to check the nature of the singularity of f at p:

Proposition 2.21. *The function f has a removable singularity at p if and only if any one of its Laurent series has no negative terms. The function f has a pole at p if and only if any one of its Laurent series has finitely many (but not zero) negative terms. The function f has an essential singularity at p if and only if any one of its Laurent series has infinitely many negative terms.*

This characterization of the nature of the singularity is possible because the value $\{\min n \mid c_n \neq 0\}$ does not depend of the choice of the chart ϕ. Hence we can define the order of f at p.

Definition 2.22. *Let f be a meromorphic function at p. Consider its Laurent series with respect to a local chart $\phi : U \to V$: $\sum_n c_n(z-z_0)^n$. The order of f at p, denoted by $ord_p(f)$ is:*

$$ord_p(f) := \min\{n \mid c_n \neq 0\}$$

Lemma 2.23. *Let f be a meromorphic function at p. Then f is holomorphic at p if and only if $ord_p(f) \geq 0$. In this case $f(p) = 0$ if and only if $ord_p(f) > 0$. f has a pole at p if and only if $ord_p(f) < 0$. f has neither a zero nor a pole at p if and only if $ord_p(f) = 0$.*

Note that the order of a meromorphic f function at a point $p \in X$ is related with the definition of multiplicity given before. In fact, consider the meromorphic function f as a holomorphic map F from X to $\mathbb{C}_\infty$. Then

(a) If $p \in X$ is a zero of f, then $mult_p(F) = ord_p(f)$.

(b) If $p \in X$ is a pole of f, then $mult_p(F) = -ord_p(f)$.

(c) If $p \in X$ is neither a zero nor a pole of f, then $mult_p(F) = -ord_p(f - f(p))$.

From the definition of the degree of a holomorphic map we can extract the following:
Let f be a non-constant meromorphic function on a compact Riemann surface X. Let $F : X \to \mathbb{C}_\infty$ be its associated holomorphic map. Let $\{x_i\}$ be the points of X mapping to 0 and $\{y_j\}$ be the points of X mapping to ∞. The $x_i's$ are exactly the zeroes of f and the $y_j's$ are its poles. Let d be the degree of F. By definition of d, we have

$$d = \sum_i mult_{x_i}(F) = \sum_j mult_{x_j}(F).$$

We explained before that the only points of X where f has nonzero order are its zeroes and poles. Moreover we have

$$mult_{x_i}(F) = ord_{x_i}(f) \quad \text{and} \quad mult_{y_j}(F) = -ord_{y_j}(f)$$

Hence

Proposition 2.24. *Let f be a non-constant meromorphic function on a compact Riemann surface X. Then*

$$\sum_p ord_p(f) = \sum_i ord_{x_i}(f) + \sum_j ord_{y_j}(f) = \sum_i mult_{x_i}(F) - \sum_j mult_{x_j}(F) = 0.$$

Note that this proposition means that a meromorphic function on a compact Riemann surface has as many zeroes as poles.

2.1.4.2 Meromorphic functions on the Riemann sphere

Meromorphic functions on the Riemann sphere $\mathbb{C}_\infty$ can be easily described.
Let $R : \mathbb{C} \to \mathbb{C}$ be a rational complex function. The function R is a meromorphic function on $\mathbb{C}$ and can be extended to a meromorphic function on $\mathbb{C}_\infty$ by defining $R(\infty) := \lim_{z\to\infty} R(z)$. If $R(\infty) = \infty$ then ∞ is a pole of R and if $R(\infty) \in \mathbb{C}$ then R is holomorphic in ∞.
Conversely, let R be any meromorphic function on $\mathbb{C}_\infty$. Since $\mathbb{C}_\infty$ is a compact topological space, R has just a finite number of poles $\{p_1, \dots, p_n\}$. Suppose that ∞ is not a pole of R. Since R is a meromorphic function of $\mathbb{C}_\infty$, the restriction $R_{|\mathbb{C}}$ of R to $\mathbb{C}$ is a meromorphic function of $\mathbb{C}$ with poles in $\{p_1, \dots, p_n\}$. Let $h_1, \dots, h_n$ be the principal parts of $R_{|\mathbb{C}}$. Hence, we can consider the holomorphic function $g := R_{|\mathbb{C}} - (h_1 + \cdots + h_n)$. Since the h_i are rational function of $\mathbb{C}$, they can be viewed as meromorphic functions of $\mathbb{C}_\infty$ with poles in p_i. Thus the map g can be extended as a holomorphic function of $\mathbb{C}_\infty$. But $\mathbb{C}_\infty$ is a compact Riemann surface and holomorphic functions of $\mathbb{C}_\infty$ are the constants. Thus g is a constant and R is a rational function (since the h_i's are rational functions).
If R has a pole in ∞, consider the meromorphic function $\frac{1}{R}$ which does not have a pole in ∞. By the same justification we prove that $\frac{1}{R}$ is a rational function and then R is a rational function too.
We just gave the proof of the following proposition:

Proposition 2.25. *Meromorphic functions on the Riemann sphere $\mathbb{C}_\infty$ are the rational functions:*

$$\mathcal{M}(\mathbb{C}_\infty) = \mathbb{C}(z)$$

2.1.4.3 Meromorphic functions on tori

Fix a complex number τ in the upper half plane and consider the lattice $L := \mathbb{Z} + \mathbb{Z}\tau$. We will see later that we can always consider lattices of this form. We form the complex torus $X := \mathbb{C}/L$.
Let f be a meromorphic function on $\mathbb{C}$. We call f doubly periodic (with respect to L) if

$$f(z + n + m\tau) = f(z), \quad \forall n, m \in \mathbb{Z},\ z \in \mathbb{C}$$

Such a function f descends to the quotient $X = \mathbb{C}/L$ to a function $\bar{f} \in \mathcal{M}(X)$. Conversely, if $g \in \mathcal{M}(X)$ we get by defining $f(z) := g(z + L)$ a function $f \in \mathcal{M}(\mathbb{C})$ which is doubly periodic and satisfies $\bar{f} = g$. Hence the doubly periodic functions are the meromorphic functions of the torus.

There are two remarkable doubly periodic functions: the Weierstrass $\wp$-function and its derivative function $\wp'$:

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega \in L \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

$$\wp'(z) := -\sum_{\omega \in L} \frac{2}{(z-\omega)^3}.$$

The function $\wp$ has poles of order 2 in each point of the lattice L. Thus $\wp$ is a meromorphic function on X with a pole of order 2 in 0.
The function $\wp'$ has poles of order 3 in each point of the lattice L. Thus $\wp'$ is a meromorphic function on X with a pole of order 3 in 0.
Note that the function $\wp$ is even and that the function $\wp'$ is odd. Hence since $\wp'$ is doubly periodic, we have

$$\wp'(1/2) = \wp'(\tau/2) = \wp'(1/2 + \tau/2) = 0.$$

Moreover, as X is a compact Riemann surface and as $\wp'$ has a pole of order 3 in 0, the points $1/2$, $\tau/2$ and $1/2 + \tau/2$ are the only zeros of $\wp'$.

These two doubly periodic functions are algebraically related by the following relation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

with

$$g_2 = 60 \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^4}$$

$$g_3 = 140 \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^6}$$

The g_2, g_3 are called the Eisenstein series. For a fixed lattice they are constants. It is possible to show that the discriminant function $\Delta(\tau) := g_2^3(\tau) - 27g_3^2(\tau)$ is never equal to zero.

For a fixed lattice, consider the polynomial equation $4t^3 - g_2 t - g_3 = 0$ and note e_1, e_2 and e_3 these roots. As the discriminant $\Delta = g_2^3 - 27g_3^2$ is not zero, no two of these roots are equal. Moreover the following relations occur:

$$e_1 + e_2 + e_3 = 0$$

$$g_2 = -4(e_1 e_2 + e_1 e_3 + e_2 e_3) = 2(e_1^2 + e_2^2 + e_3^2)$$

$$g_3 = 4e_1 e_2 e_3.$$

Since $1/2, \tau/2$ and $1/2 + \tau/2$ are the zeros of $\wp'$ and since $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$, we get that

$$\wp(1/2) = e_1 \qquad \wp(\tau/2) = e_2 \qquad \wp(1/2 + \tau/2) = e_3.$$

Note that since e_1, e_2, e_3 are the roots of the polynomial equation $4t^3 - g_2 t - g_3 = 0$, the differential equation $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ is equivalent to

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3).$$

Using $\wp$ and $\wp'$ we can describe the set of meromorphic functions on X:

Theorem 2.26. *The field of meromorphic functions on the torus X can be given as*

$$\mathcal{M}(X) = \mathbb{C}(\wp, \wp')$$

That means that each meromorphic function on X is a rational function of $\wp$ and $\wp'$.

More details about the Weierstrass $\wp$-function can be found in the book of Martin Schlichenmaier [30] or in the the book of Farkas and Kra [8].

Another approach to describe the meromorphic functions on the torus X are via theta-functions. We define

$$\theta(z) := \sum_{n\in\mathbb{Z}} e^{\pi i[n^2\tau+2nz]}.$$

This series converges absolutely and uniformly on compact subsets of $\mathbb{C}$. Hence θ is a holomorphic function on $\mathbb{C}$

Note that $\theta(z+1) = \theta(z)$ for every $z \in \mathbb{C}$. We need to know how θ transforms under translation by τ. An easy computation shows that

$$\theta(z+\tau) = e^{-\pi i[\tau+2z]}\theta(z).$$

Thus it is clear that z_0 is a zero of θ if and only if $z_0 + L$ are zeroes of θ. In fact the only zeros of θ are the points $1/2 + \tau/2 + L$ and these zeroes are simple.

So if we consider the function

$$\theta^{(x)}(z) := \theta(z - 1/2 - \tau/2 - x)$$

we get a function with simple zeroes at the points $x + L$. Moreover we have

$$\theta^{(x)}(z+1) = \theta^{(x)}(z) \quad \text{and} \quad \theta^{(x)}(z+\tau) = -e^{-2\pi i(z-x)}\theta^{(x)}(z)$$

Now we consider the ratio

$$R(z) := \frac{\prod_{i=1}^{m} \theta^{(x_i)}(z)}{\prod_{j=1}^{n} \theta^{(y_j)}(z)}$$

The function R is a meromorphic function on $\mathbb{C}$ with n simple poles at the y_j's and m simple zeroes at the x_j's. Moreover R is obviously periodic, i.e., $R(z+1) = R(z)$ but

$$R(z+\tau) = (-1)^{m-n}e^{-2\pi i[(m-n)z+\sum_j y_j - \sum_i x_i]}R(z).$$

Therefore to obtain a doubly periodic function we need $m = n$ (which it is not a surprising in regards of Proposition 2.24) and $\sum_i x_i - \sum_j y_j \in \mathbb{Z}$.

We have proved the following:

Theorem 2.27. *Fix an integer d and choose two disjoint sets of d complex numbers $\{x_i\}$ and $\{y_j\}$ such that $\sum_i x_i - \sum_j y_j$ is an integer. Then the ratio of the translated theta functions*

$$R(z) := \frac{\prod_i \theta^{(x_i)}(z)}{\prod_j \theta^{(y_j)}(z)}$$

is a meromorphic L-periodic function on $\mathbb{C}$, and descends to a meromorphic function on $X = \mathbb{C}/L$.

We remark that the ratio R has simple zeroes at the points $x_i + L$ and simple poles at the points $y_j + L$.
Moreover, let f be a meromorphic function on X and let $\{p_i\}$ and $\{q_j\}$ be the two sets of its zeroes and poles respectively. Suppose that $\sum_i p_i = \sum_j q_j$ in the quotient space X. We can choose two sets $\{x_i\}$ and $\{y_j\}$ of complex numbers with $\bar{x}_i = p_i$ and $\bar{y}_j = q_j$ such that $\sum_i x_i = \sum_j y_j$. Thus by considering the ratio function $R(z) := \prod_i \theta^{(x_i)}(z)/\prod_j \theta^{(y_j)}(z)$, we get a meromorphic function on X with zeroes at the p_i's and poles at the q_j's. Thus the function f/R is a holomorphic function on the compact Riemann surface X. So f/R is a constant, and f is equal to R, up to a constant.
In fact we are able to prove that any meromorphic function on the torus X verifies the following condition: let $\{p_i\}$ be the set of its zeroes and $\{q_j\}$ be the set of its poles, then $\sum_i p_i = \sum_j q_j$.
In fact suppose that $\sum_i p_i \neq \sum_j q_j$ for a meromorphic function $f \in \mathcal{M}(X)$. Choose two points p_0 and q_0 such that $\sum_{i=0}^d p_i = \sum_{i=0}^d q_i$ and then form the ratio of translated theta-functions $R(z) := \prod_{i=0}^d \theta^{(x_i)}(z)/\prod_{i=0}^d \theta^{(y_j)}(z)$ as above. Consider the meromorphic function $g := R/f$ and note that g has just a zero at p_0 and a pole at q_0.
Let $G : X \to \mathbb{C}_\infty$ be the holomorphic map to the Riemann sphere which corresponds to the meromorphic function g. Since g has a single simple zero and a single simple pole, G has degree one. Hence G is an isomorphism by Proposition 2.10. But that is not possible since X has genus one and $\mathbb{C}_\infty$ genus zero.
This contradiction shows that we must have $\sum_i p_i = \sum_j q_j$.

We just proved two important facts:

Proposition 2.28. *On a complex torus, there is no meromorphic function with a single simple pole.*

Proposition 2.29. *Any meromorphic function on a complex torus is given by a ratio of translated theta-functions (up to a multiplicative constant).*

More details about theta functions can be found in the book of Farkas and Kra [8] and in the book of Rick Miranda [21]

2.2 Meromorphic functions with prescribed poles on a compact Riemann surface.

In this section we discuss the situation of meromorphic functions with prescribed poles on a compact Riemann surface.

Let X be a compact Riemann surface. Let $\Gamma := \{P_1, P_2, ..., P_k\}$ $(k \geq 2)$ be a set of distinct points. We consider the algebra $\mathcal{M}(X, \Gamma)$ of global meromorphic functions of X which are holomorphic on $X \setminus \Gamma$. What we mean is the set of meromorphic functions on X such that poles can only appear at the points in Γ.

These algebras are a particular case of a more general situation: the algebras of Krichever-Novikov type.

2.2.1 Algebras of Krichever-Novikov type.

Let us give a short definition of algebras of Krichever-Novikov type. More details can be found in [34–36].

Let $\mathcal{K}$ be the canonical line bundle of the Riemann surface X. Its associated sheaf of local sections is the sheaf of holomorphic differentials. For every $\lambda \in \mathbb{Z}$ consider the bundle $\mathcal{K}^{\otimes\lambda}$, with the usual convention: $\mathcal{K}^0 := \mathcal{O}$ is the trivial bundle, and $\mathcal{K}^{-1} := \mathcal{K}^*$ is the holomorphic tangent line bundle (its associated sheaf is the sheaf of holomorphic vector fields). Denote by $\mathcal{F}^\lambda$ the vector space of global meromorphic sections of $\mathcal{K}^\lambda$ which are holomorphic on $X \setminus \Gamma$.

Locally, sections of $\mathcal{F}^\lambda(\Gamma)$ look like

$$f(z) = \alpha(z) dz^\lambda, \qquad \text{with} \quad dz^\lambda := (dz)^{\otimes\lambda}$$

where α is a local meromorphic function without poles outside of Γ.

Hence if $\lambda = 0$ we get the above set of functions $\mathcal{M}(X, \Gamma)$ or just $\mathcal{M}(\Gamma)$ if X is fixed. Other special cases are of particular interest: $\lambda = 1$ which is the case of differentials, and $\lambda = -1$ which is the vector field case. The set $\mathcal{F}^{-1}(\Gamma)$ is usually denoted by $\mathcal{L}(\Gamma)$. By multiplying sections by functions we again obtain sections. Thus the space $\mathcal{M}(\Gamma)$ becomes an associative algebra and the spaces $\mathcal{F}^\lambda(\Gamma)$ become $\mathcal{M}(\Gamma)$-modules. The vector fields in $\mathcal{L}(\Gamma)$ operate on $\mathcal{F}^\lambda(\Gamma)$ too by:

$$\left(\alpha(z)\frac{d}{dz}\right) \cdot \left(\beta(z) dz^\lambda\right) := \left(\alpha(z)\frac{d\beta}{dz}(z) + \lambda\beta(z)\frac{d\alpha}{dz}(z)\right) dz^\lambda.$$

The space $\mathcal{L}(\Gamma)$ becomes a Lie algebra and the space $\mathcal{F}^\lambda(\Gamma)$ becomes a Lie module over $\mathcal{L}(\Gamma)$.

2.2.2 Almost-graded structure.

For infinite dimensional algebras and their representation theory a graded structure is usually of importance to obtain structure results. A typical example is given by the Witt algebra W. In our more general context the algebras will almost never be graded. But it was observed by Krichever and Novikov in the two-point case that a weaker concept, an almost-graded structure (they call it a quasi-graded structure), will be enough to develop an interesting theory of representations (highest weight representations, Verma modules, etc.).

Definition 2.30. *a) Let $\mathcal{L}$ be an algebra (associative or Lie) admitting a direct decomposition as vector space $\mathcal{L} = \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_n$. The algebra $\mathcal{L}$ is called an almost-graded algebra if* $\dim \mathcal{L}_n < \infty$ *and there are constants R and S with*

$$\mathcal{L}_n \cdot \mathcal{L}_m \subseteq \bigoplus_{h=n+m+R}^{n+m+S} \mathcal{L}_h, \quad \forall n, m \in \mathbb{Z} \tag{2.1}$$

The elements of $\mathcal{L}_n$ are called homogeneous elements of degree n.

b) Let $\mathcal{L}$ be an almost-graded (associative or Lie) algebra and $\mathcal{F}$ an $\mathcal{L}$-module with decomposition $\mathcal{F} = \bigoplus_{n\in\mathbb{Z}} \mathcal{F}_n$ as vector space . The module $\mathcal{F}$ is called an almost-graded module if $\dim \mathcal{F}_n < \infty$ *and there are constants R' and S' with*

$$\mathcal{L}_m \cdot \mathcal{F}_n \subseteq \bigoplus_{h=n+m+R'}^{n+m+S'} \mathcal{F}_h, \quad \forall n, m \in \mathbb{Z} \tag{2.2}$$

The elements of $\mathcal{F}_n$ are called homogeneous elements of degree n.

In the previous definition the homogeneous spaces $\mathcal{L}_n$ and $\mathcal{F}_n$ are finite-dimensional. Thus we can find adapted bases $\{A_{n,p} \mid n \in \mathbb{Z},\ p = 1, \ldots, k_n\}$ of $\mathcal{L}$ and $\{f_{n,p} \mid n \in \mathbb{Z},\ p = 1 \ldots, h_n\}$ of $\mathcal{F}$ such that

$$\mathcal{L}_n = \mathrm{vect}(A_{n,p} \mid p = 1, \ldots, k_n) \qquad \mathcal{F}_n = \mathrm{vect}(f_{n,p} \mid p = 1, \ldots, h_n).$$

Such bases are called almost-graded basis of $\mathcal{L}$ and $\mathcal{F}$ adapted to the decompositions $\mathcal{L} = \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_n$ and $\mathcal{F} = \bigoplus_{n\in\mathbb{Z}} \mathcal{F}_n$.

In practice for a given algebra $\mathcal{L}$ we try to find a basis $\{A_{n,p} \mid n \in \mathbb{Z},\ p = 1, \dots, k_n\}$ of $\mathcal{L}$ such that $A_{n,p} \cdot A_{m,q} \in \mathrm{vect}(A_{h,p} \mid h = n+m+R, \dots, , n+m+S,\ p = 1, \dots, k_h)$. Thus by defining

$$\mathcal{L}_n := \mathrm{vect}(A_{n,p} \mid p = 1, \dots, k_n)$$

we give an almost-graded structure of $\mathcal{L}$.

Let us return now to the case of the three spaces $\mathcal{M}(\Gamma)$, $\mathcal{L}(\Gamma)$ and $\mathcal{F}^\lambda(\Gamma)$ defined in the section 2.2.1.

Theorem 2.31. *Associated to any splitting of Γ into two non-empty subsets, $\Gamma := I \dot{\cup} O$, one can introduce for $\mathcal{M}(\Gamma)$, $\mathcal{L}(\Gamma)$ and $\mathcal{F}^\lambda(\Gamma)$ a decomposition into*

$$\mathcal{M}(\Gamma) = \bigoplus_{n \in \mathbb{Z}} \mathcal{M}_n, \quad \mathcal{L}(\Gamma) = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n, \quad \mathcal{F}(\Gamma) = \bigoplus_{n \in \mathbb{Z}} \mathcal{M}_n$$

such that $\mathcal{M}(\Gamma)$, $\mathcal{L}(\Gamma)$ and $\mathcal{F}^\lambda(\Gamma)$ are almost-graded with respect to the decomposition. In all cases, the lower shifts in the degree of the result can be made to zero.

It is very important to stress the fact that the almost-graded structure depends of the splitting of Γ into two non-empty subsets.
This theorem is a central result for the Krichever-Novikov algebras. The difficult proof can be found in the works of Martin Schlichenmaier [30, 36] and is essentially based on the Riemann-Roch Theorem. Explicit generators and explicit almost-graded basis can be found for these algebras and modules [34, 35]. We give here an idea of the way how to construct these generators for the case of the algebra $\mathcal{M}(\Gamma)$.

Suppose that $\Gamma = I \dot{\cup} O$ with $I = \{P_1, \dots, P_k\}$ and $O = \{Q\}$. An almost-graded basis of $\mathcal{M}(\Gamma)$ is given by elements $A_{n,p},\ n \in \mathbb{Z}, p = 1, \dots, k$ which are essentially fixed by

$$\mathrm{ord}_{P_i}(A_{n,p}) = (n+1) - \delta_i^p,\ \ i = 1, \dots, k$$

and complementary condition for the point Q of O. Then the almost-graded structure is given by the decomposition $\mathcal{M}(\Gamma) = \bigoplus_{n \in \mathbb{Z}} \mathcal{M}(\Gamma)_n$ with

$$\mathcal{M}(\Gamma)_n := \mathrm{vect}(A_{n,p} \mid p = 1, \dots, k).$$

For example, for the two points situation (i.e. $I = P$ and $O = Q$ with P, Q in generic position) the conditions are

$$\mathrm{ord}_P(A_n) = n \qquad \mathrm{ord}_Q(A_n) = -n - g$$

with some modification at Q needed for small $|n|$ values, e.g. we take always $A_0 = 1$.

Until now we always considered a set Γ containing at least two distinct elements. This condition was necessary to fulfil the conditions of the previous theorem: split the set Γ into two non-empty subsets.
How is the situation for a one point set Γ? Can the algebra $\mathcal{M}(X,\Gamma)$ also be equipped with an almost-graded structure?
The answer is yes but the gradation will depend on the choice of another point. Suppose that $\Gamma := \{Q\}$. Choose now a point $P \neq Q$ in X. Hence we know by Theorem 2.31 that the algebra $\mathcal{M}(X, \{P\}\dot{\cup}\{Q\})$ admits an almost-graded structure

$$\mathcal{M}(X, \{P\}\dot{\cup}\{Q\}) = \bigoplus_{n\in\mathbb{Z}} \mathcal{M}_n.$$

In fact this almost-graded structure verifies the following:

1. $\mathcal{M}_+ := \bigoplus_{n\geq 0} \mathcal{M}_n$ is an almost-graded subalgebra of $\mathcal{M}(X, \{P\}\dot{\cup}\{Q\})$.
2. $\mathcal{M}_+ = \mathcal{M}(X, \{Q\})$

$\mathcal{M}_+$ is the subalgebra of $\mathcal{M}(X, \{P\}\dot{\cup}\{Q\})$ consisting of functions also holomorphic in P. Hence $\mathcal{M}(X, \{Q\})$ admits an almost-graded structure depending of the choice of the reference point P.

2.2.3 Case of genus 0

Let X be the Riemann sphere and $\Gamma := \{P, Q\}$ be a set of two distinct points of X. In this case there is only one splitting: $\Gamma = \{P\}\dot{\cup}\{Q\}$. We can give here an explicit almost-graded basis of $\mathcal{M}(X,\Gamma)$; the set of global meromorphic function on X which are holomorphic on $X \setminus \Gamma$.
If $P, Q \neq \infty$ we define

$$A_n(z) := (z-P)^n \cdot (z-Q)^{-n} \qquad \forall n \in \mathbb{Z}.$$

If $Q = \infty$ we define

$$A_n(z) := (z-P)^n \qquad \forall n \in \mathbb{Z}.$$

If $P = \infty$ we define

$$A_n(z) := (z-Q)^{-n} \qquad \forall n \in \mathbb{Z}$$

In every case, the set $\{A_n,\ n \in \mathbb{Z}\}$ is a basis of $\mathcal{M}(X,\Gamma)$. Hence

$$\mathcal{M}(\Gamma) = \bigoplus_{n\in\mathbb{Z}} \mathcal{M}_n$$

where $\mathcal{M}_n := \mathrm{vect}(A_n)$. Moreover

$$A_n \cdot A_m = A_{n+m}.$$

Hence this is even a graded basis of $\mathcal{M}(X,\Gamma)$.

In this example, clearly the subalgebra $\mathcal{M}_+$ is exactly the algebra $\mathcal{M}(X,\{Q\})$ and it admits a graded structure.

2.2.4 Case of genus 1

We use the notation of the section 2.1.4.3.

Let $L = \mathbb{Z} + \tau\mathbb{Z}$ be a lattice with $\Im(\tau) > 0$ and let X be the complex torus $\mathbb{C}/L$. We give here an explicit example of an almost-graded structure for a two points set Γ. We suppose that $\Gamma = \{1/2\}\dot\cup 0$. Then we define the following meromorphic functions:

$$A_{2k} := (\wp - e_1)^k, \qquad A_{2k+1} := \frac{1}{2}\wp'(\wp - e_1)^{k-1}.$$

Since $\wp(1/2) = e_1$ and since $\wp$ is even, the function $(\wp - e_1)$ has a pole of order 2 in 0 and a zero of order 2 in 1/2. Hence

$$\mathrm{ord}_0(A_{2k}) = -2k, \qquad \mathrm{ord}_{1/2}(A_{2k}) = 2k.$$

The function $\wp'$ has a pole of order 3 in 0 and has a zero of order 1 in $1/2, \tau/2$ and $1/2 + \tau/2$. Hence

$$\mathrm{ord}_0(A_{2k+1}) = -(2k+1), \qquad \mathrm{ord}_{1/2}(A_{2k+1}) = 2k-1.$$

The set $\{A_n,\ n \in \mathbb{Z}\}$ is a basis of $\mathcal{M}(X,\Gamma)$. Hence by setting $\mathcal{M}_n := \mathrm{vect}(A_n)$ we get the decomposition

$$\mathcal{M}(X,\Gamma) = \bigoplus_{n\in\mathbb{Z}} \mathcal{M}_n.$$

Moreover, if n or m is even we have obviously

$$A_n \cdot A_m = A_{n+m}.$$

If $n = 2k+1$ and $m = 2k'+1$ then using $e_1 + e_2 + e_3 = 0$ we get

$$\begin{aligned}
A_n \cdot A_m &= \frac{1}{4}\wp'^2(\wp - e_1)^{k+k'-2} \\
&= \frac{1}{4} \cdot 4(\wp - e_1)(\wp - e2)(\wp - e_3)(\wp - e_1)^{k+k'-2} \\
&= (\wp - e_1)(\wp - e_1 + (e_1 - e_2))(\wp - e_1 + (e_1 - e_3))(\wp - e_1)^{k+k'-2} \\
&= [(\wp - e_1)^3 + (2e_1 - e_3 - e_2)(\wp - e_1)^2 + (e_1 - e_2)(e_1 - e_3)(\wp - e_1)](\wp - e_1)^{k+k'-2} \\
&= (\wp - e_1)^{k+k'+1} + 3e_1(\wp - e_1)^{k+k'} + (e_1 - e_2)(2e_1 + e_2)(\wp - e_1)^{k+k'-1} \\
&= A_{n+m} + 3e_1 A_{n+m-2} + (e_1 - e_2)(2e_1 + e_2)A_{n+m-4}
\end{aligned}$$

Hence that defines an almost-graded structure.

As above we remark that the subalgebra $\mathcal{M}_+$ is exactly the algebra $\mathcal{M}(X, \{0\})$. Thus we have here an almost-graded structure of $\mathcal{M}(X, \{0\})$ depending of the point $1/2$.

2.3 Group Action on Riemann Surfaces

All details of this section can be found in the book of Rick Miranda [21].

2.3.1 Generalitites

Let G be a finite group and X a Riemann surface.
A left action of G on X is a map $G \times X \to X : (g, p) \mapsto g.p$ which satisfies

a) $(gh).p = g.(h.p)$ for $g, h \in G$ and $p \in X$

b) $e.p = p$ for $p \in X$ and e the unit element of G.

The orbit of a point $p \in X$ is the set $G(p) := \{g.p \mid \forall g \in G\}$. The stabilizer of a point $p \in X$ is the subgroup $G_p := \{g \in G \mid g.p = p\}$. We recall that two points in the same orbit have conjugate stabilizer. More precisely: $G_{g.p} = gG_pg^{-1}$. Moreover we have the following relation:

$$|G(p)| \cdot |G_p| = |G|.$$

The kernel of an action of G on X is the subgroup $K := \{g \in G \mid g.p = p \ \forall p \in X\}$. Obviously, K is the intersection of all stabilizer subgroups and K is a normal subgroup. Moreover the quotient group G/K acts on X with trivial kernel and with the same orbits as for the action of G. So we usually assume that the kernel is trivial. Such an action is called an effective action.

For each $g \in G$, the map $p \mapsto g.p$ is a bijection. The action is said to be continuous if for all $g \in G$ this bijection is continuous. The action is said to be holomorphic if for all $g \in G$ the bijection is holomorphic. In the holomorphic case, this bijection is necessary an analytic automorphism of X.

The quotient space X/G is defined as the set of orbits. The natural projection $\pi : X \to X/G$ sends a point $p \in X$ to its orbit $G(p)$. The usual way to give a topology to such a quotient space is to declare a subset $U \subset X/G$ to be open if and only if $\pi^{-1}(U)$ is open in X. Clearly the projection π is then continuous. If the action is continuous, then π is an open map.

Here we give some details about stabilizer subgroups of an effective and holomorphic action:

Proposition 2.32. *Let G be a group (not necessary finite) acting holomorphically and effectively on a Riemann surface X, and fix a point $p \in X$. If the stabilizer subgroup G_p is finite then it is a finite cyclic group.*
Obviously, if G is a finite group, all stabilizer subgroups are finite cyclic subgroups.

Proposition 2.33. *Let G be a finite group acting holomorphically and effectively on a Riemann surface X. The points of X with nontrivial stabilizer subgroups are discrete.*
If X is a compact Riemann surface, these points form a finite set.

Our goal is to put a complex structure on X/G such that π is a holomorphic map. This is done by the following construction which can be found in [21]

Proposition 2.34. *Let G be a finite group acting holomorphically and effectively on a Riemann surface X. Fix a point $p \in X$. Then there is an open neighborhood U of p such that:*

a) U is invariant under the stabilizer G_p: $g \cdot p \in U$ for every $g \in G_p$ and $p \in U$;

b) $U \cap (g \cdot U) = \emptyset$ for every $g \notin G_p$;

c) the natural map $\alpha : U/G_p \to X/G$, induced by sending a point in U to its orbit, is a homeomorphism onto an open subset of X/G;

d) no point of U except p is fixed by any element of G_p.

Proof. See the book of Rick Miranda [21] (p77). □

The above proposition helps us to define charts on X/G. We just have to define charts on U/G_p and transport these to X/G via the map α. More details can be found in the book of Rick Miranda [21].

Theorem 2.35. *Let G be a finite group acting holomorphically and effectively on a Riemann surface X. Then there exists a complex structure on X/G such that X/G is a Riemann surface and that the projection $\pi : X \to X/G$ is holomorphic. Moreover π is of degree $|G|$ and $mult_p(\pi) = |G_p|$ for any $p \in X$.*

Proposition 2.36. *A group G acting holomorphically and effectively on a Riemann surface X is isomorphic to a group of automorphisms of X.*

Hence in the following we will identify each group acting holomorphically and effectively on a Riemann surface with a group of automorphism.

We suppose now that X is a compact Riemann surface. Note that in this case, X/G is compact too.
In the following we explain the link between stabilizer subgroups and the ramification points of the quotient map π.
Let $\pi : X \to X/G$ be the projection map and y a point of X/G. The set of the pre-images of y is a single orbit and for each point x of this orbit, we have $mult_x(\pi) = |G_x|$. Since stabilizer subgroups of points of the same orbit are conjugated, $mult_x(\pi)$ is a constant

integer r. The set $\pi^{-1}(y)$ contains exactly $s = |G|/r$ elements.
The set of ramification points of π corresponds exactly to the set of point with nontrivial stabilizer subgroup. This set is a disjoint finite union of orbits. Moreover, these orbits correspond to the branch points.

We apply the Hurwitz's formula to the quotient map π. Let $y_1, ..., y_k$ be the k branch points in X/G of π and r_i be the multiplicity of the $|G|/r_i$ points in $\pi^{-1}(y_i)$, with $r_i \geq 2$. Then

$$2g(X)-2 = |G|(2g(X/G)-2)+\sum_{i=1}^{k}\frac{|G|}{r_i}(r_i-1) = |G|[2g(X/G)-2+\sum_{i=1}^{k}(1-\frac{1}{r_i})] \quad (2.3)$$

Set $R := \sum_{i=1}^{k}(1-\frac{1}{r_i})$. Then we get

$$2g(X)-2 = |G|[2g(X/G)-2+R]. \quad (2.4)$$

This equation is very important and gives us many pieces of information about the finite groups which can act on a Riemann surface. Moreover the quantity $R = \sum_{i=1}^{k}(1-\frac{1}{r_i})$ is clearly important in studying the actions. Some special values of R will help us in the next sections. The following lemma is elementary and can be shown by a direct check.

Lemma 2.37. *Suppose that k integer $r_1, \ldots, r_k$ with $r_i \geq 2$ for each i are given. Let $R = \sum_{i=1}^{k}(1-\frac{1}{r_i})$. Then*

$$a)\ R < 2 \Longleftrightarrow k,\ \{r_i\} = \begin{cases} k=1,\ \textit{any } r_1 \\ k=2,\ \textit{any } r_1, r_2 \\ k=3,\ \{r_i\} = \{2,2,\textit{any } r_3\} \textit{ or} \\ k=3,\ \{r_i\} = \{2,3,3\} \textit{ or } \{2,3,4\} \textit{ or } \{2,3,5\} \end{cases}$$

$$b)\ R = 2 \Longleftrightarrow k,\ \{r_i\} = \begin{cases} k=3,\ \{r_i\} = \{2,3,6\} \textit{ or } \{2,4,4\} \textit{ or } \{3,3,3\}; \\ k=4,\ \{r_i\} = \{2,2,2,2\} \end{cases}$$

We now consider the situation of the Riemann sphere and of the tori.

2.3.2 The situation of the Riemann sphere.

Using Identity (2.4) we are able to determine the finite groups acting holomorphically and effectively on the Riemann surface $\mathbb{C}_\infty$.
Suppose that G is a finite group acting holomorphically and effectively on $\mathbb{C}_\infty$. Since $\mathbb{C}_\infty$ has genus 0, $\mathbb{C}_\infty/G$ has genus 0 too. So the Hurwitz formula in this case says that

$$-2 = |G|[-2 + R].$$

We see that if G is not the trivial group (i.e. $|G| > 1$) then $R \neq 0$ and there must be ramification points. Moreover since $|G| > 0$ we must have $R < 2$.
We use the same notation as above.
Suppose that $k = 1$, then $R = 1 - \frac{1}{r}$ for some $r \geq 2$. So $0 < R < 1$ and $2 > 2 - R > 1$; hence $|G| = 2/(2 - R)$ will not be an integer. That is not possible, thus k cannot be 1.

Suppose now that $k = 2$. From Lemma 2.37 any r_1, r_2 is possible. In fact not all possibilities can occur and we show that r_1 and r_2 must be equal. To see this suppose that the branch points are y_1 and y_2. Consider a small loop γ in $\mathbb{C}_\infty/G$ around y_1 which starts and ends at a point y_0 . This loop γ may be lifted to a curve in $\mathbb{C}_\infty$ starting at any of the $|G|$ points in the fiber of π over y_0. The permutation of this fiber of π given by sending a point p in the fiber to the endpoint of the lift of γ which starts at p, is of order r_1. Similar considerations apply to a small loop around y_2 give a permutation of order r_2. However since $\mathbb{C}_\infty/G \cong \mathbb{C}_\infty$, these two loops are homotopic; hence the permutations must have the same order, so $r_1 = r_2 = r$.
In this case $|G| = 2/(2-R) = r$. This is achieved by the cyclic group $\mathbb{Z}_r$ of order r, acting on $\mathbb{C}_\infty$ by multiplying the coordinate z by r^{th} roots of unity.

In case $k = 3$, we see that:

$$if \quad \{r_i\} = \{2, 2, r\}, \quad then \quad |G| = 2r;$$

$$if \quad \{r_i\} = \{2, 3, 3\}, \quad then \quad |G| = 12;$$

$$if \quad \{r_i\} = \{2, 3, 4\}, \quad then \quad |G| = 24;$$

$$if \quad \{r_i\} = \{2, 3, 5\}, \quad then \quad |G| = 60$$

The first case is achieved by the action of a dihedral group $\mathbb{D}_r$. The latter cases are achieved by actions of $\mathbb{T}$, $\mathbb{O}$, and $\mathbb{I}$. These are the famous "platonic solid actions", which are groups acting on the sphere leaving either a tetrahedron (the 2,3,3 case), a cube and a octahedron (the 2,3,4 case), or a dodecahedron and a icosahedron (the 2,3,5 case) invariant.

Let us give explicitly the action of these groups in terms of finite groups of automorphism. Note that we give for each case an explicit example of the realization of the action. Other realizations can be given by conjugating these examples. For more details, see the paper of Lombardo and Mikhailov [20]

1. The group $\mathbb{Z}_N$, $N \in \mathbb{N}^*$:
   It is the group generated by the transformation

   $$\sigma(z) = \Omega z, \quad \Omega = \exp\left(\frac{2\pi i}{N}\right), \quad \mathbb{Z}_N = \{\sigma^n / n = 0, 1, \dots, N-1\}$$

   There are two points which are fixed by elements of $\mathbb{Z}_N$: 0 and ∞. The orbits are:

   $$\mathbb{Z}_N(0) = \{0\}, \quad \mathbb{Z}_N(\infty) = \{\infty\}.$$

   The stabilizer groups are

   $$(\mathbb{Z}_N)_0 = (\mathbb{Z}_N)_\infty = \mathbb{Z}_N.$$

2. The dihedral group $\mathbb{D}_N$:
   It is of order $2N$ and generated by two transformations:

   $$\sigma_s(z) = \Omega z, \quad \sigma_t(z) = \frac{1}{z}, \quad \Omega = \exp\left(\frac{2\pi i}{N}\right)$$

   They verify $\sigma_s^N = \sigma_t^2 = (\sigma_s \sigma_t)^2 = id$ and we have:

   $$\mathbb{D}_N = \{\sigma_s^n, \sigma_s^n \sigma_t / n = 0, \dots, N-1\}$$

   We give the non-generic orbits (i.e. the orbits containing fixed points). If N is odd:

   $$\begin{aligned} \mathbb{D}_N(0) &= \{0, \infty\}, \quad |(\mathbb{D}_N)_0| = N \\ \mathbb{D}_N(1) &= \left\{1, \Omega, \dots, \Omega^{N-1}\right\}, \quad |(\mathbb{D}_N)_1| = 2 \\ \mathbb{D}_N(-1) &= \left\{-1, -\Omega, \dots, -\Omega^{N-1}\right\}, \quad |(\mathbb{D}_N)_{-1}| = 2 \end{aligned}$$

   If N is even, then $\mathbb{D}_N(1) = \mathbb{D}_N(-1)$ but:

   $$\mathbb{D}_N(i) = \left\{i, i\Omega, \dots, i\Omega^{N-1}\right\}, \quad |(\mathbb{D}_N)_i| = 2$$

3. The group $\mathbb{T}$ (note that $\mathbb{T} \cong A_4$):
   It is of order 12 and generated by two transformations:

$$\sigma_s(z) = -z, \quad \sigma_t(z) = \frac{z+i}{z-i}.$$

We have $\sigma_s^2 = \sigma_t^3 = (\sigma_s \sigma_t)^3 = id$ and

$$\mathbb{T} = \{\sigma_t^n, \sigma_t^n \sigma_s \sigma_t^m | n, m = 0, 1, 2\}.$$

There are three non-generic orbits. One coming from the two fixed points of the transformation σ_s; 0 and ∞:

$$\mathbb{T}(0) = \{0, \infty, \pm 1, \pm i\}, \quad |(\mathbb{T})_0| = 2$$

The transformation σ_t has two fixed points, $\gamma_1 = \dfrac{1+i}{1+\sqrt{3}} = \omega + i\bar{\omega}$ and $\gamma_2 = \dfrac{1+i}{1-\sqrt{3}} = i\omega + \bar{\omega}$ with $\omega = \exp\left(\dfrac{2i\pi}{3}\right)$ and:

$$\mathbb{T}(\gamma_1) = \{\pm(\omega + i\bar{\omega}), \pm(\omega - i\bar{\omega})\}, \quad |(\mathbb{T})_{\gamma_1}| = 3$$
$$\mathbb{T}(\gamma_2) = \{\pm i(\omega + i\bar{\omega}), \pm i(\omega - i\bar{\omega})\}, \quad |(\mathbb{T})_{\gamma_2}| = 3$$

4. The group $\mathbb{O}$ (note that $\mathbb{O} \cong S_4$):
   It is of order 24 and generated by two transformations:

$$\sigma_s(z) = iz, \quad \sigma_t(z) = \frac{z+1}{z-1}$$

These generators verify $\sigma_s^4 = \sigma_t^2 = (\sigma_s \sigma_t)^3 = id$ and we have

$$\mathbb{O} = \{\sigma_s^n, \sigma_s^n \sigma_t \sigma_s^m, \sigma_s \sigma_t^2 \sigma_s | n, m = 0, 1, 2, 3\}$$

There are three non-generic orbits:

$$\mathbb{O}(0) = \mathbb{T}(0) = \{0, \infty, \pm 1, \pm i\}, \quad |\mathbb{T}_0| = 2$$

The above point γ_1 is a fixed point of σ_t, hence:

$$\mathbb{O}(\gamma_1) = \mathbb{T}(\gamma_1) \bigcup \mathbb{T}(\gamma_2)$$

The point $\delta = \exp(i\pi/4)$ id a fixed point of $z \mapsto i/z$, hence:

$$\mathbb{O}(\delta) = \left\{\pm\delta, \pm\bar{\delta}, i^n(1 + \delta + \bar{\delta}), i^n(1 - \delta - \bar{\delta}) / n = 0, 1, 2, 3\right\}$$

5. The group $\mathbb{I}$ (note that $\mathbb{I} \cong A_5$):
   It is of order 60 and generated by two transformation:

$$\sigma_s(z) = \epsilon z, \quad \sigma_t(z) = \frac{(\epsilon^2 + \epsilon^3)z + 1}{z - \epsilon^2 - \epsilon^3}, \quad \epsilon = \exp\left(\frac{2\pi i}{5}\right).$$

   The generators verify $\sigma_s^5 = \sigma_t^2 = id$ and we have

$$\mathbb{I} = \{\sigma_s^n, \sigma_s^n \sigma_t \sigma_s^m, \sigma_s^n \sigma_t \sigma_s^2 \sigma_t \sigma_s^m, \sigma_s^n \sigma_t \sigma_s^2 \sigma_t \sigma_s^3 \sigma_t | n, m = 0, 1, 2, 3, 4\}.$$

   There are three non-generic orbits:

$$\mathbb{I}(0) = \left\{0, \infty, \epsilon^{k+1} + \epsilon^{k-1}, \epsilon^{k+2} + \epsilon^{k-2} / k = 0, 1, 2, 3, 4\right\},$$

   The transformation $\sigma_s^2 \sigma_t \sigma_s^2(z) = \dfrac{(1+\bar{\epsilon})z + 1}{z - 1 - \epsilon}$ has two fixed points:

$$\mu_1 = \frac{3 + \sqrt{5} + \sqrt{6(5+\sqrt{5})}}{4} = 1 - \omega\epsilon - \bar{\omega}\bar{\epsilon},$$

$$\mu_2 = \frac{3 + \sqrt{5} - \sqrt{6(5+\sqrt{5})}}{4} = 1 - \bar{\omega}\epsilon - \omega\bar{\epsilon}$$

   The orbits $\mathbb{I}(\mu_1) = \mathbb{I}(\mu_2)$ contain 20 points which are the roots of

$$z^{20} - 228z^{15} + 494z^{10} + 228z^5 + 1 = 0.$$

   The point i is a fixed point of the transformation $\sigma_s^2 \sigma_t \sigma_s^3 \sigma_t \sigma_s^2 \sigma_t(z) = -1/z$. The orbit $\mathbb{I}(i)$ contains 30 points which are the roots of

$$z^{30} + 522z^{25} - 10005z^{20} - 10005z^{10} - 522z^5 + 1 = 0.$$

Definition 2.38. *An homography of $\mathbb{C}_\infty$ is a map from $\sigma : \mathbb{C}_\infty \to \mathbb{C}_\infty$ such that*

$$\sigma(z) = \frac{az + b}{cz + d}$$

with $ad - bc \neq 0$, $\sigma(-d/c) = \infty$, $\sigma(\infty) = a/c$ if $c \neq 0$ and $\sigma(\infty) = \infty$ if $c = 0$.
The set of homographies is a group for the usual composition denoted by $\mathcal{H}$.
The homographies are also called fractional linear transformations.

Let $\sigma_1(z) = (a_1z+b_1)/(c_1z+d_1)$ and $\sigma_2(z) = (a_2z+b_2)/(c_2z+d_2)$ be two homographies. The composition $\sigma_1 \circ \sigma_2$ is

$$\sigma_1 \circ \sigma_2(z) = \frac{(a_1a_2 + b_1c_2)z + (a_1b_2 + b_1d2)}{(c_1a_2 + d_1c_2)z + (c_1b_2 + d_1d2)}$$

Hence the inverse of σ_1 is $\sigma_1^{-1}(z) = (d_1z - b_1)/(-c_1z + a_1)$

It is now clear that the homographies are related with the invertible $2x2$ matrix. In fact there is the following surjective group morphism:

$$\begin{array}{rccc} \varphi: & GL(2,\mathbb{C}) & \longrightarrow & \mathcal{H} \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & z \mapsto \dfrac{az+b}{cz+d} \end{array}$$

Moreover an homography is invariant by dividing a, b, c, d by $\sqrt{ad-bc}$. Hence we can always choose a, b, c, d such that $ad - bc = 1$. Thus there is also the following surjective group morphism:

$$\begin{array}{rccc} \varphi: & SL(2,\mathbb{C}) & \longrightarrow & \mathcal{H} \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & z \mapsto \dfrac{az+b}{cz+d} \end{array}$$

By the Isomorphism theorem we get :

$$\mathcal{H} \cong PSL(2,\mathbb{C}) \cong PGL(2,\mathbb{C})$$

where $PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\pm Id$ and $PGL(2,\mathbb{C}) = GL(2,\mathbb{C})/\mathbb{C}^* Id$.

Proposition 2.39.

$$Aut(\mathbb{C}_\infty) = \mathcal{H}$$

Proof. Of course an homography is an automorphism of $\mathbb{C}_\infty$.
Let σ be in $Aut(\mathbb{C}_\infty)$. We said before that a holomorphic map from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$ can be viewed as an meromorphic function of $\mathbb{C}_\infty$. Hence as σ is an automorphism of $\mathbb{C}_\infty$, it is in particular a holomorphic map from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$ and can identified with a meromorphic function of $\mathbb{C}_\infty$.
We proved before that the set of meromorphic functions of $\mathbb{C}_\infty$ is the set of the rational functions $\mathbb{C}(z)$. Thus $\sigma(z) = P(z)/Q(z)$ where P and Q are two polynomial functions. Since σ is an automorphism, it has degree one. That implies that as a meromorphic function σ must have a single simple zero and a single simple pole. So the polynomial

function P and Q must be respectively of the form $az+b$ and $cz+d$. Note that $a=0$ correspond the case of a zero in ∞ and $b=0$ to a pole in ∞. Finally to avoid σ to be a constant, P and Q must be non-proportional. Hence $ad-cb\neq 0$. That shows that σ is an homography of $\mathbb{C}_\infty$. □

We give now the classification of the finite groups of $Aut(\mathbb{C}_\infty)$.

Definition 2.40. *Let A and B be two finite subgroups of $Aut(\mathbb{C}_\infty)$. The groups A and B are said to be conjugated if there exists $\sigma\in Aut(\mathbb{C}_\infty)$ such that:*

$$A=\sigma B\sigma^{-1}.$$

The conjugation is an equivalence relation for the finite subgroups of $Aut(\mathbb{C}_\infty)$. Of course two conjugated groups are isomorphic.

Theorem 2.41. *Up to conjugation, the only finite subgroups of $Aut(\mathbb{C}_\infty)$ are*

$$\mathbb{Z}_N,\quad \mathbb{D}_N,\quad \mathbb{T},\quad \mathbb{O},\quad \mathbb{I}$$

with $N\in\mathbb{Z}$.

For the proof see [16].

2.3.3 The situation of the tori.

Proposition 2.42. *Let X and Y be two complex tori given by lattices L and M respectively. Then any holomorphic map $F:X\to Y$ is induced by a map $G:\mathbb{C}\to\mathbb{C}$ of the form $G(z)=\gamma z+a$, where γ is a constant such that $\gamma L\subset M$. That means the holomorphic map G is a lift of F to the universal coverings of the tori (i.e. $\mathbb{C}$).*
The map F sends zero to zero if and only if $a\in M$. The map F is an isomorphism if and only if $\gamma L=M$.

The proof can be found in the book of Rick Miranda [21].

There are some special lattices:
The square lattices which have orthogonal generators of the same length. The hexagonal

lattices which have generators of the same length separated by an angle of $\pi/3$.

Proposition 2.43. *Let $X = \mathbb{C}/L$ be a complex torus. Any holomorphic map $F : X \to X$ fixing 0 is induced by the multiplication by some $\gamma \in \mathbb{C}$. If F is an automorphisms, then either:*

1. *L is a square lattice and γ is a 4^{th} root of unity;*
2. *L is a hexagonal lattice and γ is a 6^{th} root of unity; or*
3. *L is neither square nor hexagonal and $\gamma = \pm 1$.*

Using the two above proposition, we are able to describe all automorphisms of complex tori. Let $F : X \to X$ be an automorphism of the complex torus X. The map F is induced by a linear map $G : \mathbb{C} \to \mathbb{C}$ of the form $G(z) = \gamma z + a$ where γ is either a 4^{th} root of unity, a 6^{th} root of unity, or $\gamma = \pm 1$, depending to the form of the lattice L. Note that the only automorphisms without fixed points are translations.

Proposition 2.44. *Let X be a complex torus and G a finite group acting holomorphically and effectively on X. Then the Riemann surface X/G is of genus 1 if and only if G is a finite group of translations of X.*
If G contains at least one automorphism fixing at least one point of X; then X/G is of genus 0. Moreover the possible ramification for the quotient map π are:

$$k = 3, \{r_i\} = \{2, 3, 6\}, \{2, 4, 4\}, \text{ or } \{3, 3, 3\}$$

or

$$k = 4, \{r_i\} = \{2, 2, 2, 2\}$$

This means there are k ramification points with multiplicities r_i; $i = 1, \ldots, k$ (i.e. k nontrivial orbits with stabilizer subgroups of order r_i).

Proof. If X is a Riemann surface of genus 1 and G a finite group acting holomorphically and effectively on X, we get from Equation 2.4:

$$R = 2 - 2g(X/G)$$

As $R \geq 0$ it follows that $g(X/G)$ is at most one and $R = 0$ or $R = 2$. Moreover $R = 0$ if and only if $g(X/G) = 1$ and $R = 2$ if and only of $g(X/G) = 0$.
Note that $R = 0$ means that there are no ramification points for the map π i.e. that there

are no points in X with nontrivial stabilizer subgroups. According to the description of the automorphisms of a complex torus, it is clear that the only automorphism with no fixed points are translations. Hence G can be identified as a finite group of translations of X.
If $R = 2$, then X/G is of genus zero and there are $k \geq 1$ ramification points for π. Remember that $r_i \geq 2$ and that $R := \sum_{i=1}^{k}(1 - \frac{1}{r_i})$. As $\frac{1}{2} \leq \left(1 - \frac{1}{r_i}\right) \leq 1$, it's clear that $3 \leq k \leq 4$. A direct check shows that $R = 2$ is only possible for $\{r_i\} = \{2, 3, 6\}, \{2, 4, 4\},$ or $\{3, 3, 3\}$ for $k = 3$ and $\{r_i\} = \{2, 2, 2, 2\}$ for $k = 4$.

□

Using the results on holomorphic maps between complex tori we can give the classification of complex tori.
First we remark that every complex torus is isomorphic to a complex torus X_τ defined by a lattice L_τ generated by 1 and τ, where τ is a complex number with positive imaginary part. In fact, if a lattice L is generated by ω_1 and ω_2, then using $\gamma := 1/\omega_1$ maps L into the lattice generated by 1 and ω_2/ω_1. If the ratio ω_2/ω_1 is in the upper half-plane, then this is τ; otherwise we take the equivalent generator $\tau := -\omega_2/\omega_1$.
Now, in order to classify complex tori we ask when X_τ and $X_{\tau'}$ are isomorphic. This is the case if there exists a complex number γ such that $\gamma L_\tau = L_{\tau'}$. Since L_τ and $L_{\tau'}$ are both generated by 1 and respectively τ and τ', this is equivalent to having the two numbers γ and $\gamma\tau$ generating $L_{\tau'}$. So we need that γ and $\gamma\tau$ lie in $L_{\tau'}$: there must be integers a, b, c, d such that $\gamma = c\tau' + d$ and $\gamma\tau = a\tau' + b$. That gives $\tau = (a\tau' + b)/(c\tau' + d)$. Moreover for γ and $\gamma\tau$ to generate $L_{\tau'}$, we must have the determinant $ad - bc$ equal to ± 1. In fact, since both τ, τ' are in the upper half plane it must be equal to 1. These two conditions are clearly sufficient, and we have proven the following:

Proposition 2.45. *Two complex tori X_τ and $X_{\tau'}$ are isomorphic if and only if there is a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2, \mathbb{Z})$ such that $\tau = (a\tau' + b)/(c\tau' + d)$.*

2.3.4 Higher genus situation.

Let X be a compact Riemann surface of genus $g \geq 2$.

Theorem 2.46 (Hurwitz [9])**.**

1. *$Aut(X)$ is always finite.*
2. *$|Aut(X)| \leq 84(g-1)$.*
3. *A generic Riemann surface of genus $g \geq 3$ does not have any non-trivial automorphisms.*

Remark: "Generic" is a loose expression for the following fact, if you take a completely arbitrary Riemann surface of genus $g \geq 3$ then it will have with very high chance no non-trivial automorphism.
Mathematically more precise is that the moduli space compact Riemann surfaces of genus $g >= 2$ has (complex) dimension $3g-3$ and the set of Riemann surfaces with nontrivial automorphism for genus $g >= 3$ is a "subscheme of codimension" at least 1.

Despite the Result 3 of the theorem above there are very interesting Riemann Surfaces with nontrivial automorphism groups.

1. **Hurwitz Riemann surfaces.**
   A Riemann surface of genus $g \geq 2$ which has exactly the automorphisms group of order $84(g-1)$ is called a Hurwitz Riemann surface.

2. **Hyperelliptic Riemann surfaces**.
   They generalize tori (also called elliptic surfaces) and are compact Riemann surfaces which are a twofold covering of the Riemann sphere. Interchanging the covering sheets gives an automorphism which is of order 2 (an involution).
   In the case of genus 2 all Riemann surface are hyperelliptic.
   If we take as subgroup of the automorphism group of an hyperelliptic Riemann surface the subgroup generated by the involution we obtain as quotient the Riemann sphere and as quotient map the covering map defining the hyperelliptic Riemann surface. Moreover if we check (2.4) we see that the number of branch points is exactly $2g+2$. As the ramification indices are always exactly 2, over each branch point there lies exactly one ramification point. As they constitute the set of fixed points of the involution, the hyperelliptic involution has exactly $2g+2$ fixed points.
   This result is also true for genus one. Here the involution is $z \mod L \mapsto -z \mod L$ and the fixed points are the points $1/2, \tau/2, (1+\tau)/2$.

2.4 Automorphic algebras

2.4.1 G-invariant functions

Let X be an arbitrary compact Riemann surface. Let G be a finite group acting holomorphically on X. Associated to this left action on X there is a left action on $\mathcal{M}(X)$, the set of meromorphic function of X,

$$G \times \mathcal{M}(X) \to \mathcal{M}(X) : (g, f) \mapsto g \cdot f$$

where the meromorphic function $g \cdot f$ is defined by

$$g \cdot f(p) := f(g^{-1} \cdot p)$$

Definition 2.47. *A meromorphic function on X is called G-invariant if for all $g \in G$, $g.f = f$.*
The set of G-invariant function of X is denoted by $\mathcal{M}_G(X)$

Definition 2.48. *There is a natural projection from $\mathcal{M}(X)$ to $\mathcal{M}_G(X)$:*

$$\begin{aligned} \langle\ \rangle_G : \mathcal{M}(X) &\to \mathcal{M}_G(X) \\ f &\mapsto \langle f \rangle_G := \frac{1}{|G|} \sum_{g \in G} g.f \end{aligned}$$

The following is immediate:

Proposition 2.49. *Let f be a function of $\mathcal{M}(X)$.*

a $\langle f \rangle_G$ *is a G-invariant function.*

b *If f is G-invariant* $\langle f \rangle_G = f$

c $\langle \langle f \rangle_G \rangle_G = \langle f \rangle_G$

Proposition 2.50. *Let N be a normal subgroup of G. Then*

$$\langle f \rangle_G = \langle \langle f \rangle_N \rangle_{G/N}$$

Proof. We proof first that the projection $\langle\ \rangle_{G/N}$ is well defined for $f \in \mathcal{M}_N(\Gamma)$. Suppose that $\bar{g} = \bar{h}$ in the quotient group G/N, so that $g^{-1}h \in N$. Then $g^{-1}h \cdot f = f$ since $f \in \mathcal{M}_N(X)$ and so $h \cdot f = g \cdot f$. That proves that $\langle\ \rangle_{G/N}$ is well defined for $f \in \mathcal{M}_N(\Gamma)$. And now:

$$\begin{aligned}
\langle\langle f\rangle_N\rangle_{G/N} &= \frac{|N|}{|G|}\sum_{\bar{g}\in G/N} g\cdot\langle f\rangle_N = \frac{|N|}{|G|}\sum_{\bar{g}\in G/N} g\cdot\frac{1}{|N|}\sum_{h\in N} = \frac{|N|}{|G|}\frac{1}{|N|}\sum_{\substack{\bar{g}\in G/N\\ h\in N}} gh\cdot f\\
&= \frac{1}{|G|}\sum_{\substack{\bar{g}\in G/N\\ l\in\bar{g}}} l\cdot f \quad \text{since } \bar{g} = \{gh, h\in N\}\\
&= \frac{1}{|G|}\sum_{l\in G} l\cdot f = \langle f\rangle_G
\end{aligned}$$

$\square$

Proposition 2.51. *Consider the kernel subgroup K of the action of G on X. Then*

$$\langle f\rangle_K = f,\ \forall f \in \mathcal{M}(X)$$

and

$$\langle f\rangle_G = \langle f\rangle_{G/K}$$

Hence $\mathcal{M}_G(X) = \mathcal{M}_{G/K}(X)$.

Due to the previous proposition it is enough to consider holomorphic and effective actions on X. Recall that in this case, such a group acting on X can be identified with a subgroup of automorphisms of X.

From now on, an action will be holomorphic and effective.

Let us give some properties of G-invariant functions.

Proposition 2.52. *Let f be a G-invariant function.*
For any $x \in X$, $ord_x(f) = ord_{g.x}(f)$ for all $g \in G$. This means that if f has a pole or zero in x then f has a pole or zero of the same order in $g.x$ for all $g \in G$. Likewise if f is holomorphic in x, then f is holomorphic in $g.x$ for all $g \in G$.

Proof. Let K be the order of f in x (i.e. $K := ord_x(f)$). We choose a local coordinate z centered around x and expand the function f as a Laurent series around x:

$$f(z) = \sum_{n\geq K} a_n z^n, \quad a_K \neq 0.$$

Let g be an element of G. We write the automorphism g^{-1} in local coordinates as a Laurent series. We choose a local coordinate z' centered around $g.x$ and the local coordinate z centered around x (the same local coordinate as above). Since g^{-1} is an automorphism hence its Laurent series is

$$g^{-1}(z') = \sum_{n\geq 1} b_n z^n, \quad b_1 \neq 0.$$

By composition of the two above Laurent series we get the Laurent series of the function $f \circ g^{-1}$ in the local coordinate z' around $g\dot{x}$. We get

$$f \circ g^{-1}(z') = \sum_{n\geq K} a_n \left(\sum_{m\geq 1} b_m z^m \right)^n = \sum_{n\geq K} c_n z^n,$$

for some $c_n \in \mathbb{C}$ with $c_K \neq 0$. Hence the order of the function $f \circ g^{-1}$ in $g.x$ is K. But since f is a G-invariant function we have

$$f \circ g^{-1} = f.$$

So

$$\mathrm{ord}_{g\cdot x} f = \mathrm{ord}_{g\cdot x} f \circ g^{-1} = K = \mathrm{ord}_x f.$$

□

Proposition 2.53. *The algebras $\mathcal{M}_G(X)$ and $\mathcal{M}(X/G)$ are isomorphic.*

Proof. Let us give the isomorphism. Consider

$$\begin{aligned} \Phi : \mathcal{M}(X/G) &\to \mathcal{M}_G(X) \\ \bar{f} &\mapsto f \end{aligned}$$

where $f(x) := \bar{f}(\bar{x})$ for all $\bar{x} \in X/G$. Said differently, $f = \bar{f} \circ \pi$ where π is the usual quotient map from X to X/G.

Note that f is meromorphic since π is a holomorphic map and $\bar{f}$ a meromorphic function. Moreover f is G-invariant: $f(g.x) = \bar{f}(\overline{g.x}) = \bar{f}(\bar{x}) = f(x)$ for all $g \in G$ and $x \in X$.

It is easy to verify that Φ is a morphism of algebras. To show that this morphism is an isomorphism we give the inverse of the map:

$$\Phi^{-1} : \mathcal{M}_G(X) \to \mathcal{M}(X/G)$$
$$f \mapsto \bar{f}$$

where $\bar{f}(\bar{x}) := f(x)$. The function $\bar{f}$ is well defined since f is G-invariant. We just have to show that the function $\bar{f}$ is a meromorphic function of X/G. We know that there is just a finite set of points of X with non trivial stabilizer group. We denote this set by S. Then for each point p of $X/G \backslash \pi(S)$, the group G_p is of order 1 and then $mult_p(\pi) = 1$. This means that π is local homeomorphism in p. So it is clear that the function $\bar{f}$ is meromorphic in $X \setminus S$. As the poles of $\bar{f}$ are isolated points, we can find for each point $s \in S$ an open set U_s such that $\bar{f}$ is holomorphic in $U_s \setminus \{\pi(s)\}$. Then $\pi(s)$ is a singularity of $\bar{f}$. But $\pi(s)$ cannot be an essential singularity because in this case f should have an essential singularity in each points of $\pi^{-1}(\pi(s))$, and that is not true. Hence $\pi(s)$ is an removable singularity or a pole and hence $\bar{f}$ is meromorphic on X/G.

□

Let Γ be a finite set of points of a Riemann surface X. We consider the Krichever-Novikov algebra $\mathcal{M}(X, \Gamma)$ of global meromorphic functions on X which are holomorphic on $X \setminus \Gamma$.

Definition 2.54. *The set of global meromorphic functions on X which are holomorphic on $X \backslash \Gamma$ and G-invariant is a subalgebra of $\mathcal{M}_G(X)$ and of $\mathcal{M}(X, \Gamma)$ denoted by $\mathcal{M}_G(X, \Gamma)$.*

Proposition 2.55. *1. $\mathcal{M}_G(X, \Gamma) = \mathcal{M}_G(X) \cap \mathcal{M}(X, \Gamma)$.*

2. *The set Γ can always be split into two subsets Γ_1 and Γ_2 such that Γ_1 contains all the full orbits and Γ_2 the other points and we have*

$$\mathcal{M}_G(X, \Gamma) = \mathcal{M}_G(X, \Gamma_1).$$

3. *Suppose that $\Gamma_1 = \bigcup_{i=1}^{N} G(P_i)$. Hence*

$$\mathcal{M}_G(X, \Gamma) = \mathcal{M}_G(X, \Gamma_1) \cong \mathcal{M}(X/G, \pi(\Gamma_1)).$$

Note that $\pi(\Gamma_1) = \pi(P_1) \cup \cdots \cup \pi(P_N)$.

Proof. 1. It is obvious by the definition of $\mathcal{M}_G(X, \Gamma)$.

2. According to Proposition 2.52 a non-constant G-invariant function cannot have a pole at a point P without having poles at each point of the orbits $G(P)$. This clearly proves the assertion 2.

3. It is just by restriction of the isomorphism of Proposition 2.53.

□

We can remark that if $\Gamma_1 = \emptyset$, then $\mathcal{M}_G(X, \Gamma) = \mathcal{M}_G(X, \emptyset) = \mathcal{O}(X) = \mathbb{C}$.

Hence without restriction it is enough to consider sets Γ which are the union of full orbits.

Corollary 2.56. *Let Γ be a union of full orbits.*
The algebra $\mathcal{M}_G(X, \Gamma)$ is a Krichever-Novikov algebra.
Hence the algebra $\mathcal{M}_G(X, \Gamma)$ admits almost-graded structures. There are three situations:

1. *If $\Gamma = \emptyset$, then $\mathcal{M}_G(X, \Gamma) = \mathbb{C}$ and the almost-graded decomposition is trivial.*

2. *If $\Gamma = \bigcup_{i=1}^{N} G(P_i)$ with $N > 1$, then associated to any splitting of Γ into two disjoint sets of orbits, $\mathcal{M}_G(X, \Gamma)$ admits an almost-graded decomposition.*

3. *If $\Gamma = G(Q)$ consists of one G-orbit, then for any choice of orbit $G(P)$ with $P \notin G(Q)$, $\mathcal{M}_G(X, \Gamma)$ admits an almost-graded decomposition coming from the algebra $\mathcal{M}_+$ of the algebra $\mathcal{M}_G(X, G(P) \cup G(Q))$.*

Proof. It is obvious since $\mathcal{M}_G(X, \Gamma) \cong \mathcal{M}(X/G, \pi(\Gamma))$ and since $\mathcal{M}(X/G, \pi(\Gamma))$ is a KN-algebra. □

Note that $\mathcal{M}_G(X, \Gamma)$ is a subalgebra of the Krichever-Novikov algebra $\mathcal{M}(X, \Gamma)$ and is isomorphic to the Krichever-Novikov algebra $\mathcal{M}(X/G, \pi(\Gamma))$.

2.4.2 G-invariant current algebras

Let $\mathcal{L}$ be a finite dimensional Lie algebra. The classical current algebra associated to $\mathcal{L}$ is the algebra $\bar{\mathcal{L}} := \mathbb{C}[z, z^{-1}] \otimes \mathcal{L}$ with Lie structure

$$[z^n \otimes x, z^m \otimes y] = z^{n+m} \otimes [x, y], \quad x, y \in \mathcal{A}, \;\; n, m \in \mathbb{Z}$$

The algebra $\bar{\mathcal{L}}$ can be described as the Lie algebra of $\mathcal{L}$-valued meromorphic functions on the Riemann sphere which are holomorphic outside 0 and ∞. Starting from this description, the natural extension to a higher genus compact Riemann surface X is to replace $\mathbb{C}[z, z^{-1}]$ by an associative algebra of meromorphic functions on X.
More generally if $\mathcal{L}$ is a Lie algebra and $\mathcal{A}$ an associative, commutative and unital algebra then the tensor vector space $\mathcal{A} \otimes \mathcal{L}$ is a Lie algebra with the bracket

$$[a \otimes x, b \otimes y] := ab \otimes [x, y].$$

In this way we define some Lie algebras:

Definition 2.57. *Let $\mathcal{L}$ be a finite dimensional Lie algebra, X a Riemann surface and Γ a finite set of points of X. We define the following Lie algebras*

1. $\mathcal{L}(X) := \mathcal{M}(X) \otimes \mathcal{L}$
2. $\mathcal{L}(X, \Gamma) := \mathcal{M}(X, \Gamma) \otimes \mathcal{L}$

Consider now any finite group $\mathcal{G}$ acting on $\mathcal{L}(X)$.

3. $\mathcal{L}_{\mathcal{G}}(X) := \{a \in \mathcal{L}(X) \mid g.a = a, \ \forall g \in \mathcal{G}\}$. *This is just the set of $\mathcal{G}$-invariant elements of $\mathcal{L}(X)$.*
4. $\mathcal{L}_{\mathcal{G}}(X, \Gamma) := \{a \in \mathcal{L}(X, \Gamma) \mid g.a = a, \ \forall g \in \mathcal{G}\}$. *Note that* $\mathcal{L}_{\mathcal{G}}(X, \Gamma) = \mathcal{L}_{\mathcal{G}}(X) \cap \mathcal{L}(X, \Gamma)$.
5. $\mathcal{L}_{\mathcal{G}}^0 := \langle \mathcal{L} \rangle_{\mathcal{G}} = \{a \in \mathcal{L} \cong \mathbb{C} \otimes \mathcal{L} \mid g.a = a, \ \forall g \in \mathcal{G}\}$

The following facts are immediate:

1. $\mathcal{L}(X, \Gamma)$, $\mathcal{L}_{\mathcal{G}}(X)$ and $\mathcal{L}_{\mathcal{G}}(X, \Gamma)$ are subalgebras of $\mathcal{L}(X)$.
2. $\mathcal{L}_{\mathcal{G}}(X, \Gamma)$ is a subalgebra of $\mathcal{L}_{\mathcal{G}}(X)$.
3. $\mathcal{L} \cong \mathbb{C} \otimes \mathcal{L}$ is a subalgebra of $\mathcal{L}(X)$.
4. $\mathcal{L}_{\mathcal{G}}^0$ is a subalgebra of $\mathcal{L}$.
5. The algebra $\mathcal{L}(X)$ is a natural $\mathcal{M}(X)$-module: $g \cdot (f \otimes a) := gf \otimes a$ for $g, f \in \mathcal{M}(X)$ and $a \in \mathcal{L}$.
6. Likewise the algebra $\mathcal{L}(X, \Gamma)$ is naturally a $\mathcal{M}(X, \Gamma)$ module.

7. As for G-invariant functions, for any element $a \in \mathcal{L}(X)$ we define

$$\langle a \rangle_{\mathcal{G}} := \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} g \cdot a.$$

We have $\mathcal{L}_{\mathcal{G}}(X) = \langle \mathcal{L}(X) \rangle_{\mathcal{G}}$.

Proposition 2.58. *For any splitting of Γ into two subsets I and O, the Lie algebra $\mathcal{L}(X, \Gamma)$ admits a natural almost-graded structure coming from the almost-grading of the associative algebra $\mathcal{M}(X, \Gamma)$.*
For a single point set $\Gamma = \{Q\}$, the Lie-algebra $\mathcal{L}(X, \Gamma)$ admits a natural almost-graded structure depending of a second reference point P.

Proof. Let $\mathcal{M}(X, \Gamma) := \bigoplus_{n \in \mathbb{Z}} \mathcal{M}_n$ be the almost-graded decomposition of $\mathcal{M}(X, \Gamma)$ of Theorem 2.31. Then

$$\mathcal{L}(X, \Gamma) = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n$$

where $\mathcal{L}_n := \mathcal{M}_n \otimes \mathcal{L}$. Since $\mathcal{L}$ is supposed to be a finite-dimensional Lie algebra, the subspaces $\mathcal{L}_n$ are finite-dimensional too. This decomposition gives obviously an almost-graded structure to $\mathcal{L}(X, \Gamma)$.
If $\Gamma = \{Q\}$, as above we choose a reference point P and we consider the almost-graded structure $\mathcal{M}(X, \{P\} \dot\cup \{Q\}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{M}_n$ associated to the splitting $\Gamma = \{P\} \dot\cup \{Q\}$. Thus the algebra $\mathcal{M}(X, \Gamma)$ is equal to the algebra $\mathcal{M}_+ := \bigoplus_{n \geq 0} \mathcal{M}_n$. That gives an almost-graded structure to the algebra $\mathcal{M}(X, \Gamma)$. Hence the decomposition

$$\mathcal{L}(X, \Gamma) := \bigoplus_{n \geq 0} \mathcal{L}_n$$

with $\mathcal{L}_n := \mathcal{M}_n \otimes \mathcal{L}$, $\forall n \geq 0$ gives as above an almost-graded structure to the Lie algebra $\mathcal{L}(X, Q)$. $\square$

Proposition 2.59. *Let $\mathcal{G}$ be a finite group acting on $\mathcal{L}(X)$ and $\mathcal{N}$ a normal subgroup of $\mathcal{G}$. Then for all $a \in \mathcal{L}(X)$*

$$\langle a \rangle_{\mathcal{G}} = \langle \langle a \rangle_{\mathcal{N}} \rangle_{\mathcal{G}/\mathcal{N}} .$$

Proof. It is exactly the same proof as for Proposition 2.50. $\square$

The elements of $\mathcal{L}(X)$ can be described as the $\mathcal{L}$-valued meromorphic functions on the Riemann surface X. For this we have to define what we exactly mean by "meromorphic", "poles", "holomorphic" for the elements of $\mathcal{L}(X)$.

Definition 2.60. *Let $a \in \mathcal{L}(X)$. Let $\{e_1, \ldots, e_n\}$ be a basis of the Lie algebra $\mathcal{L}$. Thus $a = \sum_{i=1}^{n} f_i e_i$ where $f_i \in \mathcal{M}(X)$ (for convenience we dropped the sign $\otimes$) . For all point p in X, we choose a local coordinate z and we expand each function f_i as a Laurent series around p. Thus a is locally written as*

$$a(z) = \sum_{n \geq M} a_n z^n$$

where $a_n \in \mathcal{L}$, $\forall n \in \mathcal{L}$ and with $a_M \neq 0$. This series is called the Laurent series of a at the point p.
Using the Laurent series of an element $a \in \mathcal{L}(X)$ we can define the order of a in a point $p \in X$ by

$$ord_p a = M$$

As for functions the order is well-defined (it does not depend on the chosen coordinate) and p is a pole of a if $M < 0$, a is holomorphic in p if $M \geq 0$ and p is a zero of a if $M > 0$.

2.4.3 Groups acting on $\mathcal{L}(X)$

Here we give a natural way to get a finite group acting on the tensor Lie algebra $\mathcal{L}(X)$. We explain how we can reduce to some groups which are isomorphic to finite groups acting on a Riemann surface X and which are acting simultaneously on $\mathcal{M}(X)$ and $\mathcal{L}$. This method can be found in the paper of Lombardo and Mikhailov [20] for the genus 0 case but it is still true for higher genus Riemann surfaces.

The first restriction is to consider subgroups of the direct product group $Aut(X) \times Aut(\mathcal{L})$:

Proposition 2.61. *The direct product group $Aut(X) \times Aut(\mathcal{L})$ is a subgroup of $Aut(\mathcal{L}(X))$.*

Proof. The elements of $Aut(X) \times Aut(\mathcal{L})$ can be identified with elements of $Aut(\mathcal{L}(X))$. Let $(g, \sigma) \in Aut(X) \times Aut(\mathcal{L})$. We define the automorphism

$$\begin{aligned}(g, \sigma) : \mathcal{L}(X) &\to \mathcal{L}(X) \\ f \otimes a &\mapsto g \cdot f \otimes \sigma(a)\end{aligned}$$

where $g \cdot f$ is the action defined in the section 2.4.1.
This is clearly a Lie algebra morphism since g is associated to a morphism of $\mathcal{M}(X)$ and σ a morphism of $\mathcal{L}$. Moreover the inverse map of (g, σ) is just the map (g^{-1}, σ^{-1}). $\square$

We can now reduce our discussion to finite subgroups of $Aut(X) \times Aut(\mathcal{L})$ acting simultaneously on $\mathcal{M}(X)$ and $\mathcal{L}$. This is the topic of the two following results:

Theorem 2.62 ([20])**.** *Consider two groups A and B and $\mathcal{G}$ a subgroup of the direct group $A \times B$. We define:*

$$U_1 := \mathcal{G} \cap (A \times id), \qquad U_2 := \mathcal{G} \cap (id \times B), \qquad K := U_1 \cdot U_2$$

Then:

1. *U_1, U_2 and K are normal subgroups of $\mathcal{G}$*
2. *$\pi_i(U_i)$ is a normal subgroup of $\pi_i(\mathcal{G})$ for $i = 1, 2$ (where π_1 and π_2 are the projections on A and B).*
3. *There is two isomorphisms*

$$\psi_1 : \mathcal{G}/K \to \pi_1(\mathcal{G})/\pi_1(K), \qquad \psi_2 : \mathcal{G}/K \to \pi_2(\mathcal{G})/\pi_2(K)$$

4. *$\mathcal{G}/K \cong diag\,(\psi_1(\mathcal{G}/K) \times \psi_2(\mathcal{G}/K)) := \{(\psi_1(g), \psi_2(g)) \mid g \in \mathcal{G}/K\}$*

Proposition 2.63 ([20])**.** *Let $\mathcal{G} \subset Aut(X) \times Aut\mathcal{A}$ be a finite group. Consider the normal subgroups U_1, U_2 and K of the previous theorem. Then*

$$\mathcal{L}_{\mathcal{G}}(X) = \langle \mathcal{L}(X) \rangle_{\mathcal{G}} = \left\langle \langle \langle \mathcal{L}(X) \rangle_{U_1} \rangle_{U_2} \right\rangle_{\mathcal{G}/K} = \left\langle \langle \langle \mathcal{L}(X) \rangle_{U_2} \rangle_{U_1} \right\rangle_{\mathcal{G}/K}.$$

Note that in the paper of Lombardo and Mikhailov [20] the Riemann surface X is always supposed to be the Riemann sphere but the proposition is even true for any Riemann surface.

Using Theorem 2.62 and Proposition 2.63 we can conclude the following: Let $\mathcal{G} \subset Aut(X) \times Aut\mathcal{L}$ be a finite group. The group U_1 consists of the automorphism of $\mathcal{G}$ which are acting just on $\mathcal{M}(X)$. Thus averaging over U_1 is equivalent to replace in $\mathcal{M}(X) \otimes \mathcal{L}$ the associative algebra $\mathcal{M}(X)$ by the subalgebra $\mathcal{M}_{\pi_1(U_1)}(X)$. In the same way,

averaging over U_2 is equivalent to replacing the Lie algebra $\mathcal{L}$ by the Lie algebra $\mathcal{L}_{\pi_2(\mathcal{G})}$ of the $\pi_2(\mathcal{G})$-invariant elements of $\mathcal{L}$. Hence new effects (beyond the already discussed action on $\mathcal{M}(X)$ and the automorphisms of $\mathcal{L}$ separately) will only appear if we have simultaneous transformation and we might even assume the subgroup K to be trivial to study these effects. But from Theorem 2.62, if K is trivial then $\mathcal{G} \cong \pi_1(\mathcal{G}) \cong \pi_2(\mathcal{G})$.
Hence, if $\mathcal{G}$ is finite in this case it will be isomorphic to a finite subgroup G of $Aut(X)$ and finally (again using theorem 2.62) $\mathcal{G} = diag(G, \psi(G))$, where $\psi : G \to Aut(\mathcal{L})$ is a group monomorphism.
We study more precisely this situation in the following.

We have similar results as for the G-invariant function:

Proposition 2.64. *Let $\mathcal{G}$ be a finite group acting on $\mathcal{L}(X)$ and G the corresponding finite group of $Aut(X)$.*
Let a be in $\mathcal{L}_{\mathcal{G}}(X)$
For any $x \in X$, $ord_x(a) = ord_{g.x}(a)$ for all $g \in G$. If a is holomorphic in x, then a is holomorphic in $g.x$ for all $g \in G$.

Proof. That follows directly from Proposition 2.52. □

Proposition 2.65. *Let $\mathcal{G}$ be a finite group acting on $\mathcal{L}(X)$ and G the corresponding finite group of $Aut(X)$. Let Γ be a finite set of point of X. We split Γ into Γ_1 and Γ_2 such that Γ_1 contains all the full orbits and Γ_2 the other points.*

1. $\mathcal{L}_{\mathcal{G}}(X, \Gamma) = \mathcal{L}_{\mathcal{G}}(X, \Gamma_1)$
2. *If* $\Gamma_1 = \emptyset$, *then* $\mathcal{L}_{\mathcal{G}}(X, \Gamma) = \langle \mathcal{L} \rangle_{\mathcal{G}}$.
3. *The Lie algebra* $\mathcal{L}_{\mathcal{G}}(X)$ *is a* $\mathcal{M}_G(X)$*-module.*
4. *The Lie algebra* $\mathcal{L}_{\mathcal{G}}(X, \Gamma)$ *is a* $\mathcal{M}_G(X, \Gamma)$*-module.*
5. *For* $f \in \mathcal{M}_G(X)$ *and* $a \in \mathcal{L}(X)$

$$\langle f \cdot a \rangle_{\mathcal{G}} = f \cdot \langle a \rangle_{\mathcal{G}}$$

Proof. The first assertion is a direct consequence of the previous proposition: a $\mathcal{G}$-invariant element with a pole in p has necessarily a pole in each points of the orbit $G(p)$.
The other assertions are obvious. □

We remark that as before it is enough to consider sets Γ of full orbits.

2.4.4 Almost-graded structures $\mathcal{G}$-invariant current algebras

Let $\mathcal{L}$ be a finite-dimensional Lie algebra, X a compact Riemann surface, G a finite group of $Aut(X)$ and $\psi : G \to Aut(\mathcal{L})$ a group monomorphism.
We consider the average $\langle\ \rangle_{\mathcal{G}} : \mathcal{L}(X) \to \mathcal{L}(X)$ defined as

$$\langle f \otimes x \rangle_{\mathcal{G}} := \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} g \cdot (f \otimes x) = \frac{1}{|G|} \sum_{g \in G} g \cdot f \otimes \psi(g) \cdot x$$

for all $f \in \mathcal{M}(X)$ and $x \in \mathcal{L}$.
Since $\big\langle \langle\ \rangle_{\mathcal{G}} \big\rangle_{\mathcal{G}} = \langle\ \rangle_{\mathcal{G}}$, this is a projection of the vector space $\mathcal{L}(X)$.

The projection $\langle\ \rangle_{\mathcal{G}}$ can be restricted to a projection of the vector space $\mathcal{L}$:

$$\langle x \rangle_{\mathcal{G}} = \frac{1}{|G|} \sum_{g \in G} \psi(g) \cdot x, \qquad \forall x \in \mathcal{L}.$$

Hence

$$\mathcal{L} = \mathrm{Im}\,\langle\ \rangle_{\mathcal{G}} \oplus \mathrm{Ker}\,\langle\ \rangle_{\mathcal{G}}\,.$$

Note that the above restricted average is exactly the usual average $\langle\ \rangle_G$ of the faithful representation ψ of the finite group G, and the subspace $\mathrm{Im}\,\langle\ \rangle_{\mathcal{G}}$ is exactly the G-module $\mathcal{L}^G$ of the invariant element of $\mathcal{L}$. From now on we will use the notation $\langle\ \rangle_G$ to denote the restricted average to $\mathcal{L}$ and $\mathcal{L}^G$ to denote the space of the invariant elements of $\mathcal{L}$. From now on this decomposition will be noted as

$$\mathcal{L} = \mathcal{L}^G \oplus \mathrm{Ker}\,\langle\ \rangle_G\,.$$

The following properties are obvious:

1. $\mathcal{L}^G := \{x \in \mathcal{L} \mid \psi(g) \cdot x = x,\ \forall g \in G\}$.
2. $\psi(g) \circ \langle\ \rangle_G = \langle\ \rangle_G \circ \psi(g) = \langle\ \rangle_G\,, \quad \forall g \in G$.
3. $\mathcal{L}^{\mathcal{G}}$ and $\mathrm{Ker}\,\langle\ \rangle_G$ are both G-modules.

Proposition 2.66.

1. *$\mathcal{L}^G$ is a Lie subalgebra of $\mathcal{L}$.*
2. *$[\mathcal{L}, Ker\langle\ \rangle_G] \subset Ker\langle\ \rangle_G$.*
3. *Let $f \in \mathcal{M}(X)$ and $x \in \mathcal{L}^G$*
$$\langle f \otimes x\rangle_{\mathcal{G}} = \langle f\rangle_G \otimes x$$
In particular $f \otimes x$ is $\mathcal{G}$-invariant if and only if $f \in \mathcal{M}_G(X)$.
4. *Let $f \in \mathcal{M}(X)$ and $y \in Ker\langle\ \rangle_G$*
$$\langle f \otimes y\rangle_{\mathcal{G}} \in \mathcal{M}(X) \otimes Ker\langle\ \rangle_G\,.$$
In particular if f is a constant map, $\langle f \otimes y\rangle_{\mathcal{G}} = 0$.

Proof. 1. $\mathcal{L}^G$ is a vector subspace of $\mathcal{L}$ and since for all $g \in G$ the map $\psi(g)$ is an automorphism of Lie algebra, we have
$$\psi(g)\cdot[x,y] = [\psi(g)\cdot x, \psi(g)\cdot y] = [x,y], \quad \forall x,y \in \mathcal{L}^G.$$
So $[\mathcal{L}^G,\mathcal{L}^G] \subseteq \mathcal{L}^G$ and $\mathcal{L}^G$ is a Lie subalgebra of $\mathcal{L}$.

2. Let $x \in \mathcal{L}$ and $y \in \mathrm{Ker}\langle\ \rangle_G$. We have
$$\langle[x,y]\rangle_G = \frac{1}{G}\sum_{g\in G}\psi(g)\cdot[x,y] = \frac{1}{G}\sum_{g\in G}[\psi(g)\cdot x, \psi(g)\cdot y] = \frac{1}{G}\sum_{g\in G}[\psi(g)\cdot x, 0] = 0.$$
Hence $[x,y] \in \mathrm{Ker}\langle\ \rangle_G$.

3. Since $x \in \mathcal{L}^G$, $\psi(g)\cdot x = x$ for all $g \in G$. Hence
$$\langle f \otimes x\rangle_{\mathcal{G}} = \frac{1}{|G|}\sum_{g\in G} g\cdot f \otimes \psi(g).x = \frac{1}{|G|}\sum_{g\in G} g\cdot f \otimes x = \langle f\rangle_G \otimes x.$$

4. Since $\mathrm{Ker}\langle\ \rangle_G$ is a G-module, $\psi(g)\cdot y \in \mathrm{Ker}\langle\ \rangle_G$ for all $g \in G$. So
$$\langle f \otimes y\rangle_{\mathcal{G}} = \frac{1}{|G|}\sum_{g\in G} g\cdot f \otimes \psi(g)\cdot y \in \mathcal{M}(X) \otimes \mathrm{Ker}\langle\ \rangle_G\,.$$

□

Let N be the dimension of $\mathcal{L}$ and let $K(\leq N)$ be the dimension of $\mathcal{L}^G$.
Let $\{e_1,\ldots,e_N\}$ be a basis of $\mathcal{L}$ such that $\{e_1,\ldots,e_K\}$ is a basis of the algebra $\mathcal{L}^G$ and $\{e_{K+1},\ldots,e_N\}$ a basis of the vector space $\mathrm{Ker}\langle\ \rangle_G$. Any element a of $\mathcal{L}(X)$ is written as

$$a=\sum_{i=1}^{N} f_i\otimes e_i$$

with $f_i\in\mathcal{M}(X)$, $\forall i=1,\ldots,N$.

Proposition 2.67. *Let $a\in\mathcal{L}_{\mathcal{G}}(X)$. In the above decomposition $a=\sum_{i=1}^{N} f_i\otimes e_i$, for $i=1,\ldots,K$ the functions f_i are G-invariant (i.e. $f_i\in\mathcal{M}_G(X)$).*

Proof. Since $a\in\mathcal{L}_{\mathcal{G}}(X)$ we have

$$a=\langle a\rangle_{\mathcal{G}}\,.$$

Hence

$$\sum_{i=1}^{N} f_i\otimes e_i=\sum_{i=1}^{N}\langle f_i\otimes e_i\rangle_{\mathcal{G}}\,.$$

From point 4 of Proposition 2.66 we have

$$\langle f_i\otimes e_i\rangle\in\mathcal{M}(X)\otimes\mathrm{Ker}\langle\ \rangle_G=\mathrm{vect}_{\mathcal{M}(X)}(e_{K+1},\ldots,e_N),\quad \forall i=K+1,\ldots,N.$$

Moreover from the point 3 of the Proposition 2.66 we have

$$\langle f_i\otimes e_i\rangle=\langle f_i\rangle_G\otimes e_i,\qquad \forall i=1,\ldots,K.$$

Hence

$$\langle f_i\otimes e_i\rangle\in\mathcal{M}(X)\otimes\mathcal{L}^{\mathcal{G}}=\mathrm{vect}_{\mathcal{M}(X)}(e_1,\ldots,e_K),\quad \forall i=1,\ldots,K.$$

So in $\sum_{i=1}^{N} f_i\otimes e_i=\sum_{i=1}^{N}\langle f_i\otimes e_i\rangle_{\mathcal{G}}$ we have

$$\sum_{i=K+1}^{N}\langle f_i\otimes e_i\rangle_{\mathcal{G}}=\sum_{i=K+1}^{N} f_i\otimes e_i$$

and

$$\sum_{i=1}^{K}\langle f_i\otimes e_i\rangle_{\mathcal{G}}=\sum_{i=1}^{K} f_i\otimes e_i.$$

Since $\langle f_i\otimes e_i\rangle=\langle f_i\rangle_G\otimes e_i$ for $i=1,\ldots,K$, we get

$$\sum_{i=1}^{K}\langle f_i\rangle_G\otimes e_i=\sum_{i=1}^{K} f_i\otimes e_i$$

and so

$$\langle f_i \rangle_G = f_i \quad \forall i = 1, \dots, K.$$

Hence for $i = 1, \dots, K$ the function f_i are in $\mathcal{M}_G(X)$. $\square$

2.4.4.1 The case of the Riemann sphere

The situation on the Riemann sphere has been already studied in a paper of Lombardo and Mikhailov [20]. In fact in this paper many examples are given. We give here a more general result.

We use the same notation as above but here the Riemann surface X is the Riemann sphere $\mathbb{C}_\infty$ and G is a finite group of $Aut(\mathbb{C}_\infty)$.
Let $Q \in X$ be a point with a trivial stabilizer group G_Q. Let P be a second point of X such that $G(P) \cap G(Q) = \emptyset$. Let $\Gamma := G(P) \dot\cup G(Q)$. Recall that the algebra $\mathcal{M}_G(X, \Gamma)$ is isomorphic to the algebra $\mathcal{M}(X/G, \{\pi(P)\} \dot\cup \{\pi(Q)\})$. A graded basis of $\mathcal{M}(X/G, \{\pi(P)\} \dot\cup \{\pi(Q)\})$ is

$$\bar{A}_k(z) = \left(\frac{z - \pi(P)}{z - \pi(Q)} \right)^k, \quad \forall k \in \mathbb{Z}.$$

The corresponding graded basis of $\mathcal{M}_G(X, \Gamma)$ is

$$A_k(z) = \left(\prod_{g \in G} \frac{z - g \cdot P}{z - g \cdot Q} \right)^k, \quad \forall k \in \mathbb{Z},$$

with

$$A_k \cdot A_l = A_{k+l}, \quad \forall k, l \in \mathbb{Z}$$

Let $\{e_1, \dots, e_N\}$ be the above basis of $\mathcal{L}$. We define the following elements:

$$\begin{aligned}
e_i^k &= A_k \otimes e_i, \quad \forall k \in \mathbb{Z}, \ i = 1, \dots, K; \\
e_i^1 &= \left\langle \frac{1}{(z - Q)} \otimes e_i \right\rangle_{\mathcal{G}}, \quad i = K+1, \dots, N; \\
e_i^k &= A_{k-1} \cdot e_i^1, \quad \forall k \in \mathbb{Z}, \ i = K+1, \dots, N;
\end{aligned}$$

It is clear that the e_i^k are elements of $\mathcal{L}_{\mathcal{G}}(X, \Gamma)$.

Proposition 2.68. *For all $i = 1 \dots N$ and $k \in \mathbb{Z}$ the elements $e_i^k(z)$ are non-zero and linearly independent.*

Proof. Suppose first that $i = 1, \dots, K$. Then $e_i^k = A_k \otimes e_i$ is clearly a non-zero element. Suppose now that $i = K+1, \dots, N$. If the $e_i^1(z)$ are non zero, then the $e_i^k(z)$ are non-zero. So we just have to prove that the $e_i^1(z)$ are non-zero. Remember that

$$e_i^1(z) := \frac{1}{|G|} \sum_{g \in G} \left(g \cdot \frac{1}{z - Q} \right) \otimes \psi(g) \cdot e_i$$

For all $g \in G$ the function $g \cdot \frac{1}{z-Q}$ is a global meromorphic function with single simple poles in $g \cdot Q$. Since Q is supposed having a trivial stabilizer subgroup, hence for $g \neq h$ in G, $g \cdot Q \neq h \cdot Q$. Hence in the sum, the functions $g \cdot \frac{1}{z-Q}$ are functions with a single simple pole in different points. Moreover for all $g \in G$ we have $\psi(g) \cdot e_i \neq 0$ since $\psi(g)$ is an automorphism of $\mathcal{L}$. So the sum cannot be null and e_i^1 is non-zero.

Let us show that the $e_i^k(z)$ are linearly independent. We have $\mathcal{L} = \mathcal{L}^G \oplus \operatorname{Ker} \langle\ \rangle_G$. Hence

$$\mathcal{L}_{\mathcal{G}}(X, \Gamma) = \left(\mathcal{M}(X, \Gamma) \otimes \mathcal{L}^G \right) \bigoplus \left(\mathcal{M}(X, \Gamma) \otimes \operatorname{Ker} \langle\ \rangle_G \right)$$

Moreover we know that the e_i^k for $i = 1, \dots, K$ are in $\mathcal{M}(X, \Gamma) \otimes \mathcal{L}^G$ and that the e_i^k for $i = K+1, \dots, N$ are in $\mathcal{M}(X, \Gamma) \otimes \operatorname{Ker} \langle\ \rangle_G$. Hence we can show separately that the e_i^k for $i = 1, \dots, K$ and the e_i^k for $i = K+1, \dots, N$ are linearly independent.

1. For $i = 1, \dots, K$:
   Let $\{\lambda_{i,k}, i = 1, \dots, K, k \in F\}$ be a finite set of complex numbers (here F is a finite subset of $\mathbb{Z}$). Suppose that

   $$\sum_{k \in F} \sum_{i=1 \dots K} \lambda_{i,k} e_i^k = 0.$$

   We have

   $$\begin{aligned} \sum_{k \in F} \sum_{i=1 \dots K} \lambda_{i,k} e_i^k &= \sum_{k \in F} \sum_{i=1 \dots K} \lambda_{i,k} A_k \otimes e_i \\ &= \sum_{i=1 \dots K} \left(\sum_{k \in F} \lambda_{i,k} A_k \right) \otimes e_i \end{aligned}$$

   Hence $\sum_{k \in F} \sum_{i=1 \dots K} \lambda_{i,k} e_i^k = 0$ if and only if $\sum_{k \in F} \lambda_{i,k} A_k = 0$ for all $i = 1, \dots, K$.
   But each A_k is of order k in P, so each $\lambda_{i,k}$ has to be zero.
   Hence the e_i^k for $i = 1, \dots, K$ are linearly independent.

2. For $i = K+1, \dots, N$:

   The element e_i^1 can be written:

$$e_i^1 = \sum_{l=K+1}^{N} a_l e_l$$

with $a_l \in \mathcal{M}(X)$. The meromorphic entries a_l are linear sums of functions $g \cdot \frac{1}{z-Q}$. Moreover only a_i will have pole at Q of order 1 (besides maybe at other points of $G(Q)$). The a_j, $j \neq i$ will be holomorphic at Q. If $\langle . \mid . \rangle$ is the standard scalar product in $\mathcal{L}$ extended to $\mathcal{L}(X)$ such that the e_i are unitary, this can be resumed by:

$$\mathrm{ord}_Q \left(\langle e_i^1 \mid e_i \rangle \right) = -1 \text{ and } \mathrm{ord}_Q \left(\langle e_i^1 \mid e_j \rangle \right) = 0, \ \forall j \neq i.$$

So since $\mathrm{ord}_Q (A_k) = -k$ and since $e_i^k = A_{k-1} e_i^1$ we have:

$$\mathrm{ord}_Q \left(\langle e_i^k \mid e_i \rangle \right) = -k \text{ and } \mathrm{ord}_Q \left(\langle e_i^k \mid e_j \rangle \right) = -k+1, \ \forall j \neq i$$

Let $\{\lambda_{i,k}, i = K+1, \dots, N, k \in F\}$ be a finite set of complex numbers (here F is a finite subset of $\mathbb{Z}$). Suppose that

$$\sum_{k \in F} \sum_{i=K+1}^{N} \lambda_{i,k} e_i^k = 0$$

We choose a local coordinate around Q and expand e_i^k as a Laurent series:

$$e_i^k(z) = b_{-k}^i e_i z^{-k} + \sum_{l > -k} B_l^i z^l, \ b_{-k}^i \neq 0$$

where $b_{-k}^i \in \mathbb{C}^*$ and $B_l^i \in \mathrm{vect}(e_{K+1}, \dots, e_{i-1}, e_{i+1}, \dots, e_N)$

Let M be the maximal element of F. We expand the linear sum $\sum_{k \in F} \sum_{i=K+1}^{N} \lambda_{i,k} e_i^k$ as a Laurent series in the same coordinate around Q:

$$\left(\sum_{k \in F} \sum_{i=K+1}^{N} \lambda_{i,k} e_i^k \right)(z) = \left(\sum_{i=K+1}^{N} \lambda_{i,M} b_{-M}^i e_i \right) z^{-M} + \sum_{l > -M} D_l z^l$$

with $D_l \in \mathcal{L}$.

Hence if $\sum_{k \in F} \sum_{i=K+1}^{N} \lambda_{i,k} e_i^k = 0$, we have necessarily

$$\sum_{i=K+1}^{N} \lambda_{i,M} b_{-M}^i e_i = 0.$$

Since $\{e_i, i = K+1, \ldots, N\}$ are linearly independent and since $b^i_{-M} \neq 0$ for $i = K+1, \ldots, N$, we get

$$\lambda_{i,M} = 0, \ \forall i \in \{K+1, \ldots, N\}.$$

We can now repeat this process for each "new" maximal element of F. That proves

$$\lambda_{i,k} = 0, \ \forall i \in \{K+1, \ldots, N\}, \forall k \in F.$$

Thus the e_i^k for $i = K+1, \ldots, N$ are linearly independent.

□

Theorem 2.69. *The set*

$$\left\{e_i^k, \ \ i = 1, \ldots, K, \ k \in \mathbb{N}; \ e_i^k, \ i = K+1, \ldots, N, \ k \in \mathbb{N}^*\right\}$$

is a basis of $\mathcal{L}_G(X, G(Q))$.

Proof. We proved in the previous proposition that the e_i^k are linearly independent. We just have to show that any element a of $\mathcal{L}_{\mathcal{G}}(X, G(Q))$ is generated by the above e_i^k.
The element a can be written as

$$a = \sum_{i=1}^{N} f_i \otimes e_i.$$

We proved before that since a is $\mathcal{G}$-invariant, the functions f_i for $i = 1, \ldots, K$ are in $\mathcal{M}_G(X)$. Moreover, since $a \in \mathcal{L}_{\mathcal{G}}(X, G(Q))$ the functions f_i are in $\mathcal{M}_G(X, G(Q))$. But the functions A_k form a basis of the algebra $\mathcal{M}_G(X, \Gamma)$ and in particular the functions A_k for $k \geq 0$ form a basis of $\mathcal{M}(X, G(Q))$. Hence for all $i = 1, \ldots, K$ the function f_i is written as

$$f_i = \sum_{l \in \mathbb{N}} c_l A_l$$

with a finite number of $c_l^i \neq 0$.
Hence

$$\sum_{i=1}^{K} f_i \otimes e_i = \sum_{i=1}^{K} \sum_{l \in \mathbb{N}} c_l^i A_l \otimes e_i = \sum_{i=1}^{K} \sum_{l \in \mathbb{N}} c_l^i e_i^l.$$

We prove here that $\{e_i^k, \ \ i = 1, \ldots, K, \ \ k \in \mathbb{N}\}$ is a basis of $\mathcal{M}(X, G(Q)) \otimes \mathcal{L}^G = \mathrm{vect}_{\mathcal{M}(X)}(e_i, \ i = 1, \ldots, K)$.

Consider now the element of $\mathcal{L}_{\mathcal{G}}(X, \{Q\})$

$$b := a - \sum_{i=1}^{K} \sum_{l\in\mathbb{N}} c_l^i e_i^l = \sum_{i=K+1}^{N} f_i \otimes e_i.$$

Suppose that b has a pole in Q with $\mathrm{ord}_Q(a(z)) = L$, $L < 0$. We choose a local coordinate around Q and expand b into a Laurent series:

$$b = \sum_{k\geq L} b_k z^k, \ \ b_k \in \mathcal{L}, \ \ a_L \neq 0.$$

Suppose that the lower element $b_L = \sum_{i=K+1}^{N} d_L^i e_i$ with $d_L^i \in \mathbb{C}$.
As in the proof of the previous proposition, we consider the Laurent series of the e_i^{-L} for $i = K+1, \ldots, N$ around Q:

$$e_i^{-L}(z) = b_L^i e_i z^L + \sum_{l>L} B_l^i z^l, \ \ b_k^i \neq 0$$

where $b_L^i \in \mathbb{C}^*$ and $B_l^i \in \mathcal{L}$.
We consider the element

$$b'(z) := b(z) - \sum_{i=K+1}^{N} \frac{d_L^i}{b_L^i} e_i^L(z).$$

By writing the Laurent series of b' around Q we see that $\mathrm{ord}_Q(b') > L$. Moreover the element b' is $\mathcal{G}$-invariant since it is the linear sum of $\mathcal{G}$-invariant elements. Thus by Theorem 2.64 we know that $\mathrm{ord}_Q(b') = \mathrm{ord}_{g\cdot Q}(b') > L$, $\forall g \in \mathcal{G}$.
We can repeat this method and get an element

$$b - \sum_{k=1}^{-L} \sum_{i=K+1}^{N} \frac{d_k^i}{b_k^i} e_i^k$$

of $\mathcal{L}_{\mathcal{G}}(X, G(Q))$ which is holomorphic in the points of $G(Q)$. More precisely we know that $b - \sum_{k=1}^{-L} \sum_{i=K+1}^{N} \frac{d_k^i}{b_k^i} e_i^k$ is a holomorphic element of $\mathcal{M}(X) \otimes \mathrm{Ker}\langle\ \rangle_G = \mathrm{vect}_{\mathcal{M}(X)}(e_{K+1}, \ldots, e_N)$. But since the e_i are not $\mathcal{G}$-invariant the only holomorphic element of $\mathcal{M}(X) \otimes \mathrm{Ker}\langle\ \rangle_G$ is 0. Hence

$$b - \sum_{k=1}^{-L} \sum_{i=K+1}^{N} \frac{d_k^i}{b_k^i} e_i^k = 0.$$

Since $b := a - \sum_{i=1}^{K} \sum_{l\in\mathbb{N}} c_l^i e_i^l$ we have

$$a - \sum_{i=1}^{K} \sum_{l \in \mathbb{N}} c_l^i e_i^l - \sum_{k=1}^{-L} \sum_{i=K+1}^{N} \frac{d_k^i}{b_k^i} e_i^k = 0.$$

That prove that $\left\{ e_i^k, \quad i = 1, \ldots, K, \; k \in \mathbb{N}; \; e_i^k, \; i = K+1, \ldots, N, \; k \in \mathbb{N}^* \right\}$ is a basis of $\mathcal{L}_G(X, G(Q))$.

□

We introduce the following decomposition of $\mathcal{L}_{\mathcal{G}}(X, G(Q))$:

$$\mathcal{L}_{\mathcal{G}}(X, G(Q)) = \bigoplus_{k \geq 0} \mathcal{L}_{\mathcal{G}}^k$$

where $\mathcal{L}_{\mathcal{G}}^k = \mathrm{vect}_{\mathbb{C}}(e_i^k \mid i = 1, \ldots, N)$ for all $k > 0$ and $\mathcal{L}_{\mathcal{G}}^0 = \mathrm{vect}_{\mathbb{C}}(e_i^0 \mid i = 1, \ldots, K)$.
We remark that $\mathcal{L}_{\mathcal{G}}^0 = \mathcal{L}^G = \mathrm{vect}_{\mathbb{C}}(e_i \mid i = 1, \ldots, K)$.

Theorem 2.70. *The decomposition*

$$\mathcal{L}_{\mathcal{G}}(X, G(Q)) = \bigoplus_{k \geq 0} \mathcal{L}_{\mathcal{G}}^k$$

is an almost-graded decomposition of the Lie algebra $\mathcal{L}_{\mathcal{G}}(X, G(Q))$ and

$$[\mathcal{L}_{\mathcal{G}}^k, \mathcal{L}_{\mathcal{G}}^l] \subseteq \mathcal{L}_{\mathcal{G}}^{k+l-2} \oplus \mathcal{L}_{\mathcal{G}}^{k+l-1} \oplus \mathcal{L}_{\mathcal{G}}^{k+l}.$$

Proof. First it is easy to check that for all $k, l \in \mathbb{N}$:

$$A_l \cdot \mathcal{L}_{\mathcal{G}}^k \subseteq \mathcal{L}_{\mathcal{G}}^{k+l}.$$

We check now the almost-grading under the bracket.
For $i, j \in \{1, \ldots, K\}$ and $k, l \in \mathbb{N}$:
We proved before that $\mathcal{L}^G$ is a Lie subalgebra of $\mathcal{L}$. Hence $[e_i, e_j] \in \mathcal{L}^G = \mathrm{vect}_{\mathbb{C}}(e_n \mid n = 1, \ldots, K)$.
Hence:

$$[e_i^k, e_j^l] = A_k \cdot A_l \otimes [e_i, e_j] = A_{k+l} \otimes [e_i, e_j] \in \mathrm{vect}_{\mathbb{C}}(e_n^{k+l} \mid n = 1, \ldots, K) \subset \mathcal{L}_{\mathcal{G}}^{k+l}$$

For $i \in \{1, \ldots, K\}$, $j \in \{K+1, \ldots, N\}$ and $k \in \mathbb{N}$, $l \in \mathbb{N}^*$:
We proved before that $[\mathcal{L}^G, \mathrm{Ker}\langle\,\rangle_G] \subset \mathrm{Ker}\langle\,\rangle_G$ and that $e_j^l \in \mathrm{vect}_{\mathcal{M}(X)}(e_n \mid n = K+1, \ldots, N)$. Hence it is clear that

$$[e_i, e_j^1] \in \mathrm{vect}_{\mathbb{C}}(e_n^1 \mid n = K+1, \ldots, N).$$

So
$$[e_i^k, e_j^l] = A_k \cdot A_{l-1}[e_i, e_j^1] \in \mathrm{vect}_{\mathbb{C}}(e_n^{k+l} \mid n = K+1, \dots N) \subset \mathcal{L}_{\mathcal{G}}^{k+l}.$$

For $i, j \in \{K+1, \dots, N\}$ and $k, l \in \mathbb{N}^*$:
The bracket $[e_i^1, e_j^1]$ is a $\mathcal{G}$ invariant element with poles of order 2 at the points of $G(Q)$. So we have
$$[e_i^1, e_j^1] \in \mathcal{L}_{\mathcal{G}}^0 \oplus \mathcal{L}_{\mathcal{G}}^1 \oplus \mathcal{L}_{\mathcal{G}}^2.$$

Hence
$$[e_i^k, e_j^l] = A_{k-1} \cdot A_{l-1}[e_i^1, e_j^1] \in \mathcal{L}_{\mathcal{G}}^{k+l-2} \oplus \mathcal{L}_{\mathcal{G}}^{k+l-1} \oplus \mathcal{L}_{\mathcal{G}}^{k+l}.$$

□

Theorem 2.71. *The algebra $\tilde{\mathcal{L}}_{\mathcal{G}}(X, \Gamma) := vect_{\mathbb{C}}(e_i^k \mid i = 1, \dots N,\ k \in \mathbb{Z})$ is a Lie subalgebra of $\mathcal{L}_{\mathcal{G}}(X, \Gamma)$ and admits an almost-graded decomposition*
$$\tilde{\mathcal{L}}_{\mathcal{G}}(X, \Gamma) := \bigoplus_{k \in \mathbb{Z}} \tilde{\mathcal{L}}_{\mathcal{G}}^k$$
where $\tilde{\mathcal{L}}_{\mathcal{G}}^k := vect_{\mathbb{C}}(e_i^k \mid i = 1, \dots N)$. Moreover
$$[\mathcal{L}_{\mathcal{G}}^k, \mathcal{L}_{\mathcal{G}}^l] \subseteq \mathcal{L}_{\mathcal{G}}^{k+l-2} \oplus \mathcal{L}_{\mathcal{G}}^{k+l-1} \oplus \mathcal{L}_{\mathcal{G}}^{k+l}.$$

Proof. It is just an extension of the previous proof: our arguments are available for negative k. Hence the vector space $\tilde{\mathcal{L}}_{\mathcal{G}}(X, \Gamma)$ is closed under the bracket. So it is a Lie subalgebra which inherits the almost-graded structure. □

Remark: Even if the e_i^k for positive integers k generate the algebra $\mathcal{L}_{\mathcal{G}}(X, G(Q))$, the negative ones do not generate the algebra $\mathcal{L}_{\mathcal{G}}(X, G(P))$. Hence the algebra $\tilde{\mathcal{L}}_{\mathcal{G}}(X, \Gamma)$ is just a Lie subalgebra of $\mathcal{L}_{\mathcal{G}}(X, \Gamma)$. It is not clear if the full algebra $\mathcal{L}_{\mathcal{G}}(X, \Gamma)$ admits or not an almost-graded decomposition.

2.4.4.2 Examples

Here the Riemann surface X is still the Riemann sphere $\mathbb{C}_\infty$.
We choose here for $\mathcal{L}$ the Lie algebra $sl(2, \mathbb{C})$ of the traceless matrices of order 2×2.

The usual basis of $sl(2,\mathbb{C})$ is $\{x, y, h\}$ where

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The bracket is the commutator and we have

$$[x, y] = h, \quad [x, h] = -2x, \quad [y, h] = 2y.$$

The automorphisms of $sl(2,\mathbb{C})$ are given by the conjugation by an invertible matrix:

$$Aut(sl(2,\mathbb{C})) = \{a \mapsto MaM^{-1} \mid M \in GL(2,\mathbb{C})\}$$

Note that Q and cQ with $c \neq 0$ give the same automorphism.

1. Our first example can be found in details in the paper of Lombardo and Mikahailov [20].
   Let G be the dihedral group $\mathbb{D}_2$. We said before that it is a group of order 4 and generated by the transformations

   $$\sigma_s(z) = -z, \qquad \sigma_t(z) = \frac{1}{z}$$

   Since an automorphism of $sl(2,\mathbb{C})$ is given by a invertible matrix up to a multiplicative constant, a monomorphism $\psi : \mathbb{D}_2 \to Aut(sl(2,\mathbb{C}))$ is nothing but a faithful projective representation of $\mathbb{D}_2$ into $PGL(2,\mathbb{C})$: $\tilde{\psi} : \mathbb{D}_2 \to PGL(2,\mathbb{C})$. Here there is just one class of equivalent faithful representations given by the representation $\tilde{\psi}$ such that

   $$S := \tilde{\psi}(\sigma_s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad T := \tilde{\psi}(\sigma_t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

   Hence our group $\mathcal{G} = diag(G, \psi(G))$ is generated by two transformations

   $$g_s : \mathcal{L}(X) \to \mathcal{L}(X); a(z) \to Sa(-z)S^{-1}, \qquad g_t : \mathcal{L}(X) \to \mathcal{L}(X); a(z) \to Ta(1/z)T^{-1}$$

   and the group average is

   $$\langle a(z) \rangle_{\mathcal{G}} = \frac{1}{4}\left(a(z) + Sa(-z)S^{-1} + Ta(1/z)T^{-1} + TSa(-1/z)S^{-1}T^{-1}\right).$$

   We easily check that

   $$\langle x \rangle_G = \langle y \rangle_G = \langle h \rangle_G = 0.$$

In this example the subalgebra $\mathcal{L}^G$ is reduced to zero. We take as a basis of $\mathrm{Ker}\langle\ \rangle_G$ the basis $\{x, y, h\}$. Let Q be an elements with a trivial stabilizer subgroup and P an element such that $G(P) \cap G(Q) = \emptyset$. Hence, for P only the elements of $\mathbb{C}_\infty \setminus \{0, \infty, \pm 1, \pm i\}$ are allowed. Let $\Gamma := G(P) \cup G(Q)$
We give here the explicit almost-graded basis of the algebra $\tilde{\mathcal{L}}_{\mathcal{G}}(X, \Gamma)$ (Theorem 2.71). Of course by Theorem 2.70, this also gives an almost-graded basis of $\mathcal{L}_{\mathcal{G}}(X, G(Q))$.

We compute the element e_x^1, e_y^1 and e_h^1:

$$e_x^1(z) = \left\langle \frac{x}{z-Q} \right\rangle_{\mathcal{G}} = \begin{pmatrix} 0 & \frac{z}{2(z^2-P^2)} \\ \frac{z}{2(1-z^2P^2)} & 0 \end{pmatrix},$$

$$e_y^1(z) = \left\langle \frac{y}{z-Q} \right\rangle_{\mathcal{G}} = \begin{pmatrix} 0 & \frac{z}{2(1-z^2P^2)} \\ \frac{z}{2(z^2-P^2)} & 0 \end{pmatrix},$$

$$e_h^1(z) = \left\langle \frac{y}{z-Q} \right\rangle_{\mathcal{G}} = \frac{P(1-z^4)}{2(z^2-P^2)(1-z^2P^2)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here the elements A_k (which are a graded basis of $\mathcal{M}_G(X, \Gamma)$) are fixed by the condition $res_P A_1 = 1$. Hence

$$A_1(z) = \alpha \frac{(z^2-Q^2)(1-z^2Q^2)}{(z^2-P^2)(1-z^2P^2)}, \quad \alpha = \frac{2P(P^4-1)}{(Q^2-P^2)(1-Q^2P^2)},$$

$$A_k = A_1^k.$$

And the element e_i^k are

$$e_x^k = A_{k-1} e_x^1, \quad e_y^k = A_{k-1} e_y^1, \quad e_h^k = A_{k-1} e_h^1, \quad k \in \mathbb{Z}.$$

For $n, m \in$:

$$[e_x^k, e_y^l] = \frac{1}{16}\left(a e_h^{k+l-1} + e_h^{k+l}\right),$$

$$[e_h^k, e_x^l] = \frac{1}{16}\left(b e_x^{k+l-2} - c e_h^{k+l-1} + 2 e_x^{k+l}\right),$$

$$[e_h^k, e_y^l] = \frac{1}{16}\left(-b e_y^{k+l-2} + c e_x^{k+l-1} - 2 e_y^{k+l}\right).$$

where

$$a = \frac{2Q^2(1-P^4)}{P(Q^2-P^2)(1-P^2Q^2)}, \qquad b = \frac{4P(1+Q^4-4Q^2P^2+P^4+P^4Q^4)}{(1-P^4)(Q^2-P^2)(1-P^2Q^2)},$$

$$c = \frac{8P}{1-P^4}.$$

2. We consider now an example with the subalgebra $\mathcal{L}^G \neq \{0\}$. Let $G = \mathbb{Z}_2$ the finite group generated by the transformation $\sigma(z) = -z$. As in the first example, a monomorphism $\psi : G \to Aut(sl(2, \mathbb{C}))$ is equivalent to a faithful projective representation $\tilde{\psi} : \mathbb{Z}_2 \to PGL(2, \mathbb{C})$. $\tilde{\psi}$ is determined by the image of σ. Since σ is a transformation of order 2, the matrix $S := \tilde{\psi}(\sigma)$ is of order 2. Let we choose $S = \tilde{\psi}(\sigma_s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
Therefore the group $\mathcal{G} = diag(G, \psi(G))$ is generated by the transformation

$$g : \mathcal{L}(X) \to \mathcal{L}(X);\ a(z) \to Sa(-z)S^{-1}$$

and the average $\langle\ \rangle_{\mathcal{G}}$ is

$$\langle a(z) \rangle_{\mathcal{G}} = \frac{1}{2}\left(a(z) + Sa(-z)S^{-1}\right).$$

We have

$$\langle x \rangle_{\mathcal{G}} = \frac{1}{2}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = 0,$$

$$\langle y \rangle_{\mathcal{G}} = \frac{1}{2}\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = 0,$$

$$\langle h \rangle_{\mathcal{G}} = \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = h.$$

Hence $\mathcal{L}^G = \mathrm{vect}_{\mathbb{C}}(h)$ and a basis of $\mathrm{Ker}\langle\ \rangle_G$ is $\{x, y\}$. Let Q be a point with a trivial stabilizer subgroup (i.e. $Q \neq 0, \infty$) and P a point such that $G(P) \cap G(Q) = \emptyset$; let $\Gamma := G(P) \cup G(Q)$.
We give here the explicit almost-graded basis of the algebra $\tilde{\mathcal{L}}_{\mathcal{G}}(X, \Gamma)$ (Theorem 2.71). Of course by Theorem 2.70, this also gives an almost-graded basis of $\mathcal{L}_{\mathcal{G}}(X, G(Q))$.
A graded basis of $\mathcal{M}_G(X, \Gamma)$ is

$$A_k(z) := \left(\frac{z^2 - P^2}{z^2 - Q^2}\right)^k, \quad k \in \mathbb{Z}$$

if $P \neq \infty$, and

$$A_k(z) := \left(\frac{1}{z^2 - Q^2}\right)^k, \quad k \in \mathbb{Z}$$

if $P = \infty$.

Now we define

$$e_x^1(z) := \left\langle \frac{x}{z-Q} \right\rangle_{\mathcal{G}} = \frac{-z}{z^2-Q^2}x,$$

$$e_y^1(z) := \left\langle \frac{y}{z-Q} \right\rangle_{\mathcal{G}} = \frac{-z}{z^2-Q^2}y,$$

and for $k \in \mathbb{Z}$

$$\begin{aligned} e_x^k(z) &:= A_{k-1}e_x^1(z), \\ e_y^k(z) &:= A_{k-1}e_y^1(z), \\ e_h^k(z) &:= A_k h. \end{aligned}$$

The brackets of these elements are:

$$\begin{aligned} [e_x^k, e_h^l] &= -2e_x^{k+l} \\ [e_y^k, e_h^l] &= 2e_y^{k+l} \\ [e_x^k, e_y^l] &= ae_h^{k+l} + be_h^{k+l-1} + ce_h^{k+l-2} \end{aligned}$$

where

$$a = \frac{Q^2}{P^4-2P^2Q^2-Q^4}, \qquad b = \frac{-P^2-Q^2}{P^4-2P^2Q^2-Q^4}, \qquad c = \frac{P^2}{P^4-2P^2Q^2-Q^4}$$

if $P \neq \infty$, and

$$a = Q^2, \quad b = 1, \quad c = 0$$

if $P = \infty$.

Note that for $P = \infty$ the almost-grading is just

$$[\mathcal{L}_{\mathcal{G}}^k, \mathcal{L}_{\mathcal{G}}^l] \subseteq \mathcal{L}_{\mathcal{G}}^{k+l-1} \oplus \mathcal{L}_{\mathcal{G}}^{k+l}.$$

2.4.4.3 The case of the complex tori

Let $X = \mathbb{C}/L$ be a complex torus. Let $G \neq \{id\}$ be a finite group of automorphism of X. We suppose here that G contains at least one automorphism fixing at least one point of X. We know that in these situations X/G is isomorphic to the Riemann sphere. Let $\pi : X \to X/G$ be the canonical projection.

Let $\{P, Q\}$ be two points which are not fixed by the elements of G and such that the orbits $G(P)$ and $G(Q)$ are disjoints. We consider the set $\Gamma := G(P) \cup G(Q)$. Note that

$\pi(\Gamma) = \pi(P) \cup \pi(Q)$.
We saw above that the algebra $\mathcal{M}_G(X,\Gamma)$ is isomorphic to $\mathcal{M}(X/G,\pi(\Gamma))$. We gave an explicit basis of $\mathcal{M}(X/G,\pi(\Gamma))$:

$$\tilde{A}_k(z) := (z-\pi(P))^k(z-\pi(Q))^{-k}$$

The corresponding basis of $\mathcal{M}_G(X,\Gamma)$ can be expressed in terms of translated theta-function: in fact $\Gamma = \{g\cdot P, g\cdot Q,\ g\in G\}$ and since the group G is not a group of translation we have $\sum_{g\in G} g\cdot P = \sum_{g\in G} g\cdot Q = 0$. In fact we can consider (up to a translation) that the automorphism of G are fixing 0. And by Proposition 2.43 the orbit of a point R consists of the points ζR where ζ runs through all N^{th} roots of unity, with suitable N. As their sum equals zero the sum over the full orbit will be 0 too.
Let $\{p_g,\ g\in G\}$ and$\{q_g,\ g\in G\}$ be the sets of complex numbers which satisfy conditions of theorem 2.27. Then consider

$$A_k(z) := \left(\prod_{g\in G}\theta^{(p_g)}(z)\right)^k \cdot \left(\prod_{g\in G}\theta^{(q_g)}(z)\right)^{-k}$$

The set $\{A_n,\ n\in\mathbb{Z}\}$ is a graded basis of the algebra $\mathcal{M}_G(X,\Gamma)$.

Let $\mathcal{L}$ be a finite-dimensional Lie algebra and $\{e_1,\dots,e_N\}$ a basis of $\mathcal{L}$ such that $\{e_1,\dots,e_K\}$ is a basis of $\mathcal{L}^G$ and $\{e_K,\dots,e_N\}$ is a basis of $\mathrm{Ker}\langle\ \rangle_G$.

Let $f\in\mathcal{M}(X)$ be a meromorphic function with a single pole in P and Q and holomorphic elsewhere. Such a function always exists but is not unique. We introduce the following elements of $\mathcal{L}_{\mathcal{G}}(X,\Gamma)$:

$$\begin{aligned} e_i^k &= A_k\otimes e_i, \quad \forall k\in\mathbb{Z},\ i=1,\dots,K;\\ e_i^1 &= \langle f\otimes e_i\rangle_{\mathcal{G}}, \quad i=K+1,\dots,N;\\ e_i^k &= A_{k-1}\cdot e_i^1, \quad \forall k\in\mathbb{Z},\ i=K+1,\dots,N; \end{aligned}$$

Note that
$$\langle f\otimes e_i\rangle_{\mathcal{G}} = \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} g\cdot(f\otimes e_i) = \frac{1}{|G|}\sum_{g\in G}(g\cdot f)\otimes\psi(g)\cdot e_i)$$

Moreover it's clear that the elements $e_i^k(z)$ are $\mathcal{G}$ invariant.

Proposition 2.72. *For all $i=1\dots n$ and $k\in\mathbb{Z}$ the elements $e_i^k(z)$ are non-zero and linearly independant.*

Proof. It is the same proof as the proof of Proposition 2.68. □

Let $\mathcal{L}_{\mathcal{G}}^{\leq 1}$ be the subvector space of $\mathcal{L}(X,\Gamma)$ of the $\mathcal{G}$-invariant elements which are holomorphic in $X \setminus G(P) \cup G(Q)$ with poles in $G(P)$ and $G(Q)$ of order maximum 1. This vector space contains the elements e_i^1 for $i = 1, \dots, n$ and the elements e_i^0 for $i = 1, \dots, K$. Since these elements are linearly independent we complete them to a basis of $\mathcal{L}_{\mathcal{G}}^{\leq 1}$. In fact since the holomorphic elements of $\mathcal{L}_{\mathcal{G}}(X,\Gamma)$ are all generated by the e_i^0 for $i = 1, \dots, K$, just some elements with poles of order 1 left. Let $\epsilon_1, \dots, \epsilon_M$ be these elements.

Theorem 2.73. *The set*

$$\left\{\epsilon_1, \dots, \epsilon_M; e_i^k,\ i = 1 \dots N,\ k \in \mathbb{Z}\right\}$$

is a basis of $\mathcal{L}_G(X,\Gamma)$.

Proof. We proved in the previous proposition that the e_i^k are linearly independent. It is obvious that $\left\{\epsilon_1, \dots, \epsilon_M; e_i^k,\ i = 1, \dots, N,\ k \in \mathbb{Z}\right\}$ is a set of linearly independent elements. Thus we just have to show that any element of $\mathcal{L}_{\mathcal{G}}(X,\Gamma)$ is generated by the ϵ_j and the e_i^k.

Let $a \in \mathcal{L}_{\mathcal{G}}(X,\Gamma)$. The element a can be written as

$$a = \sum_{i=1}^{N} f_i \otimes e_i.$$

We proved before that since a is $\mathcal{G}$-invariant, the functions f_i for $i = 1, \dots, K$ are in $\mathcal{M}_G(X)$. Moreover, since $a \in \mathcal{L}_{\mathcal{G}}(X,\{Q\})$ the functions f_i are in $\mathcal{M}_G(X,\{Q\})$. But the functions A_k form a basis of the algebra $\mathcal{M}_G(X,\Gamma)$ and in particular the functions A_k for $k \geq 0$ form a basis of $\mathcal{M}(X,\{Q\})$. Hence for all $i = 1, \dots, K$ the function f_i is written as

$$f_i = \sum_{l \in \mathbb{N}} c_l A_l$$

with a finite number of $c_l^i \neq 0$.

Hence

$$\sum_{i=1}^{K} f_i \otimes e_i = \sum_{i=1}^{K} \sum_{l \in \mathbb{N}} c_l^i A_l \otimes e_i = \sum_{i=1}^{K} \sum_{l \in \mathbb{N}} c_l^i e_i^l.$$

We prove here that $\{e_i^k,\ i = 1, \dots, K,\ k \in \mathbb{N}\}$ is a basis of $\mathcal{M}(X,\{Q\}) \otimes \mathcal{L}^G = \mathrm{vect}_{\mathcal{M}(X)}(e_i,\ i = 1, \dots, K)$.

Consider now the element of $\mathcal{L}_{\mathcal{G}}(X, \{Q\})$

$$b := a - \sum_{i=1}^{K} \sum_{l \in \mathbb{N}} c_l^i e_i^l = \sum_{i=K+1}^{N} f_i \otimes e_i.$$

Suppose that b has a pole in Q with $\mathrm{ord}_Q(a(z)) = L$, $L < 0$. We choose a local coordinate around Q and expand b into a Laurent series:

$$b = \sum_{k \geq L} b_k z^k, \; b_k \in \mathcal{L}, \; a_L \neq 0.$$

Suppose that the lower element is $b_L = \sum_{i=K+1}^{N} d_L^i e_i$ with $d_L^i \in \mathbb{C}$.
It is easy to check that for $i = K+1, \ldots, N$ and $j \neq i$

$$\mathrm{ord}_Q \left\langle e_i^k \mid e_i \right\rangle = -k, \quad \mathrm{ord}_Q \left\langle e_i^k \mid e_j \right\rangle = -k+1, \quad \forall k \geq 1,$$

$$\mathrm{ord}_P \left\langle e_i^k \mid e_i \right\rangle = 0, \quad \mathrm{ord}_P \left\langle e_i^k \mid e_j \right\rangle = 0, \quad \forall k \geq 2,$$

We consider the Laurent series of the e_i^{-L} for $i = K+1, \ldots, N$ around Q:

$$e_i^{-L}(z) = b_L^i e_i z^L + \sum_{l > L} B_l^i z^l, \; b_k^i \neq 0$$

where $b_L^i \in \mathbb{C}^*$ and $B_l^i \in \mathcal{L}$.
We consider the element

$$b'(z) := b(z) - \sum_{i=K+1}^{N} \frac{d_L^i}{b_L^i} e_i^L(z).$$

By writing the Laurent series of b' around Q we see that $\mathrm{ord}_Q(b') > L$. Moreover the element b' is $\mathcal{G}$-invariant since it is the linear sum of $\mathcal{G}$-invariant elements. Thus by theorem 2.64 we know that $\mathrm{ord}_Q(b') = \mathrm{ord}_{g \cdot Q}(b') > L, \; \forall g \in \mathcal{G}$.
We can repeat this method and get an element

$$b' = b - \sum_{k=2}^{-L} \sum_{i=K+1}^{N} \frac{d_k^i}{b_k^i} e_i^k$$

of $\mathcal{L}_{\mathcal{G}}(X, \Gamma)$ which has poles of order at most 1 at the points of $G(Q)$.
By using the elements e_i^k for $i = K+1, \ldots, N$ with $k \leq 0$ we can reduce as well to the order 1 the order of the poles at the points of $G(P)$ of the element b'. Finally we get an element b' which have poles of order 1 at the points of $G(P)$ and $G(Q)$. Hence $b' \in \mathcal{L}_{\mathcal{G}}^{\leq 1}$.
That ends the proof. $\square$

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Résumés de la thèse

Structures Lie-admissibles sur les algèbres de type Witt et les algèbres automorphes.

L'algèbre de Witt a été intensivement étudiée. Elle est présente dans de nombreux domaines des Mathématiques. Cette thèse est l'étude de deux généralisations de l'algèbre de Witt: les algèbres de type Witt et les algèbres de Krichever-Novikov. Dans une première partie on s'intéresse aux structures Lie-admissibles sur les algèbres de type Witt. On donne toutes les structures troisième-puissance associatives et flexibles Lie-admissibles sur ces algèbres. De plus, on étudie les formes symplectiques qui induisent un produit symétrique-gauche.

Dans une seconde partie on étudie les algèbres automorphes. Partant d'une surface de Riemann compacte quelconque, on considère l'action d'un sous-groupe fini du groupe des automorphismes de la surface sur des algèbres d'origines géométriques comme les algèbres de Krichever-Novikov. Plus précisément nous faisons le lien entre la sous-algèbre des éléments invariants sur la surface et l'algèbre sur la surface quotient. La structure presque-gradue des algèbres de Krichever-Novikov induit une presque-graduation sur ces sous-algèbres de certaines algèbres de Krichever- Novikov.

Lie-admissible structures on Witt type algebras and automorphic algebras.

The Witt algebra has been intensively studied and arise in many research fields in Mathematics. We are interested in two generalizations of the Witt algebra: the Witt type algebras and the Krichever-Novikov algebras. In a first part we study the problem of finding Lie-admissible structures on Witt type algebras. We give all third-power associative Lie-admissible structures and flexible Lie-admissible structures on these algebras. Moreover we study the symplectic forms which induce a graded left-symmetric product.

In the second part of the thesis we study the automorphic algebras. Starting from arbitrary compact Riemann surfaces we consider the action of finite subgroups of the automorphism group of the surface on certain geometrically defined Lie algebras as the Krichever-Novikov type algebras. More precisely, we relate for G a finite subgroup of automorphism acting on the Riemann surface, the invariance subalgebras living on the surface to the algebras on the quotient surface under the group action. The almost-graded Krichever-Novikov algebras structure on the quotient gives in this way a subalgebra of a certain Krichever-Novikov algebra (with almost-grading) on the original Riemann surface.