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par

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sous la direction de Daniel Matignon

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Problème d'existence de feuilletage tendu dans les 3-variétés

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A mon petit Titouan,
A mon grand Amour,
A mon tout petit,

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SOMMAIRE

Cette thèse s'articule en trois chapitres. Les deux derniers peuvent être lu indépendamment, bien qu'on utilise dans le Chapitre 3 un résultat qui est démontré au Chapitre 2.

Le premier Chapitre est une introduction aux différents problèmes que l'on s'est posés au cours de ce travail. On y présente les principaux résultats ainsi que le contexte dans lesquels ils se placent.

Le Chapitre 2 fait l'objet d'un article soumis, c'est pourquoi il est rédigé en anglais. On y étudie les feuilletages non-tendus sans composantes de Reeb, ainsi que les feuilletages tendus. On remarquera que cela amène à comprendre qu'au voisinage d'une feuille torique, le feuilletage possède toujours une composante de tourbillonement généralisée ou de spiralement généralisée. Ceci nous permettra de donner une condition nécessaire et suffisante pour qu'un feuilletage soit tendu en terme d'orientation transverse des feuilles toriques. Dans cette thèse nous avons ajouté la Section 2.8.3 pour expliquer le lien avec la construction de Gabai [1983].

Le Chapitre 3 fait l'objet d'un article co-écrit avec Daniel Matignon, qui a été soumis, c'est pourquoi il est aussi rédigé en anglais. Il traite de l'existence de feuilletage tendu dans les 3-sphères d'homologies fibrées de Seifert. On y montre que toutes les 3-sphères d'homologie entière fibrées de Seifert (sauf $\mathbb{S}^3$ et la sphère d'homologie de Poincaré) possèdent un feuilletage tendu. On y exhibe aussi une infinité de sphères d'homologie rationnelle (non-entière) fibrées de Seifert possédant un feuilletage tendu, et une infinité n'en admettant pas.

Dans cette thèse nous avons encore ajouté le Lemme 3.5.1 pour expliquer la caractérisation des sphères d'homologie entière et rationnelle.

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Chapitre 1

Introduction

Depuis plus de cent ans, la classification topologique des surfaces est bien connue. Par contre la classification des variétés de dimension trois (compactes connexes orientables) reste inachevée, bien que la résolution de la conjecture de la Géométrisation de Thurston ait été une grande avancée. Une des piste privilégiée consiste à s'intéresser aux plongements de surfaces dans les 3-variétés.

Pour ce faire, on étudie les plongements de surfaces incompressibles (dont le π_1 est injectif) dans les variétés de dimension trois.
Par exemple, avant la célèbre preuve de Perelman, Thurston [1982, 1986b] donnait déjà une démonstration topologique de la Géométrisation des 3-variétés dans le cas où la variété est *irréductible* (toute sphère plongée borde une 3-boule) et possède une surface essentielle (incompressible et non parallèle au bord).
Rappelons que la conjecture de la Géométrisation consiste à dire que toute 3-variété compacte est soit hyperbolique, soit une variété de Seifert, soit elle possède un tore incompressible, ce qui montre le rôle prédominant des tores plongés dans les 3-variétés, qui comme nous le verrons, sera confirmé par l'étude des feuilletages.

Ainsi, en appliquant la célèbre JSJ-décomposition canonique (Jaco et Shalen [1979], Johannson [1979]) le long de tores incompressibles, on peut voir une 3-variété compacte connexe irréductible et orientée, comme étant découpée par un nombre fini de tores essentiels, et chaque sous-variété de dimension 3 bordée par ces tores est soit hyperbolique, soit une variété de Seifert. Cela donne déjà une bonne approche des 3-variétés.
Une restriction à cette visualisation est pour les variétés à géométrie Sol (admettant un revêtement fini par un fibré en tores sur le cercle avec homéomorphisme de recollement hyperbolique), pour qui la décomposition JSJ éclate le long d'un tore incompressible donnant une variété n'admettant pas de géométrie à volume fini, alors que sans l'éclater, ces variétés admettent la géométrie Sol qui est bien sûr à volume fini. Pour plus de détails, voir l'article de Scott [1983].
Ces variétés à géométrie Sol admettent trivialement un feuilletage avec uniquement des feuilles toriques, donc on s'intéressera moins à ces variétés.

La théorie des *feuilletages* peut être vue comme l'étape suivante dans l'étude du plongement de surfaces dans les 3-variétés. En effet, elle consiste à décomposer une 3-variété en réunion disjointe de surfaces, tel que localement un voisinage ouvert soit découpé en plans horizontaux ; c'est la notion de *feuilletage* de codimension 1. Plus précisément :

Definition 1.0.1. *Un* **feuilletage** *de codimension k sur une n-variété M est une manière de décomposer M en une collection de $(n-k)$-variétés connexes par arc ; appelées* **feuilles** *du feuilletage.*
Localement, la variété est homéomorphe à $\mathbb{R}^n$, feuilleté par des "plans horizontaux" :
$\forall x \in M, \exists U$ *un voisinage ouvert de x et un homéomorphisme*

$$h : U \to]-1,1[^{n-k} \times]-\epsilon,\epsilon[^k$$

tel que les $(n-k)$-variétés s'envoient sur les "plans horizontaux" :

$$]-1,1[^{n-k} \times \{x_{n-k+1}\} \times ... \times \{x_n\}$$

Un tel voisinage U sera appelé un **voisinage distingué**.
On se limitera au cas où $n = 3$ et $k = 1$, on aura donc :

$$U \cong]-1,1[^2 \times]-\epsilon,\epsilon[$$

On parle de $\mathcal{C}^r$-feuilletage, $r \in \mathbb{N}$, lorsque les applications de changements de coordonnées sont de classe $\mathcal{C}^r$ et préservent le feuilletage.
C'est à dire qu'un feuilletage de classe $\mathcal{C}^r$ est un atlas (U_α, h_α) dont les applications de changements de coordonnées

$$t_{\alpha\beta} = h_\alpha \circ h_\beta^{-1} = (t^1_{\alpha\beta}, t^2_{\alpha\beta}, t^3_{\alpha\beta}) : h_\beta(U_\alpha \cap U_\beta) \to h_\alpha(U_\alpha \cap U_\beta)$$

sont $\mathcal{C}^r$, les $t^i_{\alpha\beta}$, $i = 1,2,3$ sont des fonctions de $(x,y,z) \in \mathbb{R}^3$ et $t^3_{\alpha\beta}$ ne dépend que de z.

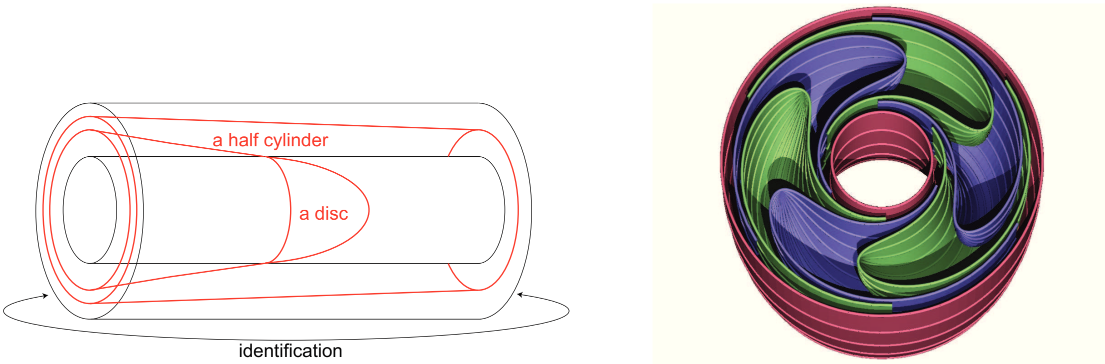
Sauf mention du contraire, nous nous restreindrons à l'étude des feuilletage de classe $\mathcal{C}^2$ (de codimension 1 sur les 3-variétés).

Historiquement, Reeb [1952] a introduit cette notion avec le point de vue des formes différentielles, pendant sa thèse, sous la direction de Ehresmann. Il mit alors en évidence un feuilletage particulier de $\mathbb{S}^3$, obtenu en recollant deux *composantes de Reeb*.
On rappelle qu'une ***composante de Reeb*** est un feuilletage de $\mathbb{D}^2 \times \mathbb{S}^1$ par une feuille torique au bord et des plans à l'intérieur comme dans la Figure 1.1 (voir encore le Chapitre 2).

Il dégagea encore la notion de *procédé de tourbillonement* qui consiste à effacer d'une 3-variété, un tore solide feuilleté trivialement et de le remplacer par une composante de Reeb sans changer la variété (voir le Chapitre 2, et la Figure 1.2).

Le premier problème qui s'est alors posé est celui de l'existence de feuilletages dans les 3-variétés qui seront toujours ici supposées compactes connexes orientables.

Thurston [1976] donna la preuve de l'existence de feuilletage de codimension 1 dans toutes les 3-variétés compactes, en isotopant des distributions (champs de plans) de manière



(a) Composante de Reeb

(b) (Demi) composante de Reeb (source : Wikipedia)

FIGURE 1.1 – Deux représentations d'une composante de Reeb

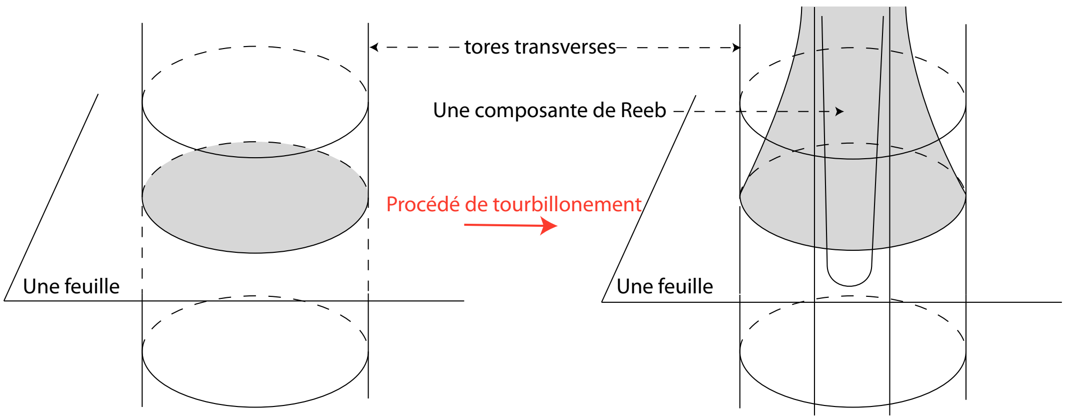


FIGURE 1.2 – Procédé de Tourbillonement

à les rendre intégrables, mais il ne donna pas de construction explicite.

Par ailleurs le procédé de tourbillonement de Reeb fit réaliser indépendamment par Lickorish [1965] et Novikov [1964] (aidé de Zieshang) comment construire explicitement un feuilletage avec composante de Reeb.
Ils partirent du fait de Lickorish [1962] et Wallace [1960] que toutes les 3-variétés compactes fermées s'obtiennent en effaçant le voisinage tubulaire d'un entrelacs de $\mathbb{S}^3$ puis en le recollant d'une manière spécifique à la 3-variété.
En voyant cet entrelacs dans le feuilletage de Reeb de $\mathbb{S}^3$ et en le rendant transverse au feuille, (par un théorème d'Alexander [1923] c'est toujours possible) ; ils effacent un voisinage de cet entrelacs feuilleté trivialement par $\mathbb{D}^2 \times \mathbb{S}^1$ puis recollent (par le bon homéomorphisme) avec le procédé de tourbillonement, des composantes de Reeb. Pour plus de détails voir le Chapitre 2.
Le cas des variétés à bord est clair en considérant le double de la variété.

Finalement les feuilletages admettant des composantes de Reeb sont trop courants pour donner des informations sur la variété sous-jacente.

Une question naturelle vient alors se poser :

Question 1.0.2. *Quelles sont les* 3*-variétés admettant un feuilletage sans composante de Reeb ?*

Rappelons tout d'abord une notion largement utilisée dans l'étude des feuilletages de codimension 1.
Un feuilletage est dit *transversalement orientable* s'il existe un champs de vecteur continue, ne s'annulant pas, transverse aux feuilles. Cela revient à pouvoir choisir de manière continue une normale aux feuilles. En fait cela est toujours possible quitte à considérer un revêtement double de la variété.

C'est le célèbre théorème de Novikov [1965] qui donne des conditions nécessaires.

Theorem 1.0.3 (Novikov [1965])**.** *Si M est une* 3*-variété compacte connexe et orientée, $M \ncong \mathbb{S}^2 \times \mathbb{S}^1$, admettant un feuilletage $\mathcal{F}$ transversalement orienté et sans composante de Reeb alors :*

- *$\pi_1(M)$ est infini ;*
- *Pour toute $F \in \mathcal{F}, \pi_1(F)$ s'injecte dans le $\pi_1(M)$, i.e les feuilles sont* incompressibles *;*
- *$\pi_2(M) = \{0\}$*

De plus en appliquant le théorème de Palmeira [1978], on obtient que sous de telles hypothèses, le revêtement universel de M est $\mathbb{R}^3$.
Par ailleurs Rosenberg [1968] a montré que sous ces conditions, M est irreductible (toute sphère plongée borde une 3-boule).
Une autre propriété importante des feuilletages sans composante de Reeb sera dégagée par Thurston [1986a], dont nous ne donneront ici qu'une version sans bord :

Theorem 1.0.4 (Thurston [1986a])**.** *Si M est une* 3*-variété fermée, compacte connexe et orientée, admettant un feuilletage $\mathcal{F}$ transversalement orienté sans composante de Reeb, alors :*
Toute feuille $F \in \mathcal{F}$ compacte est de genre minimal dans sa classe d'homologie ; i.e pout toute surface S surface plongée dans M telle que $[S] = [F]$ dans $H_2(M)$ alors :
$genre(S) \geq genre(F)$

Gabai améliora la théorie en introduisant la notion de feuilletage *tendu* (i.e toutes les feuilles sont traversées par une boucle ou un arc proprement plongé transverse au feuilletage), plus générale que feuilletage sans composante de Reeb. En effet on verra au Chapitre 2 qu'une feuille séparante, ou au bord (sous certaines conditions), ne peut être traversée par une boucle, ou un arc proprement plongé, transverse au feuilletage.

Une question plus générale, toujours ouverte se pose alors :

Question 1.0.5. *Quelles sont les* 3*-variétés admettant un feuilletage tendu ?*

Une conséquence directe du Théorème 1.0.3 et du paragraphe ci-dessus, est que ni $\mathbb{S}^3$ ni la sphère d'homologie de Poincaré n'admettent de feuilletages tendus (leur π_1 est fini).

Par contre Gabai [1983] donna une construction explicite de feuilletages tendus sur les 3-variétés ayant une homologie non-triviale, qui montre que l'hypothèse de genre minimal du Théorème 1.0.4 est fondamentale.

C'est dans ce cadre que s'inscrit cette thèse, qui s'intéresse tout d'abord (Chapitre 2) au rôle crucial des feuilles compactes, afin de mieux comprendre les feuilletages non-tendus sans composante de Reeb, et les feuilletages tendus (qui sont donc aussi sans composantes de Reeb), puis dans un second temps on s'intéressera à l'existence de feuilletage tendu dans les 3-sphères d'homologie fibrées de Seifert (Chapitre 3).

On remarquera que c'est au voisinage des feuilles toriques incompressibles que l'information est concentrée.
En effet, si une feuille torique est compressible, par le Theorème 1.0.3, la 3-variété qu'elle borde possède une composante de Reeb. Donc dans un feuilletage sans composante de Reeb, les feuilles toriques sont incompressibles.
De plus le théorème de Goodman [1975] donne que dans les 3-variétés fermées, les seules feuilles n'étant pas traversées par une transversale fermée sont les 2-tores, et donc qu'un feuilletage non-tendu possède toujours une feuille torique.
On en conclut le corollaire suivant :

Corollary 1.0.6. *Un feuilletage de classe $\mathcal{C}^1$, transversalement orienté, non-tendu et sans composante de Reeb d'une 3-variété fermée possède une feuille torique incompressible.*

Notons encore que le théorème de stabilité de Reeb [1952], nous dit que s'il existe une $\mathbb{S}^2$-feuille, alors la variété est homéomorphe à $\mathbb{S}^2 \times \mathbb{S}^1$, donc on ne considèrera jamais aux feuilles sphériques.

C'est pour cela que l'on se concentre d'abord sur les composantes de *tourbillonement* (que l'on généralise) et de *spiralement*. Cette dernière est une partie de la construction de Gabai [1983] que nous détaillons et généralisons.

Les principaux résultats sont les suivants.
Au voisinage d'une feuille torique, il existe toujours ces composantes de tourbillonement et de spirallement. Plus précisément :

Proposition 1.0.7. *Soit M une variété admettant un $\mathcal{C}^2$-feuilletage transversalement orienté $\mathcal{F}$.*
Supposons que $M \ncong T^2 \times \mathbb{S}^1$ et $M \ncong T^2 \times I$ feuilleté exclusivement par des feuilles toriques. Alors $\mathcal{F}$ contient soit une composante de spiralement généralisée soit une composante de tourbillonement généralisée si et seulement si $\mathcal{F}$ admet une feuille torique.

De plus, on montre que, lorsque toutes les feuilles toriques sont au bord ; un feuilletage est tendu si et seulement si il existe au moins une feuille torique au bord dont l'orientation transverse pointe à l'intérieur, et une autre feuille torique au bord dont l'orientation transverse pointe à l'extérieur, on dira alors que le feuilletage est *bien orienté*, sinon on dira qu'il est *mal orienté*. Plus précisément :

Theorem 1.0.8. *Soit M une 3-variété possédant un $\mathcal{C}^1$-feuilletage transversalement orienté $\mathcal{F}$.*
Supposons que le bord de M est une réunion de feuilles toriques.
Supposons encore que $\mathcal{F}$ n'admette pas de feuille torique intérieure, ni d'anneau plongé dont le feuilletage induit par $\mathcal{F}$ soit un anneau de Reeb. Alors,
$\mathcal{F}$ est tendu si et seulement si $\mathcal{F}$ est bien orienté.

Dans un second temps on se pose la question qui reste encore ouverte dans un cadre général qui est :

Question 1.0.9. *Quelles sont les 3-sphères d'homologie admettant un feuilletage tendu ?*

En effet, si la 3-variété a une homologie non-triviale, on a vu que la construction de Gabai [1983] donne un feuilletage tendu.
Parmi les sphères d'homologies, Brittenham, Naimi, et Roberts [1997] ont exhibés une infinité de variétés graphées n'admettant pas de feuilletage tendu mais admettant un feuilletage non-tendu sans composante de Reeb.
De plus, Roberts, Shareshian, et Stein [2003] ont quant à eux donné une famille infinie de variétés hyperboliques n'admettant pas de feuilletages tendu (ici tendu et sans composante de Reeb sont équivalents vu qu'il n'y a pas de tores incompressibles, voir le Corollaire 1.0.6).
Jusqu'à maintenant il n'y avait pas de résultats similaires pour les sphères d'homologie fibrées de Seifert.

Nous donnons dans cette thèse un critère arithmétique caractérisant l'existence de feuilletage tendu dans les 3-sphères d'homologie fibrées de Seifert ; en utilisant les invariants de Seifert [1933].
Celui-ci a été obtenu en comprenant dans un premier temps que l'existence de $\mathcal{C}^2$-feuilletage tendu dans les 3-sphères d'homologie fibrées de Seifert est équivalente à l'existence de feuilletage horizontal analytique. Puis nous utilisons le critère arithmétique caractérisant l'existence de ce dernier, donné par Eisenbud, Hirsch, et Neumann [1981], Jankins et Neumann [1985], Naimi [1994] en montrant que dans certains cas il est satisfait, alors que dans d'autres il ne l'est pas. En particulier on montre les théorèmes suivants qui font l'objet d'un article co-écrit avec Daniel Matignon :

Theorem 1.0.10 (Main Theorem 1)**.** *Soit M une 3-sphère d'homologie entière fibrées de Seifert. Alors M admet un feuilletage tendu analytique si et seulement si M n'est ni homéomorphe à $\mathbb{S}^3$ ni à la 3-sphère d'homologie de Poincaré.*

Theorem 1.0.11 (Main Theorem 2)**.** *Soit $n \in \mathbb{N}$.*
Soit $\mathcal{S}_n$ l'ensemble des 3-sphères d'homologie rationnelle et non-entière fibrées de Seifert avec n fibres exceptionnelles et admettant la géométrie $\widetilde{SL_2}(\mathbb{R})$. Alors, pour tout $n \geq 3$:
(i) Il existe une infinité de $M \in \mathcal{S}_n$ qui admettent un feuilletage tendu analytique ; et
(ii) Il existe une infinité de $M \in \mathcal{S}_n$ qui n'admettent pas de $\mathcal{C}^2$-feuilletage tendu.
(iii) Il existe une infinité de $M \in \mathcal{S}_3$ qui n'admettent pas de $\mathcal{C}^0$-feuilletage tendu.

Ce manuscrit s'organise de la manière suivante.
La Proposition 1.0.7 ainsi que le Théorème 1.0.8 sont démontrés dans le Chapitre 2 ; alors

que le Théorème 1.0.10 et le Théorème 1.0.11 sont prouvés dans le Chapitre 3.

Cette thèse ouvre des perspectives quant à la question de la caractérisation de l'existence de feuilletages tendus dans les sphères d'homologie graphées, en travaillant sur le recollement de variétés de Seifert à bord obtenues à partir de sphères d'homologie où l'on a enlevé un tore solide.

Un autre résultat en cours avec Daniel Matignon est de montrer que si une sphère d'homologie fibrée de Seifert possède un $\mathcal{C}^0$-feuilletage sans feuille torique alors on peut trouver un feuilletage horizontal. Ce qui montrerait avec le Théorème 1.0.11 qu'en fait, la famille exhibée n'admet pas non plus de $\mathcal{C}^0$-feuilletage tendu.

Chapter 2

Feuilles compactes et feuilletage tendu

Ce chapitre fait l'objet d'un article soumis aux Annales de l'Institut Fourrier, c'est pourquoi il est rédigé en anglais.

2.1 Introduction

In this chapter, all the manifolds M are 3-dimensional, compact, connected and irreducible. The foliations studied on M are of codimension one (i.e the leaves are 2-dimensional). Sometimes we will consider foliations on surfaces.

Since the works of Reeb [1952] and Novikov [1965], we know that all manifolds M as above admit a codimension one foliation, (see Lickorish [1965]). The construction of this foliation gives rise to a *Reeb component*. Foliations without Reeb component (or *Reebless*) are more interesting, because they give deep information on the manifold M, for example $\pi_1(M)$ is infinite or $M \cong \mathbb{S}^2 \times \mathbb{S}^1$ (see Novikov [1965]).
Gabai [1983], improved the theory by introducing the notion of *taut* foliation.
It is well known that taut foliations are Reebless; here we generalize this fact in Proposition 2.1.1 (a first version was already in Brittenham [1993a], or in Godbillon [1991] [lemma **3.8**] for closed manifolds, or manifolds with only one torus boundary component).

Proposition 2.1.1. *Let M be a 3-manifold with a transversely orientable foliation $\mathcal{F}$. If the boundary of M is a union of leaves with the same transverse orientation or if $\mathcal{F}$ contains a compact separating leaf, then $\mathcal{F}$ is not taut.*

Gabai [1983] showed that non-trivial second homology is a sufficient condition for the existence of a taut foliation. The general problem of existence of a taut foliation in homology spheres is still open, even if many works partially answer the question (see for example Brittenham, Naimi, and Roberts [1997] for graph manifolds, Chapter 3 which is a complete classification for Seifert fibered manifolds and Roberts, Shareshian, and Stein [2003] for hyperbolic manifolds).

Proposition 2.1.1 has the following corollary :

Corollary 2.1.2. *Any taut, transversely oriented foliation in a rational homology sphere, has no compact leaf.*

Proof. Indeed, suppose it admits a compact leaf. A rational homology sphere cannot admit any non-separating surface (this induces non-trivial homology). Hence this compact leaf is separating in a closed manifold. By Proposition 2.1.1, the foliation cannot be taut; a contradiction.

□

This fact is crucial for showing Theorem 1.0.10 and Theorem 1.0.11.

In Theorem 2.6.8, Goodman [1975] showed that a non-taut foliation always admits a torus leaf. We will produce examples of non-taut foliations admitting a separating torus leaf, and non-taut foliations admitting a non-separating torus leaf.

One goal of this chapter is to better understand non-taut and Reebless foliations. Note that together with Theorem 1.0.3, if a foliation of a closed 3-manifold (or with boundary leaves) is non-taut and Reebless, then it admits an incompressible torus leaf. Hence a great part of this paper studies foliations near incompressible torus leaves. Note that those non-taut and Reebless foliations never arise in hyperbolic closed manifolds (since they cannot contain incompressible tori).
In this context we study two geometric processes : *turbulization* ($\mathcal{T}_*$ component) and *spiraling* ($\mathcal{S}_*$ component) which occur near a torus leaf (or more generally near a closed compact surface).
We will see that turbulization and spiraling can give rise to non-taut Reebless foliations. Spiraling was first introduced by Gabai [1983], and here we first give a detailed definition of it; then we link it to turbulization (which was first defined by G. Reeb).
Conversely if a foliation admits a torus leaf then roughly speaking, in a regular neighborhood of this torus there is turbulization or spiraling, which is the aim of next proposition (for precise definitions see Section 3.2).

Proposition 1.0.7 *Let M be a manifold admitting a transversely oriented $\mathcal{C}^2$-foliation $\mathcal{F}$. Assume that if M is either $T^2 \times \mathbb{S}^1$ or $T^2 \times I$ then $\mathcal{F}$ contains non-torus leaves. Then $\mathcal{F}$ contains either a $\mathcal{S}_*$, or a $\mathcal{T}_*$ component, if and only if $\mathcal{F}$ admits a torus leaf.*

If all the boundary components of M (with a transversely oriented foliation $\mathcal{F}$) are torus leaves; we say that $\mathcal{F}$ has a *bad orientation* if the transverse orientation on all the torus boundary leaves is the same (all point inward or all point outward); otherwise we say that it is a *good orientation* (at least two torus leaves have opposite orientation, one inward and the other outward).
Then, we will see that the tautness of the foliation is deeply linked to the good or bad transverse orientation as suggests the following theorem.

Theorem 1.0.8 *Let M be a manifold with a transversely oriented $\mathcal{C}^1$-foliation $\mathcal{F}$.*

Assume that the boundary of M is a union of torus leaves.
Assume also that $\mathcal{F}$ contains neither a torus leaf in its interior nor an embedded annulus whose induced foliation by $\mathcal{F}$ is a Reeb annulus.
Then, $\mathcal{F}$ is taut if and only if $\mathcal{F}$ has a good orientation.

Organization of the chapter.
We organize this chapter as follows.

In Section 2.2, we recall basic definitions and notations.

Section 2.3 introduces the well-known Reeb's component, and the geometric process of turbulization in two different interesting ways.

In Section 2.4 we define the geometric process of spiraling and generalize it under certain conditions.

In Section 2.5, we first prove the equivalence of these two geometric processes in a particular case. For this, we describe $\mathcal{C}^2$-foliations near a torus leaf and prove Proposition 1.0.7.

Section 2.6 proves Proposition 2.1.1 which says that separating torus leaves or boundary leaves with the same transverse orientation are contained in a non-taut foliation.
Furthermore, we explain why each hypothesis is necessary for Proposition 2.1.1. For this, we consider the Waldhausen manifold and give an example of a taut foliation with a single boundary leaf, non-transversely orientable with non-compact leaves on Q.
Then we prove Corollary 2.6.10 of Theorem 2.6.8 which states that a non-taut foliation of a closed 3-manifold (or manifold with boundary leaves) always contains a torus leaf.
The rest of the chapter focuses on the two cases : separating torus leaves (Section 2.7) and non-separating ones (Section 2.8).

Indeed, Section 2.7 provides a collection of different non-taut foliations as follows.
There are the ones with Reeb components, that we cannot remove (example on $\mathbb{S}^3$); and the ones with Reeb components that we can remove, or non-taut and Reebless foliations; (example on T^3).

The aim of Section 2.8 is to understand why a foliation with a non-separating torus can be either taut or non-taut. We start with key examples, and we generalize by proving Theorem 1.0.8 saying that if a foliation of a manifold with torus boundary leaves does not contain embedded Reeb annuli, then it is taut if and only if it has a good orientation (at least two boundary components whose transverse orientation is opposite).

Acknowledgement.

Proposition 2.7.3 is a result of an interesting discussion with András Juhász, and the author wishes to thank him.

Perspectives.

We will see in section 2.7 some examples of non-taut and Reebless foliations, on different manifolds. One open question is the following :

Question 2.1.3. *What are the manifolds admitting a non-taut and Reebless transversely oriented foliation, but not admitting a taut foliation?*

Note that the first examples of such manifolds were found by Brittenham, Naimi, and Roberts [1997] and they are graph manifolds, (we have seen that this question is trivial for closed hyperbolic manifolds).
Note also that the examples given here of non-taut Reebless foliations, concern manifolds admitting (another) taut foliation.
In Chapter 3 we give an infinity of examples of Seifert manifolds not admitting a taut foliation, hence we should ask the following :

Question 2.1.4. *Does any member of this family admit a non-taut and Reebless foliation?*

Note also that nothing is known about the existence of taut foliations or of non-taut and Reebless foliation among non-hyperbolic manifolds admitting an hyperbolic submanifold.

2.2 Preliminaries

From now on, M will be a compact connected irreducible 3-manifold, possibly with boundary, and $\mathcal{F}$ will be a codimension one foliation on M considered up to isotopy, unless otherwise specified.
Furthermore we will let $I = [0,1]$, and denote $\mathring{X}$ the interior of X, and $\overline{X}$ the closure of X, when it makes sense, and let $T^2 \cong \mathbb{S}^1 \times \mathbb{S}^1$.
For all the following the circle $\mathbb{S}^1$ is parametrized by $\{e^{i\theta}, \theta \in]-\pi,\pi]\}$, but for more simplicity we will consider it as $\{\theta \in]-\pi,\pi]\}$.

Separating surfaces and non-separating surfaces. A properly embedded surface F in a 3-manifold M is said to be a ***separating surface*** if $M \backslash F$ is not connected; otherwise, F is said to be a ***non-separating surface*** in M. If F is a separating surface, we call ***sides of*** F the connected components of $M \backslash F$.

A 3-manifold is said to be ***reducible*** if M contains an ***essential*** 2-sphere, i.e. a 2-sphere which does not bound any 3-ball in M. Then, either M is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$, or M is a non-trivial connected sum. If M is not a reducible 3-manifold, we say that M is an ***irreducible*** 3-manifold.

Incompressible torus. An embedded torus T in M is said to be ***incompressible*** if the induced map $\pi_1(T) \to \pi_1(M)$ is injective, otherwise we say that T is compressible.
Note that in an irreducible manifold, a compressible torus is always separating, while an incompressible torus can be separating or non-separating.

Transverse orientation. Let M be a compact connected 3-manifold possibly with boundary.
Let $\mathcal{F}$ be a codimension one foliation on M.
A foliation $\mathcal{F}$ of M is ***transversely orientable***, if M admits a non-zero continuous vector field, transverse (i.e non-tangent) to all the leaves.

If we fix such a non-zero continuous vector field, then $\mathcal{F}$ is said to be ***transversely oriented***.

Reeb annulus.

First, we define a foliation of $\mathbb{R}^2$. Let $f\ :\mathbb{R}^2 \to \mathbb{R}$

$$f\ : (x, z) \mapsto (x^2 - 1) \times \exp(z)$$

f is a submersion, so it defines a foliation $\mathcal{F}$, axially symmetric about the z-axis, where :

- $f^{-1}(\{0\})$ is a union of two vertical leaf $\{x = 1\}$ and $\{x = -1\}$.

- $f^{-1}(\{c^2\})$ are leaves satifying the equation
  $z = \log(c^2) - \log(x^2 - 1)$, for $|x| > 1$.
  When $z \to +\infty$, $x^2 \to 1$ so the leaves tend toward the vertical leaves.
  When $z \to -\infty$, $x^2 \to +\infty$.
  The general shape is $-\log$.

- $f^{-1}(\{-c^2\})$ are parabolic leaves satifying the equation $z = \log(c^2) - \log(1 - x^2)$, for $|x| < 1$. They meet the z-axis for $z = \log(c^2)$.
  When $z \to +\infty$, $x^2 \to 1$, so the leaves tend toward the vertical leaves.

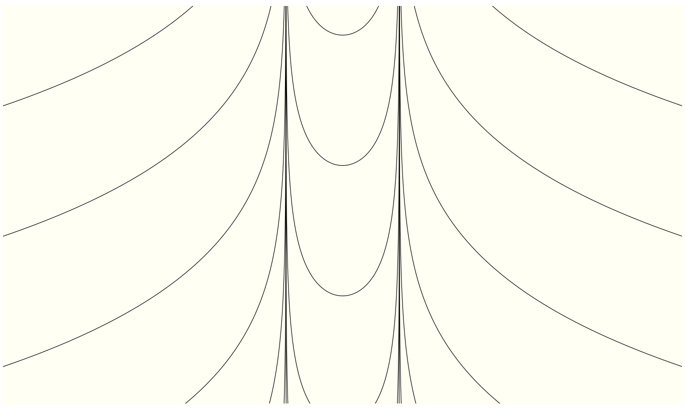


Figure 2.1: Foliation $\mathcal{F}$ of $\mathbb{R}^2$

$\mathcal{F}$ is invariant under integral translations (and in fact any translations), along the z-axis; then it induces foliations on an annulus as follows.

Consider the restricted foliation of $\mathcal{F}$ on the set $R = \{(x, z) \in \mathbb{R}^2, -1 \leq x \leq 1\}$. The annulus $R/_\sim$ where $(x, z) \sim (x, z + k), k \in \mathbb{Z}$ admits an induced foliation by $\mathcal{F}$, and is called ***Reeb annulus*** as illustrated in Figure 2.2.

Direction of rotation of a spiral foliation. Let $X = \{(x, \theta), x \in I, \theta \in]-\pi, \pi]\} \cong I \times \mathbb{S}^1$ be an annulus foliated with two circle boundary leaf and spiral leaves in the interior of X

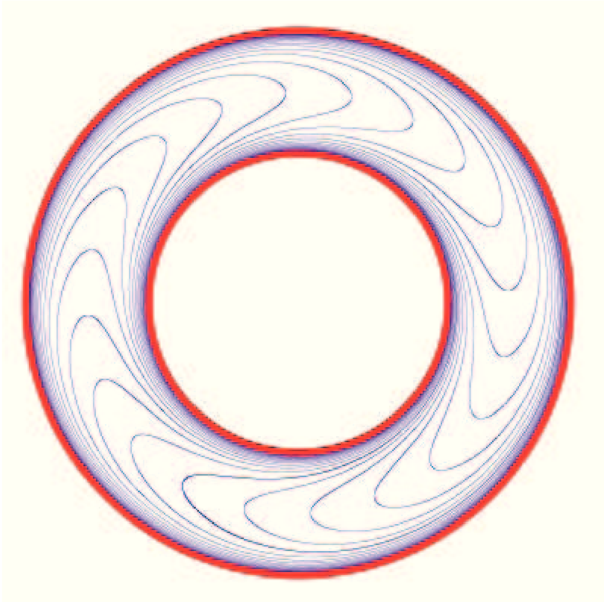


Figure 2.2: Reeb annulus (from Wikipedia)

(see Remark 2.3.6 for a definition of this foliation) we call this foliation a ***spiral foliation***. Keeping fixed the two boundary components, there are two non-isotopic such foliations drawn in Figure 2.3.

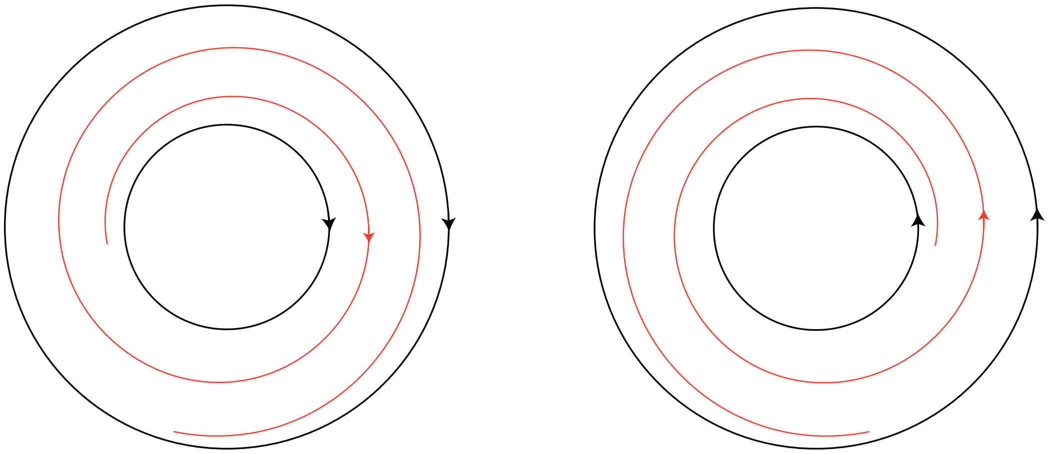
Definition 2.2.1. *Consider a foliation of X with spiral foliation. Choose any spiral leaf and orient it so that x grows in I (i.e fix the direction of rotation so that x grows).*
That induces an orientation by continuity on all the leaves of this foliation hence on the circle leaves (which is the same). If this orientation is a clockwise direction of rotation we call that foliation a **clockwise spiral foliation** *and the spiral leaves are called* **clockwise spiral**, *otherwise we call it* **anti-clockwise spiral foliation**, *and the spiral leaves are called* **anti-clockwise spiral**.

Another foliation of the annulus, denoted by $\mathcal{C}$.

Now we construct a foliation of the annulus where one boundary component is transverse to the foliation and the other is tangent. The leaves will be homeomorphic to $\mathbb{R}^+$, except one circle boundary leaf.
Consider the restricted foliation of $\mathcal{F}$ on the set $R_r = \{(x, z) \in \mathbb{R}^2, 1 \leq x \leq r\}$, for $r > 1$; (or equivalently $\{(x, z) \in \mathbb{R}^2, r \leq x \leq 1\}$, for $r < 1$).
The annulus $R_r/_\sim$ where $(x, z) \sim (x, z + k), k \in \mathbb{Z}$ admits an induced foliation by $\mathcal{F}$ that we will call $\mathcal{C}$.

$\mathcal{C}$ admits a circle boundary leaf, and the other boundary component is transverse to the foliation. All the non-compact leaves are homeomorphic to the ray $\mathbb{R}^+$ and are limiting along the circle boundary leaf as illustrated in Figure 2.4.

Taut foliation. Let $\mathcal{F}$ be a foliation of a 3-manifold M. An embedded loop, or respec-



(a) Clockwise spiral foliation (b) anti-clockwise spiral foliation

Figure 2.3: Direction of rotation of the foliations on X

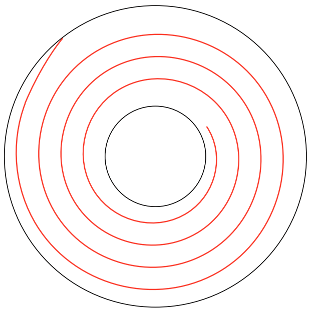


Figure 2.4: One non-compact leaf of the foliation $\mathcal{C}$

tively a properly embedded arc γ (if $\partial M \neq \emptyset$), is called ***transverse loop*** or respectively ***transverse arc*** if $\forall F \in \mathcal{F}$ such that $\gamma \cap F \neq \emptyset$, the intersection $\gamma \cap F$ is transverse.
$\mathcal{F}$ is ***taut***, if for every leaf F of $\mathcal{F}$ there exists γ an embedded transverse loop, or properly embedded transverse arc (if $\partial M \neq \emptyset$), such that $\gamma \cap F \neq \emptyset$; and if $\mathcal{F}_{|\partial M}$ contains no Reeb annulus.

The following theorem is the famous theorem of Gabai [1983] on the existence of taut foliations, which is stated here for closed 3-manifolds.

Theorem 2.2.2 (Gabai [1983])**.** *Let M be a closed 3-manifold. If $H_2(M; \mathbb{Q})$ is non-trivial then M admits a taut foliation.*

Foliated component. Suppose M admits a foliation $\mathcal{F}$, and that $\mathcal{F}_V$ is a foliation of a submanifold V of M. We say that M admits a ***foliated component*** $\mathcal{F}_V$, if the induced foliation by $\mathcal{F}$ on V is isotopic to $\mathcal{F}_V$ in M.

Foliation preserving homeomorphism. Let M be a manifold admitting a foliation $\mathcal{F}$ and N a manifold admitting a foliation $\mathcal{G}$. An homeomorphism $f : M \to N$ is a ***foliation***

preserving homeomorphism, if f sends the leaves of $\mathcal{F}$ on the leaves of $\mathcal{G}$, i.e if f preserves the leaves.

2.3 Reeb component and Turbulization

In this section we first define Reeb's component which is a foliation of a solid torus tangent to the boundary, and then we define a particular foliation of $T^2 \times I$, where one torus boundary component is a leaf and the other is transverse to the foliation, which will be called *turbulization* component.
Reeb's component and the process of turbulization were firstly defined by Reeb [1952].
Nowadays this construction is very common, and can be found for example in the notes of Brittenham [1993a].
Finally, we define *generalized turbulization.*

2.3.1 Reeb component.

First, we define a foliation of $\mathbb{R}^3$ illustrated in Figure 2.5.
Note that this is the foliation of $\mathbb{R}^2$ of Figure 2.1 in each vertical plane containing the z-axis of $\mathbb{R}^3$.

Let

$$f : \mathbb{R}^3 \to \mathbb{R}$$

$$(x, y, z) \mapsto (x^2 + y^2 - 1) \times \exp(z)$$

f is a submersion, and defines a foliation $\mathcal{F}$ of $\mathbb{R}^3$, symmetric about the z-axis, where :
– $f^{-1}(\{0\})$ is a vertical cylinder leaf C centered in 0 with radius 1.

– $f^{-1}(\{c^2\})$ are leaves homeomorphic to a cylinder, because
$x^2 + y^2 = 1 + c^2 \exp(-z)$, hence $x^2 + y^2 > 1$.
When $z \to +\infty$, $x^2 + y^2 \to 1$ so the leaves tend toward C.
When $z \to -\infty$, $x^2 + y^2 \to +\infty$, i.e. the base of the cylinder is flaring.

– $f^{-1}(\{-c^2\})$ are paraboloid leaves which intersect the z-axis for
$z = 2\log(c)$, when $x^2 + y^2 < 1$, .
When $z \to +\infty$, $x^2 + y^2 \to 1$, so the leaves tend toward C.

Let $\mathcal{F}$ be the restricted foliation on a vertical solid cylinder $\mathbb{D}^2 \times \mathbb{R}$, of radius $r \geq 1$, included in $\mathbb{R}^3$, denoted C_r. $\mathcal{F}$ is invariant under integral vertical translations (along the z-axis); hence it induces a foliation on the solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ denoted T_r, of radius r.

T_r contains T_1 which is a solid torus of radius 1 ($r \geq 1$), and a ***Reeb component*** is the induced foliation by $\mathcal{F}$ on T_1, see Figures 2.6 and 2.8.

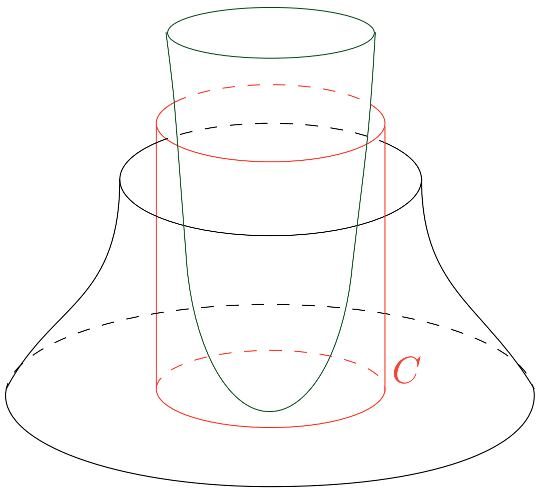


Figure 2.5: Foliation $\mathcal{F}$ of $\mathbb{R}^3$

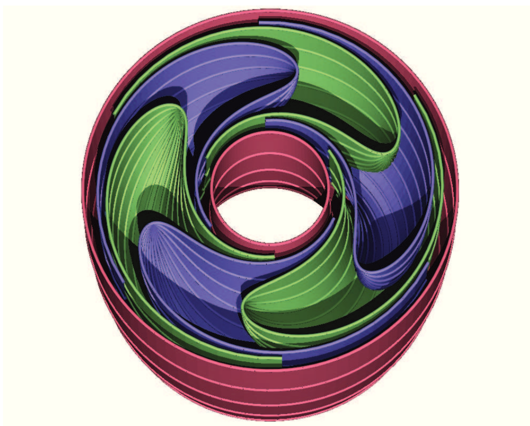


Figure 2.6: (Half) Reeb component (from Wikipedia).

Note that a Reeb annulus correspond to a 2-dimensional Reeb component.

Definition 2.3.1. *Let $\mathcal{F}$ be a foliation of a 3-manifold M. $\mathcal{F}$ is* **Reebless** *if it does not admit any Reeb component.*

2.3.2 Turbulization.

In this subsection we define in two interesting different ways the *turbulization component* denoted for all the following by $\mathcal{T}$. We talk about *turbulization* when one torus boundary is foliated by circles. So we first need the following definition.

Definition 2.3.2. *A* **circle foliation** *on a torus, (respectively on an annulus), is a foliation of T^2 (respectively of $\mathbb{S}^1 \times I$) where all the leaves are homeomorphic to $\mathbb{S}^1$. Hence, the leaves are parallel copies of an essential simple closed curve on T^2 (respectively on $\mathbb{S}^1 \times I$).*

Definition 2.3.3. *Let us call $\mathcal{T}$ the foliation induced by $\mathcal{F}$ on $T_r \backslash \mathring{T}_1$, for $r > 1$ (or equivalently on $T_1 \backslash \mathring{T}_r$, for $r < 1$). The resulting foliated manifold is homeomorphic to $T^2 \times I$, with a torus boundary leaf (∂T_1), and another torus boundary component, transverse to the foliation $\mathcal{T}$, which induces a circle foliation on it, as in Figure 2.7.*

Definition 2.3.4. *The foliation $\mathcal{T}$ is trivially transversely oriented. We will denote $\mathcal{T}^+$ (respectively $\mathcal{T}^-$), the transversely oriented foliation of $T^2 \times I$ obtained from $\mathcal{T}$, where the transverse orientation on the torus leaf points out of (respectively into) $T^2 \times I$.*

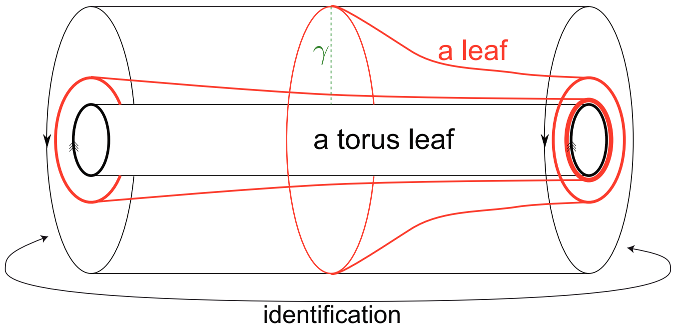


Figure 2.7: Foliation $\mathcal{T}$ of $T^2 \times I$: Turbulization

Definition 2.3.5. *Let M be a manifold with a torus boundary component T and admitting a foliation which induces on T a circle foliation.*
The process of **turbulization** *consists of pasting on T (by homeomorphism) a $T^2 \times I$ component, foliated by $\mathcal{T}$ (with the notations above).*

Roughly speaking, the process of turbulization, changes a circle foliation on a torus to a torus leaf, as in the trivial following example in Figure 2.8.

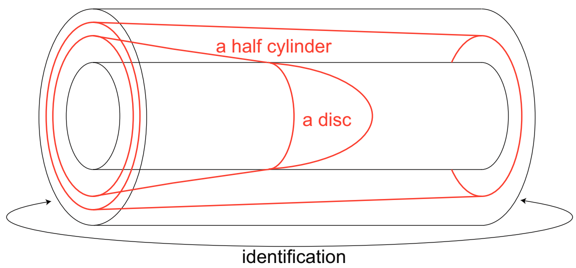


Figure 2.8: Reeb component

Remark 2.3.6. *A Reeb component contains the foliation $\mathcal{T}$.*

Let us give another definition, that we will also use later.
Let $A = \{(x, \theta), x \in [0, 1], \theta \in] - \pi, \pi]\}$ be an annulus embedded in $\mathbb{R}^3$, and consider the following foliation called $\mathcal{F}$ on $A \times I$.

Let f be a diffeomorphism of the unit interval such that $\{0\}$ and $\{1\}$ are fixed point, and f is strictly increasing.
Denote for each $x \in [0,1]$, and $z \in I$, the circle $\lambda_x^z = \{(x,\theta,z), \theta \in]-\pi,\pi]\}$ in $A \times I$.
Let us define the foliation $\mathcal{F}_f$.
The leaves of $\mathcal{F}_f$ are the annuli A_α bounded by λ_α^0 and $\lambda_{f(\alpha)}^1$ in $A \times I$, for each $\alpha \in [0,1]$, as in Figure 2.9 where we have chosen $f(t) > t$.
This foliation $\mathcal{F}_f$ is called the ***suspension foliation*** of f along λ_0^0 on $A \times I$.
More precisely $A_\alpha = \bigcup_{z\in[0,1]} \lambda^z_{[f(\alpha)-\alpha]z+\alpha}$.
Indeed the segment joining α to $f(\alpha)$ for a fixed angle $\theta \in]-\pi,\pi]\}$, with the chosen coordinates, is defined by the equation $x = [f(\alpha)-\alpha]z+\alpha$ in $A \times I$.
Obviously A_0 and A_1 are vertical leaves.

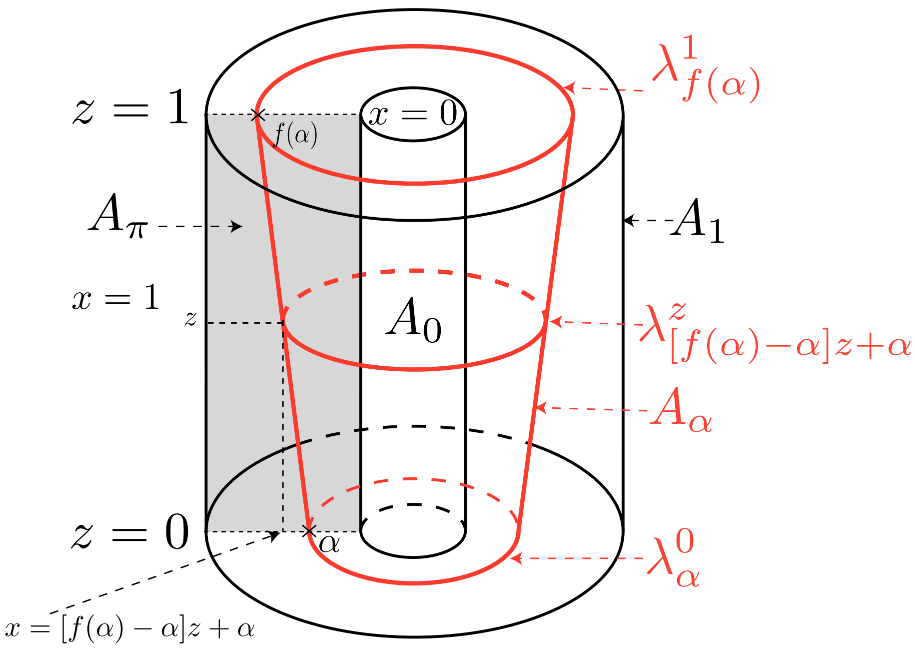
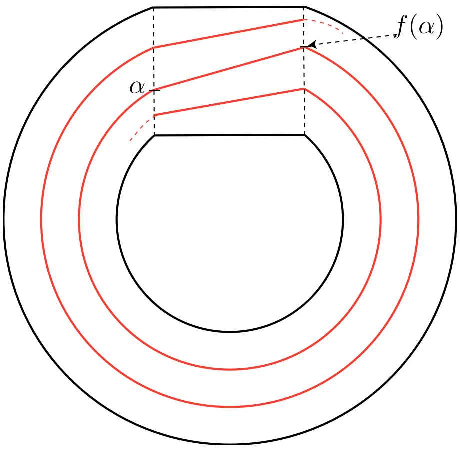


Figure 2.9: Foliation $\mathcal{F}_f$ of $A \times I$

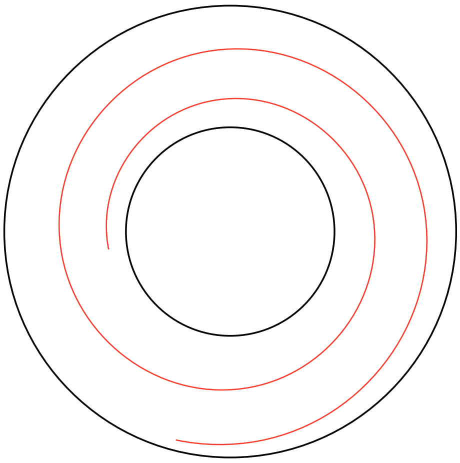
That leads us to construct a foliation on $T^2 \times I$ as follows:
Consider $T^2 \times I = (A \times I)/_\sim$, where $(x,\theta,0) \sim (x,\theta,1)$.
$\mathcal{F}_f$ induces a foliation on $T^2 \times I$ where $T^2 \times \{0\}$ and $T^2 \times \{1\}$ are torus leaves.
Now, if we choose f so that $f(t) > t$ or $f(t) < t$, for all $t \in \mathring{I}$, the foliation $\mathcal{T}$ is isotopic to the induced foliation by $\mathcal{F}_f$ on $\{(x,\theta,z), x \in [0,\frac{1}{2}], \theta \in]-\pi,\pi], z \in [0,1]\}/_\sim$, or equivalently on $\{(x,\theta,z), x \in [\frac{1}{2},1], \theta \in]-\pi,\pi], z \in [0,1]\}/_\sim$.
Indeed the torus $\{(\frac{1}{2},\theta,z), \theta \in]-\pi,\pi], z \in [0,1]\}/_\sim$ is everywhere transverse and admits a circle foliation.
Note that if $\exists t_0 \in \mathring{I}/f(t_0) = t_0$ that induces an interior torus leaf in $\mathcal{F}_f$.

Remark 2.3.7. *Note that the foliation induced by $\mathcal{F}_f$ on the transverse annulus $A_\pi = \{(x,\pi,z), x \in [0,1], z \in [0,1]\}/_\sim$ in $T^2 \times I$ is described on Figure 2.10.*

Remark 2.3.8. *Note that the choice $f(t) > t$ or $f(t) < t$ determine different senses of rotation along the torus leaf $T = \{(1, \theta, z), \theta \in] - \pi, \pi], z \in [0, 1]\}/_{\sim}$. That is the reason why we give the following definition.*



(a)



(b)

Figure 2.10: Induced foliation by $\mathcal{F}$ on A_π (isotopic representations)

Definition 2.3.9. *If $f(t) > t$ we say that $\mathcal{F}_f$ is a clockwise foliation, and if $f(t) < t$, we say that $\mathcal{F}_f$ is a anti-clockwise foliation.*

Remark 2.3.10. *Note that if $\mathcal{F}_f$ is a clockwise (respectively anti-clockwise) foliation then the induced foliation on A_π is a clockwise (respectively anti-clockwise) spiral foliation.*

Turbulization has a lot of applications; one of the most famous is the following Theorem from Lickorish [1965] and also showed independently by Novikov [1964] (helped by Zieschang) :

Theorem 2.3.11. *Every 3-manifolds admit a codimension one foliation, possibly with a Reeb component.*

Proof. (idea) We may recall that every closed 3-manifold M is obtained by deleting a tubular neighborhood of a link L in $\mathbb{S}^3$, and by gluing it back differently. Let us consider $\mathbb{S}^3$ foliated by two Reeb's component glued along their torus leaf. We can isotope L so that it meets transversely the leaves of this foliation. By choosing a thin enough tubular neighborhood of L, denoted by $N(L)$, we may assume that the induced foliation on $N(L)$ is the one by disks transverse to the boundary of $N(L)$. Then we can remove (the interior of) $N(L)$, and glue some $\mathcal{T}$ components along each boundary component of $\mathbb{S}^3 \backslash \mathring{N}(L)$, i.e we apply the process of turbulization. Then we obtain a manifold with torus boundary leaves, and it remains to glue Reeb's component along those boundary leaves by the well chosen way, to obtain M with a foliation (with Reeb components). $\square$

2.3.3 Generalized turbulization

Turbulization can be defined in a more general context.
The idea of turbulization is to extend a foliation of $T^2 \times \{1\}$ (either by circles or dense lines on the torus) in $T^2 \times I$ to obtain $T^2 \times \{0\}$ as a torus leaf.
In the preceding paragraph we have done it for a circle foliation on $T^2 \times \{1\}$, here we want to do a similar construction for a dense foliation on $T^2 \times \{1\}$. Indeed, when the foliation on a torus is $\mathcal{C}^2$, Denjoy [1932] and Siegel [1945] showed that either there is a circle leaf or the foliation is dense. When there is a circle leaf there are two cases, either this is a circle foliation, or there are spiral leaves between circles leaves. The last case will be taken in account with spiraling, while the former case has already been studied.

Let us formulate it more precisely.

One way to define generalized turbulization is as follows :
Consider $A \times I = \{(x, \theta, z), x \in [0,1], \theta \in]-\pi, \pi], z \in [0,1]\}$, ($A$ is an annulus).
For each $z \in [0,1]$ set $A_z = \{(x, \theta, z), x \in [0,1], \theta \in]-\pi, \pi]\}$.
Now foliate each A_z by the foliation $\mathcal{C}$ (of Figure 2.4), to obtain a foliation on $A \times I$ by $\mathcal{C} \times I$.

This foliation is invariant by rotation along the z-axis; hence that induces a foliation of $T^2 \times I$ by identifying A_0 to A_1 by a foliation preserving homeomorphism f from A_0 to A_1 such that $f(\{(0, \theta, 0), \theta \in]-\pi, \pi]\}) = \{(0, \theta, 1), \theta \in]-\pi, \pi]\}$, and f sends a spiral leaf to a spiral leaf, for example using any rotation.
Consider $T^2 \times I \cong (A \times I)/_\sim$ where $((x, \theta), 0) \sim (f(x, \theta), 1)$.
Note that here $T^2 \times \{x\} \cong \{(x, \theta, z), \theta \in]-\pi, \pi], z \in [0,1]\}/_\sim$, for each $x \in [0,1]$.

Definition 2.3.12. *We call this foliated component a generalized turbulization component, and we denote it by $\mathcal{T}_*(f)$, as illustrated in Figure 2.11.*

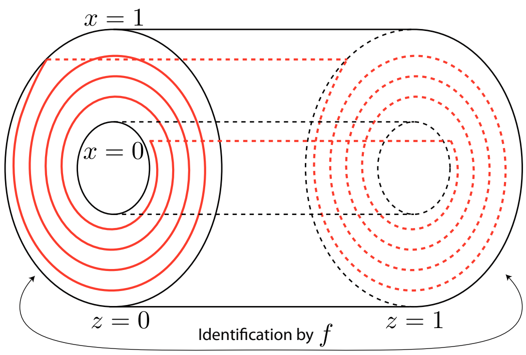


Figure 2.11: Generalized turbulization : foliation $\mathcal{T}_*(f)$

There are two crucial examples :
When f is a rotation rational rational angle, the non-compact leaves are homeomorphic

to $\mathbb{R}^+ \times \mathbb{S}^1$, and there is one compact leaf : the torus $T^2 \times \{0\}$. Moreover the leaves of the circle foliation on $T^2 \times \{1\}$ have rational slopes (recall that those are essential simple closed curves).
Then, note that $\mathcal{T}_*(f)$ (of Figure 2.11) and $\mathcal{T}$ (of Figure 2.7) are homeomorphic. When f is a rotation of irrational angle, there is no cylinder leaf, i.e all the non-compact leaves are homeomorphic to $\mathbb{R}^+ \times \mathbb{R}$, and the induced foliation on $T^2 \times \{1\}$ is dense (corresponds to irrational slopes).

2.4 Spiraling

Turbulization extends a circle foliation or a dense foliation on a torus $T^2 \times \{0\}$ in a foliation of the 3-manifold $T^2 \times I$ such that $T^2 \times \{1\}$ is a leaf (turbulization).
The goal of spiraling (here) is to extend a foliation on the torus by spirals and circles.

For this we use the construction of Gabai, who defines spiraling in a more general context which is the following.
Let S_g be a closed orientable compact genus $g \geq 1$ surface. We start with a foliation $\mathcal{F}$ on $S_g \times \{0\}$ which has a two dimensional leaf and an annulus with a one dimensional foliation tangent to its boundary.
Then, we extend it to a foliation of $S_g \times I$, where $S_g \times \{1\}$ is a leaf.

Here we explain this construction providing more details and generalize it when $g = 1$.

We first construct a foliation of $S_g \times I$ that we will call *spiraling component*. We proceed into four steps which are subsections.

- Step 1 : Notations and conventions, we fix δ and λ two essential simple closed curve on S_g such that $\#(\lambda \cap \delta) = 1$.
- Step 2 : Suspension foliation along λ.
- Step 3 : Superposition along δ.
- Step 4 : Repeating infinitely many times Step 3 (infinite induction).

Subsection 2.4.5 applies this construction to extend a foliation $\mathcal{F}$ of a manifold M with a boundary component S_g, such that $\mathcal{F}_{|S_g}$ admits a transverse annulus to a foliation of $M \cup (S_g \times I)$ such that $S_g = S_g \times \{0\}$ and $S_g \times \{1\}$ is a leaf.

Finally, Subsection 2.4.6 generalizes spiraling when $g = 1$ and when $S_g \times \{0\}$ admits Reeb annulus.

2.4.1 Step 1

We consider a closed compact surface of genus $g \geq 1$, denoted S_g, with a non-separating simple closed curve δ embedded in S_g. We set $A_\delta = \delta \times I$ a regular neighborhood of δ in S_g and we identify δ and $\delta \times \{0\}$.
Let λ be simple closed curve embedded in S_g whose geometric intersection number with

δ is one. Note that λ is non-separating. Similarly we will denote $A_\lambda = \lambda \times I$ a regular neighborhood of λ in S_g and we identify λ and $\lambda \times \{0\}$.

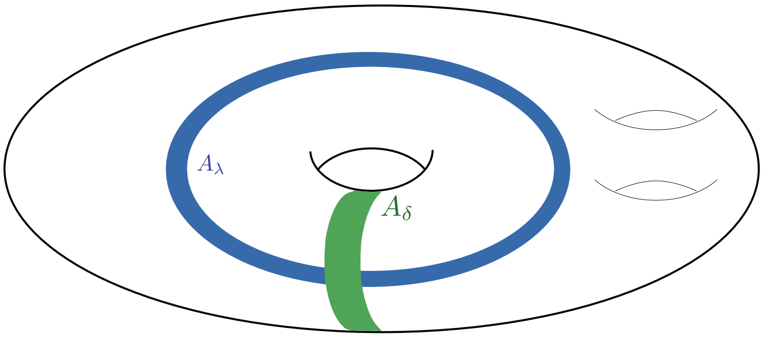


Figure 2.12: A choice of λ and δ.

Consider the product foliation on $(S_g \backslash A_\lambda) \times I$ and denote the leaves by $Q_t = (S_g \backslash A_\lambda) \times \{t\}, t \in I$.

2.4.2 Step 2

Here we construct a particular foliation of $S_g \times I$, where the two boundary components are leaves, and the interior leaves are non-compact.

Let f be a strictly increasing diffeomorphism of I such that $f(0) = 0$ and $f(1) = 1$. Consider the suspension foliation of f along $\lambda \times \{0\} \times \{0\}$, in $\lambda \times \{0\} \times I$, where the leaves are the annuli cobounded by $\lambda \times \{0\} \times \{t\}$ and $\lambda \times \{1\} \times \{f(t)\}$ in $A_\lambda \times I$, (see Figure 2.9). Now extend the product foliation on $S_g \times I$ by gluing this foliated component $A_\lambda \times I$ by the identity on $(S_g \backslash A_\lambda) \times I$, and we denote the resulting foliation of $S_g \times I$ by $\mathcal{F}_f$. To draw $\mathcal{F}_f$ more easily, we represent A_λ differently in Figure 2.13.
Note that in all the following figures we have chosen $f(t) > t$.

Note that for all $t \in \mathring{I}$, this extension adds to Q_t two annuli : one is joining Q_t to $Q_{f(t)}$ and another annulus is joining Q_t to $Q_{f^{-1}(t)}$.
Of course this extension adds an annulus to Q_t, for $t \in \{0, 1\}$, which implies that $S_g \times \{0\}$ and $S_g \times \{1\}$ are leaves of $\mathcal{F}_f$, and the other leaves tends toward those two.

To represent $\mathcal{F}_f$, we can draw a transverse cut of this foliation, i.e along $\delta \times \{0\} \times I$, as in Figure 2.14. Note that here, $\delta \times \{0\} \times I$ plays the role of A_π of Figure 2.9, see also Figure 2.15.

2.4.3 Step 3

The goal of this step is to paste together $(S_g \backslash A_\delta) \times I_0$ and $(S_g \backslash A_\delta) \times I_1$, and to extend it nicely; where $I_0 = [0, 1/2]$, and $I_1 = [1/2, 3/4]$.

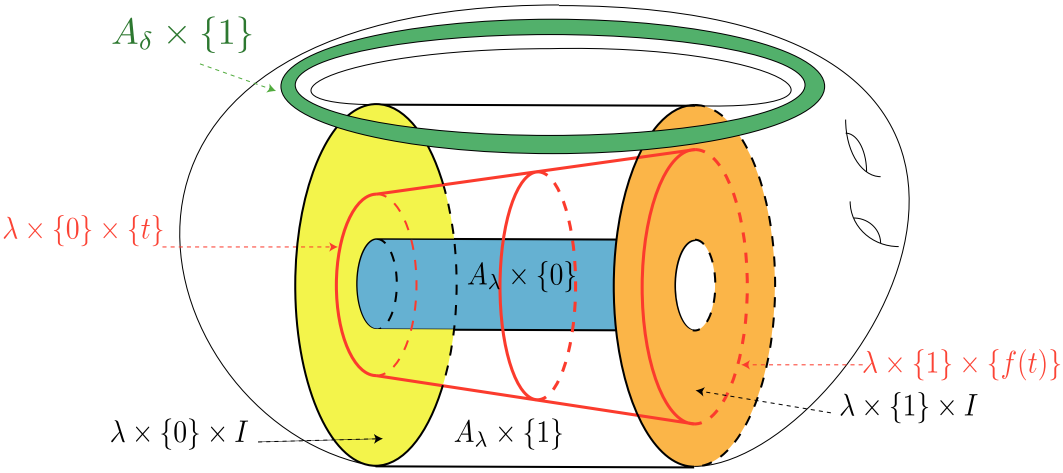


Figure 2.13: Suspension foliation along $A_\lambda \times I$ in $S_g \times I$

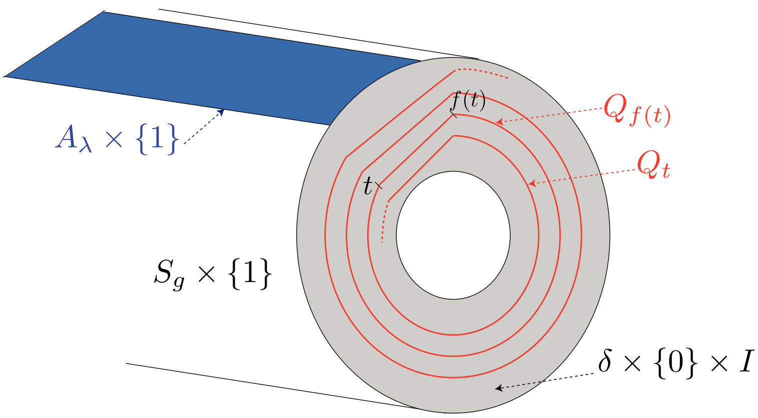


Figure 2.14: Transverse cut along $\delta \times \{0\} \times I$

Consider $(S_g \backslash A_\delta) \times I_0$ and $(S_g \backslash A_\delta) \times I_1$ both with the foliation $\mathcal{F}_f$ of Figure 2.15. Glue them along $(S_g \backslash A_\delta) \times \{1/2\}$ to obtain a manifold homeomorphic to $(S_g \backslash A_\delta) \times [0, 3/4]$. Now we do the following extension (see Figure 2.16).

We identify $\delta \times \{0\} \times I_1$ to $\delta \times \{1\} \times I_0$ by a foliation preserving homeomorphism h (i.e sending an interior leaf on an interior leaf, and sending $\delta \times \{0\} \times \{3/4\}$ on $\delta \times \{1\} \times \{1/2\}$, and $\delta \times \{0\} \times \{1/2\}$ on $\delta \times \{1\} \times \{0\}$, for example any rotation).
In particular, this amounts to gluing one annulus called $A_\delta^{3/4}$ between $\delta \times \{0\} \times \{3/4\}$ and $\delta \times \{1\} \times \{1/2\}$; and an other annulus $A_\delta^{1/2}$ between $\delta \times \{0\} \times \{1/2\}$ and $\delta \times \{1\} \times \{0\}$, which connects the compact leaves $(S_g \backslash A_\delta) \times \{0\}$, $(S_g \backslash A_\delta) \times \{1/2\}$ and $(S_g \backslash A_\delta) \times \{3/4\}$. There may exist annuli connecting the circles; which correspond to possible interior fixed points of f.
Moreover, this amounts to gluing bands $\mathbb{R} \times I$ where one boundary spiral $\mathbb{R} \times \{0\}$ lies on $\delta \times \{0\} \times I_1$ and the other boundary spiral $\mathbb{R} \times \{1\}$ lies on $\delta \times \{1\} \times I_0$, so that the foliation matches.

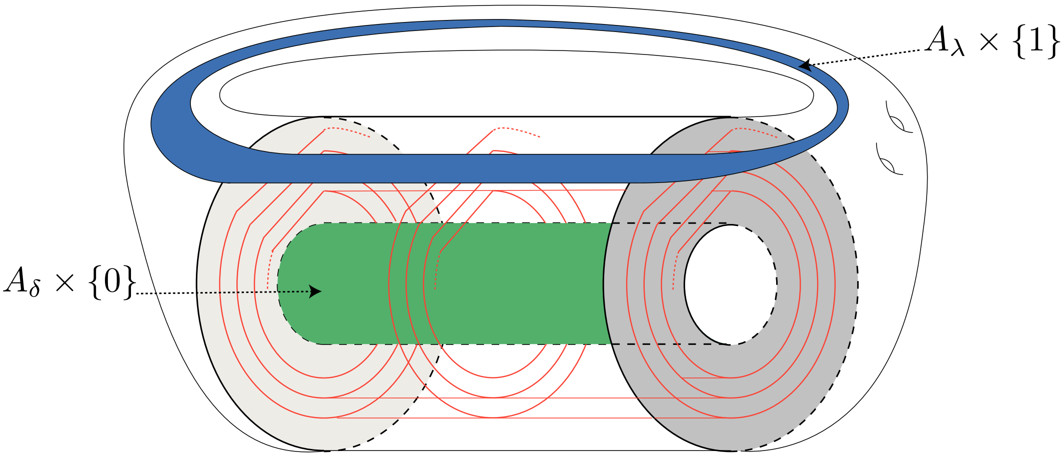


Figure 2.15: Foliation $\mathcal{F}_f$ along A_δ

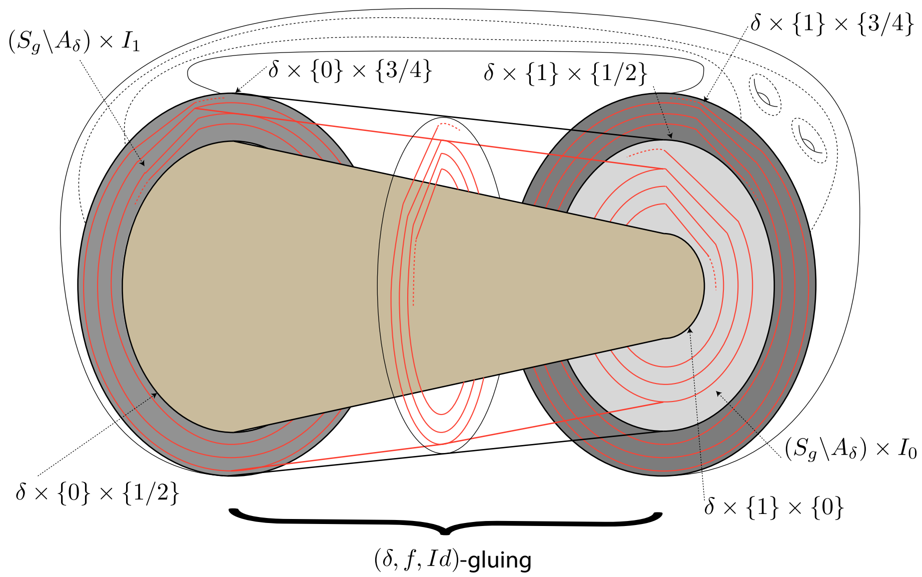


Figure 2.16: (δ, f, Id)-gluing between $(S_g \backslash A_\delta) \times I_0$ and $(S_g \backslash A_\delta) \times I_1$

We denote this extension by a (δ, f, h)***-gluing*** between $(S_g \backslash A_\delta) \times I_0$ and $(S_g \backslash A_\delta) \times I_1$, and the foliation is called $\mathcal{F}(f, I_0, I_1, h)$.

Note that we can make another choice to make this extension. Indeed, we can also identify $\delta \times \{0\} \times I_0$ and $\delta \times \{1\} \times I_1$ by a foliation preserving homeomorphism h similarly. That gives another direction of rotation along the boundary leaf.
If we do the first choice we call it a ***clockwise*** (δ, f, h)***-gluing*** otherwise if we make the second choice we call it a ***anti-clockwise*** (δ, f, h)***-gluing***. But for more simplicity when the direction of rotation does not matter we will just say a (δ, f, h)*-gluing*.

The boundary of the resulting manifold has two connected components as shown in

Figure 2.17 :

- $S_g^0 = \delta \times \{0\} \times [0, 1/2] \cup A_\delta^{1/2} \cup (S_g \backslash A_\delta) \times \{0\}$; where $\delta \times \{0\} \times [0, 1/2]$ is transverse and $A_\delta^{1/2} \cup (S_g \backslash A_\delta) \times \{0\}$ is tangent to the foliation $\mathcal{F}(f, I_0, I_1, h)$.
- $S_g^{3/4} = \delta \times \{1\} \times [1/2, 3/4] \cup A_\delta^{3/4} \cup (S_g \backslash A_\delta) \times \{3/4\}$; where $\delta \times \{1\} \times [1/2, 3/4]$ is transverse and $A_\delta^{1/2} \cup (S_g \backslash A_\delta) \times \{3/4\}$ is tangent to the foliation $\mathcal{F}(f, I_0, I_1, h)$.

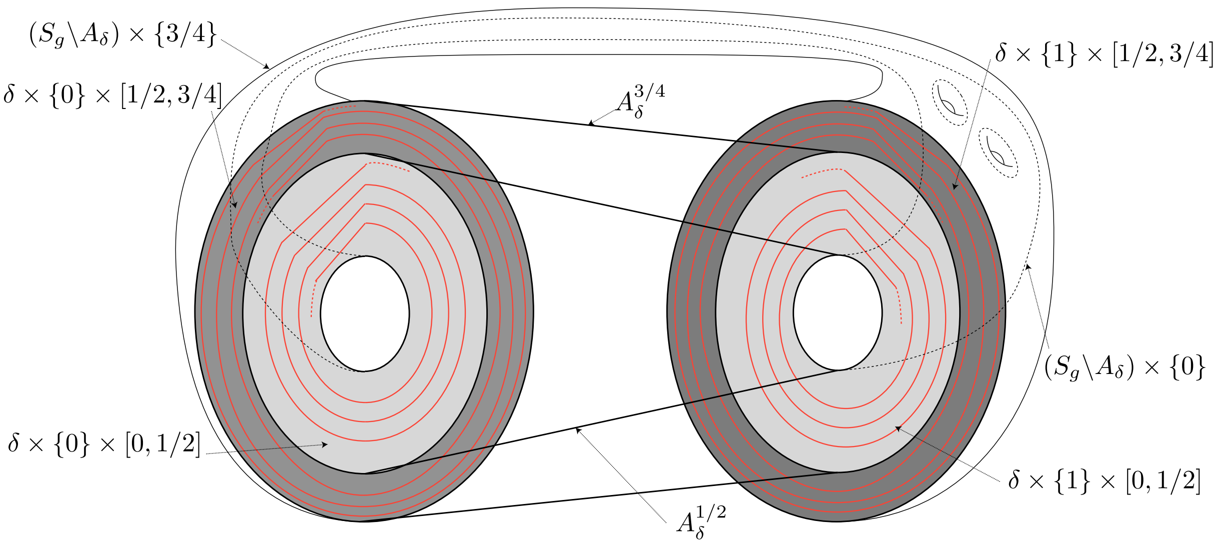


Figure 2.17: Boundary of the foliation $\mathcal{F}(f, I_0, I_1, Id)$

2.4.4 Step 4

The aim of this step is to repeat infinitely many times Step 3, in order to be a foliation of $S_g \times I$ where $S_g \times \{1\}$ is a leaf and $S_g \times \{0\}$ is foliated as $S_g^0 = \delta \times \{0\} \times [0, 1/2] \cup A_\delta^{1/2} \cup (S_g \backslash A_\delta) \times \{0\}$ defined in Step 3.

Some notations :
Set $i_0 = 0$.
Let $n \in \mathbb{N}^*$, we set :

$$i_n = \sum_{k=1}^{n} \frac{1}{2^k}$$

and $I_n = [i_n, i_{n+1}]$, for all $n \in \mathbb{N}$.
Note that $\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{1}{2^k} = 1$; hence $\overline{\bigcup_{n \in \mathbb{N}} I_n} = I$.

For all $n \in \mathbb{N}$, consider $(S_g \backslash A_\delta) \times I_n$ with the foliation $\mathcal{F}_f$ defined in Step 2.

Let $h_n, n \in \mathbb{N}$ be foliation preserving homeomorphisms between $\delta \times \{0\} \times I_{n+1}$ and $\delta \times \{1\} \times I_n$ sending an interior leaf on an interior leaf, and sending $\delta \times \{0\} \times \{i_{n+1}\}$ on

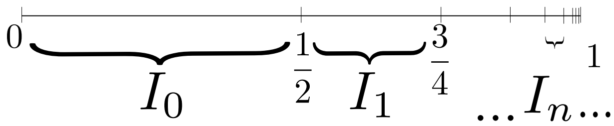


Figure 2.18: Intervals I_n

$\delta \times \{1\} \times \{i_n\}$, and $\delta \times \{0\} \times \{1/2\}$ on $\delta \times \{1\} \times \{0\}$.

For each $n \in \mathbb{N}$, apply clockwise (δ, f, h_n)-gluing (defined in Step 3) between $(S_g \backslash A_\delta) \times I_n$ and $(S_g \backslash A_\delta) \times I_{n+1}$, constructing the foliation $\mathcal{F}(f, I_n, I_{n+1}, h_n)$, and consider the closure of this manifold, to obtain $S_g \times I$ with the clockwise foliation

$$\mathcal{F}(f, h_n, n \in \mathbb{N}) = \overline{\bigcup_{n \in \mathbb{N}} \mathcal{F}(f, I_n, I_{n+1}, h_n)}$$

Since $\overline{\bigcup_{n \in \mathbb{N}} I_n} = I$, the homeomorphisms $h_n, n \in \mathbb{N}$ can be considered as a single homeomorphism h of I; so for more simplicity we denote this foliation by $\mathcal{F}(f, h)$ which will be called the clockwise foliation $\mathcal{F}(f, h)$.

We similarly define a foliation by considering only anti-clockwise (δ, f, h_n)-gluing for all $n \in \mathbb{N}$ to obtain a anti-clockwise foliation $\mathcal{F}(f, h)$.

This amounts to considering $(S_g \backslash A_\delta) \times \overline{\bigcup_{n \in \mathbb{N}} I_n}$, and to extending the foliation by pasting annuli called $A_\delta^{i_n}$ for $n \in \mathbb{N}^*$ between $\delta \times \{0\} \times \{i_n\}$ and $\delta \times \{1\} \times \{i_{n-1}\}$, and bands $\mathbb{R} \times I$ between the spirals $\mathbb{R} \times \{0\}$ on $\delta \times \{0\} \times \{i_n\}$ and the spirals $\mathbb{R} \times \{1\}$ on $\delta \times \{1\} \times \{i_{n-1}\}$ with respect to h_n.

When n tends towards the infinity, i_n tends toward 1, so we attach an annulus A_δ^1 between $\delta \times \{0\} \times \{1\}$ and $\delta \times \{1\} \times \{1\}$, hence $S_g \times \{1\}$ is a leaf.
Moreover, as in Step 3, $S_g^0 = \delta \times \{0\} \times I_0 \cup A_\delta^{1/2} \cup (S_g \backslash A_\delta) \times \{0\}$ is the second boundary component homeomorphic to $S_g \times \{0\}$, where $\delta \times \{0\} \times I_0$ is transverse and $A_\delta^{1/2} \cup (S_g \backslash A_\delta) \times \{0\}$ is tangent to the foliation $\mathcal{F}(f, h)$.

Note that the leaf starting in $\delta \times \{0\} \times \{0\}$ is homeomorphic to the half-infinite ladder as shown in Figure 2.19.

Remark 2.4.1. *Note that the induced foliation on the transverse annulus $X = \delta \times \{0\} \times \overline{\bigcup_{n \in \mathbb{N}} I_n}$ has an infinite number of circle leaves (which are $\bigcup_{n \in \mathbb{N}} \delta \times \{0\} \times \{i_n\}$). Between two consecutive such circle leaves there is the suspension foliation induced by f.*

Now we can define Spiraling.

Definition 2.4.2. *That construction of $\mathcal{F}(f, h)$ is called* **spiraling**.
We say that the nearby leaves of $S_g \times \{1\}$ in the clockwise (respectively in the anti-clockwise)

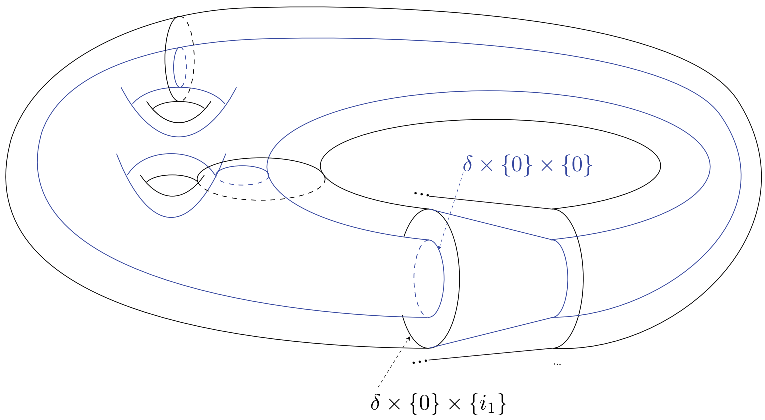


Figure 2.19: Interior leaf starting in $\delta \times \{0\} \times \{0\}$

foliation $\mathcal{F}(f,h)$ are **clockwise spiraling** *(respectively* **anti-clockwise spiraling***) along $S_g \times \{1\}$.*
We will denote the component $S_g \times I$ with the foliation $\mathcal{F}(f,h)$ by $\mathcal{S}_g(f,h)$, or when there is no ambiguity $\mathcal{S}_g$, with possibly adding the direction of rotation (clocwise or anti-clockwise). This foliation is of course transversely orientable; let $\mathcal{S}_g^+$ (respectively $\mathcal{S}_g^-$) be the foliation $\mathcal{F}(f,h)$ where the transverse orientation on the closed compact boundary leaf $S_g \times \{1\}$, points out (respectively into) $S_g \times I$.

Remark 2.4.3. *If $f = Id$ there are no spiral leaves, there are only half-infinite cylinders ($g = 1$), or half-infinite ladders ($g > 1$), and the foliation does not depend on h. In this case we will denote this foliation by $\mathcal{S}_g(Id)$.*

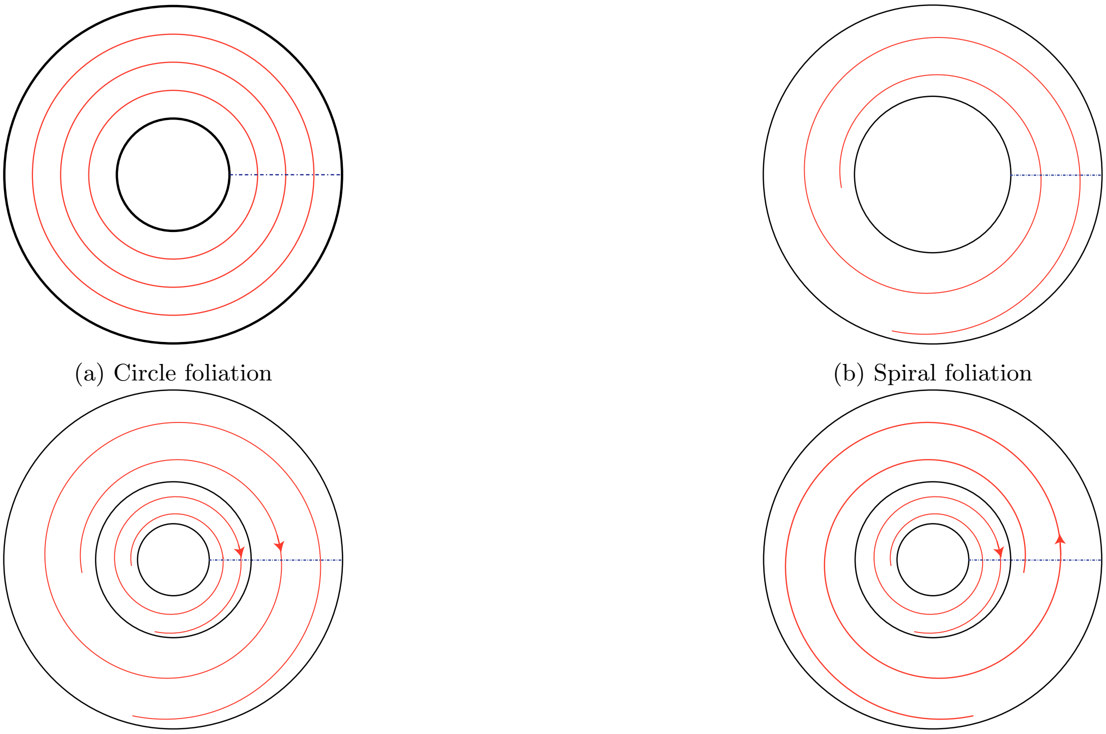
2.4.5 Attaching components of spiraling

Consider a 3-manifold M with a foliation $\mathcal{F}$, admitting a boundary component homeomorphic to S_g. Assume that $\mathcal{F}_{|S_g}$ has circle and spiral leaves contained in an annulus A with circle boundary leaves, and that $\mathcal{F}_{|S_g \backslash A}$ is a leaf.
We want to extend $\mathcal{F}$ in a neighborhood $S_g \times I$ (where $S_g = S_g \times \{0\}$), so that $S_g \times \{1\}$ is a leaf.

By the above construction it suffices to choose f such that the foliation on $\delta \times \{0\} \times [0, 1/2]$ and the foliation on A are homeomorphic, and glue $\delta \times \{0\} \times [0, 1/2]$ on A, and $S_g \backslash A$ on $(S_g \backslash A_\delta) \times \{0\} \cup A_\delta^{1/2}$ defined in Step 3, (see Figure 2.17).
It remains to make the good choice of f.

Recall that A is foliated by spirals and circles, and denote by $\mathcal{G}$ its foliation.
$\mathcal{G}$ is a foliation of an annulus with circles boundary leaves, because $S_g \backslash A$ is a leaf.
Hence $\mathcal{G}$ is isotopic to a union of the annuli of Figure 2.20.

Considering that $A = \mathbb{S}^1 \times I$, denote by $\tau = \{*\} \times I$ a transverse arc parametrized by $[0, 1]$.
Obviously, if F is a compact leaf of $\mathcal{G}$ (hence a circle) then $F \cap \tau = \{*\} \times \{x\}$, for a single



(a) Circle foliation (b) Spiral foliation

(c) Other possibility (c) Other possibility

Figure 2.20: Taut foliations of the annulus with boundary leaves.

$x \in I$, and if F is non-compact, there are infinitely many such x.
Let us call $X = \{x \in I / \exists F \in \mathcal{G}, F \cap \tau = \{*\} \times \{x\}\}$. By definition of $\mathcal{G}$, $\{0, 1\} \subset X$.
Note that possibly there exists $0 \leq a < b \leq 1$ such that $[a, b] \subset X$.
If $X = I$, $\mathcal{G}$ has trivial holonomy, i.e. $\mathcal{G}$ is a circle foliation.

Let f be a smooth, increasing map from I to I, such that $f_{|X}$ is the identity on X; and assume that f is strictly increasing out of X, and that f gives rise to the good direction of rotation. That is to say that for each spiral F of $\mathcal{G}$, there exists $x \in F \cap \tau \cap (I \backslash X)$, and we denote by I_F the connected component of $x \in I \backslash X$. If F is a clockwise spiral, we set $f(t) > t, t \in I_F$, otherwise (F is anti-clockwise) we set $f(t) < t, t \in I_F$.
Up to isotopy, f is the holonomy map of $\mathcal{G}$, but we will not use holonomy here.

By constructing the suspension foliation along λ (Step 2), we create spiral and circle leaves on $\delta \times \{0\} \times [0, 1/2]$. More precisely, we create circle leaves when f is the identity i.e when there are circles on $\mathcal{G}$, and spiral leaves out of X, with the corresponding direction of rotation; which is exactly the expected foliation (up to isotopy).

Remark 2.4.4. *Note that given a transverse orientation on the tangent part $S_g \backslash A$, say outward, (respectively inward), spiraling amounts to gluing on S_g a component $\mathcal{S}_g^+$ (respectively $\mathcal{S}_g^-$).*

2.4.6 Reeb annulus spiraling

When $g = 1$, we can also define spiraling for non-taut foliations, i.e when the induced foliation $\mathcal{G}$ on A admits Reeb annuli, as soon as it has boundary circle leaves.

We keep the previous notations of Step 4, and recall that $\overline{\bigcup_{n\in\mathbb{N}} I_n} = I$.
Indeed, here S_1 is a 2-torus that we denote T, hence $T\backslash A$ is an annulus so we have the following representation :

$I \times \mathbb{S}^1 \times I \cong \{(x,\theta,t), x \in I, \theta \in]-\pi,\pi], t \in I\}$, and denote for $t \in I$ $\delta \times \{0\} \times \{t\} = \{(0,\theta,t), \theta \in]-\pi,\pi]\}$.

Now foliate each annuli $\delta \times \{0\} \times I_n, n \in \mathbb{N}$, and $\delta \times \{1\} \times I_n, n \in \mathbb{N}$, by a foliation isotopic to $\mathcal{G}$ (the foliation is $\mathcal{G}$ up to dilatation).
Consider $T \times I \cong (I \times \mathbb{S}^1 \times I)/_\sim$
where $(0,(\theta,t)) \sim (1, h_n(\theta,t))$, for given foliation preserving homeomorphisms h_n sending $\delta \times \{0\} \times I_n$, on $\delta \times \{1\} \times I_n$, for each $n \in \mathbb{N}$, depending on the integer n such that $t \in I_n$. As above all the homeomorphisms h_n can be seen as a single homeomorphism from $\delta \times \{0\} \times I$, on $\delta \times \{1\} \times I$ since $\overline{\bigcup_{n\in\mathbb{N}} I_n} = I$.
Denote for each $t \in I$ and $x \in I$, $\delta_x^t = \{(x,\theta,t), \theta \in]-\pi,\pi]\} \cong \mathbb{S}^1$.

With those coordinate, we assume $A \cong \delta \times \{0\} \times I_0 = \{(0,\theta,t), \theta \in]-\pi,\pi], t \in I_0\} \subset T \times I$; and $T\backslash A \cong \{z \in \delta_x^{\frac{1}{2}(1-x)}, x \in [0,1]\}$; (see Figure 2.21).

In $\delta \times I \times I$ consider for all $n \in \mathbb{N}^*$, the annulus leaves denoted A^{i_n} connecting $\delta \times \{0\} \times i_n$ to $\delta \times \{1\} \times i_{n-1}$, i.e $A^{i_n} = \bigcup_{x\in I} \delta_x^{-\frac{1}{2^n}x + i_n}$ (note that $A^{1/2} = T\backslash A$).
Foliate each solid cylinder bounded by $\delta \times \{0\} \times I_{n+1} \cup A^{i_{n+1}} \cup \delta \times \{1\} \times I_n \cup A^{i_n}$, $n \in \mathbb{N}$ by a foliation isotopic to $\mathcal{G} \times I$ with respect to the foliation set on the annuli $\delta \times \{0\} \times I_n, n \in \mathbb{N}$, and $\delta \times \{1\} \times I_n, n \in \mathbb{N}$.

Remark 2.4.5. *This choice of foliated solid cylinder induces a clockwise foliation while the other choice (joining $\delta \times \{0\} \times I_n$ to $\delta \times \{1\} \times I_{n+1}$, $n \in \mathbb{N}$) induces a anti-clockwise foliation.*

When n tends towards the infinity we add the torus leaf $T \times \{1\}$, because A^1 connects $\delta \times \{0\} \times 1$ to $\delta \times \{1\} \times 1$ since $\lim_{n\to+\infty} i_n = 1$.

Definition 2.4.6. *We call that component* **generalized spiraling component***, denoted $\mathcal{S}_*(\mathcal{G},h)$, or $\mathcal{S}_*$ when there is no ambiguity.*

Note that since the identification is by a foliation preserving homeomorphism; if $\mathcal{G}$ has no Reeb annuli, $\mathcal{S}_*(\mathcal{G},h) = \mathcal{S}_1(f,h)$, where f is a suspension homeomorphism defining $\mathcal{G}$.

The two boundary components are :
$T\backslash A \cup A$, where A is transverse and $T\backslash A$ is tangent to the foliation.

The torus leaf $T \times \{1\}$.

Let us describe the induced foliation by $\mathcal{S}_*(\mathcal{G}, h)$ on the properly embedded transverse annulus $X = \delta \times \{0\} \times I$.
∂X is included in leaves, i.e that foliation admits circle boundary leaves.
It has infinitely many circles leaves in its interior. Indeed, the leaf of $\delta \times \{0\} \times \{0\}$ is an half infinite cylinder, and its intersects X in each $\delta \times \{0\} \times \{i_n\}, n \in \mathbb{N}$.
The induced foliation on each annulus $\delta \times \{0\} \times I_n, n \in \mathbb{N}$ (included in X), is $\mathcal{G}$; hence we have the following Remark.

Remark 2.4.7. *The previous construction of $\mathcal{S}_*(\mathcal{G}, h)$ when $\mathcal{G}$ admits at least one Reeb annulus contradicts part (i) of Theorem* 4.2.15 *of Hector and Hirsch [1986], which says that a foliation of an annulus tangent to the boundary can only admits finitely many Reeb components.*

Remark 2.4.8. *The induced foliation by $\mathcal{S}_*(\mathcal{G}, h)$ on $T^{1/2} = \bigcup_{x \in I} \delta_x^{1/2}$ is the annulus foliation $\mathcal{G}$ whose boundary leaves are identified.*

Note that by considering the induced foliation by $\mathcal{S}_*(\mathcal{G}, h)$ on the manifold homeomorphic to $T \times I$ bounded by $T \times \{1\}$ and $T^{1/2}$, we see that spiraling extend a given foliation on a 2-torus with at least a circle leaf (possibly with spirals and Reeb annuli) to a foliation of $T \times I$ where $T \times \{1\}$ is a torus; which was our first goal.

2.5 Foliations near torus leaves

We first prove the equivalence between trivial spiraling and turbulization (Lemma 2.5.1). Then we prove Proposition 1.0.7.

2.5.1 Equivalence between trivial spiraling and turbulization

Lemma 2.5.1. *$\mathcal{S}_1(Id)$ is isotopic to $\mathcal{T}$.*

Proof. We start from a component $\mathcal{T}$ of turbulization, and we are going to find a torus T_1 foliated by a tangent annulus and a transverse annulus, and then we can see that $\mathcal{S}_1 \subset \mathcal{T}$ up to isotopy. So $\mathcal{S}_1(Id)$ is isotopic to $\mathcal{T}$ since they foliate the same 3-manifold.

We consider $I \times \mathbb{S}^1 \times \mathbb{S}^1 \cong \{(x, \theta, z), x \in I, \theta \in]-\pi, \pi], z \in [0, 1]\}/_\sim$
where $(\theta, x, 1) \sim (\theta, x, 0)$ foliated by a $\mathcal{T}$ component.
Now, let $z_0 \in]0, 1[$, and let L_{z_0} be the leaf of $(1, \theta, z_0)$, for $\theta \in]-\pi, \pi]$.
Clearly, L_{z_0} does not depend on θ, because the foliation $\mathcal{T}$ with those coordinate is invariant by rotation around the z-axis.
Then, there exists $x_0 \in]0, 1[$ such that the point $(x_0, \theta, 1) \in L_{z_0}$, by following the leaf L_{z_0} when z grows. Note again that x_0 does not depend on $\theta \in]-\pi, \pi]$.
Let $A_h = \{(x, \theta, 0), x_0 \leq x \leq 1, \theta \in]-\pi, \pi]\}$ and $A_v = \{(1, \theta, z), \theta \in]-\pi, \pi], 0 \leq z \leq z_0\}$,

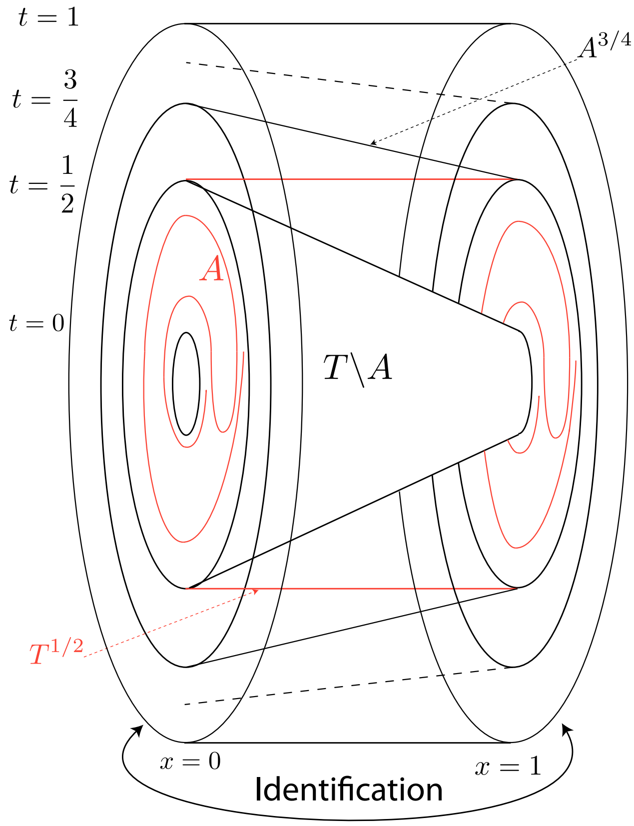


Figure 2.21: Generalized spiraling : $\mathcal{S}_*$

as in Figure 2.22.
Then set $T_1 = \{(x, \theta, z) \in L_{z_0}, x_0 \leq x \leq 1, z_0 \leq z \leq 1, \} \bigcup A_v \bigcup A_h$.
We can easily see that $A_v \bigcup A_h$ is transverse to the foliation $\mathcal{T}$, and that $\{(x, \theta, z) \in L_{z_0}, x_0 \leq x \leq 1, z_0 \leq z \leq 1, \}$ is tangent to it.
Of course between T_1 and the torus leaf, the non-compact leaves are all half infinite cylinders, which is the case when the spiraling has no holonomy. Hence the turbulization contains spiraling with trivial holonomy.

Another way of doing it is by applying Remark 2.4.8 for the component $\mathcal{S}_*(\mathcal{G}, Id) = \mathcal{S}_1(Id)$ when $\mathcal{G}$ is a circle foliation as Figure 2.23 shows it (of course in this case $T^{1/2}$ admits a circle foliation).

□

2.5.2 Proof of Proposition 1.0.7

Proof. Of course if $\mathcal{F}$ admits one of those components, $\mathcal{F}$ admits a torus leaf.
The converse is more interesting; it amounts to study the foliation in a neighborhood of a torus leaf.

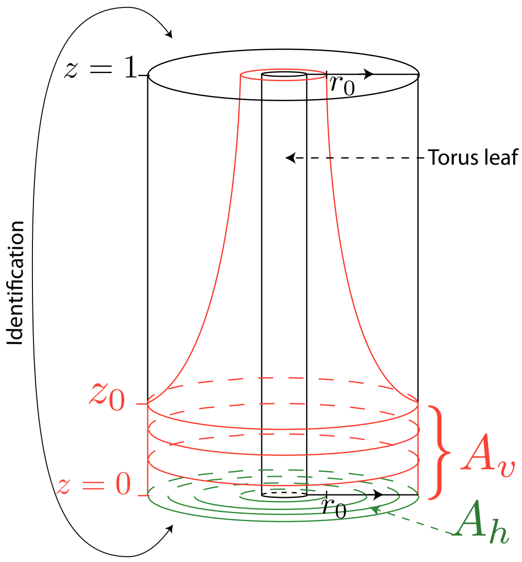


Figure 2.22:

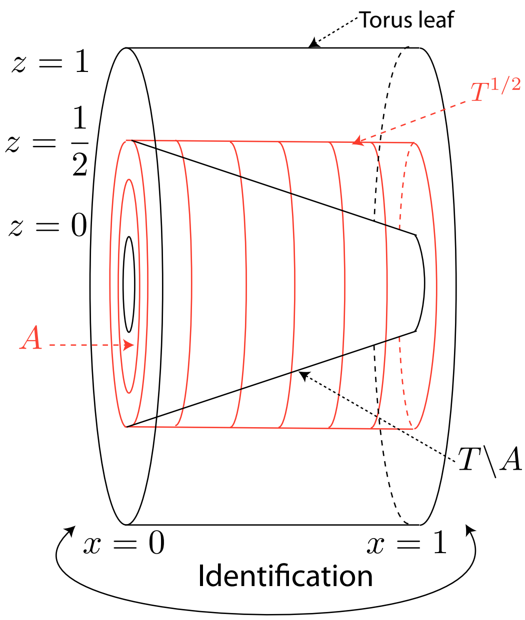


Figure 2.23:

The proof has two parts. First we choose a suitable neighborhood of a torus leaf (Claim 2.5.2). Then we recognize the foliation of $T^2 \times I$ as a $\mathcal{S}_*$ or a $\mathcal{T}_*$ component, using a properly embedded transverse annulus in this neighborhood.

Let T be the torus leaf and consider a neighborhood of T denoted by $V \cong T \times I$ where $T = T \times \{1\}$.
If T is compressible, by Theorem 1.0.3, the foliation in the 3-manifold that T bounds

admits a Reeb component and so it admits a $\mathcal{T}$ component by Remark 2.3.6.

Thus we can assume that T is incompressible, hence $T \times \{0\}$ is also incompressible. Let us choose coordinates for V.
Let $W = I \times (\mathbb{S}^1 \times I)$, i.e $W = \{(x, (\theta, z)), x \in I, \theta \in] - \pi, \pi], z \in I\}$ foliated by annulus leaves $I \times (\mathbb{S}^1 \times \{*\})$.
$V \cong W/_\sim$, where $(0, (\theta, z)) \sim (1, h(\theta, z))$ and h is a foliation preserving homeomorphism defined by $\mathcal{F}_{|V}$.
Set $X = \{(0, \theta, z), \theta \in] - \pi, \pi], z \in I\}$.

First collapse all the $T \times I$ in V whose foliation is $\{T \times \{t\}, t \in I\}$. Then we assume (since $M \ncong T^2 \times I$ and $M \ncong T^2 \times \mathbb{S}^1$ foliated by $\mathcal{T}^2 \times \{*\}$) that all the torus leaves are isolated (i.e for each torus leaf in V there exists a regular neighborhood of this torus leaf not admitting another torus leaf).

Claim 2.5.2. *We can choose inside $T \times I$ a regular neighborhood V' of a torus leaf T' such that $V' \cong T' \times I$, $T' = T' \times \{1\}$, and there is no torus leaf in $\overset{\circ}{V'}$.*

Proof. If there is no interior torus leaf in V we are done (choose $V' = V$) so we can assume that there is an interior torus leaf T_1 in V and consider a thinner regular neighborhood of T (still denoted V) where $T \times \{1\} = T$ and $T \times \{0\} = T_1$. By continuing this process either we find such a V', either this process never stops; that means that the set of torus leaves in the leaf space of $\mathcal{F}_{|V}$ is dense.
Hence we make the proof by contradiction and we suppose that such a V' does not exist. We have seen above that it means that the set of torus leaves in the leaf space of $\mathcal{F}_{|V}$ is dense (between two torus leaves there always exists another torus leaf).
Recall that we can suppose that $T \times \{0\}$ is a torus leaf and that $T \times \{1\} = T$, otherwise the claim is true.
Consider the induced foliation by $\mathcal{F}$ on X. Call C the set of circles of intersection between the torus leaves and X, and denote by $I_a^b = \{(0, 0, z), z \in [a, b]\}$ for any real such that $0 \leq a \leq b \leq 1$. Hence $C \cap I_0^1$ is dense in I_0^1.
That imposes that X admits a circle foliation. Indeed, any spirals or Reeb annulus between two circle leaves contradicts the density.
Hence, there is two cases :
V is trivially foliated by torus leaves which is impossible by assumption of collapsing.
V is foliated by cylinder leaves and torus leaves.

In the latter case any cylinder leaf contradicts the density.
Indeed, up to isotopy and up to changing the coordinates, a cylinder leaf contains the annulus $A = \{(x, \theta, (b - a)x + a), x \in I, \theta \in] - \pi, \pi]\}$ for given a and b in I.
Hence $C \cap I_a^b = \emptyset$ which contradicts the density, see Figure 2.24.

This ends the proof of Claim 2.5.2. □

By Claim 2.5.2, we may assume that V does not contain interior torus leaves.

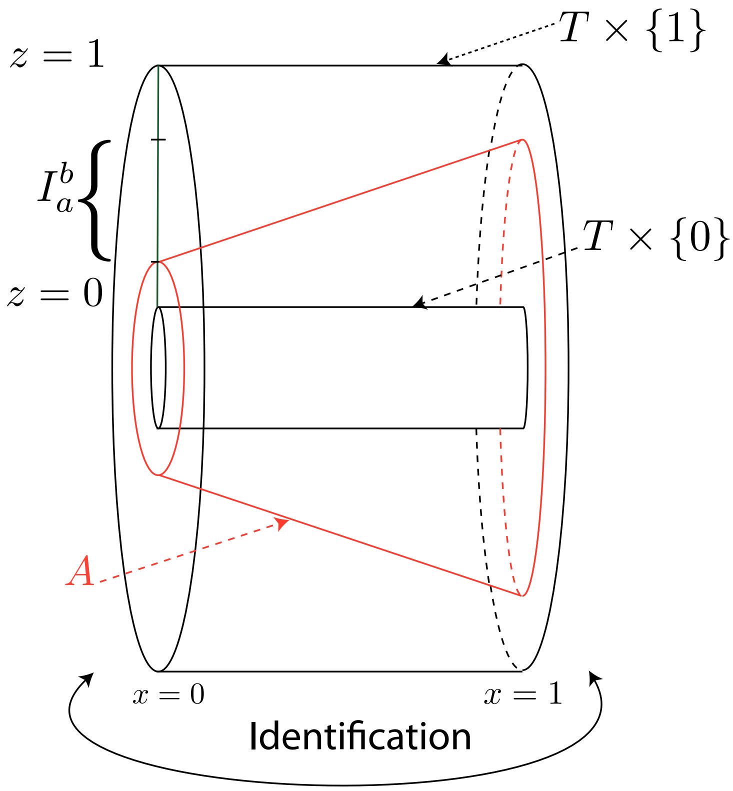


Figure 2.24: Cylinder leaf contradicts the density

If T bounds a $\mathcal{L}$ component (see Definition 2.8.1, and also Figure 2.34), or the foliation of Q (Waldhausen manifold) pictured in Figure 2.33, then $\mathcal{F}$ trivially contains a $\mathcal{T}$ component.
Hence we may assume that $\mathcal{F}_{|V}$ is different from those two foliations.

We may recall that $T \times \{0\}$ is incompressible, so by a theorem proved by Roussarie [1974] and independently by Thurston [1972] we can isotope $T \times \{0\}$ such that it is everywhere transverse to the foliation or so that it is a leaf.
Hence, up to isotopy, we can assume that all the $T \times \{t\}$ are transverse, for $t \in [0, 1[$, and so we can consider the 1-dimensional induced foliation on $T \times \{0\}$.
Since the foliation $\mathcal{F}$ is $\mathcal{C}^2$, so is the induced foliation on $T \times \{0\}$, and by a theorem of Denjoy [1932], either the induced foliation on $T \times \{0\}$ is dense (i.e all the leaves are lines); or it admits circle leaves (and some spirals limiting to those circles, or it is a circle foliation).

In the former case since the foliation on $T \times \{0\}$ is by parallel lines; one of the boundary component of X is everywhere transverse to the foliation.
Thus, there is only one circle leaf, and up to isotopy, the only $\mathcal{C}^2$-foliation of the annulus with a boundary leaf and a transverse one is $\mathcal{C}$ (see Figure 2.4).
Since $\mathcal{F}$ induces this foliation on X and since the foliation on $T \times \{0\}$ is by parallel lines, the induced foliation on the annuli $X_x = \{(x, \theta, z), \theta \in] - \pi, \pi], z \in I\}$, for each $x \in I$, is also isotopic to $\mathcal{C}$ (note that $X = X_0$).

All the foliations possible on $T^2 \times I$ are now characterized by the attachment possible between X_0 and X_1. All those foliations corresponds to a $\mathcal{T}_*$ component (see Figure 2.11).

The latter case where $T \times \{0\}$ admits at least one circle leaf, corresponds to a $\mathcal{S}_1$ or a $\mathcal{S}_*$ component depending on if there are circle foliation, spiral leaves or Reeb annuli. Recall that by Lemma 2.5.1 $\mathcal{S}_1(Id)$ is isotopic to $\mathcal{T}$.
Recall also that by Remark 2.4.8 a $\mathcal{S}_1$ or a $\mathcal{S}_*$ component can be seen as a foliation on $T^2 \times I$ where $T^2 \times \{0\}$ is everywhere transverse (with circle foliation, or admitting spiral leaves or Reeb annuli) and $T^2 \times \{1\}$ is a leaf (choose $T^2 \times \{0\}$ as $T^{1/2}$ of Remark 2.4.8). Moreover, since $\mathring{V}$ does not contain torus leaves, any circle leaf of X is included in a cylinder leaf of $\mathcal{F}_{|V}$, hence there is an infinite number of circles leaves on X.
Now given such a foliation on X and on $T \times \{0\}$, the induced foliation by $\mathcal{F}$ on W is up to isotopy (and changing the coordinates) the one of Figure 2.25.

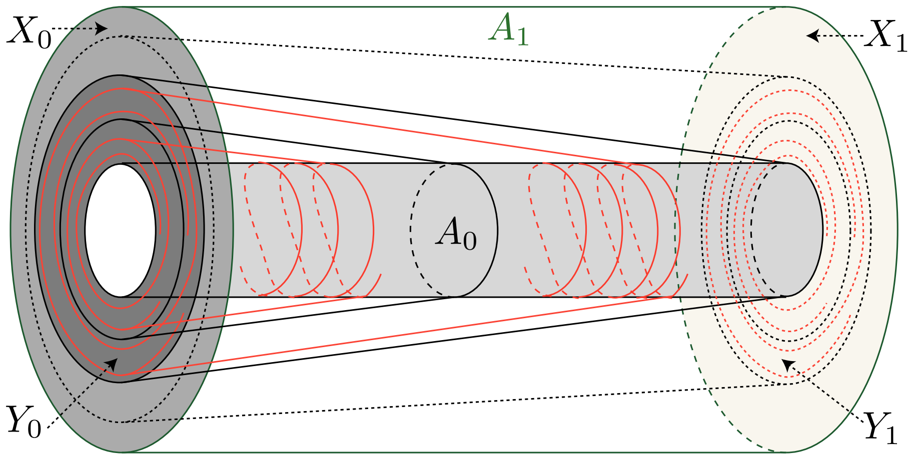


Figure 2.25: Imposed foliation on W

Indeed, it admits an annulus leaf $A_1 = \{(x, \theta, 1), x \in [0, 1[, \theta \in]-\pi, \pi]\}$, X_1 has the foliation of $X = X_0$, it admits another transverse annulus $A_0 = \{(x, \theta, 0), x \in [0, 1[, \theta \in]-\pi, \pi]\}$, with the foliation of $T \times \{0\}$ split along a circle leaf that we denote by $\mathcal{G}$.
Note that in Figure 2.25 we have chosen spiral leaves on $\mathcal{G}$ but we could have chosen circle foliation or foliation with Reeb annuli. Moreover, in W the only possibility to follow the foliation on those annuli is by following the projection of A_0 on a sub-annulus of X_0 denoted on Figure 2.25 by Y_0.
Since X_1 has the foliation of X_0, there is a sub-annulus of X_1 foliated as Y_0 denoted by Y_1. Once again, the foliation of Y_1 can be followed in W by following the projection of Y_1 on another sub-annulus of X. By continuing this process we obtain after gluing X_0 to X_1 by h, a $\mathcal{S}_1(f, h)$, or a $\mathcal{S}_*(\mathcal{G}, h)$ where f is the suspension homeomorphism defining $\mathcal{G}$ when $\mathcal{G}$ has no Reeb annuli. (Note that $f = Id$ corresponds to a circle foliation on $\mathcal{G}$).
Note that $A_0/_\sim$ corresponds to $T^{1/2}$ of Remark 2.4.8.

□

2.6 Proposition 2.1.1 and consequences

It is well known that Reeb's component (and Reeb annulus) are not taut.
Brittenham [1993a], generalized this fact to manifolds with at most one boundary component.
Here we give more details, and generalize it to manifolds with more boundary components, if we assume that the transverse orientation is the *same* on each boundary component.
This is the goal of Proposition 2.1.1 proved in Subsection 2.6.1.
In Subsection 2.6.2 we will see that the hypothesis of Proposition 2.1.1 are thin by giving interesting examples of foliations on Waldhausen manifold.
Finally in Subsection 2.6.3 we will give a partial converse of Proposition 2.1.1 which is Theorem 2.6.8 (Goodman [1975]) and Corollary 2.6.10.

2.6.1 Proof of Proposition 2.1.1

Remark 2.6.1. *Note that in an orientable manifold, if a foliation is transversely orientable, then all the leaves are orientable.*
However, the converse is not true : there exists a foliation of T^3 with all the leaves orientable but which is not transversely orientable (see the foliation $\mathcal{L}_1$ in Subsection 2.8.1). Nevertheless, this foliation is not taut and if we assume that a foliation of an orientable manifold is taut and that all the leaves are orientable then this foliation is transversely orientable.

Definition 2.6.2. *Assume that a manifold M with non-empty boundary admits a transversely orientable foliation $\mathcal{F}$ such that the boundary of M is a union of leaves. Then we say that ∂M* **has the same transverse orientation** *if the transverse orientation on those boundary leaves point all inward or point all outward.*

Proposition 2.1.1 is a direct consequence of the following proposition.

Proposition 2.6.3. *Let M be a 3-manifold with a transversely orientable foliation $\mathcal{F}$, and $n \in \mathbb{N}$.*
If the boundary of M is a union of leaves $\bigcup_{i=1...n} T_i$ with the same transverse orientation, or if $\mathcal{F}$ contains a compact separating leaf T_0, then for all $i \in \{1, ..., n\}, T_i$ does not admit a transverse loop or properly embedded transverse arc.

Proof. We are going to show that for every properly embedded arc $\gamma : I \to M$ with endpoints in a separating compact leaf or in a boundary leaf T, there exists a point of $\gamma(I)$ where the foliation is tangent to γ.

This implies the proposition, assuming first that $\partial M = \emptyset$; because any closed curve transverse to a separating compact leaf T, intersects at least two times T, hence the closed curve is a union of arcs with endpoints in T, in each side of T. So we will only study one side of T with an arc which meets T only on its endpoints.
If there is only one boundary leaf, this is exactly what we want to have.

If there are at least two compact boundary leaves, the endpoints of γ may be on two differents boundary leaves; and since the transverse orientation is the same on those two leaves, the following applies similarly.

Up to isotopy, we assume that the induced orientation by the non-zero continuous vector field is a normal vector field to the leaves noted N_x for each $x \in M$.
Recall that M is a Riemannian manifold, hence in the tangent space of M we can consider the following angles.
Let h be the map :

$$\begin{cases} h \ : \gamma(I) \to [0, \pi] \\ \\ x \mapsto (T\gamma_x, N_x) \end{cases}$$

Where $(T\gamma_x, N_x)$ is the non-oriented angle between the tangent vector to γ in x (noted $T\gamma_x$) and the normal vector N_x.
$\mathcal{F}$ is transversely oriented (i.e N is nowhere zero and continuous) and γ is embedded, so $T\gamma$ is continuous; hence, h is continuous.
Without loss of generality we can say that the transverse orientation on T is such that $h(\gamma(0)) > \pi/2$. Thus, $h(\gamma(1)) < \pi/2$, because at $\gamma(1)$ the arc gets out of T, and $N_{\gamma(1)} = N_{\gamma(0)}$, see Figure 2.26. Indeed, the leaves are orientable, and if there are at least two boundary leaves we have supposed that the transverse orientation is the same.

By the Intermediate value Theorem (h is continuous), there exists
$x \in \ \gamma(]0, 1[)$ such that $h(x) = \pi/2$. That means that $T\gamma_x$ is in the tangent plane of the leaf passing through x, i.e γ is tangent to $\mathcal{F}$ in x.
Hence $\mathcal{F}$ cannot be taut; which ends the proof of Proposition 2.1.1.

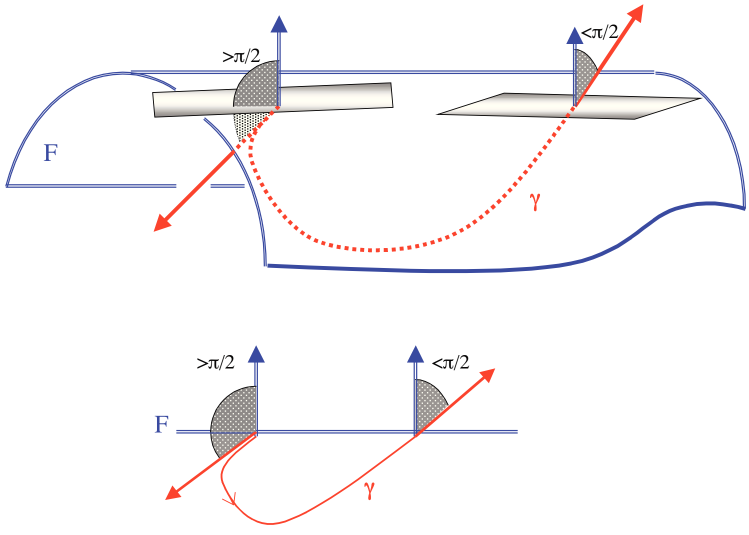


Figure 2.26:

□

Remark 2.6.4. *The above proposition similarly implies that Reeb annuli are not taut.*

Remark 2.6.5. *Note that the assumptions of Proposition 2.1.1 are sharp. Indeed :*

(1) *If there are a transverse boundary component and a tangent one, then the foliation may be taut.*
For example, the $\mathcal{T}$ component is taut, i.e we can find a properly embedded arc γ as in Figure 2.7, with an endpoint on the transverse boundary torus, and the other on the torus leaf.

(2) *When there are at least two boundary leaves without the same transverse orientation, Proposition 2.1.1 is not true.*
Trivially, the foliation $S\times I$ is taut; where S is any closed surface. But the transverse orientation on $S\times\{0\}$ is opposite to the one on $S\times\{1\}$ (one points inward and the other outward).

(3) *The assumption of transverse orientation in Proposition 2.1.1 is necessary as suggested by Lemma 2.6.6 on the Waldhausen manifold.*

2.6.2 Waldhausen manifold

Lemma 2.6.6. *Waldhausen manifold admits :*

(1) *A taut, non-transversely orientable foliation with a single torus boundary leaf, and all the leaves are compact.*
(2) *A taut, non-transversely orientable foliation with a single torus boundary leaf, and all the interior leaves are non-compact.*

Proof. Recall that Waldhausen manifold Q is the twisted product of the Klein bottle with an interval :
$Q = K\widetilde{\times}I$, where K is the Klein bottle.
Q has one torus boundary component T.
Let us represent Q as follows :
Consider $W = I\times\mathbb{S}^1\times I \cong \{(x,\theta,z), x\in[0,1], \theta\in]-\pi,\pi], z\in[0,1]\}$.

Now $Q\cong W/_{\sim}$ where $(x,\theta,0)\sim(1-x,-\theta,1)$.

Part (1) of Lemma 2.6.6 is easy to construct and is represented in Figure 2.27.
With the above representation, there is a Klein bottle leaf which is $K = \{(1/2,\theta,z),\theta\in\,]-\pi,\pi], z\in[0,1]\}/_{\sim}$,
and the other compact leaves are torus leaves which are for each $x\in[0,1]$:
$T_x = \big\{\{(x,\theta,z),\theta\in]-\pi,\pi], z\in[0,1]\}\cup\{(1-x,\theta,z),\theta\in]-\pi,\pi], z\in[0,1]\}\big\}/_{\sim}$.

This foliation is taut, because for example $\gamma=\{(x,0,1/2), x\in[0,1]\}$ is a properly embedded transverse arc, with both endpoints in the torus boundary leaf (see Figure 2.27).

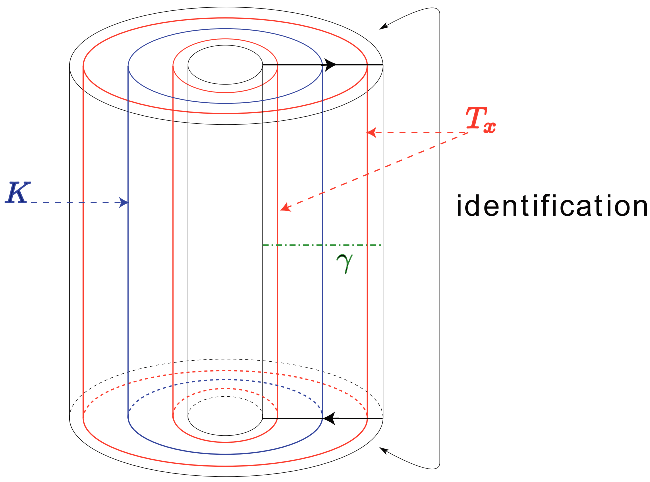


Figure 2.27: Compact foliation of Q

It is non-transversely oriented since it admits a non-orientable leaf (recall that Q is orientable).
Hence Part (1) of Lemma 2.6.6 is proven.

Part (2) of Lemma 2.6.6 needs more work and is the following construction.

For each $z \in [0,1]$, we set $A_z = \{(x,\theta,z), x \in [0,\frac{1}{2}], \theta \in]-\pi,\pi]\}$; and $A'_z = \{(x,\theta,z), x \in [\frac{1}{2},1], \theta \in]-\pi,\pi]\}$.
Consider a (clockwise) spiral foliation (see Figure 2.3) on each annulus $A_z \cup A'_z = \{(x,\theta,z), x \in I, \theta \in]-\pi,\pi]\}$. Denote the circle leaves by $C_z = \{(0,\theta,z), \theta \in]-\pi,\pi]\}$ and $C'_z = \{(1,\theta,z), \theta \in]-\pi,\pi]\}$. For each $z \in I$, we let $C_z^{1/2} = \{(\frac{1}{2},\theta,z), \theta \in]-\pi,\pi]\}$, they are all circles transverse to the foliation.
That induces on A_z and on A'_z a the foliation $\mathcal{C}$ (see Figure 2.4).

That gives a taut product foliation on $W = \bigcup_{z \in I} A_z \cup A'_z$ denoted $\widehat{\mathcal{F}}$.

Now we want to use this foliation of W to obtain by identification a foliation on Q.
We may recall that $Q \cong W/_\sim$ where $(x,\theta,0) \sim (1-x,-\theta,1)$.
So we only need to check that the foliation $\widehat{\mathcal{F}}$ in W is preserved by the identification.
More precisely, it remains to check that any leaf of A_0 is identified on any leaf of A'_1 and any leaf of A'_0 is identified on any leaf of A_1.
In particular that means that for each $\theta \in]-\pi,\pi]$, the leaf of A_0 at $(\frac{1}{2},\theta,0)$ must be identified on the leaf of A'_1 passing at $(\frac{1}{2},-\theta,1)$.

One way to do it is as follows :
Let f be a diffeomorphism such that :

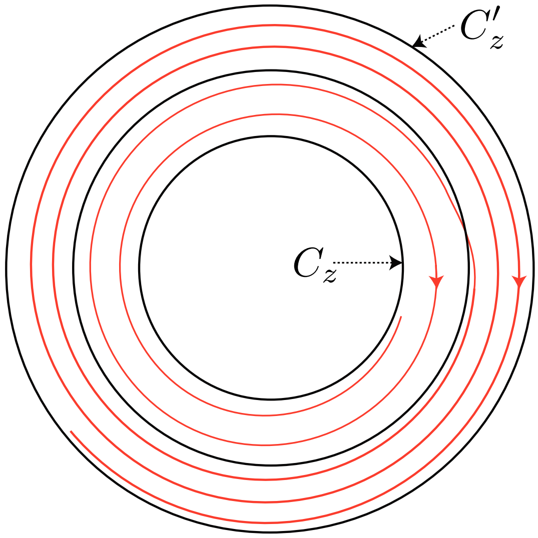


Figure 2.28: Induced foliation on $A_z \cup A'_z, z \in [0,1]$.

$$\begin{cases} f \ : A_1 \cup A'_1 \to A_0 \cup A'_0 \\ f(A_1) = A'_0, f(A'_1) = A_0 \\ \forall (1/2, \theta, 1) \in C_1^{1/2}, f(1/2,\theta,1) = (1/2, -\theta, 0) \in C_0^{1/2} \end{cases}$$

and such that f preserves the foliation, i.e f maps a half-spiral leaf on a half-spiral leaf. Note that the definition of f induces $f(C_1) = C'_0, f(C'_1) = C_0$.
We can consider $Q \cong W/_{\sim'}$ where $((x,\theta),1) \sim' (f(x,\theta),0)$ with the induced foliation by $\widehat{\mathcal{F}}$ because this foliation is preserved by the identification.
We denote by $\mathcal{F}$ this new foliation on Q.
Note that this new representation of Q with $\sim'$ is isotopic to the first one with $\sim$.

$\mathcal{F}$ is taut, because for example $\gamma = \{(x, 0, 1/2), x \in I\}$ is a properly embedded transverse arc to $\mathcal{F}$.

$\mathcal{F}$ admits a single torus boundary leaf, which is $(\bigcup_{z\in[0,1]} C_z \cup C'_z)/_{\sim'}$ and the interior leaves are non-compact (they all contain an embedded $\mathbb{R} \times I$).

The proof of Claim 2.6.7 ends the proof of Part (2) of Lemma 2.6.6.

Claim 2.6.7. *$\mathcal{F}$ is not transversely oriented.*

Proof. Let $\widehat{L}$ be the leaf of $\widehat{\mathcal{F}}$ in W containing the arc $\widehat{C} = \{(\frac{1}{2}, 0, z), z \in [0,1]\}$, and consider a regular neighborhood $\widehat{B}$ of $\widehat{C}$ in $\widehat{L}$.
Set $a_i = \partial\widehat{B} \cap A_i$, and $a'_i = \partial\widehat{B} \cap A'_i$.
$\widehat{B}/_{\sim'}$ is homeomorphic to a Mobius band since $f(a_1) = a'_0$ and $f(a'_1) = a_0$ (see Figure 2.29), and by construction is included in a leaf of $\mathcal{F}$. Since Q is oriented, $\mathcal{F}$ is non-transversely oriented.

□

In conclusion, $\mathcal{F}$ is taut, non-transversely oriented with a torus boundary leaf and with non-compact interior leaf, as in Figure 2.29, which ends the proof of Part (2) of Lemma 2.6.6.

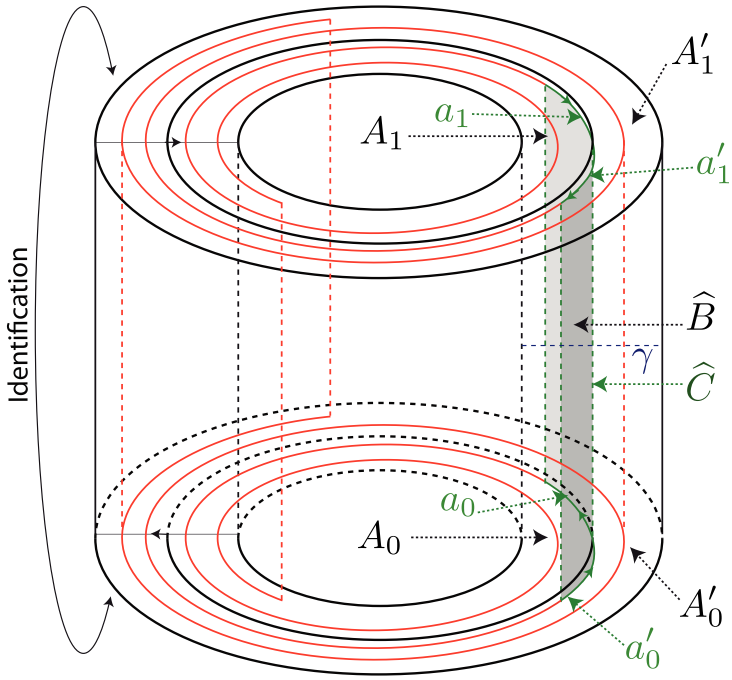


Figure 2.29: Foliation $\mathcal{F}$ on Q

□

2.6.3 Partial converse: existence of torus leaf

Now we suppose that a transversely oriented foliation is non-taut and see that it admits a torus leaf but we cannot conclude if it is separating or not, that is why it is a partial converse to Proposition 2.1.1.

Theorem 2.6.8 (Goodman [1975]). *If a leaf of a transversely orientable $\mathcal{C}^1$-foliation of a closed 3-manifold does not intersect a closed transverse curve then it is a torus leaf.*

Therefore, if a foliation is not taut then it admits a torus leaf.

Question 2.6.9. *We may wonder if it is still true when the foliation is only supposed to be $\mathcal{C}^0$.*

Corollary 2.6.10. *Consider a transversely oriented $\mathcal{C}^1$-foliation on M tangent to the boundary (possibly $\partial M = \emptyset$), then the following assertions are true.*

(1) *If a leaf does not admit a properly embedded transverse arc or transverse loop then it is a torus.*

(2) *If a leaf is separating, then it is a torus leaf.*

(3) *If the boundary components of M are a union of leaves admitting the same transverse orientation, then all are tori.*

Proof. When $\partial M \neq \emptyset$, we consider the double of M, i.e $D(M) = M \bigcup_{\partial M} M$ (the union of two copies of M with opposite orientation). Notice that any closed transverse loop passing trough a leaf in $D(M)$ would induce a closed transverse curve, or a properly embedded transverse arc in M.
Now we assume that a leaf does not admit a properly embedded transverse arc or transverse loop, and we apply Theorem 2.6.8 to M (or do $D(M)$ if $\partial M \neq \emptyset$), so this leaf is a torus, so part (1) is true.

If a separating compact surface is a leaf, then by Proposition 2.6.3, there is no transverse loop passing through it, so this is a torus by part (1), so part (2) is true.

By applying Proposition 2.6.3 and part (1), we obtain part (3). □

2.7 Separating compact leaf

As Theorem 2.6.8 says, a non-taut foliation admits a torus leaf. Then there are two possibilities. This torus leaf can be separating or non-separating. The former case is explored in this section while the latter case is the aim of Section 2.8.
Note that there are three types of non-taut foliations admitting a separating torus leaf depending on if we can modify the foliation so that it becomes taut.

- Foliations admitting a Reeb component which can be deleted to obtain another taut foliation (example in $\mathbb{S}^2 \times \mathbb{S}^1$).
- Foliations admitting a Reeb component which cannot be deleted (example in $\mathbb{S}^3$).
- Non-taut and Reebless foliations (example among graph manifolds).

2.7.1 Non-taut foliation admitting Reeb component

Consider a non-taut foliation of a manifold M containing a Reeb component R and denote by $T = \partial R$. In order to know if we can delete a neighborhood of R and replace it by a trivially foliated solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ we need to know how R is attached in M and how the foliation looks like in a neighborhood of T in $M \backslash \mathring{R}$.
This is the aim of Lemma 2.7.1. Next we will see the opposite process which consist on considering a taut foliation and adding Reeb component as Proposition 2.7.3 suggests it.

In the light of Proposition 1.0.7 in a neighborhood of T in $M \backslash \mathring{R}$ there exists either (generalised) spiraling component or (generalised) turbulization component, bounded by T or bounded by a torus leaf T' included in a neighborhood of T in $M \backslash \mathring{R}$. Hence up to deleting the foliation of $T^2 \times I$ bounded by T and T' we can consider that T bounds a (generalised) spiraling component or (generalised) turbulization component (that changes

the foliation but not the manifold M).
We want to replace the Reeb component R by a trivially foliated solid torus, i.e foliated by meridians disks $\mathbb{D}^2 \times \mathbb{S}^1$. That imposes that T must bound in $M \backslash \mathring{R}$ a turbulization component $\mathcal{T}$ (see Figure 2.7) because all the other components do not induce a circle foliation on the transverse boundary torus.
Moreover if the circles C of the circle foliation induced by $\mathcal{T}$ on the transverse torus, bound meridian disks in R then we can delete $R \cup \mathcal{T}$ and replace it by the trivially foliated solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ by gluing meridians disks on the circles C; and once again that changes the foliation but not the manifold M.

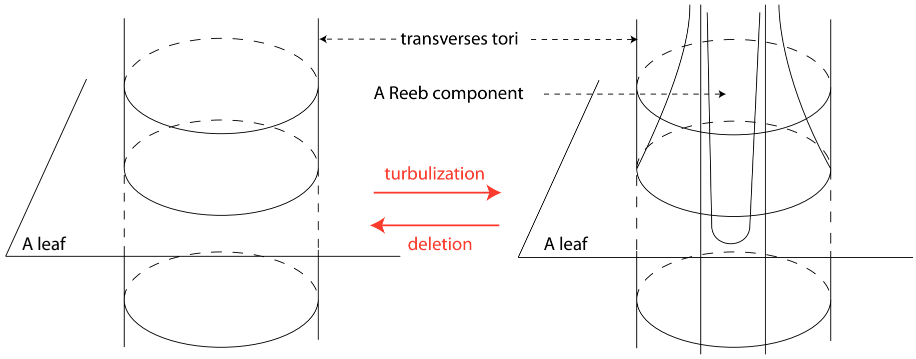


Figure 2.30: Foliation $\mathcal{F}$ on Q

Note that if the circles C do not bound meridian disks in M as in Figure 2.31 we cannot delete the Reeb component, as in the case of the Reeb foliation of $\mathbb{S}^3$ (foliation obtained by gluing two Reebs components to obtain $\mathbb{S}^3$).
Indeed the boundary of the meridians disks of a Reeb component of $\mathbb{S}^3$ are longitudes for the other boundary component.

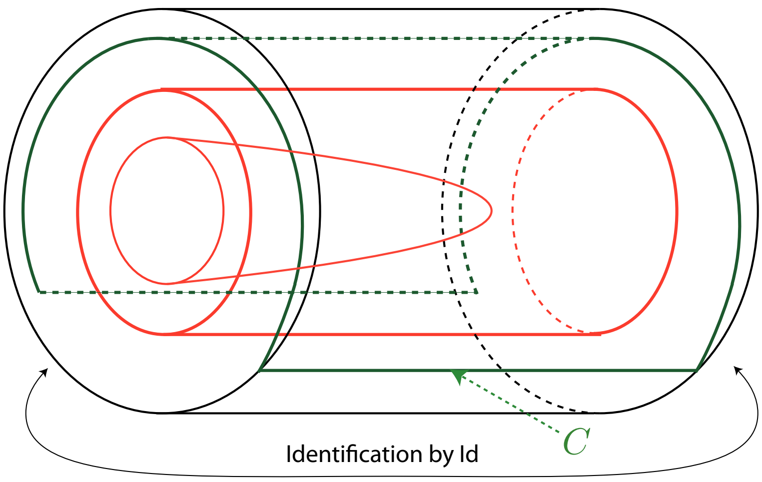


Figure 2.31: Part of the Reeb foliation of $\mathbb{S}^3$

Hence we have proved the following Lemma :

Lemma 2.7.1. *A Reeb component R can be deleted if and only if up to deleting a $T^2 \times I$, ∂R bounds a $\mathcal{T}$ component in $M \backslash \mathring{R}$ whose circles C of the circle foliation induced by $\mathcal{T}$ on the transverse torus bound disks in $\mathcal{T} \cup R$*

Remark 2.7.2. *Note that the Reeb foliation of $\mathbb{S}^2 \times \mathbb{S}^1$ (foliation obtained by gluing two Reeb components to obtain $\mathbb{S}^2 \times \mathbb{S}^1$) can be transformed by applying two times this process and we obtain the product foliation $\mathbb{S}^2 \times \mathbb{S}^1$. Indeed that gives two solid tori trivially foliated by disks glued along their circle boundary two obtain sphere leaves.*

Proposition 2.7.3. *From each transversely oriented taut foliation $\mathcal{F}$ on a closed 3-manifold M, ($M \ncong \mathbb{S}^2 \times \mathbb{S}^1$ trivially foliated), we can construct a non-taut foliation on M (with a Reeb component) and a non-taut foliation without Reeb component on $M \backslash V$, where V is a solid torus.*

Proof. By definition, there exists a closed transerve curve, say γ. Choose a small enough regular neighborhood of γ, denoted $V \cong \mathbb{D}^2 \times \mathbb{S}^1$, so that the induced foliation by $\mathcal{F}$ on V is the trivial foliation $\mathbb{D}^2 \times \mathbb{S}^1$.

Now consider $M \backslash \mathring{V}$.
By construction the foliation induced on ∂V is $(\partial \mathbb{D}^2) \times \mathbb{S}^1$.
Then we can apply the process of turbulization in $(\partial V) \times I$ by pasting a $\mathcal{T}$ component, to obtain a foliation $\mathcal{F}'$ of $M \backslash \mathring{V}$ with one torus boundary leaf.
Then, Proposition 2.1.1 implies that $\mathcal{F}'$ is not taut.

This process of turbulization gives a Reeb component if and only if $M \cong \mathbb{S}^2 \times \mathbb{S}^1$ with the product foliation.
Indeed suppose that our construction leads to a Reeb component R, i.e $M \backslash \mathring{V} = R$ so contains a $\mathcal{T}$ component. Then $M \backslash (\mathring{V} \cup \mathcal{T})$ is a solid torus foliated by $\mathbb{D}^2 \times \mathbb{S}^1$, homeomorphic to $M \backslash \mathring{V}$.
Then M is a union of two solid tori, and since V is foliated by disks, the transverse circles leaves of $\mathcal{T}$ bounds disks in V and in R, so the identification of the two solid tori pastes the boundary of the meridians disks, hence $M \cong \mathbb{S}^2 \times \mathbb{S}^1$.
The converse is trivial.

In conclusion, we have constructed a non-taut foliation on $M \backslash \mathring{V}$ without Reeb component. By gluing a Reeb component trivially to this torus leaf we obtain a non taut foliation with a Reeb component on M.
Hence we have proved Proposition 2.7.3.

□

2.7.2 Non-taut and Reebless foliations

Foliations admitting a Reeb component are not taut, but the converse is false: there are many non-taut and Reebless foliations.
There are two kinds of examples :

(1) Non-taut and Reebless foliations on manifolds admitting a taut foliation;

(2) Non-taut and Reebless foliations on manifolds without taut foliations.

Many examples for Point (1) are constructed by Proposition 2.7.3.
A simple example is the following. Consider the manifold $M = S_g \times \mathbb{S}^1$, where S_g is a closed compact surface of genus g, for $g \geq 0$, with the trivial product foliation. Note that it is taut, so we can apply Proposition 2.7.3, and we construct a non-taut and Reebless foliation on $M \backslash \mathring{V}$, where V is a solid torus (see next figure for the case where $g = 1$).
Note that by gluing two such foliations along their boundary torus leaves we obtain a separating torus leaf in a non-taut Reebless foliation.

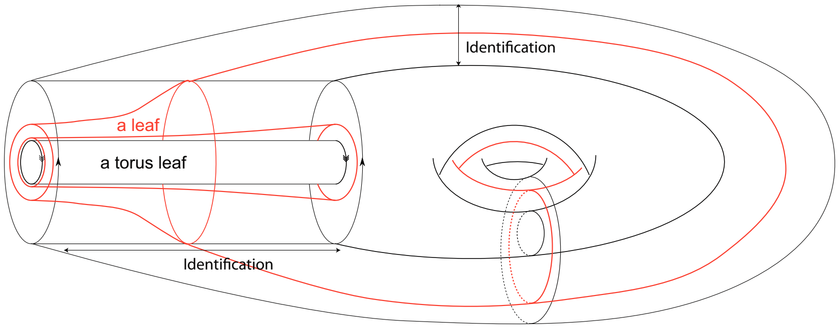


Figure 2.32: Non taut foliation on $M \backslash \mathring{V}$, when $g = 1$

Nevertheless, note that Q admits a non-taut, transversely orientable Reebless foliation, it not obtained via Proposition 2.7.3. This is the one constructed by R. Roussarie [1974], called type II_b component, and given in Figure 2.33.

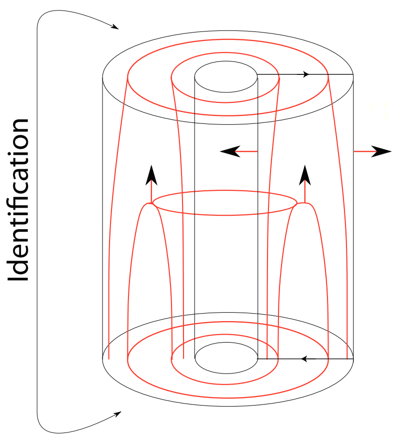


Figure 2.33: Non-taut Reebless and transversely oriented foliation on Q

Proposition 2.7.3 shows that there are a lot of non-taut, Reebless foliations, since any taut foliation on a manifold M gives rise to a non-taut Reebless foliation (on $M \backslash \mathring{V}$, where V is a solid torus).

A more interesting question is the existence of non-taut Reebless foliations in a manifold not admitting a taut foliation, (Point (2)); i.e among homology spheres by Theorem 2.2.2. Brittenham, Naimi, and Roberts [1997] gave examples of such foliations on graph manifolds.

Theorem 2.7.4 (Brittenham, Naimi, and Roberts [1997])**.** *There exist infinitely many manifolds without taut foliations admitting Reebless foliation (hence non-taut). Those are graph manifolds constructed by gluing two Seifert fibered manifold, each based on the disc with two exceptional fibers.*

Question 2.7.5. *There are infinitely many Seifert fibered homology spheres not admitting a taut foliations by Theorem 1.0.11. Do they admit a non-taut Reebless foliation?*

2.8 Non-separating torus leaf.

We have seen that a foliation with a separating torus leaf cannot be taut.
The case of non-separating torus leaves is very different since they can lie in a taut foliation or in a non-taut foliation.
The goal of Section 2.8 is to understand the reason.

In this section, we first give an example of a non-taut $\mathcal{C}^1$-foliation admitting a non-separating torus leaf, and then we give some constructions of taut and non-taut foliations admitting a non-separating torus leaf.
We will see that the key point amounts to do a *good* spiraling (opposite direction of rotation in a neighborhood of the torus leaf) to obtain a taut foliation or a *bad* spiraling (same direction of rotation in a neighborhood of the torus leaf) to obtain a non-taut foliation. Then we prove Theorem 1.0.8.
Finally we conclude by explaining that in Theorem 2.2.2, Gabai used *good* orientation.

2.8.1 Example of non-taut foliation on T^3

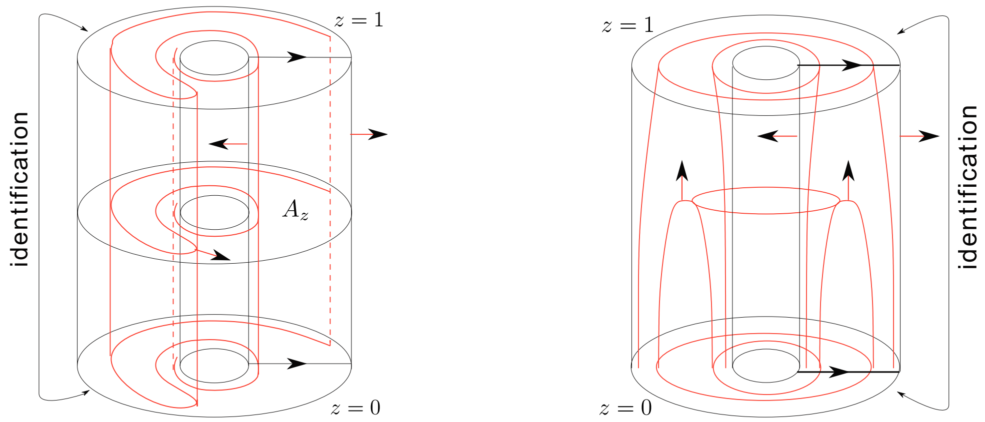
We study the well-known example of T^3 (where $T^3 \cong \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$).
Here we give two examples of non-taut foliations with non-separating torus leaves on T^3. A non-transversely oriented one ($\mathcal{L}_1$); and a transversely oriented one ($\mathcal{L}_2$).

Let us represent T^3 as follows :
Set $W = \{(x, \theta, z), x \in [0,1], \theta \in]-\pi, \pi], z \in [0,1]\} \cong I \times \mathbb{S}^1 \times I$.

Now $W/_\sim \cong T^2 \times I$ where $(x, \theta, 0) \sim (x, \theta, 1)$; and T^3 is obtained by identifying the two following torus boundary components to obtain a non- separating torus $T \subset T^3$:
$T_0 = \{(0, \theta, z), \theta \in]-\pi, \pi], z \in [0,1]\}/_\sim$ and
$T_1 = \{(1, \theta, z), \theta \in]-\pi, \pi], z \in [0,1]\}/_\sim$.

Definition 2.8.1. *Foliate each $A_z = \{(x, \theta, z), x \in [0,1], \theta \in]-\pi, \pi]\}$, for $z \in [0,1]$, by a Reeb annulus. That induces a foliation $\mathcal{L}$ on $T \times I \cong W/_\sim$ because this foliation is invariant by $\sim$.*

In this foliation $\mathcal{L}$, T_0 and T_1 are leaves, which implies a foliation $\mathcal{L}_1$ on T^3.



(a) Foliation $\mathcal{L}$ of $T \times I$. (b) Other representation of $\mathcal{L}$ (from Roussarie [1974])

Figure 2.34:

$\mathcal{L}_1$ is not taut, because any transverse loop passing through the torus leaf, would induce (after isotopy and splitting) a transverse arc on A_z, for some $z \in [0, 1]$, with endpoints in ∂A_z, which is impossible since a Reeb annulus is not taut.
Moreover, $\mathcal{L}_1$ is not transversely orientable because there is no way of extending continuously a transverse vector field on T. Indeed, $\mathcal{L}$ is transversely oriented. But on each A_z, the two boundary leaves have the same orientation. Hence on $W/_\sim \cong T^2 \times I$, the two torus boundary leaves have the same orientation (for example, in Figure 2.34, they both point outward).
Thus, when gluing the two boundary leaves by a reversing orientation homeomorphism, the transverse orientations cannot match.

Note that $\mathcal{L}_1$ does not contain any non-orientable leaf and is not transversely orientable, which is the counterexample expected in Remark 2.6.1.

Now we give the second example of non-taut foliation with non-separating torus leaf, but which is transversely oriented.

Consider the construction above, and glue with a reversing orientation homeomorphism two copies of $\mathcal{L}$, denoted by $\mathcal{L}$ and $\mathcal{L}^*$, where we add a $*$ to all the notations when we are in $\mathcal{L}^*$. The annuli A_z and A_z^* are attached along their boundary, so that the transverse orientation matches. We obtain a transversely oriented foliation $\mathcal{L}_2$ of T^3 (see the induced foliation by $\mathcal{L}_2$ on $A'_z = A_z \cup A_z^*$ for some $z \in [0, 1]$ on Figure 2.35).

In conclusion, $\mathcal{L}_2$ admits two non-separating torus leaves, is not taut and Reebless (no leaf is homeomorphic to $\mathbb{R}^2$), and is transversely orientable. Hence this is the expected foliation.
Note that by gluing together an even number of such components $\mathcal{L}$, this give an infinite number of such foliations.

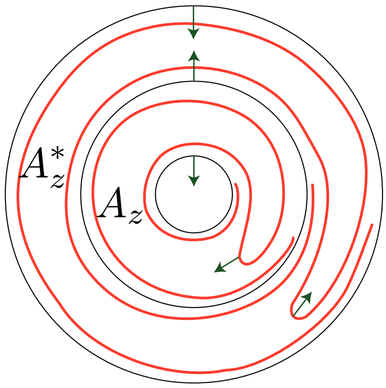


Figure 2.35: Foliation on A'_z induced by $\mathcal{L}_2$

Note that obviously T^3 admits a taut foliation, which is the product foliation; but we will see another interesting taut foliation constructed with *good* spiraling in Subsection 2.8.2, (see Figure 2.37).

2.8.2 Good orientation vs bad orientation

In this subsection we first give an example of construction where we can obtain a taut or a non-taut foliation depending on if we do a *good* or a *bad* orientation.
Theorem 1.0.8 is a generalization of this fact, so we prove it here.

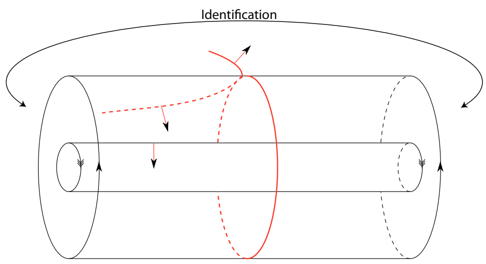
When M is a manifold with two torus boundary components, then we denote M/∂ the manifold obtained by identifying the two boundary torus components by the trivial homeomorphism.
Let us study an interesting example : $M \cong F_g \times \mathbb{S}^1/\partial$ where F_g is a twice punctured genus g compact orientable surface.

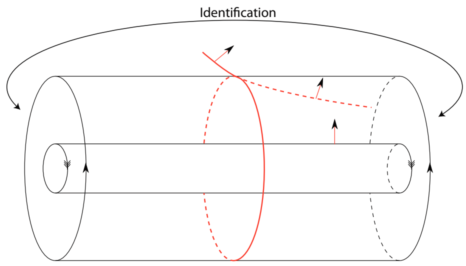
When $g = 0$, we obtain T^3. We have already seen in Subsection 2.8.1 an interesting Reebless, and non-taut foliation on T^3; here we will construct a taut one with non-compact leaves.

We set $M' = F_g \times \mathbb{S}^1$, with a fixed orientation and denote $\partial M' = T_- \cup T_+$, (a union of two tori).
We denote by T the non-separating torus resulting from the identification of T_- and T_+ in $M = M'/\partial$.

Consider on M' the product foliation $\mathcal{F}'$. That induces on T_- and T_+ a circle foliation. We want T to be a torus leaf, so we are going to apply the process of turbulization (or equivalently spiraling by Lemma 2.5.1) on T_- and T_+. This amounts to glue two $\mathcal{T}$ components, and depending on the gluing, we can construct a taut foliation (Figure 2.38), or a non-taut foliation (Figure 2.39) by gluing two copies of $\mathcal{T}$ differently on the two torus

boundary components.
In Figure 2.36 we have fixed a transverse orientation, and we explicit the two choices of turbulization. The key point is that the transverse orientation on the leaf attaching on the two transverse tori is the same, since any leaf of $\mathcal{F}'$ admits one boundary component on each transverse torus.





(a)$\mathcal{T}^+$ (b)$\mathcal{T}^-$

Figure 2.36:

Now we fix a transverse orientation on $\mathcal{F}'$.
We want to attach two components $\mathcal{T}^+$ or $\mathcal{T}^-$ on the boundary components of M'.
Let us denote T_1 and T_2 the two new torus boundary components after the pastings.
There are two choices :

(1) We glue $\mathcal{T}^+$ on one boundary component and $\mathcal{T}^-$ on the other. That induces opposite transverse orientation on T_1 and T_2, (one points inward and the other points outward), and so a taut transversely oriented foliation on M' (choose for example the arc γ in Figure 2.38).

(2) We glue $\mathcal{T}$, where $\mathcal{T} \in \{\mathcal{T}^+, \mathcal{T}^-\}$ on each boundary component, that induces the same transverse orientation on T_1 and T_2.
This foliation is transversely oriented, so by Proposition 2.1.1 it is non-taut.

Now we identify T_1 and T_2.
In the first case that induces a taut foliation admitting a non-separating torus leaf on $M = F_g \times \mathbb{S}^1/\partial$ (with non-compact leaves, but T).

In the second case that induces a non-taut, and non-transversely oriented foliation admitting a non-separating torus leaf on $M = F_g \times \mathbb{S}^1/\partial$.
Indeed, this foliation is not transversely oriented, because the transverse orientation on T_1 and T_2 is the same, and the identification T_1 and T_2 reverse the orientation, so the transverse orientation on T is not well defined.
Note that the case $g = 0$ is exactly the foliation of T^3 of Subsection 2.8.1 (see Figure 2.34).
Note also that by gluing trivially two copies of such a foliation along the boundary torus leaves, we obtain a non-taut and transversely oriented foliation with two non-separating torus leaves.

That example leads us to make the following definition :

Definition 2.8.2. *Let M be a manifold with a transversely oriented foliation $\mathcal{F}$ such that the boundary of M is a union of torus leaves.*

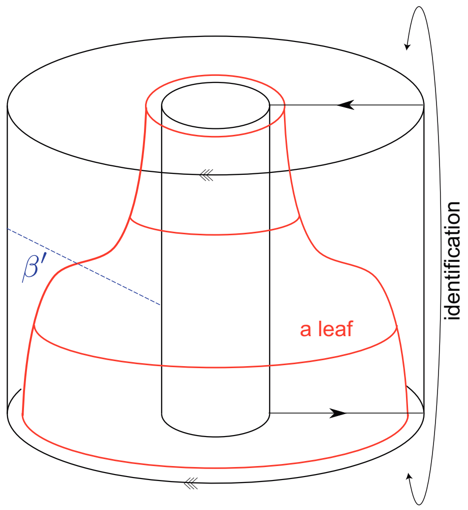


Figure 2.37: Case (1) when $g = 0$, i.e on $T \times I$

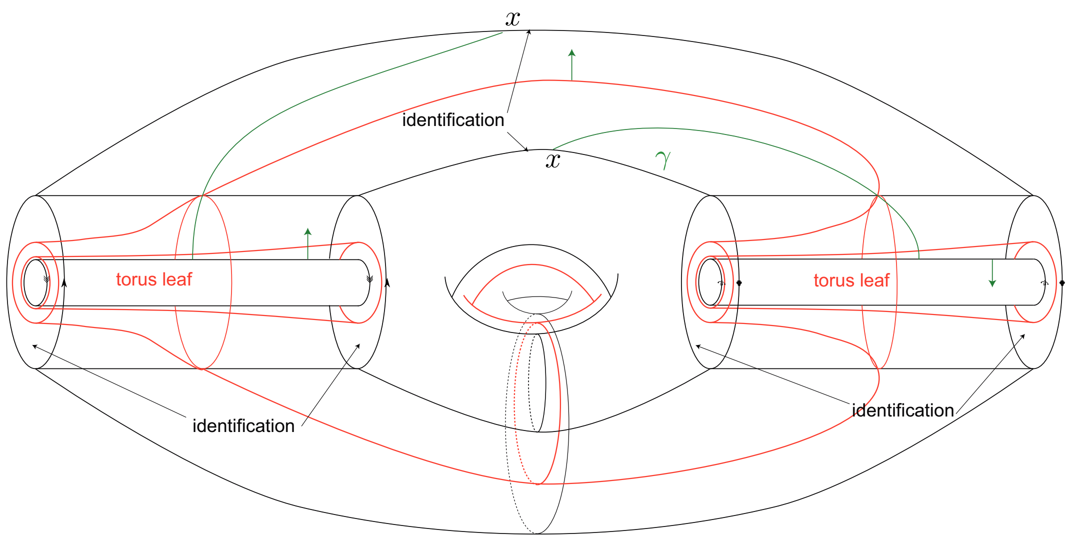


Figure 2.38: Case (1) when $g = 1$: taut foliation

We say that $\mathcal{F}$ has a **bad orientation** *if the transverse orientation on each boundary torus leaf is the same.*
Otherwise we say that $\mathcal{F}$ has a **good orientation**.

Theorem 2.8.3. *Let M be a manifold with a transversely oriented C^1-foliation $\mathcal{F}$.*
Assume that the boundary of M is a union of two torus leaves.
Assume also that $\mathcal{F}$ does not admit neither interior torus leaf, nor embedded annulus whose induced foliation by $\mathcal{F}$ is a Reeb annulus.
Then, $\mathcal{F}$ is taut if and only if $\mathcal{F}$ has a good orientation.

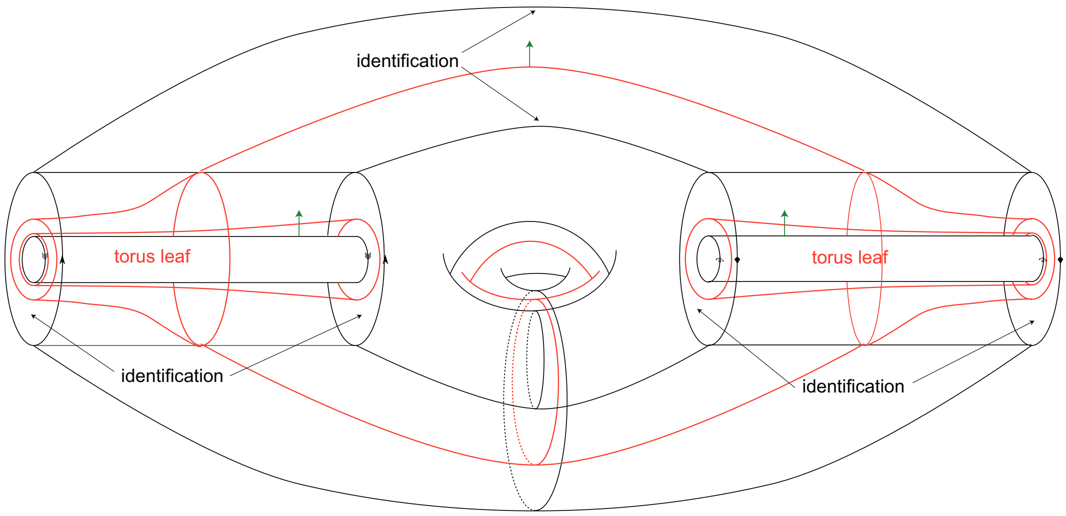


Figure 2.39: Case (2) when $g = 1$: non-taut foliation

Proof. Proposition 2.1.1 gives exactly that if $\mathcal{F}$ has a bad orientation then $\mathcal{F}$ is non-taut. This is equivalent to say that if $\mathcal{F}$ is taut then $\mathcal{F}$ has a good orientation.
It remains to show that if $\mathcal{F}$ has a good orientation then $\mathcal{F}$ is taut. This is the goal of the following.
Let us denote by T_1' and T_2' the two tori boundary leaves. Choose an embedded torus T_i in a regular neighborhood of T_i' for $i = 1, 2$, and denote by $N(T_i')$ the regular neighborhood of T_i' bounded by T_i' and T_i, for $i = 1, 2$.
By the Theorem of Roussarie [1974] and Thurston [1972] we can assume that T_1 and T_2 are transverse to $\mathcal{F}$.
Fix an orientation on M. Up to considering the opposite transverse orientation on $\mathcal{F}$, we can assume that the transverse orientation on T_1' points in M and points out of M on T_2'.
If we denote by $N = M \backslash (N(T_1') \overset{\circ}{\cup} N(T_2'))$ the oriented manifold homeomorphic to M, bounded by T_1 and T_2, and the induced foliation by $\mathcal{F}$ on N by $\mathcal{G}$, then $\mathcal{G}$ does not admit torus leaves, so by Corollary 2.6.10 it is taut.

Claim 2.8.4. *There exists a properly embedded arc $\gamma : I \to N$ transverse to $\mathcal{G}$ with an endpoint on T_1 and another on T_2.*

Proof. Since $\mathcal{G}$ is taut, either we find a properly embedded transverse arc, or a closed transverse curve to each leaf.
If there exists a properly embedded transverse arc, we note that it must have one endpoint on T_1 and another on T_2. Indeed if both endpoints are on the same boundary component then by Proposition 2.6.3, it cannot be a transverse arc.
If we find a closed transverse curve, it can be chosen so that it meets all the leaves, then we can cut this curve in two points to obtain a transverse arc, and isotope it so that the endpoints meet T_1 and T_2, and keep being transverse to $\mathcal{G}$. Indeed, it suffices to pick one leaf F_1 meeting T_1 and one (other) leaf F_2 meeting T_2, and cut the closed loop at the points it meet F_i, $i = 1, 2$, and push the endpoints to the boundary of F_i, $i = 1, 2$ by small

isotopies transverse to the leaves in a neighborhood of F_i, $i = 1, 2$. So we have proved Claim 2.8.4.

□

Therefore, up to considering $t \to \gamma(1-t)$, we can assume that $\gamma(0) \in T_1$ and $\gamma(1) \in T_2$. Moreover we make the confusion between γ and $\gamma(I)$. Let $J = [0, \epsilon]$, where $\epsilon > 0$ is small enough.
It remains to understand why we can extend γ in M to obtain a properly embedded arc transverse to $\mathcal{F}$.

Claim 2.8.5. *If $\mathcal{F}$ is transversely oriented with a good orientation, the only possibilities for $\mathcal{G}$ to be transverse to the T_i, $i = 1, 2$ are the one on Figure 2.40.*

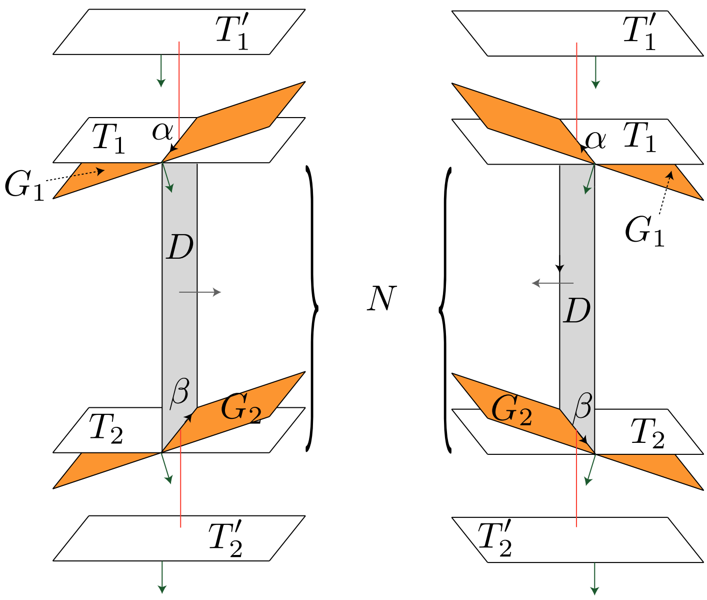


Figure 2.40: Coherent orientation

Proof. Recall that we have always assumed that M is orientable.
We can choose a small disk, denoted $D \cong \gamma \times J$ in N such that for each $t_0 \in I$, $\gamma(t_0) \times J$ is included in a leaf of $\mathcal{G}$. It admits an arc $\alpha = \gamma(0) \times J \subset \partial D$, so $\alpha \subset T_1 \cap G_1$, where $G_1 \in \mathcal{G}$, and an arc $\beta = \gamma(1) \times J \subset \partial D$, so $\beta \subset T_2 \cap G_2$, where $G_2 \in \mathcal{G}$.
Since $\mathcal{F}$ is transversely oriented, so is $\mathcal{G}$, and the transverse orientation of $\mathcal{G}$ induces an orientation on D which must be coherent, because N is oriented.
Indeed, the transverse orientation of G_1, induces an orientation on $\alpha \subset \partial G_1$, denoted $\vec{\alpha}$. This orientation induces also an orientation on D, because $\alpha \subset \partial D$.
Similarly, the transverse orientation of G_2, induces an orientation on $\beta \subset \partial G_2$, denoted $\vec{\beta}$. But since D and N are oriented, the induced orientation on ∂D imposes $\vec{\beta} = -\vec{\alpha}$.

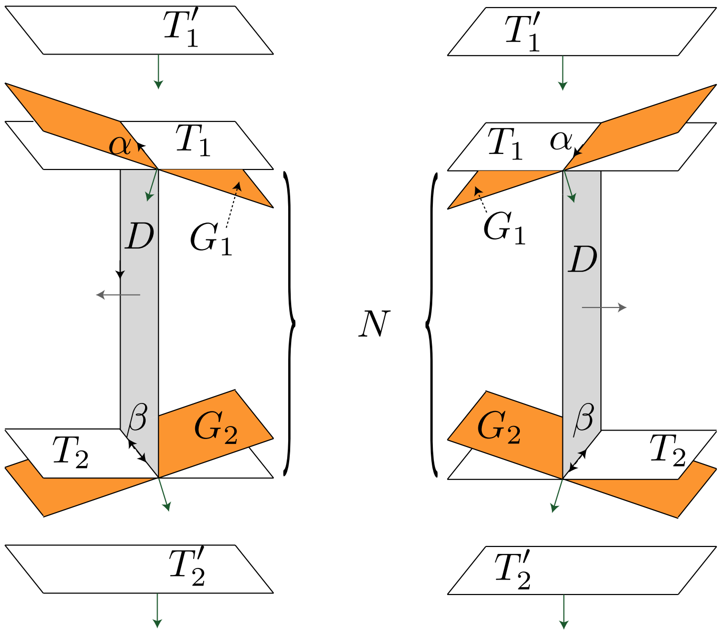


Figure 2.41: Non-coherent orientation

Moreover, there are two ways of being transverse to each T_i $(i = 1, 2)$, which gives four possibilities. Figure 2.40 showes the two possibilities with a coherent orientation, between the one induced on D and the one induced by ∂G_1 and ∂G_2.
Figure 2.41 shows the other two possibilities where the induced orientation on D by G_1 is not coherent with the induced orientation on β by G_2 in N; which ends the proof of Claim 2.8.5 □

Now we use Proposition 1.0.7 to understand the foliation of $N(T_i')$ in M, for $i = 1, 2$. Indeed, since there is no interior leaves in $\mathcal{F}$, each T_i' bounds a $\mathcal{S}_*$ or a T_* component (without embedded Reeb annulus). Hence we can easily find an extension of γ in M, as in Figure 2.40 which ends the proof of Theorem 2.8.3. □

Now we prove Theorem 1.0.8.

Proof. If there are two torus boundary leaves, then this is Theorem 2.8.3.
Otherwise, it remains to understand that for each torus leaf we can find another torus leaf with opposite orientation (we suppose there is a good orientation). So by Theorem 2.8.3, we find a properly embedded arc; which proves Theorem 1.0.8. □

Remark 2.8.6. *Note that given a manifold with a transversely oriented $\mathcal{C}^1$-foliation without embedded Reeb annulus, and with interior torus leaves, we can split along all the torus leaves. If we obtain a connected manifold we can apply Theorem 1.0.8 to know if it is taut. Of course the connectedness is crucial.*

Remark 2.8.7. *Note that Gabai's spiraling constructs a taut foliation with a non-separating torus leaf, by splitting along it, and Gabai's process imposes a good orientation by consid-*

ering the component $\mathcal{S}(f, h)$ with only a transverse annulus on one boundary component and the remaining of the boundary component of $\mathcal{S}(f, h)$ is tangent to the foliation.

2.8.3 Link with Gabai's construction

We have considered above a component of spiraling bounded by a torus everywhere transverse to the foliation, and a tangent torus, even if we defined it in Section 2.4 as a component bounded by a tangent torus and the other torus contains a transverse annulus and a tangent annulus to the foliation.
We are going to see that the second way of seeing it imposes a good orientation, and Gabai's construction imposes that there is no embedded Reeb annulus, so by Theorem 1.0.8 we construct a taut foliation.

We consider the same example as in the beginning of Subsection 2.8.2 but with a different construction.
Let us recall the definitions.
Let $M \cong F_g \times \mathbb{S}^1/\partial$ where F_g is a twice punctured genus g orientable compact surface.
We set $M' = F_g \times \mathbb{S}^1$, with a fixed orientation and denote $\partial M' = T_- \cup T_+$, (a union of two tori).
We denote by T the non-separating torus resulting from the identification of T_- and T_+ in $M = M'/\partial$.

Here, we do not consider the trivial product foliation on M'.
In the construction of Gabai, the boundary components denoted by γ corresponds to what is supposed to be transverse to the foliation, and the ones denoted by $R_\pm(\gamma)$ correspond to what is supposed to be tangent, and $\gamma \cup R_\pm(\gamma)$ is the whole boundary of the 3-manifold.
So we consider (M, γ), and (M', γ').
Here we let $R_-(\gamma') = T_-$, and $R_+(\gamma') = T_+$, and $\gamma = \gamma' = \emptyset$ because we want a torus leaf.
Fix an orientation on M' so that the orientation on T_+ points out and points in on T_-.
Set $S_1 = F_g \times \{\theta\}$, where $\theta \in \mathbb{S}^1$.

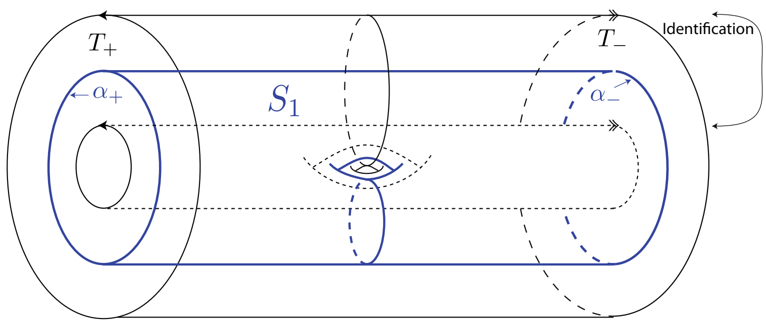


Figure 2.42: Embbeding of S_1 in M' for $g = 1$

Let us call $\alpha_- = \partial S_1 \cap T_-$, and $\alpha_+ = \partial S_1 \cap T_+$, see Figure 2.42.
Set $M_1 = M' \backslash S_1 \times \mathring{I}$; where $M_1 \cong F_g \times I$, with the induced orientation by M';
Up to changing the notations, we can assume that the orientation of $S_1^+ = S_1 \times \{0\}$ points out of M_1, and that $S_1^- = S_1 \times \{1\}$ points in M_1.

We set $R_+(\gamma_1) = S_1^+ \cup (T_+ \backslash (\alpha_+ \times \mathring{I})) \cong S_1 \cong F_g$;
$R_-(\gamma_1) = S_1^- \cup (T_- \backslash (\alpha_- \times \mathring{I})) \cong S_1 \cong F_g$;
$\gamma_1 = ((S_1^+ \cap T_-) \times I) \cup ((S_1^- \cap T_+) \times I) \cong (\mathbb{S}^1 \times I) \cup (\mathbb{S}^1 \times I)$.

That corresponds to the so-called hierarchy :

$$(M', \gamma') \overset{S_1}{\rightsquigarrow} (M_1, \gamma_1)$$

where $(M_1, \gamma_1) \cong (F_g \times I, (\partial F_g) \times I)$ and $R_+(\gamma_1) \cong F_g \times \{1\}$.

We consider the product foliation on M_1 which consists on $F_g \times I$, but which is tangent to $R_-(\gamma_1)$, and $R_+(\gamma_1)$, and transverse to γ_1, transversely oriented so that on the boundary, the transverse orientation coincide with the orientation of M_1, as illustrated in Figure 2.43.

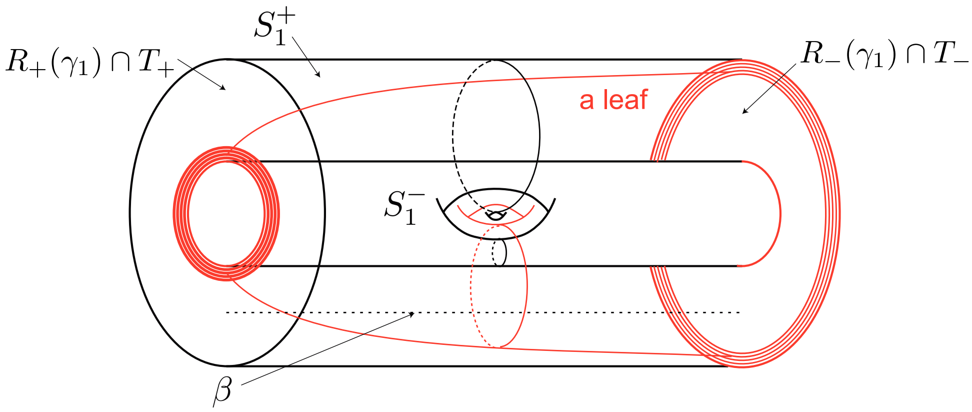


Figure 2.43: Product foliation on M_1, when $g = 1$

By gluing back S_1^+ to S_1^-, that induces a foliation $\mathcal{F}$ on M' by leaves homeomorphic to $F_g \cong S_1$, such that the two boundary tori are foliated as a union of a tangent annulus and a transverse annulus (see Figure 2.44).

Note that when we glue back S_1^+ to S_1^-, we can still consider the part of $R(\gamma_1)$ which is still on the boundary of M', and we will still denote it $R(\gamma_1)$, as we did on Figure 2.44. Then, we can paste components of spiraling on each boundary component; to extend the foliation on M' such that the two boundary tori are leaves (recall that $R_+(\gamma') = T_+$, and $R_-(\gamma') = T_-$ must be leaves); i.e we paste the foliated components $\mathcal{S}_1^+$ on the boundary component containing $R_+(\gamma_1)$ and $\mathcal{S}_1^-$ on the boundary component containing $R_-(\gamma_1)$.
Note that here we don't have the choice of $\mathcal{S}_1^+$ or $\mathcal{S}_1^-$ on each boundary component. Recall that $\mathcal{S}_1^+$ is a transversely oriented foliation of $T \times I$ and $\mathcal{S}_1^-$ is the opposite transversely oriented foliation of $T \times I$.
By Theorem 1.0.8 this foliation is taut.

Remark 2.8.8. *This example gives an interesting foliation on $M = T^3 = F_0 \times \mathbb{S}^1/\partial$, ($F_0$ is an annulus).*

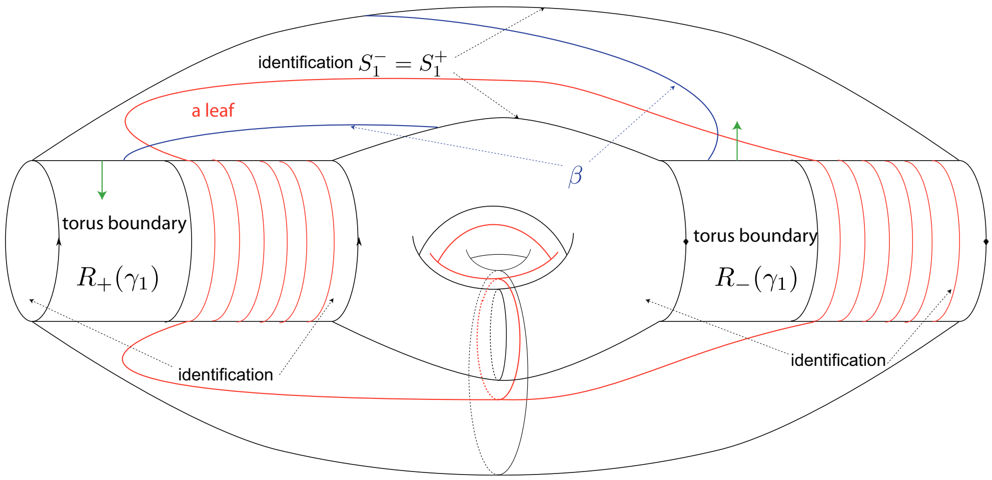


Figure 2.44: Foliation $\mathcal{F}$ by compact leaves M', when $g = 1$

Indeed this is a taut, transversely oriented foliation, admitting non-compact leaves, and a (compact) non-separating torus, shown on $T^2 \times I$ in Figures 2.45 and 2.37.

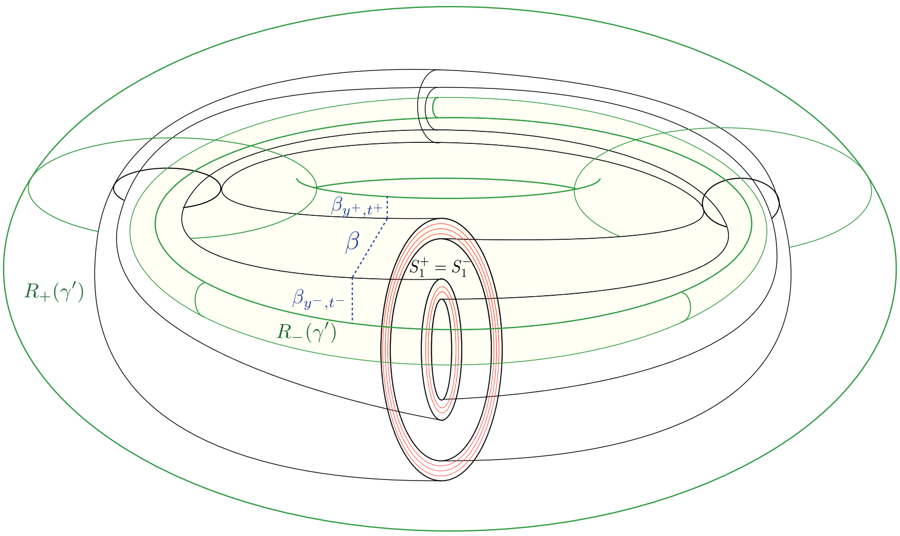


Figure 2.45: Foliation extended by spiraling on M', when $g = 0$, i.e on $T \times I$

Chapter 3

Existence de feuilletage tendu

3.1 Introduction

Ce chapitre fait l'objet d'un article co-écrit avec Daniel Matignon qui a été soumis dans le journal "Topology and its applications", c'est pourquoi il a été rédigé en anglais.

All 3-manifolds are considered compact, connected and orientable.
Taut foliations provide deep information on 3-manifolds and their contribution in understanding the topology and geometry of 3-manifolds is still in progress. The first result came from Novikov [1965], who proved that a 3-manifold which admits a taut foliation has to have infinite π_1 or $\mathbb{S}^2 \times \mathbb{S}^1$. Since then, we know by Palmeira [1978] that such manifolds have $\mathbb{R}^3$ for universal cover, and that their fundamental group are infinite Novikov [1965] and Gromov negatively curved when the manifold is also toroidal Calegari [2006]. Recently, Thurston has exhibited an approach with taut foliations towards the geometrization.

Gabai [1983], proved that a closed 3-manifold with a non-trivial second homology group admits a taut foliation. A lot of great works then concern the existence of taut foliations, see for examples Brittenham [1993a], [1993b] Clauss [1991], Roberts, Shareshian, and Stein [2003].
This paper seeks to answer the question for Seifert fibered 3-manifolds. In the following, a *non-integral homology* 3*-sphere* means a rational homology 3-sphere, which is not an integral homology 3-sphere. The results are quite different if they are integral homology 3-spheres, or non-integral homology 3-spheres.

Theorem 1.0.10 [english version] *Let M be a Seifert fibered integral homology 3-sphere. Then M admits a taut analytic foliation if and only if M is homeomorphic to neither the 3-sphere nor the Poincaré sphere.*

Concerning non-integral homology 3-spheres, the non-existence is not isolated. Of course, the 3-sphere and lens spaces do not admit a taut foliation, but for any choice of the number of exceptional slopes, there exist infinitely many which admit a taut foliation, and infinitely many which do not.

Theorem 1.0.11 [english version] *Let $n \in \mathbb{N}$.*
Let $\mathcal{S}_n$ be the set of Seifert fibered 3-manifolds whith n exceptional fibers, which are non-integral homology 3-spheres. For each $n \geq 3$:

(i) There exist infinitely many Seifert fibered manifolds in $\mathcal{S}_n$ which admit a taut analytic foliation; and
(ii) There exist infinitely many Seifert fibered manifolds in $\mathcal{S}_n$ which do not admit a taut $\mathcal{C}^2$-foliation.
(iii) There exist infinitely many Seifert fibered manifolds in $\mathcal{S}_3$ which do not admit a taut $\mathcal{C}^0$-foliation.

Actually, by considering the normalized Seifert invariant $(0; b_0, b_1/a_1, \ldots, b_n/a_n)$ of a Seifert fibered homology 3-sphere, and assuming that $b_0 \neq -1$ (nor $1-n$), then b_0 determines wether M does or does not admit a taut $\mathcal{C}^2$-foliation, see Theorem 3.4.1, which collects results in Eisenbud, Hirsch, and Neumann [1981], Jankins and Neumann [1985], Naimi [1994]. Note that there is a fiber-preserving homeomorphism of M which switches $b_0 = 1-n$ to $b_0 = -1$. Therefore, the problem remains open only for $b_0 = -1$. We will prove (see Theorem 3.7.1) that even if the 3-manifolds all are equipped with $b_0 = -1$, Theorem 1.0.11 is still true. To prove the non-existence of taut $\mathcal{C}^2$-foliations, we first prove that a taut $\mathcal{C}^2$-foliation can be isotoped to a horizontal one, and then use a characterization of horizontal foliations for Seifert fibered homology 3-spheres (see below for more details : schedule of the paper). So, the following result play a key role in the proof.

Theorem 3.1.1. *Let M be a Seifert fibered rational homology 3-sphere. Let n be the number of exceptional fibers of M. If $n > 3$ (resp. $n = 3$) then any taut $\mathcal{C}^2$-foliation (resp. $\mathcal{C}^0$-foliation) of M can be isotoped to be a horizontal foliation.*

Moreover, we will show that the geometries do not determine the existence of taut foliations on Seifert fibered rational homology 3-spheres.

Theorem 3.1.2. *Let M be a Seifert fibered rational homology 3-sphere. If M does not admit the $\widetilde{SL}_2(\mathbb{R})$-geometry, then M does not admit a taut $\mathcal{C}^2$-foliation.*

Remark 3.1.3. *There exist infinitely many such manifolds (see Section 7) but the converse is not true as says Theorem 3.7.1 : we can give infinitely many such manifolds, which admit the $\widetilde{SL}_2(\mathbb{R})$-geometry (and with $b_0 = -1$) but no taut $\mathcal{C}^2$-foliation.*

Theorem 3.1.4. *Let M be a Seifert fibered integral homology 3-sphere. If M admits the $\widetilde{SL}_2(\mathbb{R})$-geometry, then M is neither homeomorphic to the 3-sphere nor the Poincaré sphere.*

In particular (Theorem 1.0.10) M admits a taut analytic foliation.

Since a remark given by Tye Lidman, another proof of Theorem 1.0.10 is possible, even if it is much less direct. Indeed it is contained in the union of five important papers on Heegaard-Floer homology.
First, Jankins and Neumann [1985], Naimi [1994] and Lisca and Matić [2004] show that a Seifert fibered homology 3-sphere not admitting a taut foliation does not admit a transverse contact structure.

Then, Lisca and Stipsicz [2007] show that such manifolds are L-spaces.
Moreover, integral homology 3-spheres which are L-spaces have Heegaard-Floer homology isomorphic to $\mathbb{Z}$.
Recently the paper of Eftekhary [2009] (posted on Arxiv) proves that the only Seifert fibered homology 3-spheres with Heegaard-Floer homology group equal to $\mathbb{Z}$ are $\mathbb{S}^3$ and Poincaré homology 3-sphere; which implies our result.

Schedule of the paper. We organize the paper as follows.

In Section 2, we recall basic definitions and notations on Seifert fibered 3-manifolds, taut or horizontal foliations and well-known results.

Section 3 is devoted to the proof of Theorem 3.1.1, which is based on Proposition 2.1.1, which claims that a transversely oriented and taut foliation of a closed 3-manifold cannot contain a separating compact leaf. Then, a taut $\mathcal{C}^2$-foliation of a Seifert fibered homology 3-sphere cannot contain a compact leaf (see Corollary 3.3.3). Therefore, it can be isotoped to be horizontal (see Theorem 3.3.4), by collecting the works on foliations of Brittenham [1993b], Eisenbud, Hirsch, and Neumann [1981], Levitt [1978], Matsumoto [1985], Novikov [1965], Thurston [1972].

Since a horizontal foliation is clearly a taut foliation, an immediate consequence is that a Seifert fibered rational homology 3-sphere, M say, admits a taut $\mathcal{C}^2$-foliation if and only if M admits a horizontal foliation (Corollary 3.3.1). This corollary was also proved by combining results of Eliashberg and Thurston [1998], Jankins and Neumann [1985], Lisca and Matić [2004], Lisca and Stipsicz [2007], Naimi [1994], Ozsváth and Szabó [2004] (for more details, see the end of Section 3).

The goal of Section 4 is a characterization of Seifert fibered rational homology 3-spheres, which admit a taut $\mathcal{C}^2$-foliation. Since a taut $\mathcal{C}^2$-foliation can be isotoped to be horizontal, we use the characterization of Jankins and Neumann [1985], Naimi [1994] for horizontal foliations (for more details, see Section 4). This characterization gives rise to criteria to be satisfied by the Seifert invariants.

Section 5 concerns the geometries of homology 3-spheres. We will prove the following result.

Proposition 3.1.5. *Let M be a Seifert fibered rational homology 3-sphere, with n exceptional fibers. If M does not admit the $\widetilde{SL}_2(\mathbb{R})$-geometry, then the following statements all are satisfied.*

(i) $n \leq 4$.
(ii) If $n = 4$ then M admits the $\mathcal{N}il$-geometry, and is a non-integral homology 3-sphere.
(iii) If M is an integral homology 3-sphere, then M admits the $\mathbb{S}^3$-geometry and is either homeomorphic to the 3-sphere or to the Poincaré sphere.

We may note that if $n = 2$ then M is a lens space (including $\mathbb{S}^3$ and $\mathbb{S}^1 \times \mathbb{S}^2$).

We combine Proposition 3.1.5 with the criteria given by the characterization of Section 4, to prove Theorem 3.1.2.

Section 6, 7 and 8 are devoted respectively to the proof of Theorem 3.1.2, Theorem 3.7.1 and Theorem 1.0.10.

To prove Theorem 3.7.1, we first exhibit infinite families of Seifert fibered non-integral homology spheres, which admit the $\widetilde{SL}_2(\mathbb{R})$-geometry (and $b_0 = 1$). Then, we prove that they do satisfy (or do not satisfy) the criteria of the characterization described in Section 4.

To prove Theorem 1.0.10, we need to study more deeply these criteria.

Perspectives.
By Waldhausen [1967], we know that an incompressible compact surface in a Seifert fibered 3-manifold (not necessarily a homology 3-sphere) can be isotoped to be either horizontal or vertical. This is clearly not the same for foliations.

A vertical leaf is homeomorphic to either a 2-cylinder ($\mathbb{S}^1 \times \mathbb{R}$) or a 2-torus ($\mathbb{S}^1 \times \mathbb{S}^1$). Therefore, taut foliations are not necessary isotopic to vertical ones; and vice-versa, vertical foliations are not necessary isotopic to taut foliations, e.g. cylinders which wrap around two tori in a turbulization way; for more details, see chapter 2. But clearly, horizontal foliations are taut.

By Theorem 3.3.4, a taut $\mathcal{C}^2$-foliation can be isotoped to a horizontal foliation, if there is no compact leaf.

We wonder if a taut $\mathcal{C}^0$-foliation, without compact leaf, of a Seifert fibered 3-manifold can be isotoped to be horizontal and so analytic. By Brittenham, Naimi, and Roberts [1997], there exist manifolds which admit taut $\mathcal{C}^0$-foliation but not taut $\mathcal{C}^2$-foliation. Moreover, Rosenberg and Thurston [1973] have given an example of $\mathcal{C}^0$-foliation which cannot be $\mathcal{C}^0$-approximated by a $\mathcal{C}^2$-foliation.
Therefore, that seems impossible in general, but the question is still open for homology 3-spheres.

Question 3.1.6. *Let $\mathcal{F}$ be a taut $\mathcal{C}^0$-foliation, without compact leaf, of a Seifert fibered homology 3-sphere. Can $\mathcal{F}$ be isotoped to be horizontal ?*

Brittenham [1993b], answers the question when the base is $\mathbb{S}^2$ with 3 exceptional fibers, see Remark 3.3.5 for more details.

Gluing Seifert fibered 3-manifolds with boundary components along some of them (or all) give *graph manifolds*. We wonder if we can classify graph manifolds without taut foliations, with their Seifert fibered pieces and gluing homeomorphims.

Question 3.1.7. *Let M be a graph 3-manifold. What kind of obstructions are there for M not to admit a taut foliation ?*

3.2 Preliminaries

We may recall here, that all 3-manifolds are considered compact, connected and orientable. This section is devoted to recalling basic definitions and notations on Seifert fibered 3-manifolds, taut or horizontal foliations and well-known results.

Notations Let M be a 3-manifold. If M is an integral homology sphere, resp. a rational homology sphere, we say that M is a $\mathbb{Z}$HS, resp. a $\mathbb{Q}$HS. Clearly, a $\mathbb{Z}$HS is a $\mathbb{Q}$HS. If M

is a $\mathbb{Z}$HS, resp. a $\mathbb{Q}$HS, and a Seifert fibered 3-manifold, we say that M is *a $\mathbb{Z}$HS, resp. a $\mathbb{Q}$HS, Seifert fibered* 3*-manifold*.

Separating surfaces and non-separating surfaces. A properly embedded surface F in a 3-manifold M is said to be *a separating surface* if $M - F$ is not connected; otherwise, F is said to be *a non-separating surface* in M. If F is a separating surface, we call *the sides of* F the connected components of $M - F$. Note that if M is a $\mathbb{Q}$HS manifold, then M does not contain any non-separating surfaces.

A 3-manifold is said to be *reducible* if M contains an *essential* 2*-sphere*, i.e. a 2-sphere which does not bound any 3-ball in M. Then, either M is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$, or M is a non-trivial connected sum. If M is not a reducible 3-manifold, we say that M is an *irreducible* 3-manifold. We may note that all Seifert fibered 3-manifolds but $\mathbb{S}^1 \times \mathbb{S}^2$ and $\mathbb{R}P^3 \# \mathbb{R}P^3$ are irreducible 3-manifolds.

Seifert fibered 3-manifolds. We can find the first definition of Seifert fibered 3-manifolds, called *fibered spaces* in Seifert and Threlfall [1980]. We first consider fibered solid tori.

The standard solid torus V is said to be *p/q-fibered*, if V is foliated by circles, such that the core is a leaf, and all the other leaves are circles isotopic to the (p, q)-torus knot (i.e. they run p times in the meridional direction and q times in the longitudinal direction) where $q \neq 0$. A solid torus W is $\mathbb{S}^1$*-fibered* if W is foliated by circles, such that there exists a homeomorphism between W and the p/q-fibered standard solid torus V, which preserves the leaves. We may say that W is a *p/q-fibered solid torus*.

A 3-manifold M is said to be *a Seifert fibered* 3-manifold, or *a Seifert fiber space* if M is a disjoint union of simple circles, called *the fibers*, such that the regular neighborhood of each fiber is a $\mathbb{S}^1$-fibered solid torus. Let W be a p/q-fibered solid torus. If $q = 1$, we say that its core is *a regular fiber*; otherwise we say that its core is *an exceptional fiber* and q is *the multiplicity* of the exceptional fiber.

By Epstein [1972] this is equivalent to saying that M is a $\mathbb{S}^1$-bundle over a 2-orbifold.

Seifert invariants. Seifert [1933], developed numerical invariants, which give a complete classification of Seifert fibered 3-manifolds. Let M be a closed Seifert manifold based on an orientable surface of genus g, with n exceptional fibers. Let $V_1, \ldots, V_n$ be the solid tori, which are regular neighborhood of each exceptional fiber. We do not need to consider non-orientable base surface here. If we remove these solid tori, we obtain a trivial $\mathbb{S}^1$-bundle over a genus g compact surface, whose boundary is a union of 2-tori $T_1, \ldots, T_n$; where $T_i = \partial V_i$, for $i \in \{1, \ldots, n\}$. Gluing back $V_1, \ldots, V_n$ consists to assign to each of them a slope b_i/a_i : we glue V_i along T_i, such that the slope b_i/a_i on V_i bounds a meridian disk of V_i. Formally, if f and s represent respectively a fiber and a section on T_i, then the boundary of the meridian disk of V_i is attached along the slope represented by $a_i[s] + b_i[f]$ in $H_1(T_i, \mathbb{Z})$.

Clearly, $a_i \geq 2$ is the multiplicity of the core of V_i, and b_i depends on the choice of a section. Removing the regular neighborhood of a regular fiber, we obtain an integer slope b_0. Then, $g, b_0, b_1/a_1, \ldots, b_n/a_n$ completely describe M. We denote M by $M(g; b_0, b_1/a_1, \ldots, b_n/a_n)$, which is called *the Seifert invariant*.

Seifert normalized invariant and convention. New sections are obtained by Dehn twistings along the fiber (along annuli or tori); therefore a new section does not change b_i modulo a_i. Thus, we can fix b_0 so that $0 < b_i < a_i$ for $i \in \{1, \dots, n\}$.

That gives rise to the *Seifert normalized invariant* : $M(g; b_0, b_1/a_1, \dots, b_n/a_n)$; i.e. $0 < b_i < a_i$ for $i \in \{1, \dots, n\}$.

Seifert [1933] showed that $M(g; b_0, b_1/a_1, \dots, b_n/a_n)$ is fiber-preserving homeomorphic to $-M(g; -n - b_0, 1 - b_1/a_1, \dots, 1 - b_n/a_n)$ where $-M$ denotes M with the opposite orientation. In all the following, we denote by Φ this isomorphism. Therefore, we may assume that $b_0 < 0$ otherwise we switch for $-n - b_0$. For more details, see Seifert [1933] or Brittenham, Naimi, and Roberts [1997], Hatcher.

Every $\mathbb{Q}$HS Seifert fibered 3-manifold M is based on $\mathbb{S}^2$. Indeed, every non-separating curve on the base surface induces a non-separating torus in M; which cannot be in a $\mathbb{Q}$HS. Hence, the base surface of a $\mathbb{Q}$HS Seifert fibered 3-manifold is a 2-sphere.

From now on, we denote for convenience such M by $M(-b_0, b_1/a_1, \dots, b_n/a_n)$, where $b_0 > 0$ and $0 < b_i < a_i$ for $i \in \{1, \dots, n\}$. We will write :

$$M = M(-b_0, b_1/a_1, \dots, b_n/a_n).$$

Euler number. When M has a unique fibration, we denote by $e(M)$ *the Euler number of its fibration.* Note that few Seifert 3-manifolds (lens spaces and a finite number of others) do not have a unique fibration, see the web book of Hatcher for more details; all of them but lens spaces and $\mathbb{S}^3$, are not homology 3-spheres.

$$e(M) = -b_0 + \sum_{i=1}^{n} b_i/a_i.$$

Taut foliations. Let M be a 3-manifold and $\mathcal{F}$ a foliation of M. A simple closed curve γ (respectively, a properly embedded simple arc, when $\partial M \neq \emptyset$) is called *a transverse loop* (respectively a *transverse arc*) if γ is transverse to $\mathcal{F}$, i.e. γ is transverse to every leaf $F \in \mathcal{F}$, such that $\gamma \cap F \neq \emptyset$.

We say that a foliation $\mathcal{F}$ is *taut*, if for every leaf of F of $\mathcal{F}$, there exists a transverse loop, or a transverse arc if $\partial M \neq \emptyset$, γ say, such that $\gamma \cap F \neq \emptyset$.

We end this part by the famous theorem of Gabai [1983] on the existence of taut foliations, which is stated here for closed 3-manifolds.

Theorem 3.2.1 (Gabai [1983])**.** *Let M be a closed 3-manifold. If $H_2(M; \mathbb{Q})$ is non-trivial then M admits a taut foliation.*

Horizontal and vertical foliations. Let M be a Seifert fibered 3-manifold and $\mathcal{F}$ a foliation of M. We say that $\mathcal{F}$ is *horizontal* if each $\mathbb{S}^1$-fiber is a transverse loop to $\mathcal{F}$. We say that $\mathcal{F}$ is *vertical* if each leaf of $\mathcal{F}$ is $\mathbb{S}^1$-fibered, i.e. a disjoint union of $\mathbb{S}^1$-fibers.

Note that only Seifert fibered 3-manifolds are concerned by horizontal or vertical foliations. Horizontal foliations are sometimes just called *transverse foliations* to underline the fact that horizontal foliations are transverse to the $\mathbb{S}^1$-fibers.

Clearly, horizontal foliations are taut, because any transverse fiber (meeting a leaf) is the required transverse loop; so we have the following result.

Lemma 3.2.2. *A horizontal foliation is taut.*

3.3 Horizontal and taut $\mathcal{C}^2$-foliations

This section is devoted to the proof of Theorem 3.1.1; then with Lemma 3.2.2, we obtain :

Corollary 3.3.1. *Let M be a $\mathbb{Q}$HS Seifert fibered 3-manifold. Let n be the number of exceptional fibers of M. If $n > 3$ (resp. $n = 3$) then, M admits a horizontal foliation if and only if M admits a taut $\mathcal{C}^2$-foliation (resp. a $\mathcal{C}^0$-foliation).*

There exists an alternative proof (but not direct) of this corollary; see at the end of this section.

Proof of Theorem 3.1.1.
In the light of known results on foliations Brittenham [1993b], Eisenbud, Hirsch, and Neumann [1981], Levitt [1978], Matsumoto [1985], Novikov [1965], Thurston [1972] of (where Theorem 3.3.4 is their collection) it is sufficient to see that any taut foliation on a $\mathbb{Q}$HS Seifert fibered 3-manifold, has no compact leaf. Then the result follows by Corollary 3.3.3, which claims that no leaf in a taut foliation of a $\mathbb{Q}$HS can be compact. □

Corollary 3.3.3 is an immediate consequence of Proposition 3.3.2, which concerns all (compact, oriented and connected) closed 3-manifolds. Proposition 3.3.2 is a particular case of Proposition 2.1.1.

Recall that a taut foliation $\mathcal{F}$ is said to be *transversely oriented* if there exists a one-dimensional oriented foliation $\mathcal{G}$ transverse to $\mathcal{F}$. This is equivalent to say that the normal vector field to the tangent planes to the leaves of $\mathcal{F}$ is continuous (and nowhere vanishes).

Proposition 3.3.2. *A transversely oriented and taut foliation of a closed 3-manifold, cannot contain a compact separating leaf.*

For example, foliations of 3-manifolds which admit a Reeb component are not taut.

We ask if taut foliations are transversely oriented, and vice-versa. In fact, there exist taut foliations which are not transversely oriented, see chapter 2 for more details. The inverse is easy to construct, e.g a Reeb's component. We may note that there also exist foliations without non-orientable compact leaves, which are neither taut nor transversely oriented.
Here, we consider $\mathbb{Q}$HS, hence any foliation cannot contain a non-separating surface, and Proposition 3.3.2, implies there is no separating surface, hence we have the following corollary:

Corollary 3.3.3. *A transversely oriented and taut foliation of a $\mathbb{Q}$HS cannot admit a compact leaf.*

Theorem 3.3.4 (Brittenham [1993b], Eisenbud, Hirsch, and Neumann [1981], Levitt [1978], Matsumoto [1985], Novikov [1965], Thurston [1972])**.** *Let M be a $\mathbb{Q}$HS Seifert fibered 3-manifold, with n exeptional exceptional fibers (where $n \geq 3$). We assume that M admits a taut $\mathcal{C}^0$-foliation $\mathcal{F}$. Moreover, if $n > 3$, we suppose that $\mathcal{F}$ is a $\mathcal{C}^2$-foliation of M.*

If $\mathcal{F}$ does not have a compact leaf, then $\mathcal{F}$ can be isotoped to be a horizontal foliation.

Remark 3.3.5 (History on Theorem 3.3.4). *This theorem has been proved for all Seifert 3-manifolds which are not trivial bundles over the 2-torus. This is a collection of results as follows.*
The case of circle bundles over orientable surface, which is not a 2-torus is due to Thurston [1972]; it has been completed and extended to non-orientable base surface by Levitt [1978]. Eisenbud, Hirsch, and Neumann [1981], generalized it to Seifert fibered spaces, where the base surface is neither $\mathbb{S}^2$, nor the 2-torus with trivial circle bundle.
Later, Matsumoto [1985], focused on the case when the base is $\mathbb{S}^2$ with stricly more than 3 exceptional fibers.
Until then, the condition of $\mathcal{C}^r$-foliation is necessary, and implies a $\mathcal{C}^r$-isotopy, for each $r \geq 2$.
The last case (the base is $\mathbb{S}^2$ with 3 exceptional fibers) was solved by Brittenham [1993b], and the involved techniques are very different, so the author obtained a $\mathcal{C}^0$-isotopy from a $\mathcal{C}^0$-foliation.
We recall that when there are one or two exceptional fibers with base $\mathbb{S}^2$, there is no foliation without compact leaf by Novikov [1965].

Alternative proof of Corollary 3.3.1

A proof of Corollary 3.3.1 has been obtained by combining the results of Eliashberg and Thurston [1998], Jankins and Neumann [1985], Lisca and Matić [2004], Lisca and Stipsicz [2007], Naimi [1994], and, Ozsváth and Szabó [2004], in the following way.

Theorem 3.3.6. *Eliashberg and Thurston [1998], Jankins and Neumann [1985], Lisca and Matić [2004], Lisca and Stipsicz [2007], Naimi [1994], Ozsváth and Szabó [2004]] Let M be a rational Seifert fibered homology 3-sphere. The following statements are equivalent :*
(1) M is an L-space;
(2) M does not carry a transverse contact structure;
(3) M does not admit a transverse foliation;
(4) M does not admit a transversely oriented taut foliation.

This theorem is a formulation of Theorem 1.1 of Lisca and Stipsicz [2007]. The proof is mainly organized as follows.

(1) $\Rightarrow$ (2) : Ozsváth and Szabó [2004];

(2) $\Rightarrow$ (3) : Eliashberg and Thurston [1998];

(1) $\Rightarrow$ (4) : Ozsváth and Szabó [2004];

(4) $\Rightarrow$ (3) : trivial;

(3) $\Rightarrow$ (2) : Jankins and Neumann [1985], Lisca and Matić [2004], and Naimi [1994];

(2) $\Rightarrow$ (1) : Lisca and Stipsicz [2007].

To have (3) $\Rightarrow$ (4), we need to follow : (3) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (4).

We may underline that the considered taut foliations are actually $\mathcal{C}^2$-foliations, because using contact structure (see Eliashberg and Thurston [1998]). Note that there exists a taut $\mathcal{C}^0$-foliation which is not a taut $\mathcal{C}^2$-foliation in the article of Brittenham, Naimi, and Roberts [1997].

3.4 Characterization of taut $\mathcal{C}^2$-foliations

The goal of this section is to give a characterization of the existence of a taut $\mathcal{C}^2$-foliation in a $\mathbb{Q}$HS Seifert fibered 3-manifold. For this, we define the following *Property* $(*)$.

$$\textit{Property } (*) \begin{cases} (i) & \dfrac{b_1}{a_1} < \dfrac{m-\alpha}{m} \\ \\ (ii) & \dfrac{b_2}{a_2} < \dfrac{\alpha}{m} \\ \\ (iii) & \dfrac{b_i}{a_i} < \dfrac{1}{m}, \text{ for } i \in \{3,\dots,n\} \end{cases}$$

We say that *m and α satisfy Property $(*)$ for $b_1/a_1, b_2/a_2, \dots, b_n/a_n$*, if all the following statements are satisfied :

- m and α are two positive integers such that $\alpha < m$;
- $n \geq 3$ is an integer;
- a_i and b_j are positive integers for all $(i,j) \in \{1,\dots,n\}^2$, such that :
$$b_1/a_1 \geq b_2/a_2 \geq \cdots \geq b_n/a_n;$$
- $(i), (ii)$ and (iii) of Property $(*)$ all are satisfied.

When there is no confusion for the b_i/a_i's, we say for short that (m,α) *satisfies Property* $(*)$, or that *the integers α and m satisfy Property* $(*)$.

For convenience, in the following, we denote by $(i), (ii)$ and (iii) respectively, the inequalities $(i), (ii)$ and (iii) of Property $(*)$ above.

Let M be a Seifert fibered 3-manifold. In the following, we use the previous notations (see Section 2) of Seifert normalized invariant :

$$M = M(-b_0, b_1/a_1, \dots, b_n/a_n)$$

where a_i and b_j are positive integers for all $(i,j) \in \{1,\dots,n\} \times \{0,\dots,n\}$, such that $0 < b_i < a_i$. Note that the notations $M = M(-b_0, b_1/a_1, \dots, b_n/a_n)$ suppose that M contains exactly n exceptional fibers :

$$a_i \geq 2, \text{ for all } i \in \{1,2,\dots,n\}.$$

If $b_0 \notin \{1, n-1\}$ then the existence of a taut $\mathcal{C}^2$-foliation depends uniquely of b_0, as suggests the following theorem.

Theorem 3.4.1 (Eisenbud, Hirsch, and Neumann [1981], Jankins and Neumann [1985], Naimi [1994])**.** *Let n be an integer and M be a Seifert manifold based on $\mathbb{S}^2$.*
We assume that $n \geq 3$ and that $M = M(-b_0, b_1/a_1, \dots, b_n/a_n)$, where a_i and b_j are positive integers for all $(i,j) \in \{1,\dots,n\} \times \{0,\dots,n\}$. Then, all the following statements are satisfied.

(1) *If $2 \leq b_0 \leq n-2$ then M admits a horizontal foliation.*

(2) *If M admits a horizontal foliation then $1 \leq b_0 \leq n-1$.*

(3) *If M admits a horizontal $\mathcal{C}^0$-foliation, then M admits a horizontal analytic foliation.*

Corollary 3.4.2. *Let n be an integer and M be a Seifert manifold based on $\mathbb{S}^2$.
We assume that $n \geq 3$ and that $M = M(-b_0, b_1/a_1, \ldots, b_n/a_n)$, where a_i and b_j are positive integers for all $(i,j) \in \{1,\ldots,n\} \times \{0,\ldots,n\}$, and $b_0 \notin \{1, n-1\}$.
Then M admits an analytic horizontal foliation if and only if $2 \leq b_0 \leq n-2$.*

Therefore, the problem reduces to the case $b_0 = 1$; we may recall here (see Section 2) :

$$M(-1, b_1/a_1, \ldots, b_n/a_n) \cong -M(-(n-1), 1 - b_1/a_1, \ldots, 1 - b_n/a_n).$$

The following theorem is a consequence of Corollary 3.3.1 and the characterization of the existence of horizontal foliations in Seifert-fibered spaces based on $\mathbb{S}^2$, whose formulation can be found in Brittenham, Naimi, and Roberts [1997][Proposition 6].

Theorem 3.4.3. *Let $n > 2$ be an integer and $M = M(-1, b_1/a_1, \ldots, b_n/a_n)$ be a $\mathbb{Q}HS$ Seifert fibered 3-manifold; where a_i and b_j are positive integers for all $(i,j) \in \{1,\ldots,n\}^2$.*

Assume that $b_1/a_1 \geq b_2/a_2 \geq \cdots \geq b_n/a_n$.

If $n > 3$ (resp. $n = 3$) then, M admits a taut $\mathcal{C}^2$-foliation (resp. a taut $\mathcal{C}^0$-foliation) if and only if there exist two positive integers m and α such that (m, α) satisfies Property $(*)$.

We may recall that $\mathcal{P}$ denotes the Poincaré $\mathbb{Z}$HS, i.e. $\mathcal{P} = M(-1, 1/2, 1/3, 1/5)$. Note that Theorem 3.4.3 implies that $\mathcal{P}$ cannot admit a taut foliation, but this fact was already known by Theorem 1.0.3 (because its π_1 is finite). Note also that if $n \in \{1, 2\}$ then M has to be $\mathbb{S}^3$ or a lens space, which cannot admit a taut foliation.
Theorem 3.4.3 has the following corollaries, which will be useful for the next sections.

Corollary 3.4.4. *Let n be an integer and M be a $\mathbb{Q}HS$ Seifert fibered 3-manifold.
We assume that $n \geq 3$ and that $M = M(-1, b_1/a_1, \ldots, b_n/a_n)$, where a_i and b_j are positive integers for all $(i,j) \in \{1,\ldots,n\}^2$.*

*We order the rational coefficients b_i/a_i such that : $b_1/a_1 \geq b_2/a_2 \geq \cdots \geq b_n/a_n$.
If for all $i \in \{1,\ldots,n\}$, $\dfrac{b_i}{a_i} < \dfrac{1}{2}$ then M admits a taut $\mathcal{C}^2$-foliation.*

Proof. With the notations and assumptions of the theorem, if $b_i/a_i < 1/2$, for all $i \in \{1,\ldots,n\}$, then Property $(*)$ is satisfied, by choosing $m = 2$ and $\alpha = 1$. $\square$

Corollary 3.4.5. *Let n be an integer and M be a $\mathbb{Q}HS$ Seifert fibered 3-manifold.
We assume that $n \geq 3$ and that $M = M(-1, b_1/a_1, \ldots, b_n/a_n)$, where a_i and b_j are positive integers for all $(i,j) \in \{1,\ldots,n\}^2$.*

We order the rational coefficients b_i/a_i such that : $b_1/a_1 \geq b_2/a_2 \geq \cdots \geq b_n/a_n$.

If M admits a taut $\mathcal{C}^2$-foliation and $\dfrac{b_1}{a_1} \geq 1/2$, then the two following properties are both satisfied.

(1) $\dfrac{b_i}{a_i} < \dfrac{1}{2}$, *for all $i \geq 2$.*

(2) $\dfrac{b_n}{a_n} < \dfrac{1}{3}$. *In particular, $a_n \geq 4$.*

Proof. With the notations and assumptions of Theorem 3.4.3, if M admits a taut $\mathcal{C}^2$-foliation then we can find positive integers m, α such that $\alpha < m$ and Property $(*)$ is satisfied.

First, note that if $m = 2$ then $\alpha = 1$ and $b_1/a_1 < 1/2$, which is a contradiction to the hypothesis. Thus, $m \geq 3$.

Now, if $\dfrac{m-\alpha}{m} > \dfrac{1}{2}$ then $\dfrac{\alpha}{m} < \dfrac{1}{2}$, hence Property $(*)$ implies $\dfrac{b_i}{a_i} < \dfrac{1}{2}$ for $i \in \{2, \ldots, n\}$ which proves (1).

Finally, assume that $\dfrac{b_1}{a_1} \geq \dfrac{1}{2} \geq \dfrac{b_2}{a_2} \geq \dfrac{b_3}{a_3} \geq \dfrac{1}{3}$. Then $b_3/a_3 \geq 1/m$ for all $m \geq 3$, so (iii) of $(*)$ cannot be satisfied. □

3.5 Geometry

The goal of this section is to recall general results on the geometries of Seifert fibered homology 3-spheres, and prove Proposition 3.1.5.

Let n be a positive integer and $M = M(-b_0, b_1/a_1, \ldots, b_n/a_n)$ be a $\mathbb{Q}$HS Seifert fibered 3-manifold. Recall that $e(M)$ denotes the Euler number of M, see Section 3.2. The following lemma is a well-known result that we detail here.

Lemma 3.5.1. *Let $M = M(-b_0, b_1/a_1, \ldots, b_n/a_n)$ be a Seifert fibered 3-manifold. Then :*

(1) *M is a $\mathbb{Z}$HS if and only if $a_1 a_2 \ldots a_n e(M) = \varepsilon$, where $\varepsilon \in \{-1, +1\}$;*

(2) *M is a $\mathbb{Q}$HS if and only if $e(M) \neq 0$.*

Proof. We may recall that:

$$e(M) = -b_0 + \sum_{i=1}^{n} \frac{b_i}{a_i}$$

We follow the book of Saveliev [2002], and give more details.
Using Poincaré duality, it suffices to study when $H_1(M, \mathbb{Q}) = \{0\}$.

Start with the classical presentation of the fundamental group of a Seifert-fibered manifold based on $\mathbb{S}^2$ noted $M = M(-b_0, b_1/a_1, ..., b_n/a_n)$:

$$\pi_1(M) =< q_1, q_2 ..., q_n, h /\ q_i h = h q_i,\ q_i^{a_i} = h^{-b_i},\ i = 1...n,\ q_1 q_2 ... q_n = h^{-b_0} >$$

$H_1(M, \mathbb{Q})$ is the abelianization of $\pi_1(M)$, hence we can write it as a $\mathbb{Q}$-vector subspace of $\mathbb{Q}^{n+1}$:

$$H_1(M, \mathbb{Q}) = vect < q_1, q_2 ..., q_n, h >$$

with the following relations :
$a_i q_i = -b_i h$, for $i = 1...n$, and $q_1 + q_2 + ...q_n = -b_0 h$.

Let $\varphi : \mathbb{Q}^{n+1} \to H_1(M, \mathbb{Q})$ be the linear map sending the canonical base of $\mathbb{Q}^{n+1}$, noted $(e_1, e_2, ..., e_{n+1})$, on $(q_1, q_2, ..., q_n, h)$.
φ is surjective by construction.

Let $v = (1, 1, ..., 1, b_0)$, and $v_i = (0, 0..., 0, a_i, 0, ..., 0, b_i) \in \mathbb{Z}^{n+1}$ for $i = 1...n$, where a_i is the i^{th} coordinate of v_i.

We can easily see that $ker(\varphi) = vect(v, v_1, ..., v_n)$, with the relations above.
Thus $H_1(M, \mathbb{Q}) \cong \mathbb{Q}^{n+1}/ker(\varphi)$.

Therefore, we can see that $ker(\varphi) = Im(A)$ where A is the endomorphism of $\mathbb{Q}^{n+1}$ defined by the following matrix still noted A :

$$A = \begin{pmatrix} a_1 & 0 & ... & 0 & 1 \\ 0 & a_2 & ... & 0 & 1 \\ ... & ... & ... & 0 & 1 \\ 0 & ... & 0 & a_n & 1 \\ b_1 & b_2 & ... & b_n & b_0 \end{pmatrix}$$

Hence we have :
$H_1(M, \mathbb{Q}) = 0 \Leftrightarrow Im(A) = \mathbb{Q}^{n+1} \Leftrightarrow A$ is bijective $\Leftrightarrow det(A) \neq 0$.
Furthermore $det(A)$ can be easily calculated by developing along b_i's line.
We find:

$$det(A) = b_0 \prod_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \hat{a_i}$$

where $\hat{a_i} = \dfrac{a_1 a_2 ... a_n}{a_i} \in \mathbb{Z}$ for $i = 1...n$.

In conclusion we see that M is a $\mathbb{Q}$HS if and only if

$$det(A) = b_0 \prod_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \hat{a_i} \neq 0$$

which is equivalent to the inequality announced.

The particular case of $\mathbb{Z}$HS can be studied similarly, but instead of working in a vector space, we work in a $\mathbb{Z}$-module, and what we have used before is still true in a (free and finitely generated) $\mathbb{Z}$-module because all coefficients of A are integers. Hence we have:
$H_1(M, \mathbb{Z}) = 0 \Leftrightarrow Im(A) = \mathbb{Z}^{n+1} \Leftrightarrow A$ is bijective $\Leftrightarrow det(A) = \pm 1$
because now $det(A)$ needs to be invertible in $\mathbb{Z}$.
In conclusion M is a $\mathbb{Z}$HS if and only if

$$det(A) = b_0 \prod_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \hat{a_i} = \pm 1$$

which is equivalent to the identity announced.

$\square$

Remark 3.5.2. *Note that* (1) *implies that the* a_i*'s are pairwise relatively prime integers, therefore they are different.*

Then, we define the rational number χ_M as follows.

$$\chi_M = 2 - \sum_{i=1}^{n}(1 - \frac{1}{a_i}) = 2 - n + \sum_{i=1}^{n} \frac{1}{a_i}.$$

We have the following well-known result (which can be found in the paper of Scott [1983] for example).

Proposition 3.5.3. *Let n be a positive integer and $M = M(-b_0, b_1/a_1, \ldots, b_n/a_n)$ be a $\mathbb{Q}HS$ Seifert fibered 3-manifold, then the following properties all are satisfied.*

(i) $\chi_M > 0 \Leftrightarrow M$ admits the $\mathbb{S}^3$-geometry.

(ii) $\chi_M < 0 \Leftrightarrow M$ admits the $\widetilde{SL}_2(\mathbb{R})$-geometry.

(iii) $\chi_M = 0 \Leftrightarrow M$ admits the $\mathcal{N}il$-geometry.

Proposition 3.5.4. *Let n be a positive integer and $M = M(-b_0, b_1/a_1, \ldots, b_n/a_n)$ be a $\mathbb{Q}HS$ Seifert fibered 3-manifold. If M does not admit the $\widetilde{SL}_2(\mathbb{R})$-geometry then $n \leq 4$.*

Furthermore, if $n = 4$ then $M = M(-b_0, 1/2, 1/2, 1/2, 1/2)$ with $b_0 \neq 2$; so M admits the $\mathcal{N}il$-geometry and is a non-integral $\mathbb{Q}HS$.

Proof. Let n be a positive integer and $M = M(-b_0, b_1/a_1, \ldots, b_n/a_n)$ be a $\mathbb{Q}$HS. Assume that M does not admit the $\widetilde{SL}_2(\mathbb{R})$-geometry. Then, by Proposition 3.5.3, $\chi_M \geq 0$. Therefore, $n - 2 \leq \displaystyle\sum_{i=1}^{n} \frac{1}{a_i}$.

Since, $a_i \geq 2$ for all $i \in \{1, \ldots, n\}$, $n - 2 \leq \displaystyle\sum_{i=1}^{n} \frac{1}{a_i} \leq n/2 \Rightarrow n \leq 4$.

Now, assume first that $n = 4$. Then, $\displaystyle\sum_{i=1}^{4} \frac{1}{a_i} \geq 2$. On the other hand, $a_i \geq 2$ for all $i \in \{1, \ldots, 4\}$, then $\displaystyle\sum_{i=1}^{4} \frac{1}{a_i} \leq 2$, and if one $a_i > 2$ then $\displaystyle\sum_{i=1}^{4} \frac{1}{a_i} < 2$.

Therefore, $a_i = 2$ for all $i \in \{1, \ldots, 4\}$. Thus, $\chi_M = 0$ which means that M admits the $\mathcal{N}$il-geometry. Moreover, Lemma 3.5.1 (2) implies that $b_0 \neq 2$. Note that such M cannot be a $\mathbb{Z}$HS, by Remark 3.5.2. □

Corollary 3.5.5. *Let M be a $\mathbb{Z}HS$ Seifert fibered 3-manifold. Then, M has the $\widetilde{SL}_2(\mathbb{R})$-geometry or the $\mathbb{S}^3$-geometry.*

Furthermore, if M has the $\mathbb{S}^3$-geometry, then M is either homeomorphic to $\mathbb{S}^3$ or to the Poincaré sphere $\mathcal{P}$.

Proof. Let M be a $\mathbb{Z}$HS Seifert fibered 3-manifold. Assume that M does not have the $\widetilde{SL}_2(\mathbb{R})$-geometry. Note that if $n \leq 2$ then M has to be homeomorphic to $\mathbb{S}^3$. By Proposition 3.5.4, we may assume that $n = 3$ and that $a_3 > a_2 > a_1 \geq 2$ (by remark 3.5.2).

Since $\chi_M \geq 0$, $\displaystyle\sum_{i=1}^{3} \frac{1}{a_i} \geq 1$. If $a_1 \geq 3$, then $\displaystyle\sum_{i=1}^{3} \frac{1}{a_i} \leq 1/3 + 1/4 + 1/5 < 1$, which is a contradiction. Then $a_1 = 2$. If $a_2 \neq 3$ then $a_2 \geq 5$ by remark 3.5.2. Hence,

$\sum_{i=1}^{3} \frac{1}{a_i} \leq 1/2 + 1/5 + 1/7 < 1$, which is a contradiction. Therefore, $a_1 = 2$ and $a_2 = 3$. Similarly $a_3 = 5$. Since $n = 3$ and $(a_1, a_2, a_3) = (2, 3, 5)$, M has to be homeomorphic to the Poincaré sphere, which satisfies $\chi_M > 0$, so $\mathcal{P}$ has the $\mathbb{S}^3$-geometry. □

To end this section, we simply note that Proposition 3.5.4 together with Corollary 3.5.5 clearly imply Proposition 3.1.5.

3.6 Taut $\mathcal{C}^2$-foliation gives $\widetilde{SL}_2(\mathbb{R})$-geometry

Here we do the proof of Theorem 3.1.2.

We keep the previous notations. Let n be a positive integer and M be a $\mathbb{Q}$HS Seifert fibered 3-manifold, with n exceptional fibers : $M = M(-b_0, b_1/a_1, \ldots, b_n/a_n)$. Assume that M does not admit the $\widetilde{SL}_2(\mathbb{R})$-geometry. We make the proof by contradiction. Suppose that M admits a taut $\mathcal{C}^2$-foliation. We may recall that if $n \in \{1, 2\}$, then M has a finite π_1, hence M cannot admit a taut $\mathcal{C}^2$-foliation. Therefore, by Proposition 3.5.4, we have $n \in \{3, 4\}$.
Assume that $n = 4$. By Theorem 3.4.1 and Theorem 3.1.1, since M admits a taut $\mathcal{C}^2$-foliation, $b_0 \in \{1, 2, 3\}$. Moreover the cases $b_0 = 1$ and $b_0 = 3$ are equivalent (see the fiber-preserving homeomorphism Φ in Section 2).

On the other hand, Proposition 3.5.4 implies that $M = M(-b_0, 1/2, 1/2, 1/2, 1/2)$ with $b_0 \neq 2$; and Corollary 3.4.5 (1) implies that $b_0 \neq 1$.

Therefore, we may assume that $n = 3$. Similarly $b_0 \in \{1, 2\}$ and $b_0 = 1$ and $b_0 = 2$ are equivalent cases, by considering the fiber-preserving homeomorphism Φ.

So, we may assume that $b_0 = 1$. Let $M = M(-1, b_1/a_1, b_2/a_2, b_3/a_3)$,

Since M is a $\mathbb{Q}$HS Seifert fibered 3-manifold, which does not admit the $\widetilde{SL}_2(\mathbb{R})$-geometry, Proposition 3.5.3 and Lemma 3.5.1(2) give respectively :

$$\begin{cases} \text{(I1)} \quad \sum_{i=1}^{3} \frac{1}{a_i} \geq 1 \\ \\ \text{and} \\ \text{(I2)} \quad \sum_{i=1}^{3} \frac{b_i}{a_i} \neq 1 \end{cases}$$

By Corollary 3.4.5, we order the coefficients : $b_1/a_1 \geq b_2/a_2 \geq b_3/a_3$.

Let $a_{i_0} = \min(a_1, a_2, a_3)$. By (I1), $a_{i_0} \in \{2, 3\}$.

First, we prove that a_{i_0} cannot be 3. We make the proof by contradiction. Assume that $a_{i_0} = 3$, then (I1) implies that $a_i = 3$ for all $i \in \{1, 2, 3\}$. Now, for all $i \in \{1, 2, 3\}$, $b_i < a_i$ so $b_i \leq 2$. If there exists $i \in \{1, 2, 3\}$ such that $b_i = 2$ then $b_i/a_i = 2/3 > 1/2$. But for $j \neq i$, $b_j/a_j \geq 1/3$, which is a contradiction to Corollary 3.4.5 (2). Therefore, $b_i/a_i = 1/3$, for all $i \in \{1, 2, 3\}$, which contradicts (I2).

Hence, we may assume that $a_{i_0} = 2$. Then :

$$b_{i_0}/a_{i_0} = 1/2.$$

By Corollary 3.4.5 (1), $b_{i_0}/a_{i_0} = b_1/a_1$.

Then Corollary 3.4.5 (1) and (2) imply respectively that $a_3 \geq 4$ and $a_2 \geq 3$.

Now (I1) implies that $\{a_1, a_2, a_3\}$ is one of the following sets :

$$\{2,3,4\}, \{2,3,5\}, \{2,3,6\} \text{ or } \{2,4,4\}.$$

We distinguish the cases $a_2 = 3$ and $a_2 = 4$.

Case 1 : $a_2 = 3$.
Then Corollary 3.4.5 (1) implies that :

$$b_2/a_2 = 1/3.$$

Now, by Theorem 3.4.3, there exist positive integers α and m which satisfy Property $(*)$.
Now Corollary 3.4.5 (2) implies that : $\dfrac{b_3}{a_3} \in \left\{\dfrac{1}{4}, \dfrac{1}{5}, \dfrac{1}{6}\right\}$. Hence, by $(*)(iii)$, $m \leq 5$.

Since $b_1/a_1 = 1/2$, $m > 2$.

If $m = 3$ then $\alpha \in \{1, 2\}$, but in both cases $(*)(i)$ or $(*)(ii)$ cannot be satisfied. Similarly, if $m = 4$ then $\alpha \in \{1, 2, 3\}$, but in all cases $(*)(i)$ or $(*)(ii)$ cannot be satisfied.

If $m = 5$ then $a_3 = 6$ and $b_3 = 1$; otherwise $(*)(iii)$ cannot be satisfied.

Thus, $b_1/a_1 + b_2/a_2 + b_3/a_3 = 1/2 + 1/3 + 1/6 = 1$, which is in contradiction to $(I2)$, i.e. M cannot be a $\mathbb{Q}$HS.

Case 2 : $a_2 = 4$.
Then $a_2 = a_3 = 4$. Therefore Corollary 3.4.5 (1) implies that $\dfrac{b_2}{a_2} = \dfrac{b_3}{a_3} = \dfrac{1}{4}$. Therefore $(I2)$ is not satisfied, which is the final contradiction.

This ends the proof of Theorem 3.1.2.

3.7 Proof of Theorem 1.0.11

Let n be a positive integer greater than two. We keep the previous conventions and notations and denote any $\mathbb{Q}$HS Seifert fibered 3-manifolds M with its normalized Seifert invariant, by : $M = M(-b_0, b_1/a_1, b_2/a_2, \ldots, b_n/a_n)$.

Let $\mathcal{SF}_1$ be the set of all Seifert fibered 3-manifolds for which $b_0 = 1$ and which admit the $\widetilde{SL}_2(\mathbb{R})$-geometry.

We denote by $\mathcal{Q}_n$ the set :

$$\mathcal{Q}_n = \mathcal{S}_n \cap \mathcal{SF}_1.$$

Then $\mathcal{Q}_n$ is the set of non-integral $\mathbb{Q}$HS Seifert fibered 3-manifolds M with n exceptional fibers, which admit the $\widetilde{SL}_2(\mathbb{R})$-geometry and $M = M(-1, b_1/a_1, b_2/a_2, \ldots, b_n/a_n)$.

This section is devoted to prove the following result, which clearly implies Theorem 1.0.11.

Theorem 3.7.1. *Let n be a positive integer greater than two. For each n :*
(i) There exist infinitely many Seifert fibered manifolds in $\mathcal{Q}_n$ which admit a taut analytic foliation; and
(ii) There exist infinitely many Seifert fibered manifolds in $\mathcal{Q}_n$ which do not admit a taut $\mathcal{C}^2$-foliation.
(iii) There exist infinitely many Seifert fibered manifolds in $\mathcal{Q}_3$ which do not admit a taut $\mathcal{C}^0$-foliation.

Proof. The proof of Theorem 3.7.1 is an immediate consequence of the two following lemmata. Let n be a positive integer greater than two. Let $\mathcal{M}(n)$ be the family of Seifert fibered 3-manifolds M with n exceptional fibers such that $M = M(-1, \frac{1}{2}, \frac{b_2}{a_2}, \frac{b_3}{a_3}, \dots, \frac{b_n}{a_n})$ and the exceptional slopes are ordered in the following way : $\frac{1}{2} > \frac{b_2}{a_2} \geq \frac{b_3}{a_3} \geq \dots \geq \frac{b_n}{a_n}$.

Lemma 3.7.2. *Let n be a positive integer greater than two. We consider the following families of infinite Seifert fibered 3-manifolds.*

$$\mathcal{M}_1(n) = \left\{ M \in \mathcal{M}(n),\ \textit{with}\ \frac{b_2}{a_2} = \frac{3}{5},\ n > 3 \right\};$$
$$\mathcal{M}_1(3) = \left\{ M \in \mathcal{M}(3),\ \textit{with}\ \frac{b_2}{a_2} = \frac{3}{5},\ \textit{and}\ a_3 \geq 4 \right\};$$
$$\mathcal{M}_2(n) = \left\{ M \in \mathcal{M}(n),\ \textit{with}\ \frac{b_2}{a_2} = \frac{2}{5},\ \frac{b_3}{a_3} > \frac{1}{5}, n > 3 \right\};$$
$$\mathcal{M}_2(3) = \left\{ M \in \mathcal{M}(3),\ \textit{with}\ \frac{b_2}{a_2} = \frac{2}{5},\ \frac{b_3}{a_3} > \frac{1}{5},\ \textit{and}\ a_3 \geq 4 \right\}.$$

If $M \in \mathcal{M}_1(n) \cup \mathcal{M}_2(n)$, then $M \in \mathcal{Q}_n$. In particular, M is a non-integral homology 3-sphere, which admits the $\widetilde{SL}_2(\mathbb{R})$-geometry, and M does not admit a taut $\mathcal{C}^2$-foliation.

Furthermore, if $M \in \mathcal{M}_1(3) \cup \mathcal{M}_2(3)$, then $M \in \mathcal{Q}_3$, and M does not admit a taut $\mathcal{C}^0$-folitation.

Proof. First, considering Lemma 3.5.1, we may check easily that if $M \in \mathcal{M}_1(n) \cup \mathcal{M}_2(n)$ then M is a $\mathbb{Q}$HS but not a $\mathbb{Z}$HS.

Indeed if $M \in \mathcal{M}_1(n)$, then $e(M) > -1 + 1/2 + 3/5$.

If $M \in \mathcal{M}_2$(n), then $e(M) > -1 + 1/2 + 2/5 + 1/5$.

In both cases, $e(M) > 1/10$, so $e(M) \neq 0$; hence, M is a $\mathbb{Q}$HS.

On the other hand, if $e(M) = \dfrac{\varepsilon}{a_1 a_2 \dots a_n}$ (where $\varepsilon = \pm 1$) then $e(M) < \dfrac{1}{10 a_3}$;

which is a contradiction. Then, M is not a $\mathbb{Z}$HS.

Now, we check that they all have the $\widetilde{SL}_2(\mathbb{R})$-geometry.

If $n \geq 4$, then it is a direct consequence of Proposition 3.5.4.

If $n = 3$, that follows from $\sum_{i=1}^{3} \frac{1}{a_i} < 1$ (here, we need that $a_3 \geq 4$).

In conclusion, $\mathcal{M}_1(n) \cup \mathcal{M}_2(n) \subset \mathcal{Q}_n$ (for $n \geq 3$).

Finally, we check that they do not admit a taut $\mathcal{C}^2$-foliation.

If $M \in \mathcal{M}_1(n)$, Corollary 3.4.5 (1) implies that M cannot admit a taut $\mathcal{C}^2$-foliation.

If $M \in \mathcal{M}_2(n)$, then $\dfrac{b_2}{a_2}$ and $\dfrac{b_3}{a_3}$ both are greater than 1/5; therefore (iii) implies that $m \leq 4$. Thus, $\alpha \in \{1, 2, 3\}$. In all cases, (i) or (ii) cannot be satisfied.

Furthermore, by Theorem 3.1.1, if $M \in \big(\mathcal{M}_1(3) \cup \mathcal{M}_2(3)\big)$ and M admits a taut $\mathcal{C}^0$-folitation, then the foliation can be isotoped to be horizontal; which is impossible for M in $\mathcal{M}_1(3) \cup \mathcal{M}_2(3)$. □

Lemma 3.7.3. *Let n be a positive integer greater than two. Let $\mathcal{M}_3$ and $\mathcal{M}_4(n)$ be the two following families of infinite Seifert fibered 3-manifolds.*

$$\mathcal{M}_3 = \Big\{ M\big(-1, \frac{1}{2}, \frac{2}{5}, \frac{k}{7k+1}\big) \in \mathcal{M}(3),\ k \in \mathbb{Z},\ k \geq 1 \Big\};$$
$$\mathcal{M}_4(n) = \Big\{ M\big(-1, \frac{1}{2}, \frac{2}{5}, \frac{1}{10}, \frac{b_4}{10b_4+1}, \dots, \frac{b_n}{10b_n+1}\big) \in \mathcal{M}(n),\ n > 3 \Big\}.$$

If $M \in \mathcal{M}_3 \cup \mathcal{M}_4(n)$, then $M \in \mathcal{Q}_n$ and is a non-integral Seifert fibered 3-manifold, which admits the $\widetilde{SL}_2(\mathbb{R})$-geometry and a taut analytic foliation.

Proof. First, considering Lemma 3.5.1, we can check that if $M \in \mathcal{M}_3 \cup \mathcal{M}_4$, then M is a $\mathbb{Q}$HS but not a $\mathbb{Z}$HS.

Indeed, if $M \in \mathcal{M}_3$, then $e(M) > -1 + 1/2 + 2/5 + 1/8$,
i.e. $e(M) > 1/40$; so $e(M) \neq 0$ and M is a $\mathbb{Q}$HS.
If $e(M) = \dfrac{\varepsilon}{a_1 a_2 a_3}$ (where $\varepsilon = \pm 1$) then $e(M) < 1/70$;

which is not possible so M is not a $\mathbb{Z}$HS.

Similarly, if $M \in \mathcal{M}_4$, then $e(M) > -1 + 1/2 + 2/5 + 1/10 + 1/11$,
i.e. $e(M) > 1/11$; so $e(M) \neq 0$ and M is a $\mathbb{Q}$HS.
If $e(M) = \dfrac{\varepsilon}{a_1 a_2 \dots a_n}$ then $e(M) < 1/100$, which is not possible so M is not a $\mathbb{Z}$HS.

Now, we check that they all admit the $\widetilde{SL}_2(\mathbb{R})$-geometry.

If $n \geq 4$, then it is a direct consequence of Proposition 3.5.4.

If $n = 3$, that follows from $\displaystyle\sum_{i=1}^{n} \frac{1}{a_i} < 1$.

Finally, if we choose $\alpha = 3$ and $m = 7$ then (m, α) trivially satisfies Property $(*)$; which implies that they all admit a taut analytic foliation (by Theorems 3.4.1 and 3.4.3). □

End of proof of Theorem 3.7.1

□

3.8 Proof of Theorem 1.0.10

This section is almost entirely devoted to the proof of Proposition 3.8.1, which implies Theorem 1.0.10, as it will be shown below.

We may recall here (see Section 2) that if M is a Seifert fibered 3-manifold, then $M = M(-b_0, b_1/a_1, \ldots, b_n/a_n)$, where b_0 is a positive integer and $0 < b_i < a_i$ for all $i \in \{1, \ldots, n\}$. Note that n has to be greater than 2 (otherwise M cannot be a $\mathbb{Z}$HS other than $\mathbb{S}^3$).

If M is also a $\mathbb{Z}$HS, then two rational coefficients cannot be the same, see Remark 3.5.2; therefore we may re-order them so that $b_1/a_1 > b_2/a_2 > \cdots > b_n/a_n$.

Thus, two positive integers m and α satisfy Property $(*)$ (for these rational coefficients) if and only if :

$$\boxed{\alpha < m \text{ and } (i) \text{ to } (iii) \text{ of Property } (*) \text{ are all satisfied.}}$$

Proposition 3.8.1. *Let n be a positive integer and M be a $\mathbb{Z}$HS Seifert fibered 3-manifold, which is neither homeomorphic to $\mathbb{S}^3$ nor to $\mathcal{P}$.*
We assume that $M = M(-1, b_1/a_1, \ldots, b_n/a_n)$, where :
- $0 < b_i < a_i$ for all $i \in \{1, \ldots, n\}$, and;
- $b_1/a_1 > b_2/a_2 > \cdots > b_n/a_n$.
Then there exist two positive integers m and α which satisfy Property $(*)$.

PROOF OF THEOREM 1.0.10

First of all, if M is either homeomorphic to $\mathbb{S}^3$ or to the Poincaré sphere $\mathcal{P}$, then we recall that M cannot admit a taut foliation.

We assume that M is neither homeomorphic to $\mathbb{S}^3$ nor to the Poincaré sphere $\mathcal{P}$. We want to show that M always admits a taut analytic foliation.

Let $M = M(-b_0, b_1/a_1, \ldots, b_n/a_n)$, where b_0 is a positive integer and $0 < b_i < a_i$ for all $i \in \{1, \ldots, n\}$.

First, we may note that Corollary 3.4.2 claims that if $b_0 \in \{2, \ldots, n-2\}$ then M admits a horizontal analytic foliation, which is a taut $\mathcal{C}^2$-foliation. Then, we assume for the following that $b_0 \notin \{2, \ldots, n-2\}$.

On the other hand, since M is a $\mathbb{Z}$HS, Lemma 3.5.1 (1) implies that

$$b_0 = \sum_{i=1}^{n} \frac{b_i}{a_i} + \frac{\varepsilon}{a_1 a_2 \ldots a_n}, \text{ where } \varepsilon \in \{-1, +1\}.$$

Then, the property $0 < b_i/a_i < 1$ for all $i \in \{1, \ldots, n\}$, implies that $0 < b_0 < n$. By the fiber-preserving homeomorphism Φ (see Section 2) we may assume that $b_0 = 1$. Hence, Proposition 3.8.1 implies that there exists a pair of positive integers (m, α) which satisfy Property $(*)$. This implies that M admits a horizontal foliation (Theorem 3.4.3) then a taut analytic foliation (Lemma 3.2.2 and Theorem 3.4.1); which ends the proof of Theorem 1.0.10. □

The remaining of the paper is entirely devoted to the proof of Proposition 3.8.1.

Schedule of the proof of Proposition 3.8.1
The proof of Proposition 3.8.1 is organized in four steps, as follows.
Step 1 : If Proposition 3.8.1 is true for $n = 3$ then it is true for all $n \geq 3$.
Step 2 : Considering $n = 3$ gives common notations and results for the following.
Step 3 : We prove Proposition 3.8.1 for $n = 3$ and $\epsilon = -1$.
Step 4 : We prove Proposition 3.8.1 for $n = 3$ and $\epsilon = 1$.

Before starting the proof, we fix some notations and conventions for all the following of the paper.

Notations - Conventions
We keep the previous notations.

Let $M = M(-1, b_1/a_1, \ldots, b_n/a_n)$ be a $\mathbb{Z}$HS Seifert fibered 3-manifold, where $0 < b_i < a_i$ for all $i \in \{1, \ldots, n\}$.

By Lemma 3.5.1, M is a $\mathbb{Z}$HS if and only if :

$$(E1) \quad \sum_{i=1}^{n} \frac{b_i}{a_i} = 1 + \frac{\epsilon}{a_1.a_2.\ldots.a_n}, \quad \text{where } \epsilon \in \{-1, 1\}$$

Let $\hat{a}_i$ (for $i \in \{1, \ldots, n\}$), $\alpha_1, \alpha_2, a'_3, b'_3$ be the following positive rational numbers. Note that all are positive integers but α_1, α_2, which are rational numbers.

$$\boxed{\begin{array}{ll} \alpha_1 = 1 - \dfrac{b_1}{a_1} & \alpha_2 = \dfrac{b_2}{a_2} \\[2ex] \hat{a}_i = \dfrac{a_3 \ldots a_n}{a_i} & \forall i \in \{3, \ldots, n\} \\[2ex] b'_3 = \displaystyle\sum_{i=3}^{n} b_i \hat{a}_i & a'_3 = a_3 \ldots a_n \end{array}}$$

Thus,

$$\frac{b'_3}{a'_3} = \sum_{i=3}^{n} \frac{b_i}{a_i}.$$

Now, we fix the following inequalities by denoting them from (1) to (6). The former three are trivially always true. The last three are true when $n = 3$, see Claim 3.8.2; they concern Steps 2 to 4.

$$\boxed{\begin{array}{c} (1) \quad \dfrac{b_1}{a_1} > \dfrac{b_2}{a_2} > \cdots > \dfrac{b_n}{a_n} \\[2ex] (2) \quad \dfrac{b_1}{a_1} \geq \dfrac{1}{2} \qquad (3) \quad \alpha_1 \leq \dfrac{b_1}{a_1} \end{array}}$$

$$\boxed{\begin{array}{l} \text{When } n = 3 : \\[1ex] (4) \quad \dfrac{b_2}{a_2} < \dfrac{1}{2} \qquad (5) \quad \dfrac{b_3}{a_3} < \dfrac{1}{4} \qquad (6) \quad \alpha_2 > \alpha_1 - \alpha_2 \end{array}}$$

(1) up to reordering;
(2) by Corollary 3.4.4, which implies (3);
(4) to (6), by Claim 3.8.2.

When $n = 3$, $a'_3 = a_3$ and $b'_3 = b_3$, and (E_1) is equivalent to :

$$\boxed{(E2) \quad \frac{b_3}{a_3} = \alpha_1 - \alpha_2 + \frac{\epsilon}{a_1 a_2 a_3}, \quad \textit{where } \epsilon \in \{-1, 1\}.}$$

Claim 3.8.2. *If $n = 3$ then $\frac{b_2}{a_2} < \frac{1}{2}$, $\frac{b_3}{a_3} < \frac{1}{4}$ and $\alpha_2 > \alpha_1 - \alpha_2$.*

Proof. Since $b_1/a_1 \geq 1/2$, there exists a non-negative integer r_1 such that

$$2b_1 = a_1 + r_1.$$

If $b_2/a_2 + b_3/a_3 < 1/2$ then (1) implies (4) and (5). So, we may suppose that $b_2/a_2 + b_3/a_3 \geq 1/2$. Hence, there exists a non-negative integer r such that :

$$2(b_2 a_3 + a_2 b_3) = a_2 a_3 + r.$$

Then $(E2)$ implies that $\frac{r_1}{2a_1} + \frac{r}{2a_2 a_3} = \frac{\epsilon}{a_1 a_2 a_3}$, so $r_1 a_2 a_3 + r a_1 = 2\epsilon$.

Therefore, $r_1 = 0$, $r = 1$, $a_1 = 2$, and $\epsilon = +1$.
Thus, $b_1/a_1 = 1/2$ and (1) implies (4) and $a_3 b_2 > a_2 b_3$.
Then $2(b_2 a_3 + a_2 b_3) = a_2 a_3 + r$ implies $1 + a_2 a_3 > 4a_2 b_3$ and so $a_2 a_3 \geq 4a_2 b_3$, which is equivalent to $1/4 \geq b_3/a_3$.
Since $a_1 = 2$ and the a_i's are pairwise relatively prime, $1/4 > b_3/a_3$ which proves (5).

By $(E2)$, $\alpha_1 - \alpha_2 = \frac{b_3}{a_3} - \frac{\epsilon}{a_1 a_2 a_3}$.

On the other hand, (1) implies : $b_2 a_3 \geq b_3 a_2 + 1$ (since they are positive integers).

Therefore, $\alpha_2 = \frac{b_2}{a_2} \geq \frac{b_3}{a_3} + \frac{1}{a_2 a_3} > \frac{b_3}{a_3} - \frac{\epsilon}{a_1 a_2 a_3}$ which implies (6). □

3.8.1 Step 1 : From $n = 3$ to $n > 3$

We suppose that Proposition 3.8.1 is satisfied for $n = 3$. Now, we assume that $n \geq 4$ and $M = M(-1, b_1/a_1, \dots, b_n/a_n)$ is a $\mathbb{Z}$HS. We want to show that Property $(*)$ is satisfied for the rational coefficients of the Seifert invariant of M.
Let $M' = M(-1, b_1/a_1, b_2/a_2, b'_3/a'_3)$. Note that (E_1) is satisfied because M is a $\mathbb{Z}$HS; therefore M' is also a $\mathbb{Z}$HS, by the definition of b'_3/a'_3.

We separate the proof according to either $\dfrac{b'_3}{a'_3} < \dfrac{b_2}{a_2}$, or $\dfrac{b_2}{a_2} < \dfrac{b'_3}{a'_3}$.

Note that $\dfrac{b_2}{a_2} \neq \dfrac{b'_3}{a'_3}$ because the a_i's are pairwise relatively prime.

Case 1 : $\dfrac{b'_3}{a'_3} < \dfrac{b_2}{a_2}$.

First, we check that $M' \not\cong \mathcal{P}$. Indeed, otherwise $\dfrac{b'_3}{a'_3} = \dfrac{1}{5}$, so $a'_3 = a_3 \dots a_n = 5$, with $n \geq 4$; a contradiction. Then, there exist positive integers m and α such that $\alpha < m$ and :

(i) $\dfrac{b_1}{a_1} < \dfrac{m - \alpha}{m}$;

(ii) $\dfrac{b_2}{a_2} < \dfrac{\alpha}{m}$; and

(iii) $\dfrac{b'_3}{a'_3} < \dfrac{1}{m}$.

By definition, $\dfrac{b_i}{a_i} < \dfrac{b'_3}{a'_3}$ for $i \in \{3, 4, \dots, n\}$, then the same positive integers m and α satisfy Property $(*)$ for the rational coefficients $\dfrac{b_i}{a_i}$ (for $i \in \{1, \dots, n\}$).

Case 2 : $\dfrac{b_2}{a_2} < \dfrac{b'_3}{a'_3}$.

We repeat the same argument.
Similarly, $M' \not\cong \mathcal{P}$; otherwise $\dfrac{b'_3}{a'_3} = \dfrac{1}{3}$, so $a_3 \dots a_n = 3$, with $n \geq 4$; a contradiction. Then, there exist positive integers m and α such that $\alpha < m$ and :

(i) $\dfrac{b_1}{a_1} < \dfrac{m - \alpha}{m}$;

(ii) $\dfrac{b'_3}{a'_3} < \dfrac{\alpha}{m}$; and

(iii) $\dfrac{b_2}{a_2} < \dfrac{1}{m}$.

Since $b_1/a_1 > b_2/a_2 > \dots > b_n/a_n$, we obtain that $\dfrac{b_i}{a_i} < \dfrac{1}{m}$ for $i \in \{2, 3, \dots, n\}$, which implies that m and α can be chosen so that they satisfy Property $(*)$ for the rational coefficients $\dfrac{b_i}{a_i}$ (for $i \in \{1, \dots, n\}$).

3.8.2 Step 2 : General results for $n = 3$

First, note that if m and α are positive integers such that $\alpha < m$, which satisfy Property $(*)$ then, by definition of α_1 and α_2 : (i) and (ii) of Property $(*)$ are respectively equivalent to (I) and (II) bellow.

$$\begin{cases} (I) \quad \alpha < m\alpha_1 \\ \\ (II) \quad m\alpha_2 < \alpha \end{cases}$$

Let

$$\boxed{\begin{array}{c} a = a_1a_2 \text{ and} \\ b = a - b_1a_2 - b_2a_1; \\ \text{then } \dfrac{b}{a} = \alpha_1 - \alpha_2. \end{array}}$$

Let $[.]$ denote the integral value,

i.e. $[x]$ is the integer k such that $k \leq x < k+1$, for all real x.

Let $N = [a/b]$, hence

$$\boxed{N = \left[\frac{1}{\alpha_1 - \alpha_2}\right].}$$

Lemma 3.8.3. *Recall that α and m are integers. The two following properties are satisfied.*

(i) $N \geq 4$;

(ii) If $N\alpha_1 - 1 \leq \alpha \leq N\alpha_1$ and $N - 1 \leq m$, then $0 < \alpha < m$.

Proof. Proof of (i). By $(E2)$ and (5), $\alpha_1 - \alpha_2 < \dfrac{1}{4} - \dfrac{\varepsilon}{aa_3}$, i.e. $4b < a - \dfrac{4\varepsilon}{a_3}$.

Note that (5) implies that $a_3 \geq 5$ $(b_3 \geq 1)$.

Then (since a and b are positive integers) $4b \leq a$. So $N = \left[\dfrac{a}{b}\right] \geq 4$.

Proof of (ii). Let α and m such that $N\alpha_1 - 1 \leq \alpha \leq N\alpha_1$ and $N - 1 \leq m$.

Now, we can check that $0 < \alpha < m$.

The fact that $\alpha < m$ is trivial because $\alpha_1 \leq 1/2$.

Let's check that $\alpha \geq 1$.

First, note that if $b = 1$ then $N\alpha_1 - 1 = a\dfrac{(a_1 - b_1)}{a_1} - 1 = a_2(a_1 - b_1) - 1 = b_2a_1 > 1$.

Then, we assume $b > 1$.

We proceed by contradiction. Assume $\alpha = 0$, then $N\alpha_1 \leq 1$.

$$N\alpha_1 \leq 1 \Leftrightarrow \alpha_1 \leq \frac{1}{N}, \text{ which is } \frac{1}{[a/b]}.$$

Hence, (E_2) implies $\dfrac{b_2}{a_2} + \dfrac{b_3}{a_3} \leq \dfrac{1}{[a/b]} + \dfrac{\epsilon}{a_1 a_2 a_3}$.

Since $\dfrac{b_3}{a_3} < \dfrac{b_2}{a_2}$, $\dfrac{b_3}{a_3} \leq \dfrac{1}{2[a/b]} + \dfrac{\epsilon}{2a_1 a_2 a_3}$ and so $2b_3[a/b] \leq a_3 + \dfrac{\epsilon[a/b]}{a}$.

Now, $b > 1$ implies $\dfrac{[a/b]}{a} < 1$ hence $2b_3[a/b] \leq a_3$.

Furthermore $[a/b] > a/b - 1 \Rightarrow \dfrac{a}{b} - 1 < \dfrac{a_3}{2b_3}$ and so : $ab_3 - bb_3 < \dfrac{a_3 b}{2}$.

Then $ab_3 - \dfrac{a_3 b}{2} < bb_3$.

Finally, note that $(E_2) \Leftrightarrow ab_3 - a_3 b = \epsilon$, i.e. $ab_3 - \dfrac{a_3 b}{2} = \epsilon + \dfrac{a_3 b}{2}$.

Hence $\epsilon + \dfrac{a_3 b}{2} < bb_3 \Leftrightarrow \dfrac{b_3}{a_3} > \dfrac{1}{2} + \dfrac{\epsilon}{ba_3}$.

By (5) $\varepsilon = -1$ and $\dfrac{1}{4} > \dfrac{1}{2} + \dfrac{-1}{ba_3}$, i.e. $\dfrac{1}{ba_3} > \dfrac{1}{4}$, so $ba_3 < 4$.

This is a contradiction because (5) implies that $a_3 \geq 5$ and $b \geq 2$. □

Lemma 3.8.4. *Let $r = N\alpha_1 - [N\alpha_1]$, $r' = a/b - [a/b]$ and $r'' = a\alpha_1/b - [a\alpha_1/b]$.*
If $N\alpha_1 \in \mathbb{N}$, let $(\alpha, m) = (N\alpha_1 - 1, N - 1)$.
If $N\alpha_1 \notin \mathbb{N}$ and $r'\alpha_2 \leq r'' < \alpha_1 r'$, let $(\alpha, m) = ([N\alpha_1], N)$.
Otherwise, let $(\alpha, m) = ([N\alpha_1], N - 1)$.

Then the integers m and α are positive integers which satisfy (I) and (II) and $\alpha < m$.

The proof of this lemma is the main part of Step 3, but does not depend on $\varepsilon = \pm 1$. The fact that $0 < \alpha < m$ is an immediate consequence of Lemma 3.8.3.

3.8.3 Step 3 : $n = 3$ and $\epsilon = -1$

Let us consider *Property* $(**)$ bellow :

$$(**) \begin{cases} (I) & \alpha < m\alpha_1 \\ (II) & m\alpha_2 < \alpha \\ (III) & \dfrac{b}{a} < \dfrac{1}{m} \end{cases}$$

By $(E2)$: $\epsilon = -1 \Rightarrow \dfrac{b_3}{a_3} < \dfrac{b}{a}$, then Property $(**)$ implies trivially Property $(*)$, i.e. if there exist positive integers m and α, such that $\alpha < m$ which satisfy Property $(**)$, then they satisfy Property $(*)$.
We will separate the cases where $N\alpha_1 \in \mathbb{N}$ or $N\alpha_1 \notin \mathbb{N}$. If $N\alpha_1 \notin \mathbb{N}$, let

$$\boxed{r = N\alpha_1 - [N\alpha_1] \quad r' = a/b - [a/b] \quad r'' = a\alpha_1/b - [a\alpha_1/b]}$$

Claim 3.8.5. $N\alpha_1 = \dfrac{\alpha_1}{\alpha_1 - \alpha_2} - \alpha_1 r'$.

Proof. By definition of r', $N\alpha_1 = [a/b]\alpha_1 = (a/b - r')\alpha_1$.
Then $N\alpha_1 = \dfrac{\alpha_1}{\alpha_1 - \alpha_2} - \alpha_1 r'$. □

Claim 3.8.6. $N\alpha_1 = [\dfrac{\alpha_1}{\alpha_1 - \alpha_2}] + r'' - \alpha_1 r'$.

Proof. By Claim 3.8.5 $\dfrac{\alpha_1}{\alpha_1 - \alpha_2} - \alpha_1 r' = N\alpha_1$.

Moreover, $\dfrac{\alpha_1}{\alpha_1 - \alpha_2} = \dfrac{a\alpha_1}{b} = [\dfrac{\alpha_1}{\alpha_1 - \alpha_2}] + r''$, by definition of r''. □

Claim 3.8.7. *If* $[N\alpha_1] = [\dfrac{\alpha_1}{\alpha_1 - \alpha_2}] - 1$ *then* $r'' = r + \alpha_1 r' - 1$.

Proof. First, we may note that $N\alpha_1 = [N\alpha_1] + r$, by definition of r.

Assume that $[N\alpha_1] = [\dfrac{\alpha_1}{\alpha_1 - \alpha_2}] - 1$.

By Claim 3.8.5, $\dfrac{\alpha_1}{\alpha_1 - \alpha_2} - \alpha_1 r' = N\alpha_1 = [N\alpha_1] + r = [\dfrac{\alpha_1}{\alpha_1 - \alpha_2}] - 1 + r$.

Hence $\dfrac{\alpha_1}{\alpha_1 - \alpha_2} = [\dfrac{\alpha_1}{\alpha_1 - \alpha_2}] + \alpha_1 r' + r - 1$.

So $r'' = r + \alpha_1 r' - 1$, by definition of r''. □

We want to find positive integers m and α such that $\alpha < m$, and which satisfy Property $(**)$. First, we consider separately the case $b = 1$.

Lemma 3.8.8. *If* $b = 1$ *then* $m = a - 1$ *and* $\alpha = a_1 b_2$ *satisfy property* $(*)$ *and* $0 < \alpha < m$.

Proof. Assume that $b = 1$ and let $m = a - 1$ and $\alpha = a_1 b_2$. First, we can check that $0 < \alpha < m$ because $a_1 b_2 \leq a_1(a_2 - 1) < a_1 a_2 - 1$. Now, we want to check successively (I) to (III).

$(I) \Leftrightarrow \alpha < m\alpha_1$.

$$m\alpha_1 = (a_1 a_2 - 1)\frac{a_1 - b_1}{a_1} > a_2(a_1 - b_1) - 1.$$

Since $b = 1$, $a_2(a_1 - b_1) - 1 = a_1 b_2 < m\alpha_1$.

$(II) \Leftrightarrow m\alpha_2 < \alpha$.

$$m\alpha_2 = (a_1a_2 - 1)\frac{b_2}{a_2} = a_1b_2 - \frac{b_2}{a_2} < \alpha.$$

$(III) \Leftrightarrow \dfrac{b}{a} < \dfrac{1}{m}$.

Since $b = 1$ and $m = a - 1$, (III) is direct. □

In the following of this section, we assume that $b \neq 1$. We distinguish the three following cases.

Case A : $N\alpha_1 \in \mathbb{N}$. Then $(\alpha, m) = (N\alpha_1 - 1, N - 1)$.
Case B : $N\alpha_1 \notin \mathbb{N}$ and $r'\alpha_2 > r''$ or $r'' \geq \alpha_1 r'$. Then $(\alpha, m) = ([N\alpha_1], N)$.
Case C : $N\alpha_1 \notin \mathbb{N}$ and $r'\alpha_2 \leq r'' < \alpha_1 r'$. Then $(\alpha, m) = ([N\alpha_1], N - 1)$.

First, we prove (III) of Property $(**)$. Then Lemma 3.8.4 concludes this step. Note that, for $\varepsilon = 1$, Lemmata 3.8.8 and 3.8.4 imply that (I) to (III) are true, but (III) does not imply (iii).

Furthermore, we may note that $b \neq 1$ if and only if $\left[\frac{a}{b}\right] < \frac{a}{b}$ because a and b are positive coprime integers (since a_1 and a_2 are so).

Lemma 3.8.9. *We assume that $b \neq 1$. If the integers α and m are chosen as in Lemma 3.8.4 (according to Cases A, B or C) then $\dfrac{b}{a} < \dfrac{1}{m}$.*

Proof. Let α and m be integers as in Cases A, B and C successively.

Assume that Case A or Case C is satisfied.

Then $m = N - 1$. Therefore (III) is trivial, because $m = N - 1 = [a/b] - 1 < a/b$, so $1/m > b/a$.

Assume now that Case B is satisfied.

Then $(III) \Leftrightarrow b/a < 1/N \Leftrightarrow N < a/b$, which is satisfied because $N = [a/b]$ and $b \neq 1$. □

Proof of Lemma 3.8.4

We may recall that the proof does not depend on $\varepsilon = \pm 1$.

We only have to show that the considered integers in Cases A, B and C satisfy (I) and (II). We may recall that $0 < \alpha < m$ by Lemma 3.8.3.

$$\boxed{\text{Case } A: \ N\alpha_1 \in \mathbb{N},\ (\alpha, m) = (N\alpha_1 - 1, N - 1)}$$

$(I) \Leftrightarrow \alpha < m\alpha_1$.

So, $(I) \Leftrightarrow N\alpha_1 - 1 < (N - 1)\alpha_1 \Leftrightarrow \alpha_1 < 1$ which is true because $0 < \dfrac{b_1}{a_1} < 1$.

$(II) \Leftrightarrow m\alpha_2 < \alpha$.

$(II) \Leftrightarrow (N-1)\alpha_2 < N\alpha_1 - 1 \Leftrightarrow 1-\alpha_2 < N(\alpha_1-\alpha_2)$.

Therefore, $(II) \Leftrightarrow \dfrac{1}{\alpha_1-\alpha_2} - N < \dfrac{\alpha_2}{\alpha_1-\alpha_2}$.

But recall that $N = [\dfrac{1}{\alpha_1-\alpha_2}]$, hence $\dfrac{1}{\alpha_1-\alpha_2} - N < 1$. Thus, (II) follows from Claim 3.8.2 (6).

$$\boxed{\textsc{Case } B : r'' \geq \alpha_1 r' \text{ or } r'' < r'\alpha_2,\ (\alpha, m) = ([N\alpha_1], N)}$$

$(I) \Leftrightarrow \alpha < m\alpha_1$.
(I) is trivially satisfied : $(I) \Leftrightarrow [N\alpha_1] < N\alpha_1$.

$(II) \Leftrightarrow m\alpha_2 < \alpha$.
$(II) \Leftrightarrow N\alpha_2 < [N\alpha_1] \Leftrightarrow N\alpha_2 < N\alpha_1 - r$.

Then $(II) \Leftrightarrow r < N(\alpha_1-\alpha_2) \Leftrightarrow r < (a/b - r')(\alpha_1-\alpha_2)$, by definition of r'.

Recall that $b/a = \alpha_1 - \alpha_2$, so

$(II) \Leftrightarrow r < 1 - r'(\alpha_1-\alpha_2) \Leftrightarrow r + r'(\alpha_1-\alpha_2) < 1$.

We want to prove that $r + r'(\alpha_1-\alpha_2) < 1$.

Assume first, that $r'' \geq \alpha_1 r'$.
Then Claim 3.8.6 implies that : $[N\alpha_1] = [\dfrac{\alpha_1}{\alpha_1-\alpha_2}]$.

By Claim 3.8.5 $\dfrac{\alpha_1}{\alpha_1-\alpha_2} - \alpha_1 r' = [N\alpha_1] + r$, then $[N\alpha_1] = \dfrac{\alpha_1}{\alpha_1-\alpha_2} - \alpha_1 r' - r$.

Thus $\dfrac{\alpha_1}{\alpha_1-\alpha_2} = [\dfrac{\alpha_1}{\alpha_1-\alpha_2}] + \alpha_1 r' + r$; so $r'' = \alpha_1 r' + r < 1$.

Now, we can see that : $r + r'(\alpha_1-\alpha_2) < r + \alpha_1 r' < 1$, which proves (II).

Now, we may assume that $r'' < r'\alpha_2$.
By the previous work, we may assume that $r'' < \alpha_1 r'$.
Then Claim 3.8.6 implies that $[N\alpha_1] = [\dfrac{\alpha_1}{\alpha_1-\alpha_2}] - 1$.
Therefore, Claim 3.8.7 implies that $r'' = r + \alpha_1 r' - 1$.
Recall that we want to show that $r + r'(\alpha_1-\alpha_2) < 1$.
Since $r'' < r'\alpha_2$, we obtain :
$r + r'(\alpha_1-\alpha_2) = r + \alpha_1 r' - r'\alpha_2 < r + \alpha_1 r' - r''$.
Here, $r + \alpha_1 r' - r'' = 1$, which gives the required inequality.

$\boxed{\text{CASE } C: r'' < \alpha_1 r' \text{ AND } r'' \geq r'\alpha_2,\ (\alpha, m) = ([N\alpha_1], N-1)}$

$(I) \Leftrightarrow \alpha < m\alpha_1$.
$(I) \Leftrightarrow [N\alpha_1] < (N-1)\alpha_1 \Leftrightarrow \alpha_1 < r$.

Since $r'' < \alpha_1 r'$, by Claim 3.8.6 : $[N\alpha_1] = [\dfrac{\alpha_1}{\alpha_1 - \alpha_2}] - 1$.

Then, by Claim 3.8.7 : $r'' = r + \alpha_1 r' - 1$.

Thus $(I) \Leftrightarrow \alpha_1 < r'' - \alpha_1 r' + 1 \Leftrightarrow \alpha_1 r' - r'' < 1 - \alpha_1$.

Hence, $(I) \Leftrightarrow \alpha_1 r' - r'' < \dfrac{b_1}{a_1}$, because $1 - \alpha_1 = \dfrac{b_1}{a_1}$.

On the other hand, $\alpha_1 r' - r'' < \alpha_1 - r''$ and $\alpha_1 - r'' \leq \alpha_1 \leq \dfrac{b_1}{a_1}$ by (3).
Therefore (I) is satisfied.

$(II) \Leftrightarrow m\alpha_2 < \alpha$.
$(II) \Leftrightarrow (N-1)\alpha_2 < [N\alpha_1]$.

By Claim 3.8.6 and the definition of r'', and since $r'' < \alpha_1 r'$:

$[N\alpha_1] = [\dfrac{\alpha_1}{\alpha_1 - \alpha_2}] - 1 = \dfrac{\alpha_1}{\alpha_1 - \alpha_2} - r'' - 1$.

Moreover, by the definition of r' : $N\alpha_2 = \dfrac{\alpha_2}{\alpha_1 - \alpha_2} - \alpha_2 r'$.

Hence $(II) \Leftrightarrow N\alpha_2 < [N\alpha_1] + \alpha_2 \Leftrightarrow \dfrac{\alpha_2}{\alpha_1 - \alpha_2} - \alpha_2 r' < \dfrac{\alpha_1}{\alpha_1 - \alpha_2} - r'' - 1 + \alpha_2$.

Therefore, $(II) \Leftrightarrow r'' - r'\alpha_2 < \alpha_2$.

On the other hand, $\alpha_2 > \alpha_1 - \alpha_2$ by (6) and $r'\alpha_2 \leq r'' < \alpha_1 r'$.

Then, $r'' - r'\alpha_2 < r'(\alpha_1 - \alpha_2) < r'\alpha_2 < \alpha_2$, which proves that (II) is satisfied.

Proof of Lemma 3.8.4

□

In conclusion, Lemma 3.8.8 solve the case $b = 1$. If $b \neq 1$, then for the α and m chosen as in Lemma 3.8.3, we get that $0 < \alpha < m$ and Lemmata 3.8.4 together with 3.8.9 show that they satisfy $(I), (II)$ and (III). Therefore, Property $(*)$ is satisfied for $n = 3$ and $\varepsilon = -1$.

3.8.4 Step 4 : $n = 3$ and $\epsilon = 1$

Recall that $a = a_1a_2$ and $b = a - b_1a_2 - b_2a_1$.

We assume that $\epsilon = 1$ then $(E2)$ gives :

$$(E3) \quad \frac{b_3}{a_3} = \frac{b}{a} + \frac{1}{a_1a_2a_3}$$

so :

$$(7) \quad ab_3 - ba_3 = 1.$$

Then (Bezout relation) there exists a unique pair of positive coprime integers (u, v) such that :

$$(8) \begin{cases} au - bv = 1; \\ 0 < u \leq b & \text{and} \\ 0 < v \leq a \end{cases}$$

Now, (7) implies that there exists $p \in \mathbb{N}$ such that

$$\begin{cases} b_3 = u + bp & \text{and} \\ a_3 = v + ap \end{cases}$$

Moreover, for all $p \in \mathbb{N}$, we have :

$$(9) \quad \frac{u}{v} \geq \frac{u + bp}{v + ap} > \frac{u + b(p+1)}{v + a(p+1)} > \frac{b}{a}$$

We want to find positive integers α and m such that $\alpha < m$ and satisfy Property $(*)$. We consider separately the three following cases.

Case I : $u \neq 1$.
Case II : $u = 1$ and $b = 1$.
Case III : $u = 1$ and $b \neq 1$.

$$\boxed{\text{Case I} : u \neq 1}$$

We will choose the integers α and m as in Lemma 3.8.4, so $m \in \{N - 1, N\}$. By (9), if $\frac{u}{v} < \frac{1}{m}$, then (iii) of Property $(*)$ is satisfied. Therefore, Lemma 3.8.4 and the following lemma conclude Case I.

Lemma 3.8.10. *If $N - 1 \leq m \leq N$ and $u \neq 1$, then $\frac{u}{v} < \frac{1}{m}$.*

Proof. Assume that $N - 1 \leq m \leq N$ and $u \neq 1$. First, note that $b \neq 1$, because $0 < u \leq b$ implies that if $b = 1$ then $u = 1$.

We make the proof by contradiction. So, we suppose that $\frac{u}{v} \geq \frac{1}{m}$, and we look for a contradiction. Note that $v = um$ cannot happen, because u, v are coprime integers, and u and m are at least 2, by Lemma 3.8.3. Thus $\frac{u}{v} > \frac{1}{m}$.

Moreover, by Lemma 3.8.9 : $\frac{b}{a} < \frac{1}{m}$. Then

$$\frac{b}{a} < \frac{1}{m} < \frac{u}{v}$$

By (8) :

$$\frac{a}{b} - \frac{v}{u} = \frac{1}{ub}$$

we obtain

$$0 < m - \frac{v}{u} < \frac{a}{b} - \frac{v}{u} = \frac{1}{ub} < 1$$

which implies that

$$[\frac{v}{u}] = m - 1$$

Now, let

$$r' = \frac{a}{b} - [\frac{a}{b}] < 1, \text{ and } \rho = \frac{v}{u} - [\frac{v}{u}] < 1$$

We consider separately the cases $m = N$ and $m = N - 1$.

First, assume that $m = N = [\frac{a}{b}]$.

Then $[\frac{v}{u}] = [\frac{a}{b}] - 1 \Leftrightarrow \frac{a}{b} - r' - 1 = \frac{v}{u} - \rho$;

hence $\frac{1}{ub} = 1 + r' - \rho \Rightarrow 1 + r' - \rho < \frac{1}{b}$ because $u \neq 1$.

Thus $r' < \frac{1}{b}$ because $\rho < 1$.

Nevertheless, $r' = \frac{a}{b} - [\frac{a}{b}]$, a and b are coprime, and $a > b$.

Hence $a = bk + l$, where $k \in \mathbb{N}^*$, and $1 \leq l \leq b - 1$;

so r' can be written $r' = k + \frac{l}{b} - [k + \frac{l}{b}] = \frac{l}{b} \Rightarrow r' \geq \frac{1}{b}$; which is a contradiction.

Now, assume that $m = N - 1 = [\frac{a}{b}] - 1$.

Then $[\frac{v}{u}] = [\frac{a}{b}] - 2 \Leftrightarrow \frac{a}{b} - r' - 2 = \frac{v}{u} - \rho$;

hence $\frac{1}{ub} = 2 + r' - \rho$.

This implies that $\frac{1}{ub} > 1$ because r' and ρ lies in $[0, 1[$.

On the other hand, $\frac{1}{ub} \leq \frac{1}{4}$, because $b \geq 2$ and $u \geq 2$.

These are in a contradiction.

(Lemma 3.8.10)

□

$\boxed{\text{CASE II : } u = 1 \text{ AND } b = 1}$

We assume that $u = b = 1$. Then $au - bv = 1$ gives $v = a - 1$.

We consider separately the cases where $\frac{b_1}{a_1} > \frac{1}{2}$ or $\frac{b_1}{a_1} = \frac{1}{2}$.

First, assume that $\frac{b_1}{a_1} > \frac{1}{2}$.

Let $m = a-2$, and $\alpha = a_2(a_1-b_1)-1 = a-a_2b_1-1 = b_2a_1$ (because $b = a-a_2b_1-a_1b_2 = 1$. Then $0 < \alpha < m$.

We want to check $(I), (II)$ and (iii).

$(I) \Leftrightarrow \alpha < m\alpha_1 \Leftrightarrow a_2(a_1 - b_1) - 1 < (a_1 - b_1)a_2 - 2\alpha_1 \Leftrightarrow \frac{1}{2} < \frac{b_1}{a_1}$ (which is satisfied here).

$(II) \Leftrightarrow m\alpha_2 < \alpha \Leftrightarrow (a-2)\alpha_2 < a_2(a_1 - b_1) - 1 \Leftrightarrow 1 - 2\alpha_2 < a_2(a_1 - b_1) - a\alpha_2$; which is satisfied because $a_2(a_1 - b_1) - a\alpha_2 = a_2(a_1 - b_1) - a_1b_2 = b = 1$.

By (9), (iii) is satisfied if $\frac{u}{v} < \frac{1}{m}$; which is true because $\frac{u}{v} = \frac{1}{a-1}$ and $\frac{1}{m} = \frac{1}{a-2}$.

Now, assume that $\frac{b_1}{a_1} = \frac{1}{2}$.

Then $a_1 = 2$ and $b_1 = 1$. Since $1 = b = a_2(a_1 - b_1) - a_1b_2$, $a_2 = 1 + 2b_2$. So :

$$\frac{b_2}{a_2} = \frac{b_2}{1 + 2b_2}$$

and by $(E2)$: $\frac{b_3}{a_3} = \frac{1}{2} - \frac{b_2}{1 + 2b_2} + \frac{1}{2(2b_2 + 1)a_3}$.

Thus, $\frac{b_3}{a_3} = \frac{(2b_2 + 1)a_3 - 2b_2a_3 + 1}{2(2b_2 + 1)a_3}$, i.e.

$$\frac{b_3}{a_3} = \frac{a_3 + 1}{2(2b_2 + 1)a_3}$$

We consider separately the cases $b_2 = 1$ and $b_2 > 1$.
Assume first $b_2 = 1$.

Then $a_2 = 3$ so $\frac{b_3}{a_3} = \frac{a_3 + 1}{6a_3}$.

Therefore, we can check easily that $\alpha = 2$ and $m = 5$ satisfy Property $(*)$.

(i) $\frac{b_1}{a_1} = \frac{1}{2} < \frac{m - \alpha}{m}$, which is $\frac{3}{5}$;

(ii) $\frac{b_2}{a_2} = \frac{1}{3} < \frac{\alpha}{m}$, which is $\frac{2}{5}$; and

(iii) $\dfrac{b_3}{a_3} = \dfrac{a_3+1}{6a_3} < \dfrac{1}{m}$ (which is $\dfrac{1}{5}$) if and only if $a_3 > 5$.

By (5) $a_3 \geq 5$, but if $a_3 = 5$ then $M \cong \mathcal{P}$, so $a_3 > 5$.

Now, we assume that $b_2 \geq 2$.
Let $\alpha = 2b_2 - 1$ and $m = 4b_2 - 1$.
Since $b_2 \geq 2 :\ 0 < \alpha < m$. We want to check $(i), (ii)$ and (iii).

(i) $\dfrac{b_1}{a_1} = \dfrac{1}{2} < \dfrac{m-\alpha}{m}$, which is $\dfrac{2b_2}{4b_2-1}$; so (i) is satisfied.

(ii) $\dfrac{b_2}{a_2} = \dfrac{b_2}{2b_2+1} < \dfrac{\alpha}{m}$, which is $\dfrac{2b_2-1}{4b_2-1}$;

and $\dfrac{b_2}{2b_2+1} < \dfrac{2b_2-1}{4b_2-1}$ if and only if $4b_2^2 - b_2 < 4b_2^2 - 1$, i.e. $b_2 > 1$;

so (ii) is satisfied.

(iii) $\dfrac{b_3}{a_3} = \dfrac{a_3+1}{2(2b_2+1)a_3} < \dfrac{1}{m}$, which is $\dfrac{1}{4b_2-1}$.

Then, (iii) is satisfied if and only if :
$(a_3+1)(4b_2-1) < (4b_2+2)a_3$ i.e. $4b_2 < 3a_3 + 1$.

Since $\dfrac{b_3}{a_3} = \dfrac{a_3+1}{2(2b_2+1)a_3}$, $b_3 = \dfrac{a_3+1}{2(2b_2+1)} \geq 1$ (because b_3 is a positive integer).

So $a_3 + 1 \geq 4b_2 + 2$; thus (iii) is satisfied.

$\boxed{\text{CASE III : } u = 1 \text{ AND } b \neq 1}$

We assume $u = 1$ and $b \geq 2$. Then $a - bv = 1$ by (8).

Claim 3.8.11. *If* $\dfrac{b_2}{a_2} < \dfrac{1}{v}$ *then* $m = v$ *and* $\alpha = 1$ *satisfy Property* $(*)$.

Proof. Assume that $\dfrac{b_2}{a_2} < \dfrac{1}{v}$. To prove that $m = v$ and $\alpha = 1$ satisfy Property $(*)$, it remains to prove that $\dfrac{b_1}{a_1} < \dfrac{v-1}{v}$. Indeed, (II) and (iii) are trivially satisfied because $\dfrac{b_3}{a_3} < \dfrac{b_2}{a_2} < \dfrac{1}{v}$.

By (8) : $1 + \dfrac{1}{av} = \dfrac{b_1}{a_1} + \dfrac{b_2}{a_2} + \dfrac{1}{v}$.
But $\dfrac{b_2}{a_2} > \dfrac{1}{av}$, otherwise $b_2 < \dfrac{1}{a_1 v}$; which is impossible.
Therefore, $\dfrac{b_1}{a_1} = 1 + \dfrac{1}{av} - \dfrac{b_2}{a_2} - \dfrac{1}{v} < 1 - \dfrac{1}{v}$, so $\dfrac{b_1}{a_1} < \dfrac{v-1}{v}$. $\square$

Hence, in the following, we assume that $\frac{b_2}{a_2} > \frac{1}{v}$ (note that the equality is impossible because the integers are coprime).

Let α be the integer such that $(v-1)\alpha_1 - 1 \leq \alpha < (v-1)\alpha_1$, and $m = min(v-1, M)$, where M is the positive integer such that $\frac{\alpha}{\alpha_2} - 1 \leq M < \frac{\alpha}{\alpha_2}$. Then :

$$\boxed{\begin{gathered} \alpha = (v-1)\alpha_1 - r, \text{ where } 0 < r \leq 1 \\ M = \frac{\alpha}{\alpha_2} - r', \text{ where } 0 < r' \leq 1 \\ \text{and } m = \min(M, v-1). \end{gathered}}$$

First, we will check that $m > \alpha > 0$, then we will show that the integers m and α satisfy Property $(*)$.

Claim 3.8.12. *The integers m and α satisfy :* $1 \leq \alpha < m$

Proof. First, we check that $\alpha \geq 1$, where $\alpha = (v-1)\alpha_1 - r$, $0 < r \leq 1$.

We show that $(v-1)\alpha_1 > 1$, then $\alpha > 0$. Since $\alpha \in \mathbb{N}$, $\alpha \geq 1$.

By (8) : $\alpha_1 = \frac{b_2}{a_2} + \frac{1}{v} - \frac{1}{a_1 a_2 v}$.

Since $\frac{b_2}{a_2} > \frac{1}{v}$, $\alpha_1 > \frac{2}{v} - \frac{1}{a_1 a_2 v}$, i.e. $\alpha_1 > \frac{2a_1a_2 - 1}{a_1 a_2 v}$.

Therefore $v > \frac{2a_1a_2 - 1}{a_1a_2\alpha_1}$, so $(v-1)\alpha_1 > \frac{a_1a_2(2-\alpha_1) - 1}{a_1a_2}$.

Finally, recall that $1 - \alpha_1 = \frac{b_1}{a_1}$.

Thus, $(v-1)\alpha_1 > \frac{a_1a_2(1 + \frac{b_1}{a_1}) - 1}{a_1a_2}$, i.e. $(v-1)\alpha_1 > \frac{a_1a_2 + a_2b_1 - 1}{a_1a_2}$.

Since $a_2 \geq 3$, $(v-1)\alpha_1 > \frac{a_1a_2 + 2}{a_1a_2} > 1$.

Now, we check that $m > \alpha$.

If $m = v - 1$, this is trivial.

So, we may assume that $m = \frac{\alpha}{\alpha_2} - r'$, where $0 < r' \leq 1$.

Therefore, $m = \alpha(\frac{1}{\alpha_2} - \frac{r'}{\alpha})$.

Since $\alpha \geq 1$, $\frac{r'}{\alpha} \leq r' \leq 1$, so $m \geq \alpha(\frac{1}{\alpha_2} - 1)$.

Finally, (4) implies that $\alpha_2 < \frac{1}{2}$ and so that $m > \alpha$. $\square$

To show that α and m satisfy Property $(*)$, we need the following claim.

Claim 3.8.13. $\dfrac{\alpha_1-\alpha_2}{\alpha_1}+\alpha_1<1-\dfrac{1}{a}$.

Proof. We first consider that $\dfrac{b_1}{a_1}=\dfrac{1}{2}$.

Then $\alpha_1=\dfrac{1}{2}$ and $a=2a_2$; so $1-\dfrac{1}{a}=\dfrac{2a_2-1}{2a_2}$.

On the other hand, $\dfrac{\alpha_1-\alpha_2}{\alpha_1}+\alpha_1=\dfrac{3}{2}-2\alpha_2=\dfrac{3a_2-4b_2}{2a_2}$.

Then, here : $\dfrac{\alpha_1-\alpha_2}{\alpha_1}+\alpha_1<1-\dfrac{1}{a}$ if and only if

$$a_2-4b_2<-1.$$

Now, (6) implies $\alpha_2>\dfrac{\alpha_1}{2}$, i.e. $4b_2>a_2$, so $a_2-4b_2\leq -1$.

We are going to show that $a_2\neq 4b_2-1$ by contradiction.
First, note that since $b_1/a_1=1/2$, $a=2a_2$ and $b=a_2-2b_2\neq 1$.

On the other hand, since $a-bv=1$, $v=\dfrac{a-1}{b}=\dfrac{2a_2-1}{a_2-2b_2}$.

If $a_2=4b_2-1$, then $v=\dfrac{8b_2-3}{2b_2-1}$.

Now, $v=4+\dfrac{1}{2b_2-1}\in\mathbb{N}$ implies that $b_2=1$, $v=5$ and $a_2=3$. Then $b=3-2=1$; which is a contradiction.

Therefore, $a_2<4b_2-1$; which is the required inequality.

Now, we assume that $\dfrac{b_1}{a_1}>\dfrac{1}{2}$, so $2b_1-a_1>0$.

Then $a_1-b_1<a_1b_2(2b_1-a_1)$,

so $(a_1-b_1)+a_1^2b_2<2a_1b_1b_2$,

and $(a_1-b_1)-2a_1b_1b_2+2a_1^2b_2<a_1^2b_2$. Therefore :

$$(\star)\quad (a_1-b_1)(1+2a_1b_2)<a_1^2b_2.$$

On the other hand, (6) implies that $2\alpha_2>\alpha_1$, i.e. $2a_1b_2>a_2(a_1-b_1)$.
Hence, $2a_1b_2(a_1-b_1)+(a_1-b_1)>a_2(a_1-b_1)^2+(a_1-b_1)$, i.e.

$$(2a_1b_2+1)(a_1-b_1)>a_2(a_1-b_1)^2+(a_1-b_1).$$

Therefore, by the inequality $(\star)$:

$$a_2(a_1-b_1)^2+(a_1-b_1)<a_1^2b_2.$$

So $\dfrac{a_1-b_1}{a_1^2a_2}<\dfrac{a_1^2b_2-a_2(a_1-b_1)^2}{a_1^2a_2}$; i.e. $\dfrac{\alpha_1}{a}<\alpha_2-\alpha_1^2$.

Thus, we obtain $\dfrac{1}{a}<\dfrac{\alpha_2}{\alpha_1}-\alpha_1$, which gives the required inequality. $\square$

Now, we will show successively that α and m satisfy (iii), (II) and (I) of Property $(*)$.

- *α and m satisfy (iii) : $\dfrac{b_3}{a_3}<\dfrac{1}{m}$.*

This is trivially satisfied because by (9), $\dfrac{b_3}{a_3}\leq\dfrac{1}{v}$, and $m\leq v-1$.

- *α and m satisfy (II) : $m\alpha_2<\alpha$.*

Since $m\leq M$, $m\alpha_2\leq\alpha-r'\alpha_2<\alpha$ (because $r'>0$) then (α,m) trivially satisfies (II).

- *α and m satisfy (I) : $\alpha<m\alpha_1$.*

Since $r>0$, $(v-1)\alpha_1-r<(v-1)\alpha_1$. Hence, $\alpha<m\alpha_1$ if $m=v-1$. Thus, we may assume that $m=M\leq v-2$.

So, we want to show that $(v-1)\alpha_1-r<(\dfrac{\alpha}{\alpha_2}-r')\alpha_1$. Now :

$$\begin{aligned}
(v-1)\alpha_1-r<(\frac{\alpha}{\alpha_2}-r')\alpha_1 \quad&\Leftrightarrow\quad v-\frac{r}{\alpha_1}<\frac{(v-1)\alpha_1-r}{\alpha_2}-r'+1\\
&\Leftrightarrow\quad v\alpha_1\alpha_2-r\alpha_2<v\alpha_1^2-\alpha_1^2-r\alpha_1-r'\alpha_1\alpha_2+\alpha_1\alpha_2\\
&\Leftrightarrow\quad r(\alpha_1-\alpha_2)+r'\alpha_1\alpha_2+\alpha_1(\alpha_1-\alpha_2)<v\alpha_1(\alpha_1-\alpha_2)\\
&\Leftrightarrow\quad v(\alpha_1-\alpha_2)>\alpha_1-\alpha_2+r\frac{\alpha_1-\alpha_2}{\alpha_1}+r'\alpha_2
\end{aligned}$$

Recall that $\dfrac{b}{a}=\alpha_1-\alpha_2$ and $a-bv=1$, so :

$$v(\alpha_1-\alpha_2)=\frac{vb}{a}=\frac{a-1}{a}=1-\frac{1}{a}.$$

Therefore, α and m satisfy (I) if and only if

$$1-\frac{1}{a}>\alpha_1-\alpha_2+r\frac{\alpha_1-\alpha_2}{\alpha_1}+r'\alpha_2.$$

Since r and r' both lie in $]0,1]$:

$$\alpha_1-\alpha_2+r\frac{\alpha_1-\alpha_2}{\alpha_1}+r'\alpha_2<\alpha_1-\alpha_2+\frac{\alpha_1-\alpha_2}{\alpha_1}+\alpha_2,$$

i.e. $\alpha_1 - \alpha_2 + r\dfrac{\alpha_1 - \alpha_2}{\alpha_1} + r'\alpha_2 < \alpha_1 + \dfrac{\alpha_1 - \alpha_2}{\alpha_1} < 1 - \dfrac{1}{a}$, by Claim 3.8.13.

Hence, α and m satisfy (I), which ends the proof of Proposition 3.8.1.

Chapter 4

Conclusion et perspectives

Dans cette thèse nous nous sommes concentrés sur le problème toujours ouvert de l'existence de feuilletage tendu.
Le premier travail a été de bien comprendre ce qu'était un feuilletage tendu (tourbillonement, spiralement), et le rôle particulier des feuilles toriques, qui fait l'objet du Chapitre 2.
Le second travail a été de classifier les sphères d'homologie fibrées de Seifert qui admettent un feuilletage tendu.
Les perspectives sont alors les suivantes :
Un travail en cours avec Daniel Matignon montre l'existence d'un $\mathcal{C}^2$-feuilletage sans feuille compacte étant donné un $\mathcal{C}^0$-feuilletage sans feuille compacte sur une sphère d'homologie fibrées de Seifert, qui permettrait de généraliser le Théorème 1.0.11.
Une autre perspective plus large, est de comprendre alors l'existence de feuilletage tendu parmi les sphères d'homologies dont la JSJ-décomposition admet un morceaux fibré de Seifert, par exemple les variétés graphées.

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Résumé

Dans cette thèse, on étudie les C^2-feuilletages de codimension 1, dans les 3-variétés compactes connexes et orientables. Il est bien connu que l'on peut construire explicitement sur de telles variétés un feuilletage qui possède des *composantes de Reeb*. Vient alors le problème crucial d'existence des *feuilletages tendus* (toujours ouvert).

Rappelons qu'un feuilletage tendu n'admet pas de composante de Reeb, mais que la réciproque est fausse.
La première partie de ce travail, consiste à comprendre la différence entre un feuilletage non-tendu sans composante de Reeb et un feuilletage tendu. On verra que l'orientation transverse des feuilles toriques joue un rôle crucial, en donnant une condition nécessaire et suffisante sur cette orientation transverse pour qu'un feuilletage soit tendu. Pour cela on étudiera de près les procédés géométriques de *tourbillonement* et de *spiralement*, et on montrera qu'ils apparaissent toujours au voisinage d'une feuille torique.

La seconde partie de ce travail se concentre sur le problème d'existence de feuilletages tendu.
Rappelons que depuis les travaux de D. Gabai [1983], on sait que si une 3-variété admet une homologie non-triviale, alors elle admet un feuilletage tendu. Mais le problème d'existence est toujours ouvert parmi les sphères d'homologies, et on s'intéresse ici à celles qui sont fibrées de Seifert. On montre que toutes les sphères d'homologie entière fibrées de Seifert sauf $\mathbb{S}^3$ et la sphère d'homologie de Poincaré admettent un feuilletage tendu. Par contre, parmi les sphères d'homologie rationnelle non-entière, fibrées de Seifert, il existe une infinité de telles variétés qui admettent un feuilletage tendu, et une infinité qui n'en admettent pas.

Mots Clés : Feuilletages tendus, Composante de Reeb, Sphère d'homologie, Variétés de Seifert, Tourbillonement, Spirallement.

Abstract

In this thesis, we study codimension 1, $\mathcal{C}^2$-foliations, in compact, connected and orientable 3-manifolds. It is well known that we can explicitly construct on such manifolds a foliation admitting *Reeb components*. Then comes the crucial problem of existence of taut foliation (still opened).

Recall that a taut foliation does not admit a Reeb component, but the converse is false.
The first step of this work focuses on the difference between a non-taut and Reebless foliation, and a taut foliation. We will understand that the transverse orientation of the torus leaves plays a key-role, by giving a necessary and sufficient condition on the transverse orientation, for a foliation to be taut. For this, we will study the geometric processes of *turbulization* and *spiraling* with generalizations, and we see that they always appear in a neighborhood of a torus leaf.

The second step of this work is concentrated on the problem of existence of taut foliations.
Recall that since the work of D. Gabai [1983], we know that if a 3-manifold has non-trivial homology, then it admits a taut foliation. This problem is still opened among homology spheres and we focus here on Seifert fibered ones. We show that all Seifert fibered integral homology spheres (but $\mathbb{S}^3$ and Poincaré homology sphere) admit a taut foliation. Nevertheless, among Seifert fibered rational (and non-integral) homology spheres, there exist infinitely many which admit a taut foliation and infinitely many which do not admit one.

Keywords : Taut foliations, Reeb component, Homology sphere, Seifert manifolds, Turbulization, Spiraling.