Numéro d'ordre : 89-2010 Année 2010

Université Claude Bernard - Lyon 1

Institut Camille Jordan - CNRS UMR 5208
École doctorale Infomaths

Thèse de l'université de Lyon

pour l'obtention du

Diplôme de doctorat
Spécialité : mathématiques
(arrêté du 7 août 2006)

présentée par

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Quelques contributions à l'étude des marches aléatoires en milieu aléatoire

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soutenue publiquement le 25 juin 2010 devant le jury composé de

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QUELQUES CONTRIBUTIONS À L'ÉTUDE DES MARCHES ALÉATOIRES EN MILIEU ALÉATOIRE

L. Tournier

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Résumé. — Les marches aléatoires en milieu aléatoire ont suscité un vif intérêt au cours de ces dernières années, tant en sciences appliquées, comme moyen notamment d'affiner des modèles par une prise en compte des fluctuations de l'environnement ambiant, qu'en mathématiques, de par la multiplicité et la richesse des comportements qu'elles présentent.

Cette thèse est dédiée à l'étude de divers aspects de la transience des marches aléatoires en milieu aléatoire. Elle est composée de deux parties, la première consacrée au cas des environnements de Dirichlet sur $\mathbb{Z}^d$ (où $d \geq 1$), la seconde au régime sous-diffusif sur $\mathbb{Z}$.

Les marches aléatoires en milieu de Dirichlet peuvent être vues de façon équivalente comme des marches aléatoires renforcées par arêtes orientées, ce qui en fait un moyen d'étude de ces dernières et un exemple naturel d'environnement. Certaines spécificités de cette loi permettent de plus d'obtenir des résultats sensiblement plus précis que ce qui est connu dans le cas général. On démontre ainsi tout d'abord une caractérisation de l'intégrabilité des temps de sortie de parties finies de graphes quelconques, qui permet de raffiner un critère de balisticité dans $\mathbb{Z}^d$. On prouve également que les marches aléatoires en environnement de Dirichlet sont transientes directionnellement, avec probabilité positive, dès que les paramètres ne sont pas symétriques. En dimension 1, la thèse se focalise sur le rôle des vallées profondes de l'environnement, en fournissant une nouvelle preuve du théorème de Kesten-Kozlov-Spitzer dans le cas sous-diffusif basée sur l'étude fine du comportement de la marche. Outre une meilleure compréhension de l'émergence de la loi limite, cette preuve a l'avantage de fournir la valeur explicite de ses paramètres.

Abstract. — Random walks in random environment have raised a great interest in the last few years, both among applied scientists, notably as a way to refine models by taking fluctuations of the surrounding environment into account, and among mathematicians, because of the variety and wealth of behaviours they display.

This thesis aims at the study of miscellaneous aspects of the transience of random walks in random environment. A first part is dedicated to Dirichlet environments on $\mathbb{Z}^d$ (where $d \geq 1$) and a second one to the subdiffusive regime on $\mathbb{Z}$.

Random walks in Dirichlet environment arise naturally as an equivalent model for oriented-edge reinforced reinforced random walks. Its specificities also allow for sensibly sharper results than in the general case. We thus prove a characterization of the integrability of exit times out of finite subsets of arbitrary graphs, which enables us to refine a ballisticity criterion on $\mathbb{Z}^d$. We also prove that these random walks are transient with positive probability as soon as the parameters are non-symmetric. In dimension 1, the thesis focuses on the role of the deep valleys of the environment. We give a new proof of Kesten-Kozlov-Spitzer theorem in the subdiffusive regime based on a fine study of the behaviour of the walk. Together with a better understanding of the origin of the limit law, this proof also provides its explicit parameters.

REMERCIEMENTS

Ce travail de thèse doit beaucoup à toutes les personnes qui m'ont entouré au fil de ces années. Je tiens à leur exprimer ici mes plus sincères remerciements.

À Christophe Sabot, tout d'abord, qui a encadré ma thèse après mon stage de master 2, et m'a fait entrer dans le monde de la recherche. Sa confiance, ses conseils et suggestions m'ont toujours été précieux.

À Yueyun Hu et Pierre Tarrès, qui ont accepté d'être rapporteurs de ma thèse. Je les remercie pour la qualité de leur relecture et l'intérêt qu'ils ont porté à mon travail.

À Stéphane Attal, Francis Comets et Nathanaël Enriquez qui me font l'honneur de faire partie de mon jury de soutenance.

À mes co-auteurs, notamment Olivier Zindy que je n'ai pas encore cité.

Aux responsables du projet ANR MEMEMO « Marches aléatoires, Milieux aléatoires, Renforcement », dont les workshops, colloques et groupes de travail ont rythmé ma thèse.

À mes collègues et amis doctorants, Alexander, Alina, Amélie, Antoine, Frédéric, Gaëlle, Ion, Jean, Mickaël, Nicolas, Thomas, Yoann,… pour n'en citer que quelques-uns, et à mes amis de Lyon, de Paris, d'Orléans, que je n'oublie pas.

À ma famille, à mes parents, enfin, pour leur soutien et leur affection.

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CHAPITRE 1

INTRODUCTION

Les marches aléatoires constituent un modèle simple pour une large classe de phénomènes de transport tels que la diffusion de la chaleur ou le mouvement d'une particule dans un fluide. En pratique, le milieu traversé est toujours affecté par des irrégularités (impuretés, défauts). Considérer ce milieu comme aléatoire, autrement dit imprévisible localement mais possédant une régularité statistique sur de grandes échelles, est une façon naturelle de tenir compte de ces fluctuations afin d'étudier leur effet sur les trajectoires. Par ailleurs, il est également des modèles où le milieu présente de façon intrinsèque des fluctuations importantes que l'on peut considérer comme aléatoires ; c'est notamment le cas des modèles de déplacement le long de brins d'ADN.

Les premières occurrences de marches aléatoires en milieu aléatoire (MAMA) dans la littérature scientifique sont ainsi à mettre au compte des physiciens Temkin [**50**] et Chernov [**10**], respectivement dans des contextes de métallurgie (transitions de phase dans les alliages) et de biologie (réplication de l'ADN).

L'ajout d'une dimension d'aléa introduit des corrélations importantes entre les pas successifs de ces processus et complique singulièrement leur analyse. Mais au-delà du défi que cela représente, l'intérêt des mathématiciens pour les marches aléatoires en milieu aléatoire est motivé par la manifestation de comportements nouveaux, d'une grande richesse. On peut citer les phénomènes de localisation ou de ralentissement de la marche, interprétés par la présence de zones de « pièges » dans l'environnement.

Cette introduction vise à définir le cadre dans lequel se place l'ensemble de la thèse, en présentant le modèle étudié, quelques applications, et le contexte des principaux résultats obtenus. On pourra aussi se référer à [**2**], [**48**] ou [**53**] pour des introductions plus détaillées aux MAMA.

1.1. Modèle et motivations

1.1.1. Définition. — Soit d un entier ≥ 1. On présente ici le cadre de l'essentiel de cette thèse, à savoir celui des marches aléatoires aux plus proches voisins sur le réseau $\mathbb{Z}^d$, dans un environnement constitué de variables indépendantes identiquement distribuées (i.i.d.).

Une marche aléatoire en milieu aléatoire est définie au terme de deux étapes : dans un premier temps, la génération d'un environnement, c'est-à-dire, en chaque sommet du réseau, le tirage de probabilités de saut vers ses voisins (indépendamment entre les sommets, et selon la même loi) ; puis celle de la marche aléatoire elle-même, en fonction de l'environnement.

Introduisons quelques notations. On désigne par $\mathcal{V}$ l'ensemble des vecteurs unitaires de $\mathbb{Z}^d$; $\mathcal{V}$ représente donc l'ensemble des directions possibles à chaque pas de la marche ({gauche, droite} en dimension 1, {Nord, Sud, Est, Ouest} en dimension 2,...). L'ensemble des probabilités sur $\mathcal{V}$ est noté $\mathcal{P}$:

$$\mathcal{P} := \Big\{(p_e)_{e\in\mathcal{V}} \in [0,1]^{\mathcal{V}} \Big| \sum_{e\in\mathcal{V}} p_e = 1\Big\}.$$

Un environnement (ou milieu) est alors un élément de $\Omega := \mathcal{P}^{\mathbb{Z}^d}$, c'est-à-dire une famille $\omega = (\omega(x,\cdot))_{x\in\mathbb{Z}^d}$ de lois de probabilité sur $\mathcal{V}$.

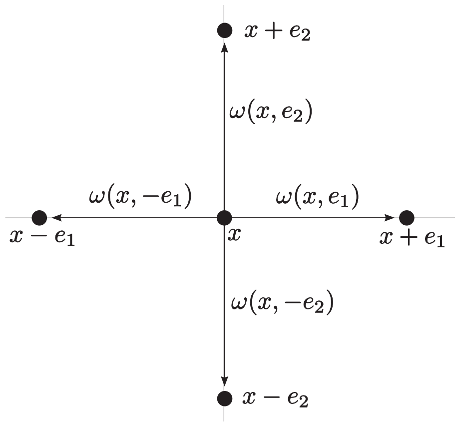

FIGURE 1.1. Environnement au point $x \in \mathbb{Z}^2$

Étant donné un tel ω et un sommet $x \in \mathbb{Z}^d$, la loi $P_{x,\omega}$ de la marche aléatoire $(X_n)_{n\in\mathbb{N}}$ dans l'environnement ω issue de x est naturellement définie par

$$P_{x,\omega}(X_0 = x) = 1$$

et, pour tout $n \geq 0$,

$$P_{x,\omega}(X_{n+1} = X_n + e | X_0, \dots, X_n) = \omega(X_n, e).$$

Afin d'introduire l'aléa de l'environnement, on se donne de plus une loi μ sur $\mathcal{P}$, ce qui permet de définir la loi $P := \mu^{\mathbb{Z}^d}$ sur Ω. Autrement dit, dans un environnement aléatoire ω de loi P, les probabilités $(\omega(x,e))_{e\in\mathcal{V}}$ sont indépendantes d'un sommet x à l'autre, et de même loi μ. Enfin, on note

$$\mathbb{P}_x := P(\,\mathrm{d}\omega) \times P_{x,\omega}$$

la loi du couple (ω, X), où

- l'environnement $\omega = (\omega(x,\cdot))_{x\in\mathbb{Z}^d}$ suit la loi P ;
- sachant ω, la marche $X = (X_n)_{n\geq 0}$ suit la loi $P_{x,\omega}$.

On s'intéresse notamment dans la suite à la loi de X sous $\mathbb{P}_x$, appelée loi *annealed* (moyennée par rapport à l'environnement) par opposition avec la loi *quenched*[(1)] $P_{x,\omega}$ (dans un environnement ω fixé).

Il est à noter que, sauf cas dégénéré, X n'est pas une chaîne de Markov sous $\mathbb{P}_x$. En effet, les transitions successives de X à partir d'un quelconque sommet $x \in \mathbb{Z}^d$ sont déterminées par la même loi $\omega(x, \cdot)$, et donc corrélées. L'observation de la trajectoire apporte progressivement de l'information sur l'environnement sous-jacent traversé et influe sur la loi des transitions futures. Plus précisément, les transitions déjà effectuées ont tendance à être favorisées, renforcées, par la suite.

1.1.2. Pièges, puits de potentiel. — Sous l'action du renforcement progressif des transitions, on peut s'attendre à un ralentissement de la trajectoire. Une seconde explication (de nature « *quenched* ») à ce phénomène tient dans la formation de pièges dans l'environnement aléatoire, c'est-à-dire de zones qui retiennent la marche un temps relativement long.

Pour tout sommet x, notons $d_\omega(x) := \sum_{e\in\mathcal{V}} \omega(x,e)e = E_{x,\omega}[X_1 - X_0]$ la dérive de l'environnement en x. On peut se représenter une forme simple de piège comme une boule B_L (centrée en 0, de rayon L) dans laquelle toutes les dérives renvoient la marche vers l'intérieur : pour tout $x \in B_L$, $d_\omega(x) \cdot x \leq 0$ (voir figure 1.2).

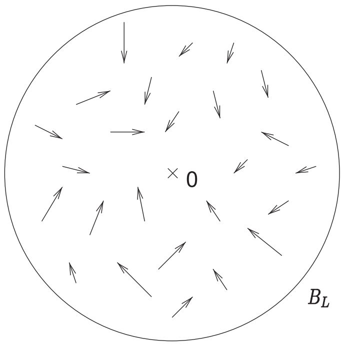

FIGURE 1.2. Piège simple (les flèches représentent les dérives moyennes)

La compréhension de la géométrie et de l'incidence des pièges en dimension ≥ 2 est encore très incomplète. La situation est fort différente en dimension 1 : d'une part, la structure linéaire empêche la marche d'éviter les pièges, et d'autre part leur étude est simplifiée par l'existence d'un *potentiel* associé à l'environnement.

(1) Ce vocabulaire vient de la métallurgie : *quenched* se traduit par « trempé » et *annealed* par « recuit ». La trempe consiste à plonger un matériau chaud dans un fluide plus froid ; c'est un refroidissement brutal de la pièce qui a pour objectif de figer la structure obtenue lors de la mise en solution. Par analogie, on utilise le même terme pour qualifier la situation à environnement fixé (« figé »), sous la loi $P_{x,\omega}$. Par opposition, le recuit est une opération de chauffage de pièces métalliques, et c'est sous ce nom que l'on désigne la situation en moyenne, sous $\mathbb{P}_x$.

Plaçons-nous en dimension 1. Comme $\omega(x,1) = 1 - \omega(x,-1)$ pour tout x, il est équivalent de définir l'environnement par les seules transitions vers la droite

$$\omega_x := \omega(x,1).$$

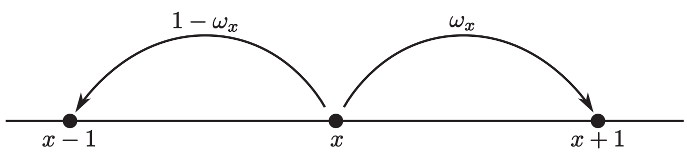

FIGURE 1.3. Environnement unidimensionnel

Soit $\omega = (\omega_x)_{x\in\mathbb{Z}}$ un environnement sur $\mathbb{Z}$. On suppose $0 < \omega_x < 1$ pour tout $x \in \mathbb{Z}$. Motivé par le formalisme utilisé en physique, le potentiel $V = (V(x))_{x\in\mathbb{Z}}$ associé à ω se définit par $V(0) = 0$ (arbitrairement) et, pour tout $x \in \mathbb{Z}$,

$$\omega_x = \frac{\mathrm{e}^{-V(x)}}{\mathrm{e}^{-V(x-1)} + \mathrm{e}^{-V(x)}}.$$

Cette définition invite à considérer $V(x)$ comme l'*énergie* du système lorsque la marche emprunte l'arête $\{x, x+1\}$. L'équation ci-dessus équivaut à $V(x)-V(x-1) = \log\frac{1-\omega_x}{\omega_x}$, ce qui permet effectivement une définition par récurrence : pour $x \geq 0$,

$$V(x) = \sum_{k=1}^{x} \log\frac{1-\omega_k}{\omega_k},$$

et une expression similaire vaut si $x \leq 0$. On constate que le potentiel V est lui-même une marche aléatoire. De plus, $V(x) > V(x-1)$ équivaut à $\omega_x > \frac{1}{2}$: la marche a tendance à se déplacer vers les potentiels décroissants. Il apparaît donc que les zones où le potentiel forme une « vallée » (ou un puits) retiennent la marche d'autant plus longtemps qu'elles sont profondes. Voir l'illustration figure 1.4. Cette intuition, associée à des formules explicites pour les temps de sortie d'une vallée, pour les probabilités de sortie d'un côté ou de l'autre, etc., joue un rôle crucial dans l'étude des MAMA en dimension 1.

Notons que la définition de V repose sur le caractère réversible de la loi $P_{x,\omega}$, qui est mis en défaut en dimension supérieure. Il est néanmoins alors possible de partir d'un potentiel aléatoire pour définir un environnement ; le modèle devient une marche aléatoire dans un environnement de conductances aléatoires, dont l'analyse repose sur des techniques très différentes de celles des MAMA.

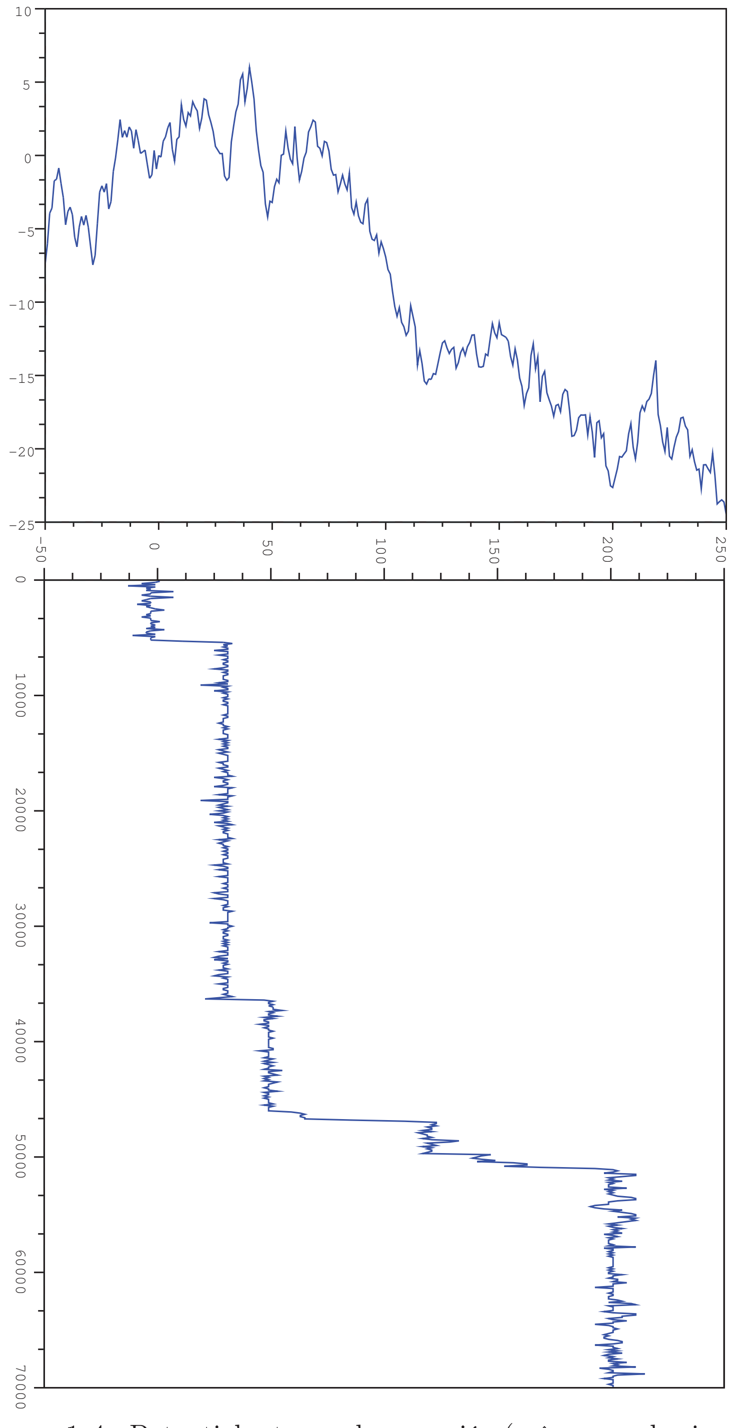

FIGURE 1.4. Potentiel, et marche associée (même axe horizontal $\mathbb{Z}$)

1.1.3. Applications. — Une motivation théorique pour les physiciens, en étudiant les MAMA, était de proposer des mécanismes statistiques pouvant justifier l'apparition de comportements sous-diffusifs (aussi appelés diffusions anormales du fait de leur lenteur) observés dans diverses situations (cf. [**8**]). Sur un plan plus appliqué, l'étude de changements de phase pour les polymères aléatoires offre des exemples marquants d'utilisation de MAMA. On en expose ici deux versions.

Hétéropolymères aléatoires. — Un polymère est une macromolécule formée par enchaînement d'un grand nombre de motifs (dits monomères). Selon la conformation de la chaîne, on distingue plusieurs phases : une phase vitreuse, où elle présente une forme régulière (typiquement une hélice) et une phase amorphe, où elle prend l'aspect d'une pelote aléatoire. À grande échelle, cela correspond respectivement à un état solide et à un état souple, caoutchouteux. Le passage du premier état au second est appelé transition vitreuse, et s'effectue à une température T_v. Au voisinage de T_v, deux phénomènes se concurrencent : la force d'attraction des liaisons entre éléments de la chaîne, qui maintient la forme d'hélice, et l'agitation thermique qui tend à les rompre.

Suivant De Gennes [**11**], supposons que la transition s'effectue uniquement par une extrêmité de la chaîne, de sorte que l'état du système est uniquement paramétré par la position X de la limite entre les phases hélice et pelote d'une certaine molécule (cf. figure 1.5) : $X = 0$ est la phase *hélice*, $X = N$ (nombre de monomères) la phase *pelote*. On peut également considérer que la différence d'énergie représentée par le passage d'un monomère d'un état à l'autre ne dépend que de la nature de ce monomère et non de celle de ses voisins ou de sa position.

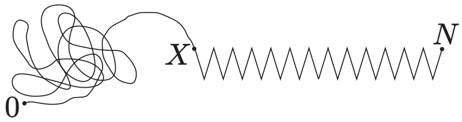

FIGURE 1.5. Schéma de la transition hélice-pelote d'un polymère

Dans le cas d'un homopolymère (un seul type de maillon), le potentiel associé est linéaire, avec une pente positive ou négative selon que $T < T_v$ (tendance à l'état vitreux) ou $T > T_v$. La position X suit alors une marche aléatoire simple biaisée (ou symétrique si $T = T_v$).

En revanche, dans le cas des hétéropolymères aléatoires, présentant par exemple deux types de monomères A et B de températures de transition vitreuse $T_v^A < T_v^B$, dans certaines proportions moyennes, le potentiel devient une marche aléatoire dont les pas sont donnés par la nature des maillons, de sorte que la position X est une MAMA. La situation devient particulièrement intéressante dans le cas $T_v^A < T < T_v^B$ car alors les accroissements du potentiel sont négatifs pour un maillon A et positifs pour B, permettant la formation de pièges conduisant à des ralentissements de la transition de phase que l'on observe effectivement.

Dézippage de l'ADN. — Une double-chaîne d'ADN est un exemple d'hétéropolymère que l'on peut considérer aléatoire, du point de vue de ses propriétés physiques, les monomères étant donnés par les quatre types de codons A, T, C et G (et leurs contreparties) ; on ne s'intéresse pas ici à sa déformation à haute température mais à la séparation des deux brins l'un de l'autre (ou « dézippage »). Remarquablement, quelque trente ans après les travaux théoriques [**10**] sur un modèle-jouet de duplication de l'ADN qui avaient initialement introduit le modèle de MAMA, les progrès des techniques de micromanipulation l'ont fait ressurgir dans un cadre expérimental.

Quelques équipes de physiciens sont parvenues, à la fin des années 1990, à séparer mécaniquement les deux brins d'une chaîne d'ADN. Un brin étant fixé à un support, l'autre peut être soumis à une force constante (voir figure 1.6). On retrouve alors un schéma similaire à celui évoqué précédemment, où la variation de la force de séparation joue le même rôle que la variation de la température : lorsque cette force est voisine de la force d'attraction entre les brins, la position de l'ouverture de la chaîne (qu'il est possible de mesurer) se déplace selon une MAMA. On renvoie à [**29**] pour plus de détails sur la modélisation, qui a été comparée avec succès aux résultats expérimentaux.

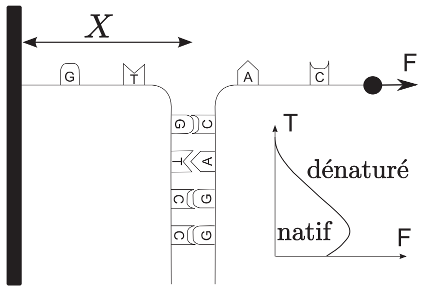

FIGURE 1.6. Schéma du dézippage d'une molécule d'ADN, d'après [**29**]. Le graphe représente la frontière entre les états natif (brins associés) et dénaturé (brins séparés) sous l'action de la température et de la force exercée.

L'étude du problème inverse (partant de l'observation du déplacement de l'ouverture, en déduire la séquence des bases) fournit également une méthode de séquençage de l'ADN à base de techniques de MAMA, voir [**4**].

1.2. Contexte : survol de quelques résultats

Comme évoqué plus haut, le cas unidimensionnel présente de nombreuses particularités, notamment la réversibilité des lois *quenched* et de nombreuses expressions explicites en fonction de l'environnement. Il en résulte une forte disparité entre la compréhension des MAMA en dimension 1 et en dimension supérieure.

1.2.1. En dimension 1. — Notons $\rho := \frac{1-\omega_0}{\omega_0}$. La caractérisation des régimes transient et récurrent et une loi des grands nombres sont donnés par le théorème suivant.

***Théorème* (Solomon [43]).** —
- *si* $E[\log\rho] < 0$, *alors* $\lim_n X_n = +\infty$, $\mathbb{P}_0$*-p.s. ;*
- *si* $E[\log\rho] > 0$, *alors* $\lim_n X_n = -\infty$, $\mathbb{P}_0$*-p.s. ;*
- *si* $E[\log\rho] = 0$, *alors* $\liminf_n X_n = -\infty$ *et* $\limsup_n X_n = +\infty$, $\mathbb{P}_0$*-p.s..*

De plus, $\mathbb{P}_0$*-p.s., la suite* $\left(\frac{X_n}{n}\right)_{n\geq 1}$ *admet une limite* v *déterministe, donnée par :*

$$v = \begin{cases} \frac{1-E[\rho]}{1+E[\rho]} > 0 & si\ E[\rho] < 1, \\ -\frac{1-E[\rho^{-1}]}{1+E[\rho^{-1}]} < 0 & si\ E[\rho^{-1}] < 1, \\ 0 & sinon. \end{cases}$$

Notons qu'un résultat $\mathbb{P}_0$-presque sûr équivaut à un résultat $P_{0,\omega}$-presque sûr pour P-presque tout environnement ω.

Une conséquence remarquable de ce théorème est l'existence de lois μ (loi de ω_0) pour lesquelles la marche aléatoire est transiente à vitesse nulle. Cette manifestation d'un ralentissement n'a pas lieu pour les marches aléatoires simples.

On dit qu'une loi sur $\mathbb{R}$ est non-arithmétique si son support engendre un sous-groupe dense de $\mathbb{R}$. Dans le régime transient, le résultat de Solomon est précisé par des théorèmes limites :

***Théorème* (Kesten-Kozlov-Spitzer [27]).** — *On suppose*

(i) *il existe* $\kappa > 0$ *tel que* $E[\rho^\kappa] = 1$ *et* $E[\rho^\kappa(\log\rho)_+] < \infty$ *;*

(ii) *la distribution de* $\log\rho$ *est non-arithmétique.*

Alors, en notant $\tau(n) := \inf\{t \geq 0 | X_t \geq n\}$ *le temps d'atteinte de* n*,*

- *si* $0 < \kappa < 1$*,*
$$\frac{\tau(n)}{n^{1/\kappa}} \xrightarrow[n]{\text{(loi)}} S_\kappa \qquad et \qquad \frac{X_t}{t^\kappa} \xrightarrow[t]{\text{(loi)}} \frac{1}{(S_\kappa)^{1/\kappa}},$$
- *si* $\kappa = 1$*,*
$$\frac{\tau(n) - A_1 u_n n\log n}{n} \xrightarrow[n]{\text{(loi)}} S_1 \qquad et \qquad \frac{X_t - (A_1)^{-1} v_t \frac{t}{\log t}}{\frac{t}{(\log t)^2}} \xrightarrow[t]{\text{(loi)}} -\frac{1}{(A_1)^2} S_1,$$
- *si* $1 < \kappa < 2$*,*
$$\frac{\tau(n) - v^{-1}n}{n^{1/\kappa}} \xrightarrow[n]{\text{(loi)}} S_\kappa \qquad et \qquad \frac{X_t - vt}{t^{1/\kappa}} \xrightarrow[t]{\text{(loi)}} -\frac{1}{v^{1+1/\kappa}} S_\kappa,$$
- *si* $\kappa = 2$*,*
$$\frac{\tau(n) - v^{-1}n}{B_1\sqrt{n\log n}} \xrightarrow[n]{\text{(loi)}} \mathcal{N} \qquad et \qquad \frac{X_t - vt}{B_1 v^{3/2}\sqrt{t\log t}} \xrightarrow[t]{\text{(loi)}} \mathcal{N},$$
- *si* $\kappa > 2$*,*
$$\frac{\tau(n) - v^{-1}n}{B_2\sqrt{n}} \xrightarrow[n]{\text{(loi)}} \mathcal{N} \qquad et \qquad \frac{X_t - vt}{B_2 v^{3/2}\sqrt{t}} \xrightarrow[t]{\text{(loi)}} \mathcal{N},$$

où $A_\kappa > 0$, $B_1, B_2 > 0$, $u_n \to 1$, $v_t \to 1$, S_κ *est une loi stable totalement asymétrique d'indice* κ *et* $\mathcal{N}$ *est la loi gaussienne standard.*

Le régime balistique (vitesse positive) se décompose donc en un régime diffusif et un régime sous-diffusif.

Dans le cas récurrent, le ralentissement est particulièrement marqué :

***Théorème* (Sinai [42]).** — *On suppose*

(i) $E[\log\rho]=0$ *et* $0<\sigma^2:=E[(\log\rho)^2]<\infty$*;*

(ii) *il existe* $\eta>0$ *tel que* $\eta<\omega_0<1-\eta$*,* P*-p.s..*

Alors il existe une fonction $b_n=b_n(\omega)$ *de l'environnement telle que, pour tout* $\varepsilon>0$*,*

$$\mathbb{P}_0\left(\left|\frac{\sigma^2 X_n}{(\log n)^2}-b_n\right|>\varepsilon\right)\underset{n}{\longrightarrow}0.$$

De plus, la suite $(b_n)_n$ *converge en loi (et donc* $\left(\frac{\sigma^2 X_n}{(\log n)^2}\right)_n$ *aussi vers la même loi).*

Le contenu de cet énoncé est double : d'une part, X_n est de l'ordre de $(\log n)^2$ (à comparer à $\sqrt{n}$ pour une marche simple), et d'autre part la position X_n est presque déterminée par le seul environnement (on parle de localisation). Cette position correspond au fond de la plus profonde vallée du potentiel que la marche a pu atteindre au temps n ; c'est d'ailleurs la preuve de ce résultat qui a initié les techniques à base de potentiel.

La preuve du théorème de Kesten, Kozlov et Spitzer repose sur l'analyse d'un processus de branchement avec immigration en milieu aléatoire lié à la marche ; ce point de vue efficace donne assez peu d'intuition quant à l'origine de la limite stable. Récemment, Enriquez, Sabot et Zindy (cf. [**17**]) ont donné une nouvelle preuve de ce théorème dans le cas où $0<\kappa<1$ par une tout autre méthode, basée sur l'étude fine du potentiel : la définition de vallées analogues à celles de Sinai permet de cerner les portions de l'environnement où la marche passe l'essentiel de son temps et ramène à considérer le temps de sortie d'une seule vallée, qui fait intervenir une variable proche d'une série de renouvellement étudiée par Kesten. Associée à l'article [**16**], qui établit le lien avec le théorème de renouvellement de Kesten, cette approche permet d'obtenir une expression explicite de la loi limite (paramètres de la loi stable) et améliore la compréhension du comportement du modèle, ce qui permet par exemple de démontrer des résultats de localisation et de vieillissement (cf. [**18**]). La localisation mise en évidence par Sinai dépendait de façon déterministe de l'environnement ; dans le cas transient à vitesse nulle, elle a lieu dans une vallée aléatoire. La profondeur de cette vallée dépend du temps, ce qui occasionne un phénomène de vieillissement : la durée des corrélations s'allonge avec l'age du processus. Ici, au temps t, la marche aléatoire passe ainsi un temps de l'ordre de t concentrée dans un petit intervalle (le fond d'une vallée) :

***Théorème* (Enriquez-Sabot-Zindy [18]).** — *Sous les hypothèses (i) et (ii) du théorème de Kesten-Kozlov-Spitzer, avec* $0<\kappa<1$*, on a, pour tout* $h>1$ *et tout* $\eta>0$*,*

$$\lim_t \mathbb{P}_0\big(|X_{th}-X_t|\le\eta\log t\big)=\frac{\sin(\kappa\pi)}{\pi}\int_0^{1/h}\frac{\mathrm{d}y}{y^{1-\kappa}(1-y)^{\kappa}}.$$

La localisation prouvée pour $0 < \kappa < 1$ fait obstacle à un éventuel résultat *quenched* (à environnement fixé) similaire au théorème de Kesten-Kozlov-Spitzer. Ceci a été également précisé par Peterson et Zeitouni **[37]** dans ce cas, et par Peterson **[36]** quand $1 < \kappa < 2$: contrairement au théorème central limite, qui vaut aussi sous $P_{0,\omega}$ pour presque tout environnement ω, comme il a été montré indépendamment (sous certaines hypothèses) par Goldsheid **[22]** et Peterson **[35]**, la limite stable non gaussienne observée sous $\mathbb{P}_0$ vient ainsi des fluctuations d'un environnement à l'autre, tandis qu'à environnement fixé il n'existe presque sûrement pas de loi limite (on peut trouver des sous-suites qui fournissent des limites différentes).

1.2.2. En dimension supérieure. — Dès que $d \geq 2$, on ne dispose plus de caractérisation des comportements transient et récurrent. L'essentiel des résultats porte sur deux domaines « antipodaux » : les environnements symétriques (en divers sens) ou presque symétriques, et la balisticité (vitesse non nulle). On se restreint ici à évoquer ce second point.

L'hypothèse suivante sera souvent requise : la loi de l'environnement est dite uniformément elliptique s'il existe $\eta > 0$ tel que, pour tout $e \in \mathcal{V}$,

$$P-\text{p.s.},\qquad \omega(0,e) > \eta. \tag{1.2.1}$$

Elle est elliptique si ceci vaut pour $\eta = 0$.

Soit $\ell \in \mathbb{R}^d \setminus \{0\}$. La marche aléatoire $X = (X_n)_n$ est dite transiente dans la direction ℓ si $X_n \cdot \ell \to_n +\infty$. La simple existence d'une loi du 0-1 pour la transience dans une direction est une question ouverte depuis l'article de Kalikow **[24]**. Définissons l'événement

$$A_\ell := \{X_n \cdot \ell \to_n +\infty\}.$$

Kalikow a démontré, dans le cas uniformément elliptique, $\mathbb{P}_0(A_\ell \cup A_{-\ell}) \in \{0,1\}$, ce qui a été étendu au cas elliptique par Zerner et Merkl **[55]**. En dimension 2, Zerner et Merkl ont de plus prouvé, sous la seule hypothèse d'ellipticité, $\mathbb{P}_0(A_\ell) \in \{0,1\}$. La validité de ce résultat en dimension ≥ 3 reste ouverte. On peut noter que des contre-exemples ont été trouvés en affaiblissant légèrement l'hypothèse d'indépendance de l'environnement, voir **[55]**, **[3]** et **[54]**.

Le principal critère explicite de balisticité a été apporté par Kalikow **[24]** : (sous une forme affaiblie ; voir théorème 2.3 pour l'énoncé général)

***Théorème* (Kalikow [24]).** — *On suppose l'environnement elliptique avec constante η (cf. (1.2.1)), et qu'il existe une direction $\ell \in \mathbb{R}^d \setminus \{0\}$ telle que*

$$E[(d_\omega(0)\cdot\ell)_+] > \frac{1}{\eta} E[(d_\omega(0)\cdot\ell)_-],$$

où $d_\omega(0) := \sum_{e\in\mathcal{V}} \omega(0,e)e = E_{0,\omega}[X_1]$ est la dérive de l'environnement moyen. Alors, $\mathbb{P}_0$-p.s.,

$$X_n \cdot \ell \to_n +\infty.$$

Sznitman et Zerner [**49**] ont montré que la même condition implique la balisticité de la marche aléatoire. Leur preuve repose sur une structure de renouvellement obtenue en découpant la trajectoire en tronçons contenus dans des tranches disjointes perpendiculaires à la direction ℓ. En notant τ_1 le premier temps de renouvellement, une majoration de $\mathbb{E}_0[\tau_1]$ permet d'appliquer la loi des grands nombres usuelle. Un raffinement de ces techniques a permis à Sznitman [**45**] d'obtenir des bornes sur la queue de la loi de τ_1 impliquant l'existence de tous ses moments et d'en déduire un théorème central limite, toujours sous la condition de Kalikow.

Dans une série d'articles (voir [**48**]), Sznitman a également introduit des conditions plus générales (T) et (T') garantissant la balisticité sous l'hypothèse d'uniforme ellipticité. Il est conjecturé (voir [**48**] p.227) que, dans le cas uniformément elliptique, la transience directionnelle implique la balisticité : il n'y aurait pas de régime transient à vitesse nulle, à la différence de la dimension 1. Intuitivement, en dimension ≥ 2, la marche peut contourner les pièges, et l'uniforme ellipticité lui donne la possibilité de le faire à moindre coût (et limite la « force » des pièges).

Notons que l'existence de pièges (au sens de la figure 1.2) est conditionnée au fait que l'enveloppe convexe du support de la loi de la dérive contienne l'origine. Si cette condition (dite *nestling*) n'est pas réalisée, la balisticité s'obtient facilement. L'intérêt de la condition de Kalikow vient de ce qu'elle permet également de traiter des situations *nestling*.

Comme on l'a déjà évoqué, les MAMA constituent des processus renforcés : les transitions déjà empruntées deviennent plus probables. On peut aussi inverser le point de vue, et partir d'une « loi de renforcement » donnant l'évolution des probabilités de transition en fonction des choix antérieurs ; il est alors possible de déterminer lesquelles de ces lois correspondent effectivement à des MAMA, cf.[**14**]. Un cas très naturel est le renforcement linéaire. Munissons les arêtes orientées $(x, x+e)$ ($x \in \mathbb{Z}^d$, $e \in \mathcal{V}$) du graphe de poids initiaux $\alpha(x,e) = \alpha_e > 0$ (les poids *initiaux* dépendent uniquement de la direction). La marche linéairement renforcée par arêtes orientées associée à ces poids, issue de 0, est alors définie par $X_0 = 0$ et, pour tout $n \in \mathbb{N}$, pour tout $e \in \mathcal{V}$,

$$P(X_{n+1} = X_n + e | X_0, \ldots, X_n) = \frac{\alpha_e + N_n(X_n, e)}{\sum_{f \in \mathcal{V}} \big(\alpha_f + N_n(X_n, f)\big)}, \tag{1.2.2}$$

où $N_n(x,e)$ est le nombre de transitions de la marche de x vers $x+e$ avant l'instant n. Comme cela peut se voir à l'aide de propriétés des urnes de Polya, la loi de cette marche coïncide avec celle d'une marche aléatoire dans un milieu suivant une loi de Dirichlet. Cette propriété confère à ces environnements une place particulière. Certains calculs explicites se trouvent de plus être possibles dans ce cas. Ainsi, en dimension 1, les constantes des lois limites s'expriment simplement. Et en dimension quelconque, une formule d'intégration par parties a permis à Enriquez et Sabot [**15**] d'obtenir un critère explicite de balisticité pour ces environnements, basé sur le critère de Kalikow :

***Théorème* (Enriquez-Sabot [15]).** — *On considère une marche en environnement de Dirichlet sur $\mathbb{Z}^d$, de paramètres $(\alpha_e)_{e\in\mathcal{V}}$. Supposons qu'une direction $e \in \mathcal{V}$ vérifie $\alpha_e > \alpha_{-e} + 1$. Alors il existe $v \in \mathbb{R} \setminus \{0\}$ tel que $v \cdot e > 0$ et, $\mathbb{P}_0$-p.s.,*

$$\frac{X_n}{n} \underset{n}{\longrightarrow} v.$$

À l'aide d'une propriété de stabilité de la loi de Dirichlet par inversion temporelle, Sabot [**40**] a également pu montrer la transience (non directionnelle) des marches aléatoires en milieu de Dirichlet en dimension ≥ 3 quels que soient les paramètres, donc notamment dans le cas symétrique.

1.3. Résultats de la thèse, organisation du mémoire

Ce mémoire expose le contenu de trois articles rédigés durant la thèse :

(i) *Integrability of exit times and ballisticity for random walks in Dirichlet environment* (publié dans l'Electronic Journal of Probability (Vol. 14, n° 16, pp. 431–451) ;

(ii) *Reversed Dirichlet environment and directional transience of random walks in Dirichlet environment*, avec C. Sabot (accepté pour publication dans les Annales de l'Institut Poincaré) ;

(iii) *Stable fluctuations for ballistic random walks in random environment on $\mathbb{Z}$*, avec N. Enriquez, C. Sabot et O. Zindy (prépublication).

Une note fournissant une preuve courte d'un théorème de Merkl et Rolles sur les marches renforcées par arêtes est également jointe en appendice B.

Aperçu du contenu des articles. —

Article (i). — On a mentionné plus haut la condition d'uniforme ellipticité (1.2.1), souvent requise dans les preuves en dimension ≥ 2. Les environnements de Dirichlet, qui ont des queues polynomiales au bord de $\mathcal{P}$, ne satisfont pas à cette propriété. En résulte un phénomène singulier : si ces queues sont assez lourdes, c'est-à-dire si les paramètres de la distribution sont assez petits, certaines parties de l'environnement deviennent des pièges (*annealed*) « forts », au sens où le temps de sortie de ces parties n'est pas intégrable. Plus généralement, les temps de sortie d'une partie bornée n'ont pas tous leurs moments finis sous $\mathbb{P}_0$. L'article (i) calcule l'exposant d'intégrabilité critique pour les temps de sortie de graphes finis par une marche aléatoire en environnement de Dirichlet.

La preuve fonctionne par récurrence, en mettant en oeuvre, à environnement fixé, une technique de quotientage du graphe initial par un sous-graphe sur lequel une uniforme ellipticité a lieu (pour une raison combinatoire) et qui se comporte donc, pour ce qui est de l'intégrabilité du temps de sortie, comme un unique sommet.

Le résultat vaut dans un cadre très général, et s'applique notamment aux sous-graphes de $\mathbb{Z}^d$, où il montre que les pièges forts minimaux sont constitués d'une unique arête. En raffinant les techniques d'Enriquez et Sabot, et à l'aide de ce critère d'intégrabilité, on obtient une version améliorée de leur critère de balisticité.

Article (ii). — Parmi les propriétés remarquables des environnements de Dirichlet, une stabilité par renversement du temps a été observée par Sabot dans [**40**], à l'aide d'un délicat changement de variable. On propose ici une preuve probabiliste de cette même propriété, inspirée par le lien avec les marches renforcées.

De plus on prouve que, dès que les poids initiaux ne sont pas symétriques, la marche aléatoire en milieu de Dirichlet est transiente dans une direction avec probabilité positive. Ceci fournit les premiers exemples non-dégénérés de MAMA transientes à vitesse nulle en dimension ≥ 2.

Article (iii). — Dans cet article consacré au cas unidimensionnel, on étend le travail d'Enriquez, Sabot et Zindy [**17**] en proposant une nouvelle preuve « à la Sinai » du théorème de Kesten-Kozlov-Spitzer dans le cas $1 \leq \kappa < 2$, avec expression explicite des paramètres de la limite. La preuve contient en fait aussi le cas $0 < \kappa < 1$.

Pour $1 < \kappa < 2$, la MAMA est balistique, et le théorème énonce une limite stable pour les fluctuations de la marche par rapport à la trajectoire moyenne (linéaire). On montre que les fluctuations du temps d'atteinte $\tau(n)$ sont essentiellement liées au temps passé dans un petit nombre (presque indépendant de n) de profondes vallées du potentiel ; celles-ci étant peu nombreuses, elles sont d'autant plus distantes entre elles que n est grand, et sont donc presque indépendantes les unes des autres.

La preuve procède en un découpage du temps d'atteinte de n entre le temps passé dans les « petites » vallées (dont on montre que les fluctuations sont négligeables) et dans les « grandes ». L'indépendance entre ces dernières est assurée par un découpage supplémentaire, sur un événement de grande probabilité. Remplacer alors les petites vallées par de nouvelles petites vallées indépendantes et de même loi permet de se ramener à un cadre i.i.d..

Au terme de cette chirurgie, on peut appliquer un théorème limite général dès lors que le temps passé dans une grande vallée appartient au domaine d'attraction d'une loi stable, ce qui est l'objet d'une dernière partie.

Organisation du mémoire. — Les articles mentionnés ci-dessus, rédigés en anglais, constituent les chapitres 4, 5 et 6. Ils sont précédés de deux introductions en français (chapitres 2 et 3) donnant des compléments sur le contexte des articles (de façon à permettre leur compréhension en se référant au minimum aux éléments de bibliographie) et quelques explications ou remarques complémentaires.

CHAPITRE 2

MARCHES ALÉATOIRES EN MILIEU DE DIRICHLET

Les chapitres 4 et 5 portent sur les marches aléatoires en milieu de Dirichlet. On rappelle ici brièvement leur lien avec les marches renforcées, et les éléments de la preuve de balisticité de [**15**] qui seront utilisés en 4.4.

NB. Dans cette introduction et les chapitres associés, on utilise les notations en usage dans les articles antérieurs sur les milieux de Dirichlet : on note $\omega(x,y)$ la probabilité de transition entre les sommets x et y dans l'environnement ω ; de plus, la loi de l'environnement est notée $\mathbb{P}$ et la loi *annealed* (loi de la MAMA) issue de x est $P_x := \mathbb{P}(\,\mathrm{d}\omega)P_{x,\omega}(\,\mathrm{d}X)$.

2.1. Loi de Dirichlet et renforcement

2.1.1. Définition et propriétés. — Soit I un ensemble fini. On note $\mathrm{Prob}(I)$ le simplexe des vecteurs de probabilité sur I. Pour $\alpha = (\alpha_i)_{i\in I} \in (0,\infty)^I$, la loi de Dirichlet de paramètre α est la loi $\mathcal{D}(\alpha)$ sur $\mathrm{Prob}(I)$ de densité

$$(x_i)_{i\in I} \mapsto \frac{\Gamma(\sum_{i\in I}\alpha_i)}{\prod_{i\in I}\Gamma(\alpha_i)}\prod_{i\in I} x_i^{\alpha_i-1}$$

par rapport à la mesure de Lebesgue $\prod_{i\neq i_0}\,\mathrm{d}x_i$ (où i_0 est un élément quelconque de I). Notons que le cas où I n'a que deux éléments correspond à la loi Beta.

Cette loi s'obtient notamment en normalisant un vecteur de variables de loi Gamma : si $X_1,\ldots,X_n$ sont indépendantes avec, pour $i = 1,\ldots,n$, X_i de loi $\Gamma(\alpha_i,1)$ (densité $\frac{1}{\Gamma(\alpha_i)}x^{\alpha_i-1}\mathrm{e}^{-x}\mathbf{1}_{\{x>0\}}$), alors le vecteur $\left(\frac{X_i}{\sum_{j\in I}X_j}\right)_{i\in I}$ suit la loi $\mathcal{D}(\alpha)$. Il en résulte des propriétés de stabilité par ajout de composantes entre elles et par restriction, cf. 4.3.1. Citons de plus une propriété bayésienne de $\mathcal{D}(\alpha)$.

Lemme 2.1. — *Soit $\alpha \in (0,+\infty)^I$, et (X,p) un couple de variables aléatoires tel que p suit la loi $\mathcal{D}(\alpha)$ et, sachant p, X suit la loi p : pour tout $i\in I$, $P(X=i|p) = p_i$. Alors, pour tout $i \in I$, $P(X=i) = \frac{\alpha_i}{\sum_{j\in I}\alpha_j}$, et la loi de p sachant $\{X=i\}$ est $\mathcal{D}(\alpha+\mathbf{1}_{\{i\}})$, c'est-à-dire une loi de Dirichlet dont le paramètre d'indice i est augmenté de 1 par rapport à α.*

Démonstration. — Soit $i\in I$. La formule pour la loi de X vient directement de $P(X=i) = E[p_i]$ et de la définition de $\mathcal{D}(\alpha)$, avec la propriété habituelle de la fonction Γ. On a alors, pour toute fonction f mesurable bornée,

$$E[f(p)|X=i] = \frac{1}{P(X=i)} E[f(p)\mathbf{1}_{\{X=i\}}] = \frac{\sum_j \alpha_j}{\alpha_i} E\Big[f(p)p_i\Big],$$

et $\frac{\sum_j \alpha_j}{\alpha_i} p_i$ est la densité de $\mathcal{D}(\alpha + \mathbf{1}_{\{i\}})$ par rapport à $\mathcal{D}(\alpha)$, d'où le deuxième point. □

Étant donné un graphe orienté $G = (V, E)$, dont les arêtes (orientées $e = (\underline{e}, \overline{e})$) sont munies de poids positifs $\alpha = (\alpha_e)_{e\in E}$, la loi de Dirichlet sur les environnements de G est naturellement la loi produit

$$\mathbb{P}^{(\alpha)} := \prod_{x\in V} \mathcal{D}((\alpha_e)_{\underline{e}=x}).$$

2.1.2. Urne de Pólya. — Une urne contient des boules de r couleurs différentes, en nombres respectifs $\alpha_1, \ldots, \alpha_r \in \mathbb{N}$. Après chaque tirage d'une boule, on la replace dans l'urne avec une boule supplémentaire de la même couleur. Le modèle s'étend immédiatement à des paramètres $\alpha_1, \ldots, \alpha_r$ réels positifs, la probabilité d'une couleur à un tirage étant proportionnelle au « nombre de boules » (réel) de cette couleur. Notons $(X_n)_n$ la suite des couleurs ($\in \{1, \ldots, r\}$) successivement tirées. Selon un résultat classique, la suite $(X_n)_n$ suit la même loi que $(Y_n)_n$ où, conditionnellement à une variable aléatoire $p = (p_1, \ldots, p_r)$ de loi $\mathcal{D}(\alpha_1, \ldots, \alpha_r)$, la suite $(Y_n)_n$ est i.i.d. de loi donnée par p.

Ce résultat vient de l'échangeabilité partielle de la suite $(X_n)_n$ *via* le théorème de De Finetti, et s'obtient aussi par un calcul direct ou encore par récurrence avec le lemme 2.1.

Dans le cas d'environnements de Dirichlet, cette propriété prend l'interprétation suivante : la loi *annealed* P_0 d'une marche aléatoire en milieu de Dirichlet de paramètres $(\alpha_e)_{e\in E}$ coïncide avec la loi d'une marche renforcée par arêtes orientées de poids initiaux $(\alpha_e)_{e\in E}$ (définie en (1.2.2) pour $\mathbb{Z}^d$; le cas général s'en déduit). En effet, cette dernière loi équivaut à la donnée d'urnes de Pólya indépendantes, en chaque sommet de G, fournissant les choix successifs d'arête de sortie.

Ainsi, tous les énoncés des chapitres 4 et 5 relatifs à la loi *annealed* (intégrabilité des temps de sortie, critères de balisticité et de transience directionnelle) portent également sur les marches renforcées par arêtes orientées.

2.2. Balisticité

2.2.1. Critère de Kalikow. — Kalikow [**24**] a prouvé l'un des tout premiers résultats sur la transience des marches aléatoires en milieu aléatoire en dimension $d \geq 2$. Son critère est basé sur l'introduction pour toute partie connexe $U \subset \mathbb{Z}^d$ et tout $z_0 \in U$ d'une chaîne de Markov auxilliaire, à valeurs dans U, qui a la remarquable propriété d'avoir même distribution de sortie de U que la marche aléatoire en milieu aléatoire sous P_{z_0}.

Soit $U \subset \mathbb{Z}^d$ fini, connexe, et $z_0 \in U$. La fonction de Green de la marche aléatoire dans $\omega \in \Omega$ tuée à sa sortie de U est, pour $x, y \in U \cup \partial U$,

$$G_U^\omega(x,y) := E_{x,\omega}\left[\sum_{n=0}^{T_U} \mathbf{1}_{\{X_n = y\}}\right],$$

où T_U est le temps de sortie de U. Supposons T_U intégrable sous P_{z_0}, ce qui équivaut à ce que $G_U^\omega(z_0, x)$ soit intégrable sous $\mathbb{P}$ pour tout $x \in U$. Cette condition est automatiquement satisfaite si $\mathbb{P}$ est uniformément elliptique (cf. (1.2.1)) car alors $E_{z_0,\omega}[T_U]$ est uniformément bornée (par compacité). Dans le cas Dirichlet, la vérification de cette hypothèse fait l'objet du théorème 4.1.

On définit l'environnement de Kalikow $\widehat{\omega}_{U,z_0}$ sur $U \cup \partial U$ (en autorisant les transitions des sommets de ∂U vers eux-mêmes) par : pour tous $x, y \in U \cup \partial U$,

$$\begin{aligned} \widehat{\omega}_{U,z_0}(x,y) &:= \frac{\mathbb{E}[G_U^\omega(z_0,x)\omega(x,y)]}{\mathbb{E}[G_U^\omega(z_0,x)]} \quad \text{si } x \in U \\ \widehat{\omega}_{U,z_0}(x,x) &:= 1 \quad \text{si } x \in \partial U. \end{aligned} \tag{2.2.1}$$

La chaîne de Markov de loi $P_{0,\widehat{\omega}_{U,z_0}}$ est appelée marche de Kalikow.

La propriété remarquable de cette définition tient dans le lemme suivant, qui permet de se ramener à un problème markovien donc *a priori* plus simple.

***Lemme 2.2* (Kalikow [24]).** — *On a, pour tout $x \in U \cup \partial U$,*

$$\mathbb{E}[G_U^\omega(z_0,x)] = G_U^{\widehat{\omega}_{U,z_0}}(z_0,x). \tag{2.2.2}$$

En particulier, les lois de T_U et de X_{T_U} sont les mêmes sous P_{z_0} et $P_{z_0,\widehat{\omega}_U}$.

Pour $x \in U$, on note $\widehat{d}_{U,z_0}(x) := d_{\widehat{\omega}_{U,z_0}}(x)$ la dérive de $\widehat{\omega}_{U,z_0}$ en x. On peut alors énoncer le principal résultat.

***Théorème 2.3* (Kalikow [24], Sznitman-Zerner [49])**
On suppose l'« hypothèse de Kalikow » satisfaite : Il existe $\varepsilon > 0$ et $\ell \in \mathbb{R}^d$ tels que, pour tout $U \subset \mathbb{Z}^d$ fini et tous $z_0, x \in U$,

$$\widehat{d}_{U,z_0}(x) \cdot \ell \geq \varepsilon.$$

Alors il existe $v \in \mathbb{R}^d$ tel que $v \cdot \ell > 0$ et, P_0-p.s., $\dfrac{X_n}{n} \underset{n}{\longrightarrow} v.$

Kalikow a démontré la transience directionnelle, et Sznitman et Zerner la balisticité. Leurs articles supposent l'uniforme ellipticité de $\mathbb{P}$, mais celle-ci peut être remplacée par la seule l'intégrabilité du temps de sortie de U sous P_0, la loi du 0-1 de Kalikow ayant été étendue au cas elliptique par Zerner et Merkl [**55**].

La preuve classique de la transience directionnelle sous la condition de Kalikow exploite le lemme précédent pour se ramener à une probabilité de sortie de bande par la chaîne de Markov auxiliaire. Donnons-en plutôt une preuve courte, due à Rassoul-Agha [**38**], qui n'utilise que la définition de la condition de Kalikow.

Preuve de la transience directionnelle. — Soit $\lambda > 0$. Soit U une partie finie de $\mathbb{Z}^d$ contenant 0. Avec la définition de $G_U^\omega(0,x)$, la condition de Kalikow se réécrit, pour $x \in U$ (avec $z_0 = 0$),

$$E_0\left[\sum_{n=0}^{T_U-1} \mathbf{1}_{\{X_n=x\}} d_\omega(X_n)\cdot\ell\right] \geq \varepsilon E_0\left[\sum_{n=0}^{T_U-1} \mathbf{1}_{\{X_n=x\}}\right].$$

En multipliant les deux membres par $\mathrm{e}^{-\lambda x\cdot\ell}$ et en sommant sur $x \in U$, on obtient

$$E_0\left[\sum_{n=0}^{T_U-1} \mathrm{e}^{-\lambda X_n\cdot\ell} d_\omega(X_n)\cdot\ell\right] \geq \varepsilon E_0\left[\sum_{n=0}^{T_U-1} \mathrm{e}^{-\lambda X_n\cdot\ell}\right]. \tag{2.2.3}$$

Par ailleurs, comme T_U est un temps d'arrêt pour la filtration $(\mathcal{F}_n)_n$ associée à X,

$$E_0\left[\sum_{n=1}^{T_U} \mathrm{e}^{-\lambda X_n\cdot\ell}\right] = \sum_{n\geq 1} E_0\big[\mathbf{1}_{\{T_U\geq n\}} E_{0,\omega}[\mathrm{e}^{-\lambda X_n\cdot\ell}|\mathcal{F}_{n-1}]\big],$$

or on a $\mathrm{e}^{-\lambda X_n\cdot\ell} = \mathrm{e}^{-\lambda X_{n-1}\cdot\ell}(1+\lambda(X_n - X_{n-1})\cdot\ell + O(\lambda^2))$ quand λ tend vers 0, où le terme $O(\lambda^2)$ est uniforme par rapport à X (car $\|X_n - X_{n-1}\| \leq 1$), de sorte que

$$E_{0,\omega}[\mathrm{e}^{-\lambda X_n\cdot\ell}|\mathcal{F}_{n-1}] = \mathrm{e}^{-\lambda X_{n-1}\cdot\ell}\big(1 - \lambda d_\omega(X_{n-1})\cdot\ell + O(\lambda^2)\big).$$

En utilisant (2.2.3), on en déduit

$$E_0\left[\sum_{n=1}^{T_U} \mathrm{e}^{-\lambda X_n\cdot\ell}\right] \leq E_0\left[\sum_{n=0}^{T_U-1} \mathrm{e}^{-\lambda X_n\cdot\ell}\right]\big(1 - \lambda\varepsilon + O(\lambda^2)\big).$$

En choisissant λ assez petit pour que le facteur entre parenthèses soit inférieur à 1, cette inégalité fournit une majoration du terme de gauche indépendante de U, d'où

$$E_0\left[\sum_{n\geq 0} \mathrm{e}^{-\lambda X_n\cdot\ell}\right] < \infty.$$

En particulier, presque sûrement la série converge donc son terme général tend vers 0, ce qui conclut. On constate aussi que le nombre de visites en 0 est intégrable. $\square$

La preuve de Sznitman et Zerner de la balisticité repose sur une structure de renouvellement : un découpage de la trajectoire en tronçons indépendants et suivant la même loi, qui permet ensuite de faire appel à la loi des grands nombres si une certaine condition d'intégrabilité est satisfaite. Donnons les grandes lignes de cette construction.

On considère tout d'abord les temps d'arrêt suivants :

$$T_u^\ell := \inf\{n \geq 0 \,|\, X_n\cdot\ell \geq u\} \text{ pour } \ell \in \mathbb{S}^{d-1} \text{ et } u > 0,$$

et

$$D := \inf\{n \geq 0 \,|\, X_n\cdot\ell < X_0\cdot\ell\}.$$

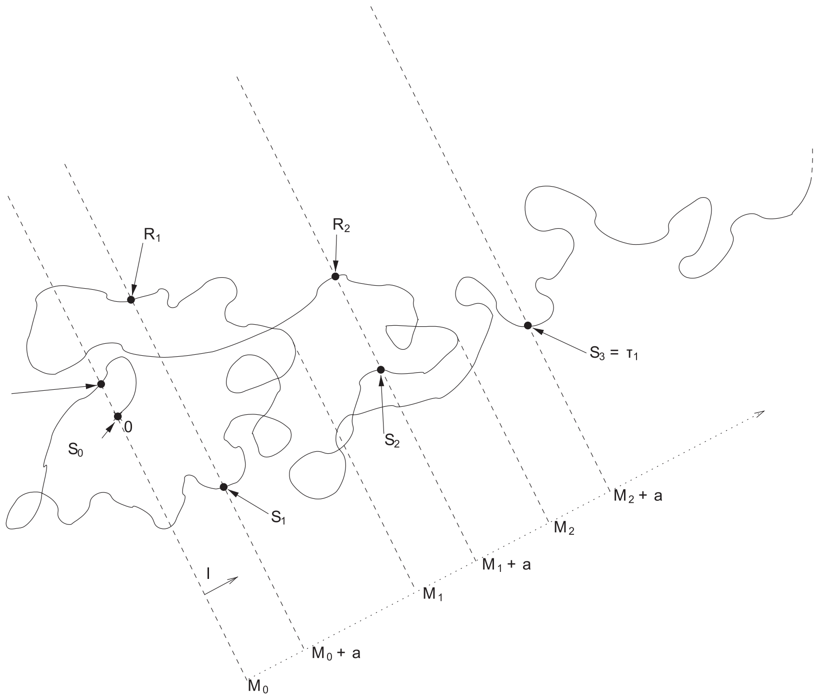

FIGURE 2.1. Définition de la structure de renouvellement

Soit $\ell \in \mathbb{S}^{d-1}$ et $a > 0$. On définit les suites $(S_k)_{k\geq 0}$ et $(R_k)_{k\geq 1}$ de temps d'arrêt (pour la filtration $(\mathcal{F}_n)_n$ associée à $(X_n)_n$) et la suite $(M_k)_{k\geq 0}$ des maxima successifs par (voir figure 2.1)

$$\begin{aligned}
&S_0 := 0,\\
&M_0 := \ell \cdot X_0,\\
\\
&S_1 := T^{\ell}_{M_0+a} = \inf\{n \geq 0 \,|\, X_n \cdot \ell \geq M_0 + a\},\\
&R_1 := \inf\{n \geq S_1 \,|\, X_n \cdot \ell < X_{S_1} \cdot \ell\},\\
&M_1 := \sup\{X_n \cdot \ell \,|\, 0 \leq n \leq R_1\}
\end{aligned}$$

et, pour $k \geq 1$, par récurrence,

$$\begin{aligned}
&S_{k+1} := T^{\ell}_{M_k+a} = \inf\{n \geq 0 \,|\, X_n \cdot \ell \geq M_k + a\},\\
&R_{k+1} := \inf\left\{n \geq S_{k+1} \,\middle|\, X_n \cdot \ell < X_{S_{k+1}} \cdot \ell\right\},\\
&M_{k+1} := \sup\{X_n \cdot \ell \,|\, 0 \leq n \leq R_{k+1}\}.
\end{aligned}$$

Dans cette construction, toutes les variables peuvent *a priori* être infinies, et on a bien sûr

$$0 = S_0 \leq S_1 \leq R_1 \leq S_2 \leq R_2 \leq \cdots ,$$

avec des inégalités strictes tant que les variables sont finies.

On introduit le temps de renouvellement relatif à $\ell \in \mathbb{S}^{d-1}$ et $a > 0$:

$$\tau_1 := S_K, \text{ où } K := \inf\{k \geq 1 \,|\, S_k < \infty, R_k = \infty\}.$$

On peut montrer que, si $X_n \cdot \ell \to_n +\infty$ p.s., alors $\tau_1 < \infty$ p.s. Cet instant est tel que, pour tout $n < \tau_1$, $X_n \cdot \ell < X_{\tau_1} \cdot \ell$ et, pour tout $n \geq \tau_1$, $X_n \cdot \ell \geq X_{\tau_1} \cdot \ell$. Ainsi les ensembles de sites visités avant et après τ_1 sont disjoints et donc associés à des probabilités de transitions indépendantes.

L'étape suivante consiste à itérer cette construction après τ_1. Considérant τ_1 comme une fonction de $X.$, on définit, sur $\{\tau_1 < \infty\}$, (voir figure 2.2)

$$\tau_2 := \tau_1(X.) + \tau_1(X_{\tau_1+\cdot} - X_{\tau_1})$$

et, par récurrence, pour tout $k \geq 1$,

$$\tau_{k+1} := \tau_k(X.) + \tau_1(X_{\tau_k+\cdot} - X_{\tau_k}),$$

avec $\tau_{k+1} = \infty$ sur $\{\tau_k = \infty\}$. Si $X_n \cdot \ell \to +\infty$ p.s., alors p.s. $\tau_k < \infty$ pour tout $k \geq 1$.

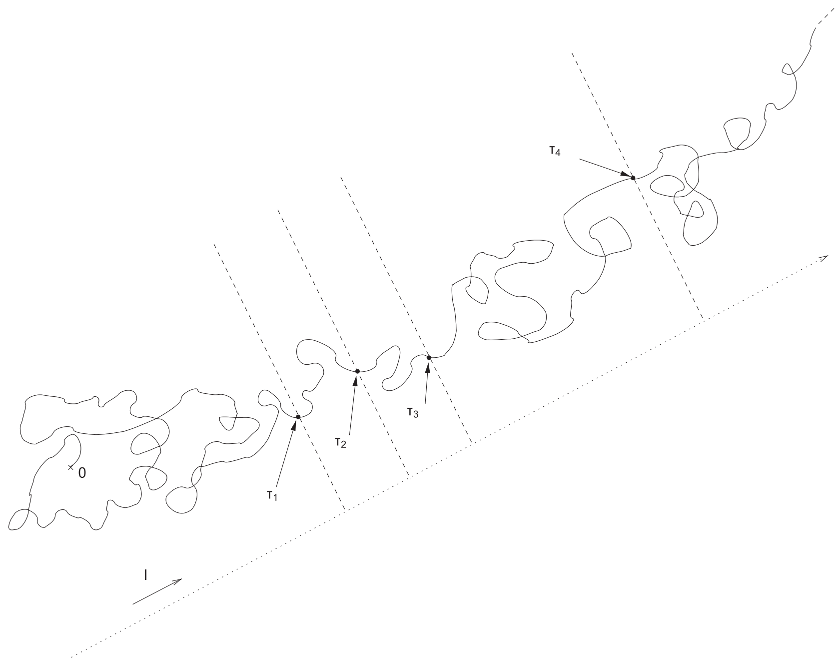

FIGURE 2.2. Itération de la construction

La propriété de renouvellement se résume par l'énoncé qui suit.

***Théorème 2.4* (Sznitman-Zerner [49]).** — *On note $A_\ell := \{X_n \cdot \ell \to_n +\infty\}$. Sous $P_0(\cdot|A_\ell)$, les variables aléatoires*

$$(X_{\tau_1}, \tau_1), (X_{\tau_2} - X_{\tau_1}, \tau_2 - \tau_1), \ldots, (X_{\tau_{k+1}} - X_{\tau_k}, \tau_{k+1} - \tau_k), \ldots$$

sont indépendantes. De plus, sous $P_0(\cdot|A_\ell)$,

$$(X_{\tau_2} - X_{\tau_1}, \tau_2 - \tau_1), \ldots, (X_{\tau_{k+1}} - X_{\tau_k}, \tau_{k+1} - \tau_k), \ldots$$

ont même loi que (X_{τ_1}, τ_1) sous $P_0(\cdot|D = \infty)$.

La balisticité résulterait alors de $E_0[\tau_1|D = \infty] < \infty$ par la loi des grands nombres usuelle. La preuve de cette majoration exploite la propriété de Kalikow pour se ramener à des estimées relatives à des chaînes de Markov, obtenues quant à elles par des techniques de martingales.

2.2.2. Formule d'intégration par parties. — La preuve du critère de balisticité d'Enriquez et Sabot [**15**] repose sur une vérification du critère de Kalikow. Un premier outil est une formule d'intégration par parties :

Lemme 2.5. — *Soit $\alpha \in (0,+\infty)^{2d}$. Pour toute fonction différentiable f sur $\mathbb{R}^{2d}$, en notant λ^α la loi de Dirichlet de paramètre α,*

$$\int f\,\mathrm{d}\lambda^\alpha = \frac{\alpha_1 + \cdot + \alpha_{2d}}{\alpha_1}\int x_1 f\,\mathrm{d}\lambda^\alpha + \frac{1}{\alpha_1}\int x_1\Big(\sum_{k=1}^{2d} x_k \frac{\partial f}{\partial x_k} - \frac{\partial f}{\partial x_1}\Big)\,\mathrm{d}\lambda^\alpha, \tag{2.2.4}$$

où les intégrales portent sur le simplexe $\mathrm{Prob}(\{1,\ldots,2d\})$.

La preuve, simple, consiste à passer à une intégrale sur $\mathbb{R}_+^{2d}$ en utilisant la représentation d'un vecteur de Dirichlet comme vecteur normalisé de variables de loi Gamma, puis à utiliser la formule d'intégration par parties usuelle et à traduire le résultat à nouveau en termes de loi de Dirichlet.

L'espérance $E[G_U^\omega(z_0,x)\omega(x,y)]$ dans la formule de Kalikow est à rapprocher du terme $\int x_1 f\,\mathrm{d}\lambda^\alpha$ ci-dessus, en voyant $G_U^\omega(z_0,x)$ comme fonction de l'environnement au site x. Pour s'abstraire du lien entre les variables $\omega(x,y)$, où y est voisin de x, et pouvoir dériver par rapport à celles-ci, on introduit un nouveau paramètre $\delta \in (0,1)$.

Soit $\delta \in (0,1)$. Pour $U \subset \mathbb{Z}^d$ fini contenant 0, et ω un environnement dans U, la fonction de Green en environnement ω tuée au taux δ et à la sortie de U est, pour $x,y \in U \cup \partial U$,

$$G_{U,\delta}^\omega(x,y) := E_{x,\omega}\Big[\sum_{n=0}^{T_U} \delta^n \mathbf{1}_{\{X_n=y\}}\Big]$$

(cela revient à dire qu'à chaque instant, la marche a une probabilité $1-\delta$ de sauter hors de U). On a

$$G_{U,\delta}^\omega(x,y) = \sum_{n\geq 0} \delta^n (\Omega_U)^n(x,y),$$

où Ω_U est la matrice indexée par $U\cup\partial U$ avec, pour tous $x,y \in U\cup\partial U$, $\Omega_U(x,y) = \omega(x,y)$ si $x \in U$, et $\Omega(x,y) = 0$ si $x \notin U$ (et la puissance n est matricielle). Cette série converge pour tout $\delta \in (0,1)$; ceci permet de petites variations des coefficients de Ω_U indépendamment entre eux. En ce sens, on peut donc considérer les dérivées de $G_{U,\delta}^\omega(x,y)$ par rapport à une variable $\omega(z,z')$. En dérivant terme à terme la série ci-dessus, on obtient :

Lemme 2.6. — *Pour toute partie U finie de $\mathbb{Z}^d$ et tous $x_1,x_2,x_4 \in U$ et $x_3 \in U\cup\partial U$ tels que $|x_3-x_2| = 1$, pour tout $\delta \in (0,1)$,*

$$\frac{\partial G_{U,\delta}^\omega(x_1,x_4)}{\partial\omega(x_2,x_3)} = \delta G_{U,\delta}^\omega(x_1,x_2) G_{U,\delta}^\omega(x_3,x_4). \tag{2.2.5}$$

Dès lors, pour tout $z \in U$, la formule d'intégration par partie appliquée à $f = G_{U,\delta}^\omega(z_0,z)$ vue comme fonction des seules variables $x_i = \omega(z,z+e_i)$, pour

$i = 1, \ldots, 2d$, donne

$$\begin{aligned}\mathbb{E}[G^{\omega}_{U,\delta}(z_0, z)] = \frac{\alpha_1 + \cdots + \alpha_{2d}}{\alpha_1}\mathbb{E}[G^{\omega}_{U,\delta}(z_0, z)\omega(z, z + e_1)] \\ + \frac{1}{\alpha_1}\mathbb{E}\Big[\omega(z, z + e_1)G^{\omega}_{U,\delta}(z_0, z)\Big(\delta\sum_{k=1}^{2d}\omega(z, z + e_k)G^{\omega}_{U,\delta}(z + e_k, z) - \delta G^{\omega}_{U,\delta}(z + e_1, z)\Big)\Big] \\ = \frac{\alpha_1 + \cdots + \alpha_{2d}}{\alpha_1}\mathbb{E}[G^{\omega}_{U,\delta}(z_0, z)\omega(z, z + e_1)] \\ + \frac{1}{\alpha_1}\mathbb{E}\Big[\omega(z, z + e_1)G^{\omega}_{U,\delta}(z_0, z)\big(G^{\omega}_{U,\delta}(z, z) - 1 - \delta G^{\omega}_{U,\delta}(z + e_1, z)\big)\Big].\end{aligned}$$

Si on définit l'environnement de Kalikow modifié $\widehat{\omega}_{U,z_0,\delta}$ en remplaçant G^{ω}_U par $G^{\omega}_{U,\delta}$ dans la définition de $\widehat{\omega}_{U,z_0}$, alors la relation ci-dessus mène à

$$\widehat{\omega}_{U,\delta}(z, z + e_1) = \frac{1}{\Sigma - 1}\left(\alpha_1 - \frac{\mathbb{E}[G^{\omega}_{U,\delta}(z_0, z)p_{\omega,\delta}(z, z + e_1)]}{\mathbb{E}[G^{\omega}_{U,\delta}(z_0, z)]}\right),$$

où $p_{\omega,\delta}(z, z+e_1)) := \omega(z, z+e_1)(G^{\omega}_{U,\delta}(z, z)-\delta G^{\omega}_{U,\delta}(z+e_1, z))$ et $\Sigma := \alpha_1+\cdots+\alpha_{2d}$. À partir de là, on renvoie à 4.4 pour la conclusion (on borne $p_{\omega,\delta}$ pour obtenir une « condition de Kalikow modifiée » avant de faire tendre δ vers 1).

2.3. Transience directionnelle

On donne simplement deux remarques sur la preuve du Theorème 5.2.

2.3.1. Un analogue en dimension 1. — Il résulte d'un article de Chamayou et Letac ([**9**], exemple 9 p.21) que si la loi μ de l'environnement est la loi Beta(α, β), où $\alpha > \beta > 0$, alors la variable $R := \sum_{n\geq 0} \mathrm{e}^{V(n)}$ (cf. (1.1.2) pour la définition de V) suit la même loi que $\frac{1}{W}$, où W est de loi Beta$(\alpha-\beta, \beta)$. Comme on a aussi $R^{-1} = P_{0,\omega}(\tau(-1) = \infty)$, on en déduit

$$P_0(\tau(-1) = \infty) = \frac{\alpha - \beta}{\alpha} = 1 - \frac{\beta}{\alpha}.$$

Ces propriétés sont à rapprocher de celles obtenues pour les probabilités de sortie de cylindre en dimension supérieure, où seule une inégalité est prouvée. Dans le cas de la dimension 1, l'autre inégalité se déduit de la transience directionnelle et fournit d'ailleurs une nouvelle preuve de la propriété de [**9**] citée plus haut.

2.3.2. Transience à vitesse nulle. — Le théorème 5.2 fournit des exemples de MAMA transientes directionnellement à vitesse nulle dans $\mathbb{Z}^d$. Mentionnons une autre famille d'exemples, due à Alexander Fribergh. On choisit $0 < \alpha < 1$. Considérons une loi μ telle que l'environnement vérifie presque sûrement, sous $\mathbb{P} := \mu^{\mathbb{Z}^d}$,

$$\omega(0, -e_1) = \alpha\omega(0, e_1) \qquad \text{et} \qquad \text{pour } i = 1, \ldots, d,\ \omega(0, \pm e_i) > 0. \tag{2.3.1}$$

Par la première condition on a, $\mathbb{P}$-p.s., pour tout n,

$$P_{0,\omega}(X_{n+1} = X_n + e_1 | X_{n+1} = X_n \pm e_1) = \frac{1}{1 + \alpha} > \frac{1}{2}.$$

Ainsi, la projection de $(X_n)_n$ sur l'axe $\mathbb{R}e_1$, en supprimant les instants où cette projection ne varie pas, est une marche aléatoire simple biaisée vers la droite. Pour conclure $X_n \cdot e_1 \to_n +\infty$ p.s., il suffit donc de voir que $(X_n \cdot e_1)_n$ n'est pas bornée ; or ceci résulte du lemme 4 de **[55]** avec la deuxième condition de (2.3.1).

Enfin, on peut facilement choisir la loi μ de sorte que le temps de sortie d'une arête dans la direction e_2 ne soit pas intégrable sous P_0 (c.-à-d. $\mathbb{E}[\frac{1}{1-\omega(0,e_2)\omega(e_2,-e_2)}] = \infty$), ce qui implique la vitesse nulle de la marche (voir la proposition 4.12 pour cette dernière implication).

CHAPITRE 3

LIMITES STABLES EN DIMENSION 1

La preuve du théorème principal du chapitre 6 repose sur deux ingrédients principaux : le comportement des sommes de variables indépendantes à queue lourde, et le temps passé dans une vallée. Suivant ces grandes lignes, on rappelle ici l'essentiel des résultats utilisés au cours du chapitre 6. Avec ce chapitre, le seul élément important de la preuve du théorème 6.1 à ne pas être démontré dans ce mémoire est le théorème de renouvellement de Kesten, sous la forme donnée dans **[16]** (voir (3.2.13)).

3.1. Lois stables, domaine d'attraction

Pour des variables réelles i.i.d. $X_1, X_2, \ldots$ de carré intégrable, le théorème central limite montre que la variable $\frac{1}{\sqrt{n}}(X_1 + \cdots + X_n - nE[X_1])$ est asymptotiquement gaussienne. De façon plus générale, on peut s'intéresser aux lois limites d'expressions de la forme

$$\frac{X_1 + \cdots + X_n - b_n}{a_n}, \tag{3.1.1}$$

où les X_i sont i.i.d. et $(a_n)_n, (b_n)_n$ sont des suites réelles déterministes. Les lois limites obtenues dans ce cadre sont appelées *lois stables*. On montre en effet facilement qu'elles vérifient une propriété de stabilité : si Z suit une loi stable, alors il existe $\alpha \in (0, 2]$ tel que, pour tout n, si $Z_1, \ldots, Z_n$ sont des copies i.i.d. de Z,

$$Z_1 + \cdots + Z_n \overset{\text{(loi)}}{=} n^{1/\alpha} Z + \beta_n,$$

où $\beta_n \in \mathbb{R}$. Le paramètre α est l'*indice* de la loi de Z. À translation et homothétie près, un seul autre paramètre (d'asymétrie) $\theta \in [0, 1]$ suffit à caractériser la loi de Z ; pour $\theta = 1$, la loi est dite *totalement asymétrique*. Une variable aléatoire X est dans le *bassin d'attraction* d'une loi stable s'il existe des suites $(a_n)_n, (b_n)_n$ telles que le quotient (3.1.1) converge en loi vers celle-ci, $X_1, X_2, \ldots$ étant des copies i.i.d. de X. On démontre (voir **[20]**) que X est dans le bassin d'attraction d'une loi stable d'indice $\alpha \in (0, 2)$ et d'asymétrie θ si, et seulement si

$$\frac{P(X > x)}{P(|X| > x)} \underset{x \to \infty}{\longrightarrow} \theta \qquad \text{et} \qquad P(|X| > x) = \frac{L(x)}{x^\alpha}, \tag{3.1.2}$$

où L est à variation lente : pour tout $t > 0$, $\frac{L(tx)}{L(x)} \to_{x\to\infty} 1$.

On n'a besoin dans la suite que d'un résultat nettement plus simple (tiré de [**13**]), qui servira doublement : la preuve du théorème 6.1 consiste à se ramener à ce théorème-ci, et le principe de cette preuve est basé sur celui de la démonstration ci-après.

Théorème 3.1. — *Soit $(X_n)_{n\geq 1}$ une famille de copies i.i.d. d'une variable aléatoire X telle que $X \geq 0$ p.s. et*

$$P(X > x) \underset{x\to\infty}{\sim} Cx^{-\alpha}, \tag{3.1.3}$$

pour des constantes $C > 0$ et $0 < \alpha < 2$. On note $S_n := X_1 + \cdots + X_n$,

$$a_n := (Cn)^{1/\alpha} \qquad \text{et} \qquad b_n := nE[X\mathbf{1}_{\{X<a_n\}}].$$

Alors

$$\frac{S_n - b_n}{a_n} \underset{n}{\longrightarrow} \mathcal{S}_\alpha, \tag{3.1.4}$$

où $\mathcal{S}_\alpha$ suit la loi stable totalement asymétrique donnée par la fonction caractéristique

$$E[\mathrm{e}^{it\mathcal{S}_\alpha}] = \exp\left(\int_1^\infty (\mathrm{e}^{itx} - 1)\alpha\frac{\mathrm{d}x}{x^{\alpha+1}} + \int_0^1 (\mathrm{e}^{itx} - 1 - itx)\alpha\frac{\mathrm{d}x}{x^{\alpha+1}}\right). \tag{3.1.5}$$

Démonstration. — Soit $\varepsilon > 0$. On découpe S_n en « petits » et « grands » termes :

$$S_n - b_n = \underbrace{\sum_{i=1}^n \big(X_i\mathbf{1}_{\{X_i\leq\varepsilon a_n\}} - E[X\mathbf{1}_{\{X\leq\varepsilon a_n\}}]\big)}_{=:\overline{S}_n(\varepsilon)} + \underbrace{\sum_{i=1}^n X_i\mathbf{1}_{\{X_i>\varepsilon a_n\}}}_{=:\widehat{S}_n(\varepsilon)} - n\underbrace{E[X\mathbf{1}_{\{\varepsilon a_n<X<a_n\}}]}_{=:\widehat{\mu}_n(\varepsilon)}.$$

On a

$$\begin{aligned} E[\overline{S}_n(\varepsilon)^2] &= n\,\mathit{Var}(X\mathbf{1}_{\{X<\varepsilon a_n\}}) \leq nE[X^2\mathbf{1}_{\{X<\varepsilon a_n\}}] \\ &= \int_0^{\varepsilon a_n} 2xP(X > x)\,\mathrm{d}x \underset{n}{\sim} \frac{2}{2-\alpha}\varepsilon^{2-\alpha}a_n^2, \end{aligned}$$

en utilisant (3.1.3), $\alpha < 2$ et la définition de a_n, d'où

$$\limsup_n E\Big[\Big(\frac{\overline{S}_n(\varepsilon)}{a_n}\Big)^2\Big] \leq \frac{2}{2-\alpha}\varepsilon^{2-\alpha}. \tag{3.1.6}$$

Comme $nP(X > \varepsilon a_n) \to_n \varepsilon^{-\alpha}$, le nombre (binomial) $K_n(\varepsilon)$ de termes dans la somme $\frac{1}{a_n}\widehat{S}_n(\varepsilon)$ converge en loi vers une variable de Poisson de paramètre $\varepsilon^{-\alpha}$. De plus, conditionnellement à $\{K_n(\varepsilon) = m\}$, ces m termes sont i.i.d. de fonction de répartition F_n^ε vérifiant, pour $x \geq 0$,

$$1 - F_n^\varepsilon(x) = P(X > xa_n | X > \varepsilon a_n) \underset{n}{\longrightarrow} \frac{\varepsilon^\alpha}{x^\alpha}\mathbf{1}_{\{x>\varepsilon\}},$$

d'où l'on déduit que leur fonction caractéristique ψ_n^ε vérifie, pour $t \in \mathbb{R}$,

$$\psi_n^\varepsilon(t) \underset{n}{\longrightarrow} \int_\varepsilon^\infty \mathrm{e}^{itx}\varepsilon^\alpha\alpha\frac{\mathrm{d}x}{x^{\alpha+1}} =: \psi^\varepsilon(t).$$

Ainsi, par convergence bornée, en notant $N(\varepsilon)$ une variable de loi Poisson$(\varepsilon^{-\alpha})$,

$$\begin{aligned} E\Big[\mathrm{e}^{it\widehat{S}_n(t)/a_n}\Big] &= E\Big[\big(\psi_n^\varepsilon(t)\big)^{K_n(\varepsilon)}\Big] \\ &\underset{n}{\longrightarrow} E\Big[\big(\psi^\varepsilon(t)\big)^{N(\varepsilon)}\Big] = \exp\big(\varepsilon^{-\alpha}(\psi^\varepsilon(t)-1)\big). \end{aligned} \tag{3.1.7}$$

La loi μ_n de $\frac{X}{a_n}$ vérifie, pour tous $0<x<y$,

$$n\mu_n([x,y]) \underset{n}{\longrightarrow} x^{-\alpha} - y^{-\alpha} = \mu([x,y]),$$

où $\mu := \mathbf{1}_{\{x>0\}}\alpha\frac{\mathrm{d}x}{x^{\alpha+1}}$, d'où (les mesures $n\mu_n$ et μ étant finies sur $[\varepsilon,1]$)

$$\frac{n\widehat{\mu}_n(\varepsilon)}{a_n} = nE\left[\frac{X}{a_n}\mathbf{1}_{\{\varepsilon<\frac{X}{a_n}<1\}}\right] \underset{n}{\longrightarrow} \int_\varepsilon^1 x\,\mathrm{d}\mu(x) = \int_\varepsilon^1 x\alpha\frac{\mathrm{d}x}{x^{\alpha+1}}. \tag{3.1.8}$$

En combinant (3.1.7) et (3.1.8), on obtient

$$E[\mathrm{e}^{it(\widehat{S}_n(\varepsilon)-n\widehat{\mu}_n(\varepsilon))/a_n}] \underset{n}{\longrightarrow} \exp\left(\int_1^\infty(\mathrm{e}^{itx}-1)\alpha\frac{\mathrm{d}x}{x^{\alpha+1}} + \int_\varepsilon^1(\mathrm{e}^{itx}-1-itx)\alpha\frac{\mathrm{d}x}{x^{\alpha+1}}\right).$$

Cette écriture de la limite montre que le terme de droite admet une limite quand ε tend vers 0, vu que $\mathrm{e}^{itx}-1-itx \sim -\frac{1}{2}t^2x^2$ quand $t\to 0^+$, et $\alpha<2$. Notons $h_n(\varepsilon)$ le terme de gauche ci-dessus, et $g(\varepsilon)$ celui de droite. On a donc, pour tout $\varepsilon>0$, $h_n(\varepsilon)\to_n g(\varepsilon)$, et $g(\varepsilon)\to_{\varepsilon\to 0} g(0)$ où $g(0)$ est la fonction caractéristique donnée dans l'énoncé. On en déduit l'existence d'une suite $\varepsilon_n \to_n 0$ telle que $h_n(\varepsilon_n)\to_n g(0)$ (choisir $\varepsilon_n = \frac{1}{k}$ pour $N_k \le n < N_{k+1}$, où N_k est tel que $N_k \ge N_{k-1}$ et pour tout $n\ge N_k$, $|h_n(\frac{1}{k}) - g(\frac{1}{k})| \le \frac{1}{k}$).

De plus, (3.1.6) donne alors $\frac{1}{a_n}\overline{S}_n(\varepsilon_n)\to_n 0$ en probabilité. Ceci conclut la preuve. □

Remarque. — *Le terme de translation* $c_n := \frac{n}{a_n}E[X\mathbf{1}_{\{X<a_n\}}]$ *vérifie* $c_n \to_n \frac{1}{1-\alpha}$ *si* $0<\alpha<1$, $c_n \sim_n C\log n$ *si* $\alpha=1$, *et* $c_n = \frac{n}{a_n}E[X] - \frac{1}{\alpha-1} + o_n(1)$ *si* $\alpha>1$. *Quitte à translater la loi limite, on peut donc enlever le centrage si* $0<\alpha<1$ *et centrer par l'espérance si* $1<\alpha<2$. *Pour une expression de la loi limite dans un autre paramétrage, voir Théorème 6.18 page 101.*

Il résulte de la preuve que la loi limite n'est due qu'à un très petit nombre de termes, ce qui constitue une différence qualitative importante par rapport au théorème central limite. Cette différence, typique des distributions « à queue lourde », transparaît d'ailleurs dans la limite du processus en temps continu

$$W_n : t \mapsto \frac{S_{\lfloor nt\rfloor} - b_{\lfloor nt\rfloor}}{a_{\lfloor nt\rfloor}}.$$

Pour des variables de carré intégrable, $(W_n)_n$ converge en loi vers un mouvement brownien, processus continu, tandis que dans le cadre du théorème précédent, $(W_n)_n$ converge en loi vers un processus de Lévy stable d'indice α, qui présente des sauts.

3.2. Étude du potentiel

On reprend les notations de l'introduction : on se donne une loi μ sur $(0,1)$, ce qui permet de définir $P = \mu^{\mathbb{Z}}$ (loi de l'environnement) ; pour tout environnement $\omega = (\omega_x)_{x\in\mathbb{Z}}$, on dispose de la loi $P_{0,\omega}$ (loi de la marche dans ω) ; enfin, $\mathbb{P}_x = P(\,\mathrm{d}\omega) \times P_{0,\omega}$ (loi de la MAMA). Soit $\omega \in (0,1)^{\mathbb{Z}}$. Le potentiel V associé à ω est tel que $V(0) = 0$ et $V(x) - V(x-1) = \log \rho_x$ pour tout $x \in \mathbb{Z}$, où $\rho_x = \frac{1-\omega_x}{\omega_x}$. On note $\rho := \rho_0$.

On suppose ici satisfaites les hypothèses du théorème de Kesten-Kozlov-Spitzer :

Hypothèses. —
(a) il existe $0 < \kappa < 2$ *tel que* $E\left[\rho^\kappa\right] = 1$ *et* $E\left[\rho^\kappa \log^+ \rho\right] < \infty$*;*
(b) la loi de $\log \rho$ *est non-arithmétique.*

Remarquons que (b) exclut en particulier les environnements déterministes ; on reviendra sur (b) en 3.3.2. Concernant l'hypothèse (a), on peut noter que la fonction

$$\varphi : s \mapsto E[\rho^s] \tag{3.2.1}$$

vérifie $\varphi(0) = \varphi(\kappa) = 1$, et est strictement convexe si $\rho \not\equiv 1$ (ce qui est garanti par (b)), donc κ est défini de manière unique. On a donc aussi $E[\log \rho] = \varphi'(0) < 0$, ce qui donne $V(x) \to \mp\infty$ p.s. quand $x \to \pm\infty$ par la loi des grands nombres.

3.2.1. Définition des excursions. — On découpe le potentiel en excursions au-dessus de son minimum passé. Les fins d'excursions sont données par la suite des temps de descente large du potentiel. Ainsi, $e_0 := 0$ et, pour tout $k \geq 0$,

$$e_{k+1} := \inf\{x > e_k | V(x) \leq V(e_k)\}.$$

Il s'agit de temps d'arrêt, de sorte que la suite des excursions $(V(e_k + x) - V(e_k))_{0\leq x\leq e_{k+1}-e_k}$ est i.i.d. par la propriété de Markov et la stationnarité de P. On étend la définition à $\mathbb{Z}_-$: si $k \leq 0$,

$$e_{k-1} := \sup\{x < e_k | \text{pour tout } y \leq x, V(y) \geq V(x)\}.$$

Pour conserver la stationnarité, il faut alors remplacer P par

$$P^{\geq 0} := P(\cdot | \text{pour tout } x \leq 0,\, V(x) \geq 0\},$$

voir lemme 6.5 (p.79) pour une preuve. Notons que prouver le théorème limite sous P ou $P^{\geq 0}$ est équivalent puisque la marche, étant transiente vers la droite dans les deux cas, ne passe qu'un temps fini à gauche de 0.

De plus, sous l'hypothèse (a), e_1 est exponentiellement intégrable : pour tout $n \in \mathbb{N}$, pour tout $\lambda > 0$,

$$P(e_1 > n) \leq P(V(n) \geq 0) = P(\mathrm{e}^{\lambda V(n)} \geq 1) \leq E[\mathrm{e}^{\lambda V(n)}] = E[\rho^\lambda]^n$$

et $E[\rho^\lambda] < 1$ si on choisit $\lambda \in (0,\kappa)$. Ainsi, il sera équivalent de montrer le théorème limite pour la suite $(\tau(n))_{n\geq 0}$, ou sa sous-suite $(\tau(e_k))_{k\geq 0}$ avec le centrage approprié (les détails de cette étape sont donnés dans l'appendice A).

Pour $k \in \mathbb{Z}$, la hauteur de la $(k+1)$-ième excursion est

$$H_k := \max\{V(x) - V(e_k) | e_k \leq x < e_{k+1}\}.$$

En particulier, $H := H_0$ est la hauteur de la première excursion. Les excursions de hauteur positive représentent des « obstacles » pour la marche aléatoire.

3.2.2. Temps dans une vallée. — On s'intéresse au temps $\tau := \tau(e_1)$ de traversée de la première excursion du potentiel. En d'autres termes, il s'agit du temps de sortie d'un puits de potentiel (ou de traversée d'une barrière de potentiel). Son évaluation présente donc un intérêt théorique et pratique important en physique, en premier lieu (historiquement) dans la théorie des vitesses de réaction chimique. Dans ce domaine, une version simple est connue sous le nom de loi d'Arrhénius (1889) donnant essentiellement l'ordre de grandeur exponentiel $\tau^{-1} \simeq A\mathrm{e}^{-\frac{H}{k_B T}}$ (où T est la température et k_B la constante de Boltzmann) ; Kramers (1940) **[28]** a précisé le résultat à l'aide d'un modèle continu de diffusion dans un potentiel (équation de Langevin), obtenant sous certaines contraintes la formule $\tau \simeq 2\pi m\gamma \ell_v \ell_b \mathrm{e}^{-\frac{H}{k_B T}}$, où γ est la viscosité, m la masse de la particule, et ℓ_v (resp. ℓ_b) une longueur caractéristique du fond de la vallée (resp. du haut de la barrière) : $V(x) \simeq \frac{1}{2}\left(\frac{x}{\ell_v}\right)^2$ près de 0, et $V(x) \simeq H - \frac{1}{2}\left(\frac{x-T_H}{\ell_b}\right)^2$ près de T_H (temps d'atteinte de la hauteur H). On s'attache maintenant à obtenir des résultats similaires rigoureux, pour notre modèle.

En décomposant le temps τ en excursions successives dans $\mathbb{Z}_+$, on écrit

$$\tau = F_1 + \cdots + F_N + G,$$

où N est le nombre de transitions de 1 à 0 avant d'atteindre e_1, les F_i sont les durées des tentatives de traversée de l'excursion soldées par un échec (c.-à-d. le temps de retour de 1 à 0, sachant que e_1 n'est pas atteint) et G la durée de la tentative réussie (temps d'atteinte de e_1 partant de 1, sachant que 0 n'est pas atteint). Ces temps sont indépendants sous $P_{0,\omega}$ et indépendants de N. Sous $P_{o,\omega}$, la variable N suit une loi géométrique de paramètre

$$q = P_{0,\omega}(\tau(e_1) < \tau^+(0)) = \frac{\omega_0}{\sum_{0 \leq x < e_1} \mathrm{e}^{V(x)}},$$

où $\tau^+(0)$ est le temps de retour en 0 ; la deuxième égalité vient d'un calcul classique sur les chaînes de Markov (la section 6.4.1 recense les formules utiles de cet ordre). Du fait du conditionnement par l'échec ou le succès, les espérances de F_i et G sous $P_{0,\omega}$ s'expriment non en fonction de V mais de h-processus associés à V. L'argument qui suit sera rendu rigoureux au chapitre prochain (section 6.7), il permet de motiver ce qui va suivre. Dans la limite des grandes valeurs de H, le nombre N diverge vers $+\infty$, tandis que la loi des F_i converge (vers un temps de retour en 0 non conditionné dans une vallée infiniment haute) et G reste petit devant N, de sorte que τ est de l'ordre de $NE_\omega[F]$ par la loi des grands nombres : τ suit approximativement une loi exponentielle de moyenne $E_\omega[N]E_\omega[F]$. On en déduit l'expression approchée (pour H grand)

$$\tau \simeq 2\mathrm{e}^H M_1 M_2 \mathbf{e},$$

où $\mathbf{e}$ est une variable exponentielle de paramètre 1 indépendante de l'environnement,

$$M_1 := \sum_{x<T_H} \mathrm{e}^{-V(x)} \qquad \text{et} \qquad M_2 := \sum_{0\le x<e_1} \mathrm{e}^{V(x)-H}.$$

Les termes dominants dans la somme définissant M_1 sont ceux d'indice proche de 0, tandis que pour M_2 ce sont ceux voisins de T_H ; on a donc une expression similaire à la formule de Kramers.

Pour déterminer la queue de τ, on est ainsi ramené à déterminer celle de $\mathrm{e}^H M_1 M_2$.

3.2.3. Queue de H. — Notons $S := \max_{k\ge 0} V(k)$. La proposition suivante est d'usage continuel dans la suite. On en donne donc une démonstration, d'après [**19**].

Proposition 3.2 **(Feller [19]-Iglehart [23])**. — *Quand $t\to\infty$,*

$$P(S>t)\sim C_F \mathrm{e}^{-\kappa t} \qquad \textit{et} \qquad P(H>t)\sim C_I \mathrm{e}^{-\kappa t}, \tag{3.2.2}$$

où

$$C_F := \frac{1-E[\mathrm{e}^{\kappa V(e_1)}]}{\kappa E[\rho^\kappa \log\rho] E[e_1]} \qquad \textit{et} \qquad C_I := (1-E[\mathrm{e}^{\kappa V(e_1)}])C_F. \tag{3.2.3}$$

Démonstration. — La preuve repose sur le théorème de renouvellement et l'introduction d'une « transformée de Girsanov » de P. Notant μ la loi de ω_0, définissons les mesures, respectivement sur $[0,1]$ et sur $[0,1]^{\mathbb{Z}}$,

$$\widetilde{\mu} := \rho^\kappa \mu = \mathrm{e}^{\kappa V(1)}\mu \qquad \text{et} \qquad \widetilde{P} := \widetilde{\mu}^{\otimes\mathbb{Z}}. \tag{3.2.4}$$

Alors on a $\int \mathrm{d}\widetilde{\mu} = E[\rho^\kappa] = 1$, de sorte que $\widetilde{\mu}$ et $\widetilde{P}$ sont des probabilités. De plus, $\int \log\rho\, \mathrm{d}\widetilde{\mu} = E[\rho^\kappa\log\rho] = \varphi'(\kappa)$ (cf. 3.2.1) et, par convexité de φ, les dérivées de φ en 0 et κ sont de signes opposés. Ainsi, sous $\widetilde{P}$, $V\to+\infty$ en $+\infty$. Notamment, le premier temps de montée stricte

$$\widetilde{e}_1 := \inf\{n\ge 0 | V(n)>0\}$$

est presque sûrement fini sous $\widetilde{P}$. Notons que, pour toute fonctionnelle mesurable positive $\psi((V(k))_{0\le k\le n})$,

$$\widetilde{E}[\psi((V(k))_{0\le k\le n})] = E[\mathrm{e}^{\kappa V(n)}\psi((V(k))_{0\le k\le n})], \tag{3.2.5}$$

comme cela se voit en considérant d'abord ψ de la forme $\psi_1(V(1)-V(0))\cdots\psi_n(V(n)-V(n-1))$. Cette relation se généralise immédiatement si n est remplacé par un temps d'arrêt T pour le processus $(V(k))_{k\ge 0}$. En particulier, on a donc

$$1 = \widetilde{P}(\widetilde{e}_1<\infty) = E[\mathrm{e}^{\kappa V(\widetilde{e}_1)}\mathbf{1}_{\{\widetilde{e}_1<\infty\}}]. \tag{3.2.6}$$

Définissons les fonctions

$$Z(t) := P(S>t) \qquad \text{et} \qquad Z^{\#}(t) := Z(t)\mathrm{e}^{\kappa t}.$$

L'événement $\{S > t\}$ est réalisé si $\widetilde{e}_1 < \infty$ et $V(\widetilde{e}_1) > t$ ou si $\widetilde{e}_1 < \infty$ et $\sup_{s \geq \widetilde{e}_1} V(t) - V(\widetilde{e}_1) > t - V(\widetilde{e}_1) > 0$, ce qui conduit à l'équation de renouvellement

$$Z(t) = P(V(\widetilde{e}_1) > t, \widetilde{e}_1 < \infty) + \int_0^t Z(t-y) L(\mathrm{d}y), \tag{3.2.7}$$

avec $L(\mathrm{d}y) := P(V(\widetilde{e}_1) \in \mathrm{d}y, \widetilde{e}_1 < \infty)$. Par (3.2.6), $L^\#(\mathrm{d}y) := \mathrm{e}^{\kappa y} L(\mathrm{d}y)$ est une probabilité sur $\mathbb{R}_+$ (c'est la loi de $V(\widetilde{e}_1)$ sous $\widetilde{P}$). De l'équation ci-dessus on déduit

$$Z^\#(t) = z^\#(t) + \int_0^t Z^\#(t-y) L^\#(\mathrm{d}y), \tag{3.2.8}$$

où $z^\#(t) := \mathrm{e}^{\kappa t} P(V(\widetilde{e}_1) > t, \, \widetilde{e}_1 < \infty)$. On calcule $\int_0^\infty z^\#(u)\,\mathrm{d}u = \frac{P(\widetilde{e}_1 = \infty)}{\kappa} < \infty$. Comme $z^\#$ est décroissante (donc directement Riemann intégrable avec ce qui précède) et $L^\#$ est une probabilité, non-arithmétique par l'hypothèse (b), l'application du théorème de renouvellement (cf. [**19**] p.363) donne

$$Z^\#(t) \underset{t \to \infty}{\longrightarrow} \frac{1}{m^\#} \int_0^\infty z^\#(u)\,\mathrm{d}u, \tag{3.2.9}$$

où $m^\# = \widetilde{E}[V(\widetilde{e}_1)]$ est l'espérance de $L^\#$. C'est le résultat annoncé, avec

$$C_F = \frac{P(\widetilde{e}_1 = \infty)}{\kappa \widetilde{E}[V(\widetilde{e}_1)]}.$$

L'expression de C_F de l'énoncé s'obtient par les identités suivantes : $\widetilde{E}[V(\widetilde{e}_1)] = \widetilde{E}[V(1)]\widetilde{E}[\widetilde{e}_1]$ (identité de Wald), $\widetilde{E}[V(1)] = E[\rho^\kappa \log \rho]$ (par définition de $\widetilde{P}$), ainsi que $E[e_1]P(\widetilde{e}_1 = \infty) = 1$ et $\widetilde{E}[\widetilde{e}_1](1 - E[\mathrm{e}^{\kappa V(e_1)}]) = 1$. Les deux dernières sont des conséquences d'une propriété de dualité : pour tout $n \in \mathbb{N}$,

$$P(e_1 \geq n) = P(n \in \{\widetilde{e}_u | u \geq 0\}), \tag{3.2.10}$$

où $(\widetilde{e}_u)_{u \geq 0}$ est la suite des temps de montée stricte de V, avec $\widetilde{e}_0 = 0$ et éventuellement $\widetilde{e}_u = \infty$ à partir d'un certain rang. Cette égalité vient du fait que, pour tout n, V et $V^* := (V(n-x) - V(n))_{x \in \mathbb{Z}}$ ont même loi et que l'évènement de gauche de (3.2.10) pour V correspond à celui de droite pour V^*. En sommant (3.2.10) pour $n \in \mathbb{N}^*$, on obtient $E[e_1] = E[\#\{\widetilde{e}_u < \infty | u \geq 0\}]$. Par la propriété de Markov, le nombre de temps de montée stricte de V finis suit une loi géométrique (sur $\mathbb{N}^*$) de paramètre $P(\widetilde{e}_1 = \infty)$, d'où finalement $E[e_1] = P(\widetilde{e}_1 = \infty)^{-1}$. En remplaçant P par $\widetilde{P}$, temps de montée stricte par temps de descente large et vice-versa, on obtient de même $\widetilde{P}(\widetilde{e}_1 \geq n) = \widetilde{P}(n \in \{e_u | u \geq 0\})$, et donc $\widetilde{E}[\widetilde{e}_1]\widetilde{P}(e_1 = \infty) = 1$. Enfin, $\widetilde{P}(e_1 < \infty) = E[\mathrm{e}^{\kappa V(e_1)}]$ car $e_1 < \infty$ P-p.s. (cf. (3.2.6)) d'où la dernière formule.

On en déduit l'équivalent de la queue de H comme suit. Notons $T_t := \inf\{n | V(n) > t\}$. Alors

$$P(S > t) = P(H > t) + P(S > t, e_1 < T_t) \tag{3.2.11}$$

d'où, en conditionnant par la valeur de $V(e_1)$,

$$\begin{aligned}\mathrm{e}^{\kappa t}P(H>t) &= \mathrm{e}^{\kappa t}P(S>t)\\ &\quad -\int_{-\infty}^{0}\mathrm{e}^{\kappa(t-y)}P(S>t-y)\mathrm{e}^{\kappa y}P(V(e_1)\in \mathrm{d}y,\, e_1<T_t).\end{aligned}$$

Pour $t\to\infty$, $\mathrm{e}^{\kappa(t-y)}P(S>t-y)$ converge uniformément vers C_F pour $y\le 0$. De plus, $P(e_1<T_t)\to 1$, et l'application $y\mapsto \mathrm{e}^{\kappa y}$ est continue bornée sur $(-\infty,0)$, donc le membre de droite ci-dessus converge vers $C_F - C_F E[\mathrm{e}^{\kappa V(e_1)}]$. Ceci conclut la preuve. □

3.2.4. Queue de $\mathrm{e}^H M_1 M_2$. — Commençons par donner deux lemmes techniques importants de **[16]**. Le second permet de majorer des moments de M_1 indépendamment de H.

Notons

$$R_- := \sum_{x\le 0}\mathrm{e}^{-V(x)}.$$

On rappellera plus bas que cette série de renouvellement de Kesten n'a de moments que jusqu'à l'ordre κ. En revanche, sachant $\{V_{|\mathbb{Z}_-}\ge 0\}$, tous sont finis :

Lemme 3.3. — *Tous les moments de R_- sont finis sous $P^{\ge 0}$: pour tout $N\in\mathbb{N}$,*

$$E^{\ge 0}[(R_-)^N]<\infty.$$

Démonstration. — En découpant en excusions, on a

$$R_- = \sum_{n\le 0}\mathrm{e}^{-V(e_n)}\sum_{e_{n-1}<k\le e_n}\mathrm{e}^{-(V(k)-V(e_n))}\le \sum_{n\le 0}\mathrm{e}^{-V(e_n)}L_n,$$

où $L_n := e_n - e_{n-1}$. Appliquons alors l'inégalité de Hölder avec les poids $p_1 = \cdots = p_N = N$:

$$E^{\ge 0}[(R_-)^N]\le \sum_{n_1,\dots,n_N\le 0}\prod_{i=1}^{N}E^{\ge 0}[(\mathrm{e}^{-V(e_{n_i})}L_{n_i})^N]^{1/N}.$$

Par l'indépendance et la stationnarité des excursions, $E^{\ge 0}[(\mathrm{e}^{-V(e_{n_i})}L_{n_i})^N]^{1/N} = E[\mathrm{e}^{NV(e_1)}]^{n_i/N}E[(e_1)^N]^{1/N}$. Par l'hypothèse (b) (voir remarque en 3.3.2), on a $E[\mathrm{e}^{NV(e_1)}]<1$; et on a vu que e_1 est exponentiellement intégrable donc ses moments sont finis. La somme ci-dessus est donc un produit de séries géométriques convergentes, ce qui conclut. □

Lemme 3.4. — *Pour tout $\eta>0$, il existe une constante $c_\eta>0$ telle que, $P^{\ge 0}$-p.s.,*

$$E^{\ge 0}\big[(M_1)^\eta\big|\,\lfloor H\rfloor\,\big]\le c_\eta.$$

Démonstration. — Soit $h\in\mathbb{N}$ tel que $P(\lfloor H\rfloor = h)>0$. Par le lemme 3.3, l'inégalité $(a+b)^\eta\le 2^\eta(a^\eta+b^\eta)$ et l'indépendance entre R_- et H, il suffit de majorer

$$E\Bigg[\Bigg(\sum_{0\le k\le T_H}\mathrm{e}^{-V(k)}\Bigg)^{\eta}\,\Bigg|\,\lfloor H\rfloor = h\Bigg]$$

uniformément par rapport à h. Cette espérance est inférieure à

$$\frac{1}{P(\lfloor H\rfloor = h)}\sum_{p=0}^{\infty} E\left[\Big(\sum_{0\le k\le p} \mathrm{e}^{-V(k)}\Big)^{\eta}\mathbf{1}_{\{\forall 0\le k<p,\, 0\le V(k)<V(p)\}}\mathbf{1}_{\{V(p)\in[h,h+1)\}}\right].$$

En faisant intervenir la « transformée de Girsanov » $\widetilde{P}$ (cf. (3.2.4)), l'expression devient (voir (3.2.5)) :

$$\frac{1}{P(\lfloor H\rfloor = h)}\sum_{p=0}^{\infty} \widetilde{E}\left[\mathrm{e}^{-\kappa V(p)}\mathbf{1}_{\{V(p)\in[h,h+1)\}}\Big(\sum_{0\le k\le p} \mathrm{e}^{-V(k)}\Big)^{\eta}\mathbf{1}_{\{\forall 0\le k<p,\, 0\le V(k)<V(p)\}}\right].$$

La condition « pour $0\le k<p$, $V(k)<V(p)$ » caractérise p comme l'un des temps de montée stricte $\widetilde{e}_u$, $u\ge 0$, de V (définis similairement aux e_u, $u\ge 0$, temps de descente large), d'où la réécriture

$$\frac{1}{P(\lfloor H\rfloor = h)}\widetilde{E}\left[\sum_{u=0}^{\infty}\mathrm{e}^{-\kappa V(\widetilde{e}_u)}\mathbf{1}_{\{V(\widetilde{e}_u)\in[h,h+1)\}}\Big(\sum_{0\le k\le \widetilde{e}_u} \mathrm{e}^{-V(k)}\Big)^{\eta}\mathbf{1}_{\{\forall 0\le k\le \widetilde{e}_u,\, 0\le V(k)\}}\right].$$

En minorant $V(\widetilde{e}_u)$ par h, et en appliquant l'inégalité de Cauchy-Schwarz, ceci est inférieur à

$$\frac{\mathrm{e}^{-\kappa h}}{P(\lfloor H\rfloor = h)}\widetilde{E}\left[\Big(N\big([h,h+1)\big)\Big)^{2}\right]^{1/2}\widetilde{E}\left[\Big(\sum_{0\le k\le \widetilde{e}_u} \mathrm{e}^{-V(k)}\Big)^{2\eta}\mathbf{1}_{\{\forall 0\le k\le \widetilde{e}_u,\, 0\le V(k)\}}\right]^{1/2},$$

où $N(A) := \#\{u\ge 0 | V(\widetilde{e}_u)\in A\}$ pour $A\subset\mathbb{R}$. Par la propriété de Markov en $T_h := \inf\{u\ge 0|V(\widetilde{e}_u)\in[h,h+1)\}$, on a $\widetilde{E}[N([h,h+1))^2]\le \widetilde{E}[N([0,1))^2]$. Or $N([0,1))$ est exponentiellement intégrable : $\widetilde{P}(N([0,1))>u)\le \widetilde{P}(V(\widetilde{e}_u)\le 1)\le \mathrm{e}\widetilde{E}[\mathrm{e}^{-V(\widetilde{e}_u)}] = \mathrm{e}\widetilde{E}[\mathrm{e}^{-V(\widetilde{e}_1)}]^u$. On a donc une majoration de la première espérance indépendante de h. Considérons la seconde espérance. L'événement $\{\forall k\ge \widetilde{e}_u,\ V(k)\ge V(\widetilde{e}_u)\}$ est indépendant de $(V(k))_{0\le k\le \widetilde{e}_u}$ et a pour probabilité $\widetilde{P}(\forall k\ge 0, V(k)\ge 0)>0$ (puisque V est transient vers $+\infty$ sous $\widetilde{P}$) donc, quitte à multiplier par une constante, on peut restreindre l'espérance à cet événement, de sorte qu'il suffit de montrer

$$\widetilde{E}\left[\Big(\sum_{k\ge 0}\mathrm{e}^{V(k)}\Big)^{2\eta}\Big|\forall k\ge 0, V(k)\ge 0\right]<\infty,$$

ce qui résulte du lemme 3.3 appliqué au potentiel $(V(-k))_{k\in\mathbb{Z}}$ sous $\widetilde{P}$.

En définitive,

$$E\left[\Big(\sum_{0\le k\le T_H}\mathrm{e}^{V(k)}\Big)^{\eta}\Big|\lfloor H\rfloor = h\right]\le C_\eta\frac{\mathrm{e}^{-\kappa h}}{P(\lfloor H\rfloor = h)},$$

où C_η ne dépend pas de h. Comme, par l'estimée d'Iglehart (3.2.2), le terme de droite converge vers $C_\eta(C_I(1-\mathrm{e}^{-\kappa}))^{-1}>0$ quand h tend vers l'infini, ce terme est majoré uniformément par une constante strictement positive, ce qui conclut. □

Conditionnellement à l'événement

$$\mathcal{I} := \{H = S\}\cap\{\forall k\le 0, V(k)\ge 0\},$$

les potentiels V et $(H - V(T_H - k))_{k\in\mathbb{Z}}$ ont même loi, de sorte que M_1 a même loi que

$$M_2' := \sum_{k\geq 0} \mathrm{e}^{V(k)-H}.$$

Or il est facile de voir que le lemme précédent reste valable si $P^{\mathcal{I}} := P(\cdot|\mathcal{I})$ remplace $P^{\geq 0}$; dès lors, il vaut aussi pour M_2' et donc $M_2 (\leq M_2')$ sous $P^{\mathcal{I}}$. En conditionnant par la partie entière de H, on obtient alors des bornes de la forme : pour tous α, β, γ, il existe $C > 0$ tel que, pour tout $h > 0$,

$$E^{\mathcal{I}}[(M_1)^\alpha (M_2)^\beta \mathrm{e}^{\gamma H} \mathbf{1}_{\{H<h\}}] \leq C E^{\mathcal{I}}[\mathrm{e}^{\gamma H} \mathbf{1}_{\{H<h+1\}}].$$

L'estimation de la dernière espérance est alors donnée par la queue de H vue plus haut (on note que $P(H > t, H = S) \sim P(H > t)$ quand $t \to \infty$) ; si $\gamma > \kappa$ par exemple, la borne ci-dessus devient $C'\mathrm{e}^{(\gamma-\kappa)h}$, voir lemme 6.4 pour un énoncé général.

On en déduit rapidement que la queue de la variable $Z := M_1 M_2 \mathrm{e}^H$ est polynomiale d'ordre κ : il existe $c_1, c_2 > 0$ telles que, pour tout $t > 0$,

$$c_1 t^{-\kappa} \leq P^{\mathcal{I}}(Z > t) \leq c_2 t^{-\kappa}. \tag{3.2.12}$$

En effet, pour tous $\gamma, h > 0$, utilisant $M_1, M_2 \geq 1$ pour la minoration,

$$\begin{aligned} P^{\mathcal{I}}(\mathrm{e}^H > t) \leq P^{\mathcal{I}}(Z > t) &\leq P^{\mathcal{I}}(H \geq h) + P^{\mathcal{I}}(Z > t,\, H < h) \\ &\leq P^{\mathcal{I}}(H \geq h) + \frac{1}{t^\gamma} E^{\mathcal{I}}[Z^\gamma \mathbf{1}_{\{H<h\}}], \end{aligned}$$

et en choisissant $h = \log t$ et $\gamma > \kappa$, le membre de droite se majore alors bien par $c_2 t^{-\kappa}$ d'après la remarque précédente ; et le membre de gauche est équivalent à $C_I P(H = S)^{-1} t^{-\kappa}$.

La preuve du théorème 6.1 requiert un énoncé plus précis :

Théorème. — *Il existe une constante C_U telle que, pour $t \to \infty$,*

$$P^{\mathcal{I}}(\mathrm{e}^H M_1 M_2' > t) \sim C_U t^{-\kappa}. \tag{3.2.13}$$

Une preuve de cet équivalent en suivant les techniques de Kesten, Kozlov et Spitzer [**27**] est donnée dans [**37**] (Théorème 1.4). Celle-ci ne fournit cependant pas d'expression de la constante C_U.

Enriquez, Sabot et Zindy [**16**] ont en revanche obtenu une interprétation probabiliste de C_U et une relation entre celle-ci et la constante intervenant dans le théorème de renouvellement de Kesten [**26**], ce qui permet d'obtenir son expression explicite dans certains cas, et une méthode d'estimation numérique en général. Esquissons brièvement le principe de leur approche.

Un délicat argument de couplage permet d'établir une « indépendance asymptotique » entre M_1, M_2' et H dans la limite des grandes valeurs de H, formalisant ainsi l'intuition que M_1 dépend essentiellement de la géométrie du « fond » de la vallée (autour de 0, où l'argument de l'exponentielle est proche de 0) et M_2' de celle du sommet de la première excursion (autour de T_H). Informellement on a alors, par l'estimée d'Iglehart, quand $t \to \infty$,

$$P^{\mathcal{I}}(\mathrm{e}^H M_1 M_2' > t) = P^{\mathcal{I}}\big(\mathrm{e}^H > \frac{t}{M_1 M_2'}\big) \sim C_I E[M^\kappa]^2 t^{-\kappa},$$

pour une certaine variable M « limite » de M_1 et M_2'. Cette identité

$$C_U = C_I E[M^\kappa]^2 \tag{3.2.14}$$

a été démontrée dans [**16**], avec

$$M := \sum_{i<0} \mathrm{e}^{-V(i)} + \sum_{i\geq 0} \mathrm{e}^{-V(i)},$$

où $V_{|\mathbb{Z}_-}$ est distribué selon $P(\cdot|\forall k \leq 0, V(k) \geq 0)$, et $V_{|\mathbb{Z}_+}$ selon $\widetilde{P}(\cdot|\forall k \geq 0, V(k) \geq 0)$, ces deux variables étant indépendantes. Cette expression de C_U est adaptée à une évaluation numérique par méthode de Monte-Carlo.

L'équivalent reste vrai en remplaçant M_2' par M_2 (voir remarque 7.1 de [**16**] ; sur $\{H = S\}$, les termes dominants sont donnés par la première excursion seulement). Le conditionnement par $\{H = S\}$, utile pour assurer la symétrie entre M_1 et M_2', devient alors accessoire dans la limite $t \to \infty$ (voir le lemme 6.3) : pour $t \to \infty$,

$$P^{\geq 0}(Z > t) \sim C_U t^{-\kappa}. \tag{3.2.15}$$

Par ailleurs, le même argument appliqué à la série de renouvellement de Kesten

$$R := \sum_{k=0}^{\infty} \mathrm{e}^{V(k)} = \mathrm{e}^H M_2'$$

donne, pour $t \to \infty$,

$$P^{\mathcal{I}}(R \geq t) \sim C_I E[M^\kappa] t^{-\kappa}$$

et, sans conditionnement (partant de $R = \mathrm{e}^S \sum_{k\geq 0} \mathrm{e}^{V(k)-S}$), le résultat de Kesten : quand $t \to \infty$,

$$P(R \geq t) \sim C_K t^{-\kappa},$$

avec

$$C_K := C_F E[M^\kappa]. \tag{3.2.16}$$

La comparaison de (3.2.14) et (3.2.16) fournit une expression de C_U en fonction de la constante plus « classique » C_K :

$$C_U = C_I \left(\frac{C_K}{C_F}\right)^2. \tag{3.2.17}$$

On rappelle que C_F et C_I sont reliées par (3.2.3). Enfin, la constante de Kesten a été rendue explicite dans certains cas : lorsque κ est entier par Goldie [**21**], et lorsque l'environnement suit une loi Beta (cas unidimensionnel des environnements de Dirichlet) par Chamayou et Letac [**9**].

3.3. Compléments

3.3.1. Principe de la preuve. — On met ici en parallèle les preuves des théorèmes 3.1 et 6.1, en complément au schéma de preuve donné en 6.3.3.

À l'aide de (3.2.15), et rendant rigoureux l'argument donnant $\tau \simeq 2\mathrm{e}^H M_1 M_2 \mathbf{e}$ (cf. 3.2.2), la proposition 6.12 prouve $\mathbb{P}_0^{\geq 0}(\tau > t) \sim C_T t^{-\kappa}$, où $C_T = 2^\kappa \Gamma(\kappa+1) C_U$. On peut dès lors appliquer le théorème 3.1 à une suite de copies i.i.d. $\widehat{Z}_1, \widehat{Z}_2, \ldots$ de τ. D'après la preuve de ce résultat, seul un petit nombre d'exceptionnellement grands termes suffit à obtenir la limite, les autres (une fois centrés) pouvant être négligés.

D'autre part, le temps d'atteinte de e_n s'écrit

$$\tau(e_n) = Z_0 + \cdots + Z_{n-1},$$

où $Z_i := \tau(e_i, e_{i+1})$. Ces n termes ont même loi sous $\mathbb{P}_0^{\geq 0}$ (et non sous $\mathbb{P}_0$) mais ne sont pas indépendants. On va néanmoins mimer la preuve du cas i.i.d. en montrant que l'on peut négliger la plupart des termes. Plutôt que de séparer les Z_i selon leur valeur, qui dépend à la fois de l'environnement et de la marche, il est plus adapté de distinguer selon les hauteurs H_i. Suivant la preuve du théorème 3.1 et l'ordre de grandeur $\tau \simeq \mathrm{e}^H$, on appelle *grandes excursions* celles dont la hauteur est supérieure à h_n définie par $\mathrm{e}^{h_n} = \varepsilon_n n^{1/\kappa}$, où on choisit (pour des raisons techniques) $\varepsilon_n := \frac{1}{\log n}$. Leur nombre est donc (par l'estimée d'Iglehart) d'ordre $(\varepsilon_n)^{-\kappa} = (\log n)^\kappa$.

Pour $0 < \kappa < 1$, la négligeabilité des temps de traversée des petites excursions, centrés ou non, s'obtient simplement par une méthode de premier moment (majoration de leur espérance $n\mathbb{E}^{\geq 0}[\tau \mathbf{1}_{\{H<h\}}]$), qui ne fait donc pas intervenir les corrélations entre les termes. Pour le cas $1 < \kappa < 2$, où le centrage est nécessaire, la preuve est rendue plus délicate par la nécessité de faire appel à la variance *annealed*, qui se décompose en $\mathbb{V}\mathrm{ar}_0(\cdot) = \mathit{Var}(E_{0,\omega}[\cdot]) + \mathbb{E}[\mathit{Var}_{0,\omega}(\cdot)]$. On évoquera le cas $\kappa = 1$ plus bas.

Reste la dépendance entre le nombre logarithmique de grands termes. Les temps de traversée ne sont corrélés que par les remontées de la marche dans les grandes excursions précédentes, dont la durée totale est rendue faible (en espérance) par le large espacement entre les grandes excursions. On formalise ceci en définissant une extrêmité gauche (notée a_i) aux vallées, où le potentiel est avec grande probabilité supérieur au sommet de l'excursion, et on montre d'une part qu'avec grande probabilité les vallées ainsi définies sont disjointes, et d'autre part que l'on peut négliger le temps passé à gauche des vallées une fois leur fond atteint. La seule partie de $\tau(e_n)$ apportant une contribution de l'ordre de $n^{1/\kappa}$ aux fluctuations est donc formée des temps de traversée des grandes excursions tronquées à gauche. Avec grande probabilité, celles-ci sont disjointes, de sorte que leurs temps de traversée coïncident avec des variables indépendantes.

En réintégrant des copies indépendantes des fractions de $\tau(e_n)$ que l'on a négligées (temps de traversée des petites excursions et temps passé à gauche des grandes vallées), qui sont tout autant négligeables (les preuves en sont simplifiées) donc sans incidence sur la limite en loi, on se ramène au strict jeu d'hypothèses du théorème 3.1.

Le cas $\kappa = 1$ présente quelques difficultés supplémentaires d'ordre technique. Dans ce cas critique, l'heuristique d'équivalence entre « τ grand » et « H grande » trouve ses limites : pour négliger les traversées des petites excursions, le premier moment est insuffisant vu la nécessité du centrage, et de plus $E[\tau^2 \mathbf{1}_{\{H<h\}}] = \infty$ (alors que, bien sûr, $E[\tau^2 \mathbf{1}_{\{\tau<\mathrm{e}^h\}}] < \infty$), ce qui exclut l'usage direct de la variance. Le temps passé dans les vallées à gauche de 0 contribue en effet sensiblement à $E[\tau^2 \mathbf{1}_{\{H<h\}}]$: la probabilité pour la marche de remonter dans la vallée à gauche de 0 est de l'ordre de $\mathrm{e}^{-H_{-1}}$ et le temps pour en sortir $\mathrm{e}^{H_{-1}}$, de sorte que l'espérance du carré du temps passé dans cette vallée est d'ordre $\mathrm{e}^{H_{-1}}$, non intégrable si $\kappa \leq 1$. La solution proposée consiste en un découpage supplémentaire : on montre que le temps passé à remonter dans les grandes vallées à partir des petites est négligeable (en probabilité) ce qui permet de se ramener à contrôler la variance du temps de traversée des petites excursions dans un environnement où les grandes ont été remplacées par des petites, éliminant ainsi le problème de non-intégrabilité. Cette difficulté du cas $\kappa = 1$ se présente aussi dans l'estimation de la queue de τ (pour montrer que $\{\tau > t\}$ implique, avec grande probabilité, $\{H > \log t - \log\log t\}$ et ainsi permettre d'analyser la forme de la vallée), et se résout de même.

3.3.2. Sur l'hypothèse de non-arithméticité. — L'hypothèse (b) (c.-à-d. que le support de la loi de $V(1) = \log\rho$ engendre un sous-groupe dense de $\mathbb{R}$) est invoquée à plusieurs reprises dans la preuve pour justifier des inégalités strictes telles que $E[\mathrm{e}^{V(e_1)}] < 1$; cet usage-ci est artificiel et pourrait être évité à l'aide de temps de descente *stricts*. Elle est aussi requise pour les estimées asymptotiques de Feller et Iglehart (3.2.2), mais des résultats similaires sont valables dans le cas arithmétique (voir par exemple [**44**] p.218).

En revanche, cette hypothèse de non-arithméticité joue un rôle essentiel dans les estimées des queues de R et Z. Supposons par exemple que $\log\rho$ soit à valeurs entières. Alors la hauteur H est entière également, de sorte que e^H prend ses valeurs le long d'une suite géométrique. D'après l'heuristique selon laquelle M_1 et M_2 dépendent peu de H, on en déduirait que $Z = M_1 M_2 \mathrm{e}^H$ se concentre approximativement autour d'une suite géométrique. Les « creux » dans la distribution empêchent alors la queue de Z d'être à variation régulière et donc dans le bassin d'attraction d'une loi stable (voir (3.1.2)).

Dans le cas arithmétique, on peut s'attendre à ce qu'une limite stable existe le long de certaines sous-suites géométriques. Un tel résultat a été obtenu par Solomon [**43**] dans un cas particulier : $P(\omega_0 = \alpha) = \gamma = 1 - P(\omega_0 = 1)$ où $\alpha, \gamma \in (0,1)$ (les sites avec $\omega_x = 1$ jouent le rôle de miroirs réfléchissant la marche vers la droite). Et par Ben Arous, Fribergh, Gantert et Hammond [**5**] dans le cas des marches aléatoires biaisées sur des arbres de Galton-Waltson. Le rôle des vallées ralentissant la marche y est joué par les branches finies de l'arbre ; le temps passé dans une telle branche est de l'ordre de e^H où H, la hauteur (ou profondeur) de ladite branche, est donc intrinsèquement à valeurs entières.

3.3.3. Principe de la preuve originale. — L'approche initiale de Kesten, Kozlov et Spitzer est basée sur l'étude de la famille de variables $(U_i^n)_{i\le n,\,0\le n}$, avec

$$U_i^n := \#\{k \le \tau(n) | X_k = i,\ X_{k+1} = i-1\}.$$

On constate que l'on a

$$\tau(n) = n + 2\sum_{i\le n} U_i^n \tag{3.3.1}$$

(en plus des n premiers pas $i \to i+1$, $i = 0, \dots, n-1$, chaque pas vers la gauche implique un pas vers la droite le long de la même arête). De plus, sous $P_{0,\omega}$, pour $i \le n$, la loi de U_i^n sachant $U_{i+1}^n, \dots, U_n^n (= 0)$ est la loi de la somme de U_{i+1}^n variables indépendantes de loi géométrique (≥ 0) de paramètre ω_i. Ainsi, en inversant l'ordre des indices, sous $\mathbb{P}_0$,

$$\sum_{i=1}^n U_i^n \overset{\text{(loi)}}{=} \sum_{t=0}^{n-1} Z_t,$$

où $Z_0 = 0, Z_1, \dots$ est un processus de branchement en milieu aléatoire avec un immigrant à chaque génération et une loi de reproduction géométrique de paramètre ω_i pour les individus présents au temps i. Comme la marche passe un temps fini sur $\mathbb{Z}_-$ du fait de sa transience, on peut négliger les indices $i \le 0$ dans (3.3.1) et être donc ramené à étudier le processus Z. En définissant la suite des temps de régénération ν_k, $k \ge 1$, où $Z_{\nu_k} = 0$, la suite des variables

$$W_k := \sum_{\nu_k \le t < \nu_{k+1}} Z_t$$

est i.i.d.. Pour appliquer le théorème 3.1 (ou le théorème central limite), il reste alors à obtenir un équivalent de la queue de W_0. Or (c'est le corps de la preuve) W_0 peut être relié approximativement à la série de renouvellement de Kesten, qui décrit ici le nombre moyen de descendants du premier immigrant, de sorte qu'intervient le théorème de Kesten pour conclure.

CHAPITRE 4

INTEGRABILITY OF EXIT TIMES AND BALLISTICITY FOR RANDOM WALKS IN DIRICHLET ENVIRONMENT

Abstract.We consider random walks in Dirichlet random environment. Since the Dirichlet distribution is not uniformly elliptic, the annealed integrability of the exit time out of a given finite subset is a non-trivial question. In this paper we provide a simple and explicit equivalent condition for the integrability of Green functions and exit times on any finite directed graph. The proof relies on a quotienting procedure allowing for an induction argument on the cardinality of the graph. This integrability problem arises in the definition of Kalikow auxiliary random walk. Using a particular case of our condition, we prove a refined version of the ballisticity criterion given by Enriquez and Sabot in [**15**].

4.1. Introduction

Since their introduction in the 70's, models of random walks in random environment have mostly been studied in the one dimensional case. Using specific features of this setting, like the reversibility of the Markov chain, Solomon [**43**] set a first milestone by proving simple explicit necessary and sufficient conditions for transience, and a law of large numbers. In contrast, the multidimensional situation is still poorly understood. A first general transience criterion was provided by Kalikow [**24**], which Sznitman and Zerner [**49**] later proved to imply ballisticity as well. Under an additional uniform ellipticity hypothesis, Sznitman ([**46**], [**47**]) could weaken this ballisticity criterion, but not much progress was made since then about the delicate question of sharpening transience or ballisticity criterions.

Another approach consists in deriving explicit conditions in more specific random environments. Among them, Dirichlet environments, first studied by Enriquez and Sabot in [**15**], appear as a natural choice because of their connection with oriented edge linearly reinforced random walks (cf. [**14**], and [**34**] for a review on reinforced processes). Another interest in this case comes from the existence of algebraic relations involving Green functions. These relations allowed Enriquez and Sabot to show that Kalikow's criterion is satisfied under some simple condition, thus proving ballistic behaviour, and to give estimates of the limiting velocity.

Defining Kalikow's criterion raises the problem of integrability of Green functions on finite subsets. While this property is very easily verified for a uniformly elliptic environment, it is no longer the case in the Dirichlet situation. In **[15]**, the condition on the environment allowed for a quick proof, and the general case remained unanswered.

The main aim of this article is to state and prove a simple necessary and sufficient condition of integrability of these Green functions in Dirichlet environment on general directed graphs. Integrability conditions for exit times are then easily deduced. The "sufficiency" part of the proof is the more delicate. It procedes by induction on the size of the graph by going through an interesting quotienting procedure.

This sharpening of the integrability criterion, along with an additional trick, allows us to prove a refined version of Enriquez and Sabot's ballisticity criterion. The condition of non integrability may also prove useful in further analysis of random walks in Dirichlet environment. Indeed, finite subsets with non integrable exit times play the role of "strong traps" for the walk. As a simple example, one can prove that the existence of such a subset implies a null limiting velocity.

Next section introduces the notations, states the results and various corollaries. Section 4.3 contains the proofs of the main result and corollary. Finally, Section 4.4 proves the generalization of Enriquez and Sabot's criterion.

4.2. Definitions and statement of the results

4.2.1. Dirichlet distribution. — Let us first recall the definition of the usual Dirichlet distribution. Let I be a finite set. The set of probability distributions on I is denoted by $\mathrm{Prob}(I)$:

$$\mathrm{Prob}(I) := \{(p_i)_{i\in I} \in \mathbb{R}_+^I : \textstyle\sum_{i\in I} p_i = 1\}.$$

Given a family $(\alpha_i)_{i\in I}$ of positive real numbers, the **Dirichlet distribution** of parameter $(\alpha_i)_{i\in I}$ is the probability distribution $\mathcal{D}((\alpha_i)_{i\in I})$ on $\mathrm{Prob}(I)$ of density

$$(x_i)_{i\in I} \mapsto \frac{\Gamma(\sum_{i\in I}\alpha_i)}{\prod_{i\in I}\Gamma(\alpha_i)} \prod_{i\in I} x_i^{\alpha_i-1}$$

with respect to the Lebesgue measure $\prod_{i\neq i_0} \mathrm{d}x_i$ (where i_0 is any element of I) on the simplex $\mathrm{Prob}(I)$. We will recall a few properties of this distribution on page 45.

4.2.2. Definition of the model. — In order to later deal with multiple edges, we define a **directed graph** as a quadruplet $G = (V, E, \mathit{head}, \mathit{tail})$ where V and E are two sets whose elements are respectively called the **vertices** and **edges** of G, endowed with two maps $\mathit{head} : e \mapsto \overline{e}$ and $\mathit{tail} : e \mapsto \underline{e}$ from E to V. An edge $e \in E$ is thought of as an oriented link from $\underline{e}$ (*tail*) to $\overline{e}$ (*head*), and the usual definitions apply. Thus, a vertex x is **connected** to a vertex y in G if there is an oriented path from x to y, i.e. a sequence $e_1, \ldots, e_n$ of edges with $\underline{e_1} = x$,

$\overline{e_k} = \underline{e_{k+1}}$ for $k = 1, \dots, n-1$, and $\overline{e_n} = y$. For brevity, we usually only write $G = (V, E)$, the tail and head of an edge e being always denoted by $\underline{e}$ and $\overline{e}$.

Unless explicitely stated otherwise (i.e. before the quotienting procedure page 49), graphs are however supposed not to have multiple edges, so that the notation (x, y) for the edge from x to y makes sense.

In the following, we consider *finite* directed graphs $G = (V \cup \{\partial\}, E)$ (without multiple edges) possessing a cemetery vertex ∂. In this setting, we always assume that the set of edges is such that

(i) ∂ is a dead end: no edge in E exits this vertex;
(ii) every vertex is connected to ∂ through a path in E.

Let $G = (V \cup \{\partial\}, E)$ be such a graph. For all $x \in V$, let $\mathcal{P}_x$ designate the set of probability distributions on the set of edges originating at x:

$$\mathcal{P}_x := \left\{ (p_e)_{e \in E, \underline{e}=x} \in \mathbb{R}_+^{\{e \in E : \underline{e}=x\}} : \textstyle\sum_{e \in E, \underline{e}=x} p_e = 1 \right\}.$$

Then the set of **environments** is

$$\Omega := \prod_{x \in V} \mathcal{P}_x \subset \mathbb{R}^E.$$

We will denote by $\omega = (\omega_e)_{e \in E}$ the canonical random variable on Ω, and we usually write $\omega(x, y)$ instead of $\omega_{(x,y)}$.

Given a family $\alpha = (\alpha_e)_{e \in E}$ of positive weights indexed by the set of edges of G, the **Dirichlet distribution on environments** of parameter α is the product measure on $\Omega := \prod_{x \in V} \mathcal{P}_x$ of Dirichlet distributions on each of the $\mathcal{P}_x$, $x \in V$:

$$\mathbb{P} = \mathbb{P}^{(\alpha)} := \prod_{x \in V} \mathcal{D}((\alpha_e)_{e \in E,\ \underline{e}=x}).$$

Note that this distribution does not satisfy the usual uniform ellipticity condition: there is no positive constant bounding $\mathbb{P}$-almost surely the transition probabilities ω_e from below.

In the case of $\mathbb{Z}^d$, we always consider translation invariant distributions of environments, hence the parameters are identical at each vertex and we only need to be given a $2d$-uplet $(\alpha_e)_{e \in \mathcal{V}}$ where $\mathcal{V} := \{e \in \mathbb{Z}^d : |e| = 1\}$. This is the usual case of i.i.d. Dirichlet environment.

The canonical process on V will be denoted by $(X_n)_{n \geq 0}$, and the canonical shift on $V^{\mathbb{N}}$ by Θ: for $p \in \mathbb{N}$, $\Theta_p((X_n)_{n \in \mathbb{N}}) := (X_{p+n})_{n \in \mathbb{N}}$.

For any environment $\omega \in \Omega$, and any vertex $x \in V$, the **quenched law** in the environment ω starting at $x \in V$ is the distribution $P_{x,\omega}$ of the Markov chain starting at x with transition probabilities given by ω and stopped when it hits ∂. Thus, for every $o \in V$, $(x, y) \in E$, $\omega \in \Omega$, $n \in \mathbb{N}$,

$$P_{o,\omega}(X_{n+1} = y | X_n = x) = \omega(x, y)$$

and

$$P_{o,\omega}(X_{n+1} = \partial | X_n = \partial) = 1.$$

The **annealed law** starting at $x \in V$ is then the following averaged distribution on random walks on G:

$$P_x(\cdot) := \int P_{x,\omega}(\cdot)\mathbb{P}(\,\mathrm{d}\omega) = \mathbb{E}[P_{x,\omega}(\cdot)].$$

We will need the following stopping times:

$$\begin{aligned}
\text{for } A \subset E, \quad & T_A := \inf\{n \geq 1 : (X_{n-1}, X_n) \notin A\},\\
\text{for } U \subset V, \quad & T_U := \inf\{n \geq 0 : X_n \notin U\},\\
\text{for } x \in V, \quad & H_x := \inf\{n \geq 0 : X_n = x\}\\
\text{and} \quad & \widetilde{H}_x := \inf\{n \geq 1 : X_n = x\}.
\end{aligned}$$

If the random variable N_y denotes the number of visits of $(X_n)_{n\geq 0}$ at site y (before it hits ∂), then the **Green function** G^ω of the random walk in the environment ω is given by

$$\text{for all } x, y \in V,\ G^\omega(x,y) := E_{x,\omega}[N_y] = \sum_{n\geq 0} P_{x,\omega}(X_n = y).$$

Due to the assumption (ii), $G^\omega(x,y)$ is $\mathbb{P}$-almost surely finite for all $x, y \in V$. The question we are concerned with is the integrability of these functions under $\mathbb{P}$, depending on the value of α.

4.2.3. Integrability conditions. — The main quantity involved in our conditions is the sum of the coefficients α_e over the edges e exiting some set. For every subset A of E, define

$$\begin{aligned}
\underline{A} &:= \{\underline{e} : e \in A\} \subset V,\\
\overline{A} &:= \{\overline{e} : e \in A\} \subset V \cup \{\partial\},\\
\underline{\overline{A}} &:= \{\underline{e} : e \in A\} \cup \{\overline{e} : e \in A\} \subset V \cup \{\partial\},\\
\partial_E A &:= \{e \in E \setminus A : \underline{e} \in \underline{A}\} \subset E,
\end{aligned}$$

and the sum of the coefficients of the edges "exiting A"

$$\beta_A := \sum_{e \in \partial_E A} \alpha_e.$$

A is said to be **strongly connected** if, for all $x, y \in \underline{\overline{A}}$, x is **connected** to y in A, i.e. there is an (oriented) path from x to y through edges in A. If A is strongly connected, then $\underline{A} = \overline{A}$.

Theorem 4.1. — *Let $G = (V \cup \{\partial\}, E)$ be a finite directed graph and $\alpha = (\alpha_e)_{e\in E}$ be a family of positive real numbers. We denote by $\mathbb{P}$ the corresponding Dirichlet distribution. Let $o \in V$. For every $s > 0$, the following statements are equivalent:*

(i) $\mathbb{E}[G^\omega(o,o)^s] < \infty$*;*

(ii) *for every strongly connected subset A of E such that $o \in \underline{A}$, $\beta_A > s$.*

Undirected graphs are directed graphs where edges come in pair: if $(x,y) \in E$, then $(y,x) \in E$ as well. In this case, the previous result translates into a statement on subsets of V. For any $S \subset V$, we denote by β_S the sum of the coefficients of the edges "exiting S":

$$\beta_S := \sum_{\underline{e}\in S,\ \overline{e}\notin S} \alpha_e.$$

Suppose there is no loop in G (i.e. no edge both exiting from and heading to the same vertex). For any strongly connected subset A of E, if we let $S := \underline{A}$, then S is connected, $|S| \geq 2$ and $\beta_S \leq \beta_A$. Conversely, provided the graph is undirected, a connected subset S of V of cardinality at least 2 satisfies $\beta_S = \beta_A$ where $A := \{e \in E : \underline{e} \in S, \overline{e} \in S\}$, which is strongly connected. This remark yields

Theorem 4.2. — *Let $G = (V \cup \{\partial\}, E)$ be a finite undirected graph without loop and $(\alpha_e)_{e\in E}$ be a family of positive real numbers. We denote by $\mathbb{P}$ the corresponding Dirichlet distribution. Let $o \in V$. For every $s > 0$, the following statements are equivalent:*

(i) $\mathbb{E}[G^\omega(o,o)^s] < \infty$;
(ii) *every connected subset S of V such that $\{o\} \subsetneq S$, $\beta_S > s$.*

In particular, we get the case of i.i.d. environments in $\mathbb{Z}^d$. Given a subset U of $\mathbb{Z}^d$, let us introduce the Green function G_U^ω of the random walk on $\mathbb{Z}^d$, in environment ω, killed when exiting U. Identifying the complement of U with a cemetery point ∂ allows to apply the previous theorem to G_U^ω. Among the connected subsets S of vertices of $\mathbb{Z}^d$ such that $\{o\} \subsetneq S$, the ones minimizing the "exit sum" β_S are made of the two endpoints of an edge. The result may therefore be stated as

Theorem 4.3. — *Let $\alpha = (\alpha_e)_{e\in\mathcal{V}}$ be a family of positive real numbers. We denote by $\mathbb{P}$ the translation invariant Dirichlet distribution on environments on $\mathbb{Z}^d$ associated with α. Let U be a finite subset of $\mathbb{Z}^d$ containing o. Let $\Sigma := \sum_{e\in\mathcal{V}} \alpha_e$. Then for every $s > 0$, the following assertions are equivalent:*

(i) $\mathbb{E}[G_U^\omega(o,o)^s] < \infty$;
(ii) *for every edge $e = (o,x)$ with $x \in U$, $2\Sigma - \alpha_e - \alpha_{-e} > s$.*

Assuming the hypothesis of Theorem 4.1 to be satisfied relatively to all vertices instead of only one provides information about exit times.

Corollary 4.4. — *Let $G = (V \cup \{\partial\}, E)$ be a finite strongly connected directed graph and $(\alpha_e)_{e\in E}$ be a family of positive real numbers. We denote by $\mathbb{P}$ the corresponding Dirichlet distribution. For every $s > 0$, the following properties are equivalent:*

(i) *for every vertex x, $\mathbb{E}[E_{x,\omega}[T_V]^s] < \infty$;*
(ii) *for every vertex x, $\mathbb{E}[G^\omega(x,x)^s] < \infty$;*
(iii) *every non-empty strongly connected subset A of E satisfies $\beta_A > s$;*
(iv) *there is a vertex x such that $\mathbb{E}[E_{x,\omega}[T_V]^s] < \infty$.*

And in the undirected case,

Corollary 4.5. — *Let $G = (V \cup \{\partial\}, E)$ be a finite connected undirected graph without loop, and $(\alpha_e)_{e\in E}$ be a family of positive real numbers. We denote by $\mathbb{P}$ the corresponding Dirichlet distribution. For every $s > 0$, the following properties are equivalent:*

(i) *for every vertex x, $\mathbb{E}[E_{x,\omega}[T_V]^s] < \infty$;*
(ii) *for every vertex x, $\mathbb{E}[G^\omega(x,x)^s] < \infty$;*
(iii) *every connected subset S of V of cardinality ≥ 2 satisfies $\beta_S > s$;*
(iv) *there is a vertex x such that $\mathbb{E}[E_{x,\omega}[T_V]^s] < \infty$.*

4.2.4. Ballisticity criterion. — We now consider the case of random walks in i.i.d. Dirichlet environment on $\mathbb{Z}^d$, $d \geq 1$.

Let $(e_1, \ldots, e_d)$ denote the canonical basis of $\mathbb{Z}^d$, and $\mathcal{V} := \{e \in \mathbb{Z}^d : |e| = 1\}$. Let $(\alpha_e)_{e\in\mathcal{V}}$ be positive numbers. We will write either α_i or α_{e_i}, and α_{-i} or α_{-e_i}, $i = 1, \ldots, d$.

Enriquez and Sabot proved that the random walk in Dirichlet environment has a ballistic behaviour as soon as $\max_{1\leq i\leq d} |\alpha_i - \alpha_{-i}| > 1$. Our improvement replaces ℓ^∞-norm by ℓ^1-norm:

Theorem 4.6. — *If $\sum_{i=1}^{d} |\alpha_i - \alpha_{-i}| > 1$, then there exists $v \neq 0$ such that, P_0-a.s., $\frac{X_n}{n} \to_n v$, and the following bound holds:*

$$\left| v - \frac{\Sigma}{\Sigma - 1} d_m \right|_1 \leq \frac{1}{\Sigma - 1},$$

where $\Sigma := \sum_{e\in\mathcal{V}} \alpha_e$, $d_m := \sum_{i=1}^{d} \frac{\alpha_i - \alpha_{-i}}{\Sigma} e_i$ is the drift in the averaged environment, and $|X|_1 := \sum_{i=1}^{d} |X \cdot e_i|$ for any $X \in \mathbb{R}^d$.

4.3. Proof of the main result

Let us first give a few comments about the proof of Theorem 4.1. Proving this integrability condition amounts to finding bounds for the tail probability $\mathbb{P}(G^\omega(o,o) > t)$ from below and above.

In order to get the lower bound, we consider an event consisting of environments with small transition probabilities out of a given subset containing o. This forces the mean exit time out of this subset to be large. However, getting a large number of returns to the starting vertex o requires an additional trick: one needs to control from below the probability of some paths leading back to o. The important yet basic remark here is that, at each vertex, there is at least one exiting edge with transition probability greater than the inverse number of neighbours at that vertex. By restricting the probability space to an event where, at each vertex, this (random) edge is fixed, we are able to compensate for the non uniform ellipticity of $\mathbb{P}$.

The upper bound is more elaborate. Since $G^\omega(o,o) = 1/P_{o,\omega}(H_\partial < \widetilde{H}_o)$, we need lower bounds on the probability to reach ∂ without coming back to o. If $\mathbb{P}$ were uniformly elliptic, it would suffice to consider a single path from o to ∂. In the present case, we construct a random subset $C(\omega)$ of E containing o where a weaker ellipticity property holds anyway: vertices in $\underline{C(\omega)}$ can be easily connected to each other through paths inside $C(\omega)$ (cf. Proposition 4.8). The probability, from o, to reach ∂ without coming back to o is greater than the probability, from o, to exit $C(\omega)$ without coming back to o and then to reach ∂ without coming back to $C(\omega)$. The uniformity property of $C(\omega)$ allows to bound $P_{o,\omega}(T_{C(\omega)} < \widetilde{H}_o)$ by a simpler quantity, and to relate the probability of reaching ∂ without coming back to $C(\omega)$ to the probability, *in a quotient graph* $\widetilde{G}$, of reaching ∂ from a vertex $\widetilde{o}$ (corresponding to $C(\omega)$) without coming back to $\widetilde{o}$. We thus get a lower bound on $P_{o,\omega}(H_\partial < \widetilde{H}_o)$ involving the same probability relative to a quotient graph. This allows us to perform an induction on the size of the graph. Actually, the environment we get on the quotient graph is not exactly Dirichlet, and we first need to show (cf. Lemma 4.9) that its density can be compared to that of a Dirichlet environment.

4.3.1. Properties of Dirichlet distributions. — Notice that if (p_1,p_2) is a random variable with distribution $\mathcal{D}(\alpha,\beta)$ under P, then p_1 is a Beta variable of parameter (α,β), and has the following tail probability:

$$(*) \qquad P(p_1 \le \varepsilon) \underset{\varepsilon\to 0}{\sim} C\varepsilon^\alpha,$$

where $C := \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)}$.

Let us now recall two useful properties that are simple consequences of the representation of a Dirichlet random variable as a normalized vector of independent gamma random variables (cf. for instance [**52**]). Let $(p_i)_{i\in I}$ be a random variable distributed according to $\mathcal{D}((\alpha_i)_{i\in I})$. Then

- **(Associativity) :** Let $I_1,\dots,I_n$ be a partition of I. The random variable $\big(\sum_{i\in I_k} p_i\big)_{k\in\{1,\dots,n\}}$ on $\mathrm{Prob}(\{1,\dots,n\})$ follows the Dirichlet distribution $\mathcal{D}((\sum_{i\in I_k}\alpha_i)_{1\le k\le n})$.
- **(Restriction) :** Let J be a nonempty subset of I. The random variable $\left(\frac{p_i}{\sum_{j\in J} p_j}\right)_{i\in J}$ on $\mathrm{Prob}(J)$ follows the Dirichlet distribution $\mathcal{D}((\alpha_i)_{i\in J})$ and is independent of $\sum_{j\in J} p_j$ (which follows a Beta distribution $B(\sum_{j\in J}\alpha_j, \sum_{j\notin J}\alpha_j)$ due to the associativity property).

Thanks to the associativity property, the asymptotic estimate $(*)$ holds as well (with a different C) for the marginal p_1 of a Dirichlet random variable $(p_1,\dots,p_n)$ with parameters $(\alpha,\alpha_2,\dots,\alpha_n)$.

4.3.2. First implication (lower bound). — Let $s>0$. Suppose there exists a strongly connected subset A of E such that $o\in\underline{A}$ and $\beta_A \le s$. We shall prove the stronger statement that $\mathbb{E}[G_A^\omega(o,o)^s] = \infty$ where G_A^ω is the Green function of the random walk in the environment ω killed when exiting A.

Let $\varepsilon > 0$. Define the event $\mathcal{E}_\varepsilon := \{\forall x \in \underline{A},\ \sum_{e\in\partial_E A,\ \underline{e}=x} \omega_e \le \varepsilon\}$. On $\mathcal{E}_\varepsilon$, one has

$$E_{o,\omega}[T_A] \ge \frac{1}{\varepsilon}. \tag{4.3.1}$$

Indeed, by the Markov property, for all $n \in \mathbb{N}^*$,

$$\begin{aligned} P_{o,\omega}(T_A > n) &= E_{o,\omega}[\mathbf{1}_{\{T_A>n-1\}} P_{X_{n-1},\omega}(T_A > 1)] \\ &\ge P_{o,\omega}(T_A > n-1) \min_{x\in\overline{A}} P_{x,\omega}(T_A > 1) \end{aligned}$$

and if $\omega \in \mathcal{E}_\varepsilon$ then, for all $x \in \overline{A} = \underline{A}$, $P_{x,\omega}(T_A > 1) \ge 1-\varepsilon$, hence, by recurrence,

$$P_{o,\omega}(T_A > n) \ge P_{o,\omega}(T_A > 0)(1-\varepsilon)^n = (1-\varepsilon)^n,$$

from which inequality (4.3.1) results after summation over $n \in \mathbb{N}$.

As a consequence,

$$\mathbb{E}[E_{o,\omega}[T_A]^s] = \int_0^\infty \mathbb{P}\left(E_{o,\omega}[T_A]^s \ge t\right) \mathrm{d}t \ge \int_0^\infty \mathbb{P}\left(\mathcal{E}_{\frac{1}{t^{1/s}}}\right) \mathrm{d}t.$$

Notice now that, due to the associativity property of section 4.3.1, for every $x \in \underline{A}$, the random variable $\sum_{e\in\partial_E A,\ \underline{e}=x} \omega_e$ follows a Beta distribution with parameters $(\sum_{e\in E\setminus A,\ \underline{e}=x} \alpha_e, \sum_{e\in A,\ \underline{e}=x} \alpha_e)$, so that the tail probability $(*)$ together with the spatial independence gives

$$\mathbb{P}(\mathcal{E}_\varepsilon) \underset{\varepsilon\to 0}{\sim} C\varepsilon^{\beta_A},$$

where C is a positive constant. Hence $\mathbb{P}\left(\mathcal{E}_{\frac{1}{t^{1/s}}}\right) \underset{t\to\infty}{\sim} Ct^{-\frac{\beta_A}{s}}$, and the assumption $\beta_A \le s$ leads to

$$\mathbb{E}[E_{o,\omega}[T_A]^s] = \infty.$$

Dividing T_A into the time spent at each point of $\underline{A}$, one has

$$\begin{aligned} E_{o,\omega}[T_A]^s = \left(\sum_{x\in\underline{A}} G_A^\omega(o,x)\right)^s &\le \left(|\underline{A}| \max_{x\in\underline{A}} G_A^\omega(o,x)\right)^s = |\underline{A}|^s \max_{x\in\underline{A}} G_A^\omega(o,x)^s \\ &\le |\underline{A}|^s \sum_{x\in\underline{A}} G_A^\omega(o,x)^s, \end{aligned} \tag{4.3.2}$$

so that there is a vertex $x \in \underline{A}$ such that $\mathbb{E}[G_A^\omega(o,x)^s] = \infty$.

Getting the result on $G_A^\omega(o,o)$ requires to refine this proof. To that aim, we shall introduce an event $\mathcal{F}$ of positive probability on which, from every vertex of $\underline{A}$, there exists a path toward o whose transition probability is bounded from below uniformly on $\mathcal{F}$.

Let $\omega \in \Omega$. Denote by $\vec{G}(\omega)$ the set of the edges $e^* \in E$ such that $\omega_{e^*} = \max\{\omega_e : e \in E, \underline{e} = \underline{e^*}\}$. If $e^* \in \vec{G}(\omega)$, then (by a simple pigenhole argument),

$$\omega_{e^*} \ge \frac{1}{n_{\underline{e^*}}} \ge \frac{1}{|E|},$$

where n_x is the number of neighbours of a vertex x. In particular, there is a positive constant κ depending only on G such that, if x is connected to y through a (simple) path π in $\vec{G}(\omega)$ then $P_{x,\omega}(\pi) \geq \kappa$. Note that for $\mathbb{P}$-almost every ω in Ω, there is exactly one maximizing edge e^* exiting each vertex.

Since A is a strongly connected subset of E, it possesses at least one spanning tree T oriented toward o. Let us denote by $\mathcal{F}$ the event $\{\vec{G}(\omega) = T\}$: if $\omega \in \mathcal{F}$, then every vertex of $\underline{A}$ is connected to o in $\vec{G}(\omega)$. One still has

$$\mathbb{P}(\mathcal{E}_\varepsilon \cap \mathcal{F}) \geq \mathbb{P}(\mathcal{E}_\varepsilon \cap \{\forall e \in T, \omega_e > 1/2\}) \underset{\varepsilon\to 0}{\sim} C'\varepsilon^{\beta_A},$$

where C' is a positive constant (depending on the choice of T). Indeed, using the associativity property and the spatial independence, this asymptotic equivalence reduces to the fact that if (p_1, p_2) has distribution $\mathcal{D}(\alpha,\beta)$ or if (p_1,p_2,p_3) has distribution $\mathcal{D}(\alpha,\beta,\gamma)$, then there is $c > 0$ such that $P(p_1 \leq \varepsilon, p_2 > 1/2) \underset{\varepsilon\to 0}{\sim} c\varepsilon^\alpha$. In the first case, this is exactly $(*)$, and in the case of 3 variables,

$$\begin{aligned}
P(p_1 \leq \varepsilon, p_2 > 1/2) &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^\varepsilon \int_{1/2}^1 x_1^{\alpha-1} x_2^{\beta-1}(1-x_1-x_2)^{\gamma-1}\,\mathrm{d}x_2\,\mathrm{d}x_1 \\
&\underset{\varepsilon\to 0}{\sim} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^\varepsilon x_1^{\alpha-1}\,\mathrm{d}x_1 \int_{1/2}^1 x_2^{\beta-1}(1-x_2)^{\gamma-1}\,\mathrm{d}x_2 \\
&= c\varepsilon^\alpha.
\end{aligned}$$

Then, as previously,

$$\mathbb{E}[E_{o,\omega}[T_A]^s, \mathcal{F}] = \int_0^\infty \mathbb{P}(E_{o,\omega}[T_A]^s \geq t, \mathcal{F})\,\mathrm{d}t \geq \int_0^\infty \mathbb{P}\left(\mathcal{E}_{\frac{1}{t^{1/s}}} \cap \mathcal{F}\right)\mathrm{d}t = +\infty,$$

and subsequently there exists $x \in \underline{A}$ such that $\mathbb{E}[G_A^\omega(o,x)^s, \mathcal{F}] = \infty$. Now, there is an integer l and a real number $\kappa > 0$ such that, for every $\omega \in \mathcal{F}$, $P_{x,\omega}(X_l = o) \geq \kappa$. Thus, thanks to the Markov property,

$$\begin{aligned}
G_A^\omega(o,x) &= \sum_{k\geq 0} P_{o,\omega}(X_k = x, T_A > k) \\
&\leq \sum_{k\geq 0} \frac{1}{\kappa} P_{o,\omega}(X_{k+l} = o, T_A > k+l) \\
&\leq \frac{1}{\kappa} G_A^\omega(o,o).
\end{aligned}$$

Therefore we get

$$\mathbb{E}[G_A^\omega(o,x)^s, \mathcal{F}] \leq \frac{1}{\kappa^s}\mathbb{E}[G_A^\omega(o,o)^s, \mathcal{F}],$$

and finally $\mathbb{E}[G_A^\omega(o,o)^s] = \infty$.

4.3.3. Converse implication (upper bound). —

Scheme of the proof. — The proof of the upper bound procedes by induction on the number of edges of the graph. More precisely, we prove the following property by induction on $n \geq 1$:

Proposition 4.7 **(Induction hypothesis)**. — *Let $n \in \mathbb{N}^*$. Let $G = (V \cup \{\partial\}, E)$ be a directed graph possessing at most n edges and such that every vertex* *is connected to ∂, and $(\alpha_e)_{e\in E}$ be positive real numbers. We denote by $\mathbb{P}$ the corresponding Dirichlet distribution. Then, for every vertex $o \in V$, there exist real numbers $C, r > 0$ such that, for small $\varepsilon > 0$,*

$$\mathbb{P}(P_{o,\omega}(H_\partial < \widetilde{H}_o) \leq \varepsilon) \leq C\varepsilon^\beta(-\ln\varepsilon)^r,$$

where $\beta := \min\{\beta_A : A$ is a strongly connected subset of E and $o \in \underline{A}\}$.

The implication (ii)⇒(i) in Theorem 4.1 results from this proposition using the integrability of $t \mapsto \frac{(\ln t)^r}{t^\beta}$ in the neighbourhood of $+\infty$ as soon as $\beta > 1$, and from the following Markov chain identity:

$$G^\omega(o,o) = \frac{1}{P_{o,\omega}(H_\partial < \widetilde{H}_o)}.$$

Let us initialize the induction. If a graph $G = (V \cup \{\partial\}, E)$ with a vertex o is such that $|E| = 1$ and o is connected to ∂, the only edge links o to ∂, so that $P_{o,\omega}(H_\partial < \widetilde{H}_o) = 1$, and the property is true (β is infinite here).

Let $n \in \mathbb{N}^*$. We suppose the induction hypothesis to be true at rank n. Let $G = (V \cup \{\partial\}, E)$ be a directed graph with $n+1$ edges, o be a vertex and $(\alpha_e)_{e\in E}$ be positive parameters. As usual, $\mathbb{P}$ is the corresponding Dirichlet distribution. In the next paragraph, we introduce the random subset $C(\omega)$ of E we will then be interested to quotient G by.

Construction of $C(\omega)$. — Let $\omega \in \Omega$. We define inductively a finite sequence $e_1 = (x_1, y_1), \ldots, e_m = (x_m, y_m)$ of edges in the following way: letting $y_0 = o$, if $e_1, \ldots, e_{k-1}$ have been defined, then e_k is the edge in E which maximizes the exit distribution out of $C_k := \{e_1, \ldots, e_{k-1}\}$ starting at y_{k-1}, that is

$$e \mapsto P_{y_{k-1},\omega}((X_{T_{C_k}-1}, X_{T_{C_k}}) = e).$$

The integer m is the least index ≥ 1 such that $y_m \in \{o, \partial\}$. In words, the edge e_k is, among the edges exiting the set $C_k(\omega)$ of already visited edges, the one maximizing the probability for a random walk starting at y_{k-1} to exit $C_k(\omega)$ through it; and the construction ends as soon as an edge e_k heads at o or ∂. The assumption that each vertex is connected to ∂ guarantees the existence of an exit edge out of $C_k(\omega)$ for $k \leq m$, and the finiteness of G ensures that m exists: the procedure ends. We set

$$C(\omega) := C_{m+1} = \{e_1, \ldots, e_m\}.$$

Note that the maximizing edges, and thus $C(\omega)$, are in fact well defined only for ω out of a Lebesgue negligible subset of Ω.

Notice that $C_1 = \varnothing$, hence $T_{C_1} = 1$ and e_1 is the edge maximizing $e \mapsto \omega_e$ among the edges starting at o. Hence e_1 is the edge of $\vec{G}(\omega)$ starting at o (cf. the proof of the first implication). More generally, for $1 \leq k \leq m$, if $y_{k-1} \notin \{x_1, \ldots, x_{k-1}\}$, then e_k is the edge of $\vec{G}(\omega)$ starting at y_{k-1}.

The main property of $C(\omega)$ is the following:

Proposition 4.8. — *There exists a constant $c > 0$ such that, for every $\omega \in \Omega$ such that $C(\omega)$ is well defined, for all $x \in \overline{C(\omega)} \setminus \{o\}$,*

$$P_{o,\omega}(H_x < \widetilde{H}_o \wedge T_{C(\omega)}) \geq c. \tag{4.3.3}$$

Proof. — Let ω be such that $C(\omega)$ is well defined. For $k = 1, \ldots, m$, due to the choice of e_k as a maximizer over E (or $\partial_E C_k$), we have

$$P_{y_{k-1},\omega}((X_{T_{C_k}-1}, X_{T_{C_k}}) = e_k) \geq \frac{1}{|E|} = \kappa.$$

For every k such that $y_k \neq o$ (that is for $k = 1, \ldots, m-1$ and possibly $k = m$), we deduce that

$$P_{y_{k-1},\omega}(H_{y_k} < \widetilde{H}_o \wedge T_{C(\omega)}) \geq P_{y_{k-1},\omega}(X_{T_{C_k}} = y_k) \geq \kappa.$$

Then, by the Markov property, for any $x \in \overline{C(\omega)} = \{y_1, \ldots, y_m\}$, if $x \neq o$,

$$P_{o,\omega}(H_x < \widetilde{H}_o \wedge T_{C(\omega)}) \geq \kappa^m \geq \kappa^{|E|} = c,$$

as expected. □

The support of the distribution of $\omega \mapsto C(\omega)$ writes as a disjoint union $\mathcal{C} := \mathcal{C}_o \cup \mathcal{C}_\partial$ depending on whether o or ∂ belongs to $\overline{C(\omega)}$. For any $C \in \mathcal{C}$, we define the event

$$\mathcal{E}_C := \{C(\omega) = C\}.$$

On such an event, the previous proposition gives uniform lower bounds inside C, "as if" a uniform ellipticity property held. Because $\mathcal{C}$ is finite, it will be sufficient to prove the upper bound separately on the events $\mathcal{E}_C$ for $C \in \mathcal{C}$.

If $C \in \mathcal{C}_\partial$, then $\partial \in \overline{C}$ and the proposition above provides $c > 0$ such that, on $\mathcal{E}_C$, $P_{o,\omega}(H_\partial < \widetilde{H}_o) \geq c$ hence, for small $\varepsilon > 0$,

$$\mathbb{P}(P_{o,\omega}(H_\partial < \widetilde{H}_o) \leq \varepsilon,\ \mathcal{E}_C) = 0.$$

In the following we will therefore work on $\mathcal{E}_C$ where $C \in \mathcal{C}_o$. In this case, C is a strongly connected subset of E. Indeed, $y_0 = y_m = o$ and, due to the construction method, y_k is connected in $C(\omega)$ to y_{k+1} for $k = 0, \ldots, m-1$.

Quotienting procedure. — We intend to introduce a quotient of G by contracting the edges of C, and to relate its Green function to that of G. Let us first give a general definition:

Definition 1. — *If A is a strongly connected subset of edges of a graph $G = (V, E, head, tail)$, the quotient graph of G obtained by contracting A to $\widetilde{a}$ is the graph $\widetilde{G}$ deduced from G by deleting the edges of A, replacing all the vertices of $\underline{A}$ by one new vertex $\widetilde{a}$, and modifying the endpoints of the edges of $E \setminus A$ accordingly. Thus the set of edges of $\widetilde{G}$ is naturally in bijection with $E \setminus A$ and can be thought of as a subset of E.*

In other words, $\widetilde{G} := (\widetilde{V}, \widetilde{E}, \widetilde{head}, \widetilde{tail})$ where $\widetilde{V} := (V \setminus \underline{A}) \cup \{\widetilde{a}\}$ ($\widetilde{a}$ being a new vertex), $\widetilde{E} := E \setminus A$ and, if π denotes the projection from V to $\widetilde{V}$ (i.e. $\pi_{|V \setminus \underline{A}} := \mathrm{id}$ and $\pi(x) := \widetilde{a}$ if $x \in \underline{A}$), $\widetilde{head} := \pi \circ head$ and $\widetilde{tail} := \pi \circ tail$.

Notice that this quotient may well introduce multiple edges.

In our case, we consider the quotient graph $\widetilde{G} := (\widetilde{V} := (V \setminus \underline{C}) \cup \{\partial, \widetilde{o}\}, \widetilde{E} := E \setminus C, \widetilde{head}, \widetilde{tail})$ obtained by contracting C to a new vertex $\widetilde{o}$.

Starting from $\omega \in \Omega$, let us define the *quotient environment* $\widetilde{\omega} \in \widetilde{\Omega}$, where $\widetilde{\Omega}$ is the analog of Ω for $\widetilde{G}$. For every edge $e \in \widetilde{E}(\subset E)$, if $e \notin \partial_E C$ (i.e. if $\widetilde{tail}(e) \neq \widetilde{o}$), then $\widetilde{\omega}_e = \omega_e$, and if $e \in \partial_E C$, then $\widetilde{\omega}_e = \frac{\omega_e}{\Sigma}$, where

$$\Sigma := \sum_{e \in \partial_E C} \omega_e.$$

This environment allows to bound the Green function of G using that of $\widetilde{G}$ in a convenient way. Notice that, from o, one way for the walk to reach ∂ without coming back to o consists in exiting C without coming back to o and then reaching ∂ without coming back to $\underline{C}$. Thus we have, for $\omega \in \mathcal{E}_C$,

$$\begin{aligned} P_{o,\omega}(H_\partial < \widetilde{H}_o) &\geq \sum_{x \in \underline{C}} P_{o,\omega}(H_x < \widetilde{H}_o \wedge T_C, H_\partial < H_x + \widetilde{H}_{\underline{C}} \circ \Theta_{H_x}) \\ &= \sum_{x \in \underline{C}} P_{o,\omega}(H_x < \widetilde{H}_o \wedge T_C) P_{x,\omega}(H_\partial < \widetilde{H}_{\underline{C}}) \\ &\overset{(4.3.3)}{\geq} c \sum_{x \in \underline{C}} P_{x,\omega}(H_\partial < \widetilde{H}_{\underline{C}}) \\ &= c\Sigma \cdot P_{\widetilde{o},\widetilde{\omega}}(H_\partial < \widetilde{H}_{\widetilde{o}}) \end{aligned}$$

where the first equality is an application of the Markov property at time H_x, and the last equality comes from the definition of the quotient: both quantities correspond to the same set of paths viewed in G and in $\widetilde{G}$, and, for all $x \in \underline{C}$, $P_{x,\omega}$-almost every path belonging to the event $\{H_\partial < \widetilde{H}_{\underline{C}}\}$ contains exactly one edge exiting from $\underline{C}$ so that the normalization by Σ appears exactly once by quotienting.

Finally, for $c' := 1/c > 0$, we have

$$\mathbb{P}(P_{o,\omega}(H_\partial < \widetilde{H}_o) \leq \varepsilon,\ \mathcal{E}_C) \leq \mathbb{P}(\Sigma \cdot P_{\widetilde{o},\widetilde{\omega}}(H_\partial < \widetilde{H}_{\widetilde{o}}) \leq c'\varepsilon,\ \mathcal{E}_C). \tag{4.3.4}$$

Back to Dirichlet environment. — It is important to remark that, under $\mathbb{P}$, $\widetilde{\omega}$ does not follow a Dirichlet distribution because of the normalization (and neither is it independent of Σ). We can however reduce to the Dirichlet situation and thus procede to induction. This is the aim of the following lemma, inspired by the restriction property of Section 4.3.1. Because its proof, while simple, is a bit tedious to write, we defer it until the Appendix page 57.

Lemma 4.9. — *Let $(p_i^{(1)})_{1\leq i\leq n_1}, \ldots, (p_i^{(r)})_{1\leq i\leq n_r}$ be independent Dirichlet random variables with respective parameters $(\alpha_i^{(1)})_{1\leq i\leq n_1}, \ldots, (\alpha_i^{(r)})_{1\leq i\leq n_r}$. Let $m_1, \ldots, m_r$ be integers such that $1 \leq m_1 < n_1, \ldots, 1 \leq m_r < n_r$, and let $\Sigma := \sum_{j=1}^r \sum_{i=1}^{m_j} p_i^{(j)}$ and $\beta := \sum_{j=1}^r \sum_{i=1}^{m_j} \alpha_i^{(j)}$. There exists positive constants*

C, C' such that, for any positive measurable function $f : \mathbb{R} \times \mathbb{R}^{\sum_j m_j} \to \mathbb{R}$,

$$E\left[f\left(\Sigma, \frac{p_1^{(1)}}{\Sigma}, \ldots, \frac{p_{m_1}^{(1)}}{\Sigma}, \ldots, \frac{p_1^{(r)}}{\Sigma}, \ldots, \frac{p_{m_r}^{(r)}}{\Sigma}\right)\right] \\ \leq C \cdot \widetilde{E}\left[f\left(\widetilde{\Sigma}, \widetilde{p}_1^{(1)}, \ldots, \widetilde{p}_{m_1}^{(1)}, \ldots, \widetilde{p}_1^{(r)}, \ldots, \widetilde{p}_{m_r}^{(r)}\right)\right],$$

where, under the probability $\widetilde{P}$, $(\widetilde{p}_1^{(1)}, \ldots, \widetilde{p}_{m_1}^{(1)}, \ldots, \widetilde{p}_1^{(r)}, \ldots, \widetilde{p}_{m_r}^{(r)})$ is sampled from a Dirichlet distribution of parameter $(\alpha_1^{(1)}, \ldots, \alpha_{m_1}^{(1)}, \ldots, \alpha_1^{(r)}, \ldots, \alpha_{m_r}^{(r)})$, $\widetilde{\Sigma}$ is bounded and satisfies $\widetilde{P}(\widetilde{\Sigma} < \varepsilon) \leq C'\varepsilon^{\beta}$ for every $\varepsilon > 0$, and these two random variables are independent.

We apply this lemma to ω by normalizing the transition probabilities of the edges exiting C. Using (4.3.4) and Lemma 4.9, we thus get

$$\begin{aligned} \mathbb{P}(P_{o,\omega}(H_\partial < \widetilde{H}_o) \leq \varepsilon,\ \mathcal{E}_C) &\leq \mathbb{P}(\Sigma \cdot P_{\widetilde{o},\widetilde{\omega}}(H_\partial < \widetilde{H}_{\widetilde{o}}) \leq c'\varepsilon,\ \mathcal{E}_C) \\ &\leq \mathbb{P}(\Sigma \cdot P_{\widetilde{o},\widetilde{\omega}}(H_\partial < \widetilde{H}_{\widetilde{o}}) \leq c'\varepsilon) \\ &\leq C \cdot \widetilde{\mathbb{P}}(\widetilde{\Sigma} \cdot P_{\widetilde{o},\omega}(H_\partial < \widetilde{H}_{\widetilde{o}}) \leq c'\varepsilon), \end{aligned} \tag{4.3.5}$$

where $\widetilde{\mathbb{P}}$ is the Dirichlet distribution of parameter $(\alpha_e)_{e\in\widetilde{E}}$ on $\widetilde{\Omega}$, ω is the canonical random variable on $\widetilde{\Omega}$ (which can be seen as a restriction of the canonical r.v. on Ω) and, under $\widetilde{\mathbb{P}}$, $\widetilde{\Sigma}$ is a positive bounded random variable independent of ω and such that, for all $\varepsilon > 0$, $\widetilde{\mathbb{P}}(\widetilde{\Sigma} \leq \varepsilon) \leq C'\varepsilon^{\beta_C}$.

Induction. — Equation (4.3.5) relates the same quantities in G and $\widetilde{G}$, allowing to finally complete the induction argument.

Because the induction hypothesis deals with graphs with simple edges, it may be necessary to reduce the graph $\widetilde{G}$ to this case. Another possibility would have been to define the random walk as a sequence of edges, thus allowing to define T_C (where $C \subset E$) for graphs with multiple edges. In the Dirichlet case, the reduction is however painless since it leads to another closely related Dirichlet distribution.

Note indeed first that, for any $\widetilde{\omega} \in \widetilde{\Omega}$, the quenched laws $P_{x,\widetilde{\omega}}$ (where $x \in \widetilde{V}$) are not altered if we replace multiple (oriented) edges by simple ones with transition probabilities equal to the sum of those of the deleted edges. Due to the associativity property of section 4.3.1, the distribution under $\widetilde{\mathbb{P}}$ of this new environment is the Dirichlet distribution with weights equal to the sum of the weights of the deleted edges. Hence the annealed laws on $\widetilde{G}$ and on the new simplified graph $\widetilde{G}'$ with these weights are the same and, for the problems we are concerned with, we may use $\widetilde{G}'$ instead of $\widetilde{G}$.

The edges in C do not appear in $\widetilde{G}$ anymore. In particular, $\widetilde{G}$ (and a fortiori $\widetilde{G}'$) has strictly less than n edges. In order to apply the induction hypothesis, we need to check that each vertex is connected to ∂. This results directly from the same property for G. We apply the induction hypothesis to the graph $\widetilde{G}'$ and to $\widetilde{o}$. It states that, for small $\varepsilon > 0$,

$$\widetilde{\mathbb{P}}(P_{\widetilde{o},\omega}(H_\partial < \widetilde{H}_{\widetilde{o}}) \leq \varepsilon) \leq C''\varepsilon^{\widetilde{\beta}}(-\ln\varepsilon)^r, \tag{4.3.6}$$

where $C'' > 0, r > 0$ and $\widetilde{\beta}$ is the exponent "β" from the statement of the induction hypothesis corresponding to the graph $\widetilde{G}'$ (it is in fact equal to the "β" for $\widetilde{G}$). As for the left-hand side of (4.3.6), it may equivalently refer to the graph $\widetilde{G}$ or to $\widetilde{G}'$, as explained above.

Considering (4.3.5) and (4.3.6), it appears that the following lemma allows to carry out the induction.

Lemma 4.10. — *If X and Y are independent positive bounded random variables such that, for some real numbers $\alpha_X, \alpha_Y, r > 0$,*

- *there exists $C > 0$ such that $P(X < \varepsilon) \le C\varepsilon^{\alpha_X}$ for all $\varepsilon > 0$ (or equivalently for small ε);*
- *there exists $C' > 0$ such that $P(Y < \varepsilon) \le C'\varepsilon^{\alpha_Y}(-\ln\varepsilon)^r$ for small $\varepsilon > 0$,*

then there exists a constant $C'' > 0$ such that, for small $\varepsilon > 0$,

$$P(XY \le \varepsilon) \le C''\varepsilon^{\alpha_X \wedge \alpha_Y}(-\ln\varepsilon)^{r+1}$$

(and $r+1$ can be replaced by r if $\alpha_X \neq \alpha_Y$).

Proof. — We denote by M_X and M_Y (deterministic) upper bounds of X and Y. We have, for $\varepsilon > 0$,

$$P(XY \le \varepsilon) = P\left(Y \le \frac{\varepsilon}{M_X}\right) + P\left(XY \le \varepsilon, Y > \frac{\varepsilon}{M_X}\right).$$

Let $\varepsilon_0 > 0$ be such that the upper bound in the statement for Y is true as soon as $\varepsilon < \varepsilon_0$. Then, for $0 < \varepsilon < \varepsilon_0$, we compute

$$\begin{aligned}
P(XY \le \varepsilon, Y > \frac{\varepsilon}{M_X}) &= \int_{\frac{\varepsilon}{M_X}}^{M_Y} P\left(X \le \frac{\varepsilon}{y}\right) P(Y \in \mathrm{d}y) \\
&\le C \int_{\frac{\varepsilon}{M_X}}^{M_Y} \left(\frac{\varepsilon}{y}\right)^{\alpha_X} P(Y \in \mathrm{d}y) \\
&= C\varepsilon^{\alpha_X} E\left[\mathbf{1}_{\{Y \ge \frac{\varepsilon}{M_X}\}} \frac{1}{Y^{\alpha_X}}\right] \\
&= C\varepsilon^{\alpha_X}\left(\int_{\frac{\varepsilon}{M_X}}^{M_Y} P(\frac{\varepsilon}{M_X} \le Y \le x)\frac{\alpha_X\,\mathrm{d}x}{x^{\alpha_X+1}} + \frac{P\big(Y \ge \frac{\varepsilon}{M_X}\big)}{{M_Y}^{\alpha_X}}\right) \\
&\le C\varepsilon^{\alpha_X}\left(\alpha_X C' \int_{\frac{\varepsilon}{M_X}}^{\varepsilon_0} x^{\alpha_Y}(-\ln x)^r \frac{\mathrm{d}x}{x^{\alpha_X+1}} + \frac{1}{{M_Y}^{\alpha_X}}\right) \\
&\le C\varepsilon^{\alpha_X}\left(\alpha_X C' \int_{\frac{\varepsilon}{M_X}}^{\varepsilon_0} x^{\alpha_Y-\alpha_X-1}\,\mathrm{d}x(-\ln\frac{\varepsilon}{M_X})^r + \frac{1}{{M_Y}^{\alpha_X}}\right) \\
&\le C''\varepsilon^{\alpha_X\wedge\alpha_Y}(-\ln\varepsilon)^{r+1}.
\end{aligned}$$

Indeed, if $\alpha_Y > \alpha_X$, the integral converges as $\varepsilon \to 0$; if $\alpha_Y = \alpha_X$, it is equivalent to $-\ln\varepsilon$; if $\alpha_Y < \alpha_X$, the equivalent becomes $\frac{1}{\varepsilon^{\alpha_X-\alpha_Y}}$. And the formula is checked in every case (note that $-\ln\varepsilon > 1$ for small ε). □

Using (4.3.6), Lemma 4.10 and (4.3.5), we get constants $c, r > 0$ such that, for small $\varepsilon > 0$,

$$\mathbb{P}(P_{o,\omega}(H_\partial < \widetilde{H}_o) \le \varepsilon, \mathcal{E}_C) \le c\varepsilon^{\beta_C \wedge \widetilde{\beta}}(-\ln \varepsilon)^{r+1}. \tag{4.3.7}$$

Let us prove that $\widetilde{\beta} \ge \beta$, where β is the exponent defined in the induction hypothesis relative to G and o (remember that $\widetilde{\beta}$ is the same exponent, relative to $\widetilde{G}'$ (or $\widetilde{G}$) and $\widetilde{o}$). Let $\widetilde{A}$ be a strongly connected subset of $\widetilde{E}$ such that $\widetilde{o} \in \underline{\widetilde{A}}$. Set $A := \widetilde{A} \cup C \subset E$. In view of the definition of $\widetilde{E}$, every edge exiting $\widetilde{A}$ corresponds to an edge exiting A and vice-versa (the only edges to be deleted by the quotienting procedure are those of C). Thus, recalling that the weights of the edges are preserved in the quotient (cf. Lemma 4.9), $\beta_{\widetilde{A}} = \beta_A$. Moreover, $o \in \underline{A}$, and A is strongly connected (so are $\widetilde{A}$ and C, and $\widetilde{o} \in \widetilde{A}$, $o \in C$), so that $\beta_A \ge \beta$. As a consequence, $\widetilde{\beta} \ge \beta$, as announced.

Then $\beta_C \wedge \widetilde{\beta} \ge \beta_C \wedge \beta = \beta$ because C is strongly connected and $o \in \underline{C}$. Hence (4.3.7) becomes: for small $\varepsilon > 0$,

$$\mathbb{P}(P_{o,\omega}(H_\partial < \widetilde{H}_o) \le \varepsilon, \mathcal{E}_C) \le c\varepsilon^{\beta}(-\ln \varepsilon)^{r+1}.$$

Summing on all events $\mathcal{E}_C$, $C \in \mathcal{C}$, this concludes the induction.

Remark. — *This proof (with both implications) gives the following more precise result: there exist $c, C, r > 0$ such that, for large enough t,*

$$c\frac{1}{t^{\min_A \beta_A}} \le \mathbb{P}(G^\omega(o,o) > t) \le C\frac{(\ln t)^r}{t^{\min_A \beta_A}},$$

where the minimum is taken over all strongly connected subsets A of E such that $o \in \underline{A}$.

4.3.4. Proof of the corollary. — We prove Corollary 4.4. Let s be a positive real number. The equivalence of (i) and (ii) results from the inequalities below: for every $\omega \in \Omega$, $x \in V$,

$$\begin{aligned} G^\omega(x,x)^s = E_{x,\omega}[N_x]^s \le E_{x,\omega}[T_V]^s &= \left(\sum_{y \in V} P_{x,\omega}(H_y < H_\partial) G^\omega(y,y)\right)^s \\ &\le |V|^s \sum_{y \in V} G^\omega(y,y)^s, \end{aligned}$$

where the second inequality is obtained by bounding the probability by 1 and proceding as in equation (4.3.2). Theorem 4.1 provides the equivalence of (ii) and (iii). The fact that (i) implies (iv) is trivial.

Let us suppose that (iii) is not satisfied: there is a strongly connected subset A of E such that $\beta_A \le s$. Let o be a vertex. If $o \in \underline{A}$, then $\mathbb{E}[E_{o,\omega}[T_V]^s] \ge \mathbb{E}[G^\omega(o,o)^s] = \infty$ thanks to Theorem 4.1; and if $o \notin \underline{A}$, there exists (thanks to strong connexity) a path π from o to some vertex $x \in \underline{A}$ which remains outside $\underline{A}$ (before x), and we recall that Theorem 4.1 proves $\mathbb{E}[G^\omega_A(x,x)^s] = \infty$ hence, thanks to the spatial independence of the environment,

$$\begin{aligned}\mathbb{E}[E_{o,\omega}[T_V]^s] &\geq \mathbb{E}[G^\omega(o,x)^s]\\ &\geq \mathbb{E}[P_{o,\omega}(\pi)^s G_A^\omega(x,x)^s] = \mathbb{E}[P_{o,\omega}(\pi)^s]\mathbb{E}[G_A^\omega(x,x)^s] = \infty,\end{aligned}$$

so that in both cases, $\mathbb{E}[E_{o,\omega}[T_V]^s] = \infty$. Thus, (iv) is not true. So (iv) implies (iii), and we are done.

Remark. — *Under most general hypotheses,* (i) *and* (ii) *are still equivalent (same proof). The equivalence of* (i) *and* (iv) *can be shown to hold as well in the following general setting:*

Proposition 4.11. — *Let $G = (V \cup \{\partial\}, E)$ be a finite strongly connected graph endowed with a probability measure $\mathbb{P}$ on the set of its environments satisfying*

– *the transition probabilities $\omega(x,\cdot)$, $x \in V$, are independent under $\mathbb{P}$;*
– *for all $e \in E$, $\mathbb{P}(\omega_e > 0) > 0$.*

If there exists $x \in V$ such that $E_x[T_V] = +\infty$, then for all $y \in V$, $E_y[T_V] = +\infty$.

Proof. — Suppose $x \in V$ satisfies $E_x[T_V] = +\infty$. We denote by A a subset of E satisfying $E_x[T_A] = +\infty$, and being *minimal* (with respect to inclusion) among the subsets of E sharing this property. Since E is finite, the existence of such an A is straightforward.

Let $y \in \underline{A}$: there is an $e \in A$ such that $\underline{e} = y$. Let us prove $E_y[T_A] = +\infty$. We have, by minimality of A, $E_x[T_{A\setminus\{e\}}] < \infty$. Let $H_e := \inf\{n \geq 1 : (X_{n-1}, X_n) = e\}$. Then

$$\begin{aligned}E_x[T_A] &= E_x[T_A, H_e < T_A] + E_x[T_A, H_e > T_A]\\ &\leq E_x[T_A, H_e < T_A] + E_x[T_{A\setminus\{e\}}],\end{aligned}$$

hence $E_x[T_A, H_e < T_A] = +\infty$. Thus, using the Markov property,

$$\begin{aligned}+\infty &= E_x[T_A - T_{A\setminus\{e\}} + 1, H_e < T_A] = E_x[T_A - (H_e - 1), H_e < T_A]\\ &\leq E_x[T_A - (H_e - 1), H_e - 1 < T_A] = E_x[T_A \circ \Theta_{H_e - 1}, H_e - 1 < T_A]\\ &= \mathbb{E}[E_{x,\omega}[E_{X_{H_e-1},\omega}[T_A], H_e - 1 < T_A]] = \mathbb{E}[E_{\underline{e},\omega}[T_A] P_{x,\omega}(H_e - 1 < T_A)]\\ &\leq E_{\underline{e}}[T_A],\end{aligned}$$

which gives $E_y[T_A] = +\infty$ as announced.

Let $z \in V$. If $z \in \underline{A}$, we have of course $E_z[T_V] \geq E_z[T_A] = +\infty$. Suppose $z \in V \setminus \underline{A}$. By strong connexity of G, one can find a simple path $e_1, \cdots, e_n$ from z to a point $y = \overline{e_n} \in \underline{A}$ such that $\underline{e_1}, \ldots, \underline{e_n} \notin \underline{A}$ (by taking any simple path from z to any point in $\underline{A}$ and stopping it just before it enters $\underline{A}$ for the first time). Then, by the Markov property and using independence between the vertices in the environment,

$$\begin{aligned}E_z[T_V] &\geq E_z[T_V, X_i = \overline{e_i} \text{ for } i = 1, \ldots, n]\\ &= \mathbb{E}[\omega_{e_1} \cdots \omega_{e_n} E_{y,\omega}[T_V + n]]\\ &\geq \mathbb{E}[\omega_{e_1} \cdots \omega_{e_n} E_{y,\omega}[T_A + n]\\ &= \mathbb{E}[\omega_{e_1}] \cdots \mathbb{E}[\omega_{e_n}](E_y[T_A] + n)\end{aligned}$$

hence $E_z[T_V] = +\infty$ because the first factors are positive and the last one is infinite *via* the first part of the proof. This concludes the proof of Proposition 4.11. $\square$

4.4. Proof of the ballisticity criterion

We now consider random walks in i.i.d. Dirichlet environment on $\mathbb{Z}^d$, $d \geq 1$. Let $(e_1, \ldots, e_d)$ denote the canonical basis of $\mathbb{Z}^d$, and $\mathcal{V} := \{e \in \mathbb{Z}^d : |e| = 1\}$. Let $(\alpha_e)_{e\in\mathcal{V}}$ be positive numbers. We will write either α_i or α_{e_i}, and α_{-i} or α_{-e_i}, $i = 1, \ldots, d$. Let us prove Theorem 4.6.

This proof relies on properties and techniques of **[15]**. Our improvement is twofold: first, thanks to the previous sections, we are able to define the Kalikow random walk under weaker conditions, namely those of the statement; second, we get a finer bound on the drift of this random walk.

Let us recall a definition. Given a finite subset U of $\mathbb{Z}^d$ and a point $z_0 \in U$ such that $\mathbb{E}[G^\omega_U(z_0, z_0)] < \infty$, the **Kalikow auxiliary random walk** related to U and z_0 is the Markov chain on $U \cup \partial_V U$ (where $\partial_V U$ is the set of the vertices neighbouring U) given by the following transition probabilities:

$$\text{for all } z \in U \text{ and } e \in \mathcal{V},\ \widehat{\omega}_{U,z_0}(z, z+e) := \frac{\mathbb{E}[G^\omega_U(z_0, z)\omega(z, z+e)]}{\mathbb{E}[G^\omega_U(z_0, z)]}$$

and $\widehat{\omega}_{U,z_0}(z, z) := 1$ if $z \in \partial_V U$. For the sake of making formal computations rigorous, Enriquez and Sabot first consider the **generalized Kalikow random walk**. Given an additional parameter $\delta \in (0, 1)$, the new transition probabilities $\widehat{\omega}_{U,z_0,\delta}(z, z+e)$ are defined like the previous ones except that, in place of $G^\omega_U(z_0, z)$, we use the Green function of the random walk under the environment ω killed at rate δ and at the boundary of U:

$$G^\omega_{U,\delta}(z_0, z) := E_{z_0,\omega}\left[\sum_{k=0}^{T_U} \delta^k \mathbf{1}_{\{X_k=z\}}\right]$$

(and we don't need any assumption on $\mathbb{P}$ anymore).

The following identity (equation (2) of **[15]**) was a consequence of an integration by part formula: for all finite $U \subset \mathbb{Z}^d$, $z \in U$, $e \in \mathcal{V}$, $\delta \in (0, 1)$,

$$\widehat{\omega}_{U,z_0,\delta}(z, z+e) = \frac{1}{\Sigma - 1}\left(\alpha_e - \frac{\mathbb{E}[G^\omega_{U,\delta}(z_0, z)p_{\omega,\delta}(z, z+e)]}{\mathbb{E}[G^\omega_{U,\delta}(z_0, z)]}\right)$$

where $p_{\omega,\delta}(z, z+e)) := \omega(z, z+e)(G^\omega_{U,\delta}(z, z) - \delta G^\omega_{U,\delta}(z+e, z))$. The Markov property for the killed random walk shows that, for all z, the components of $(p_{\omega,\delta}(z, z+e))_{e\in\mathcal{V}}$ are positive and sum up to 1: this is a probability measure. Besides, after a short computation, it can be rewritten as

$$p_{\omega,\delta}(z, z+e) = P_{z,\omega}(X_1 = z+e | H_\partial < \widetilde{H}_z),$$

which highlights its probabilistic interpretation. This remark allows us to refine the estimates of [**15**]. The drift of the generalized Kalikow random walk at z is

$$\widehat{d}_{U,z_0,\delta}(z) = \frac{1}{\Sigma - 1}\left(\sum_{i=1}^{d}(\alpha_i - \alpha_{-i})e_i - \widetilde{d}\right) = \frac{1}{\Sigma - 1}(\Sigma d_m - \widetilde{d}), \tag{4.4.1}$$

where $\widetilde{d}$ (depending on all parameters) is the expected value of the following probability measure:

$$\frac{\mathbb{E}[G^{\omega}_{U,\delta}(z_0,z)p_{\omega,\delta}(z,z+\cdot)]}{\mathbb{E}[G^{\omega}_{U,\delta}(z_0,z)]}.$$

This measure is supported by $\mathcal{V}$, hence $\widetilde{d}$ belongs to the convex hull of $\mathcal{V}$, which is the closed $|\cdot|_1$-unit ball $\mathrm{B}_{|\cdot|_1}$:

$$|\widetilde{d}|_1 \leq 1.$$

On the other hand, the assumption gives $\Sigma d_m \notin \mathrm{B}_{|\cdot|_1}$, and the convexity of $\mathrm{B}_{|\cdot|_1}$ provides $l \in \mathbb{R}_d \setminus \{0\}$ and $c > 0$ (depending only on the parameters $(\alpha_e)_{e\in\mathcal{V}}$) such that, for all $X \in \mathrm{B}_{|\cdot|_1}$,

$$\Sigma d_m \cdot l > c > X \cdot l.$$

Therefore, noting that our assumption implies $\Sigma > 1$, we have, for every finite subset U of $\mathbb{Z}^d$, every $z_0, z \in U$ and $\delta \in (0,1)$,

$$\widehat{d}_{U,z_0,\delta}(z)\cdot l = \frac{1}{\Sigma-1}(\Sigma d_m \cdot l - \widetilde{d}\cdot l) \geq \frac{\Sigma d_m \cdot l - c}{\Sigma - 1} > 0.$$

It is time to remark that Theorem 4.3 applies under our condition: the hypothesis implies $\Sigma > 1$ so that, for all i, $2\Sigma - \alpha_i - \alpha_{-i} > 1$. This guarantees the integrability of $G^{\omega}_U(z_0,z)$ and allows us to make δ converge to 1 in the last inequality (monotone convergence theorem applies because $G^{\omega}_{U,\delta}$ increases to G^{ω}_U as δ increases to 1). We get a uniform lower bound concerning the drift of Kalikow random walk:

$$\widehat{d}_{U,z_0}(z)\cdot l \geq \frac{\Sigma d_m \cdot l - c}{\Sigma - 1} > 0.$$

In other words, Kalikow's criterion is satisfied for finite subsets U of $\mathbb{Z}^d$. As underlined in [**15**], this is sufficient to apply Sznitman and Zerner's law of large numbers ([**49**]), hence there is a deterministic $v \neq 0$ such that, $\mathbb{P}$-almost surely,

$$\frac{X_n}{n} \underset{n}{\longrightarrow} v.$$

Finally, identity (4.4.1) gives

$$\left|\widehat{d}_{U,z_0,\delta}(z) - \frac{\Sigma}{\Sigma-1}d_m\right|_1 = \frac{1}{\Sigma-1}|\widetilde{d}|_1 \leq \frac{1}{\Sigma-1},$$

from which the stated bound on v results using Proposition 3.2 of [**39**]: v is an accumulation point of the convex hull of $\{\widehat{d}_{U,z_0,\delta}(z) : U \text{ finite}, z_0, z \in U\}$ when δ tends to 1, and any point of these convex hulls lies in the desired (closed, convex) ℓ^1-ball.

4.5. Concluding remarks and computer simulations

In the case of $\mathbb{Z}^d$, we have provided a criterion for non-zero limiting velocity. One may prove the following criterion as well, thanks to Theorem 4.3 (for a proof, please refer to the Appendix below):

Proposition 4.12. — *If there exists $i \in \{1, \ldots, d\}$ such that $\alpha_i + \alpha_{-i} \geq 2\Sigma - 1$, then*

$$P_0\text{-}a.s., \ \frac{X_n}{n} \underset{n}{\to} 0.$$

The question remains open whether one of these criterions is sharp. Actually, computer simulations let us think that neither is. We were especially able to find parameters such that exit times of all finite subsets are integrable and the random walk has seemingly zero speed (more precisely, X_n looks to be on the order of n^κ for some $0 < \kappa < 1$). Figure 4.1 shows some results obtained with $(\alpha_1, \alpha_{-1}, \alpha_2, \alpha_{-2}) = (0.5, 0.2, 0.1, 0.1)$. We performed 10^3 numerical simulations of trajectories of random walks up to time $n_{\max} = 10^6$ and compared the averaged values of $y_n := X_n \cdot e_1$ with $C_\alpha n^\alpha$, where C_α is chosen so as to make curves coincide at $n = n_{\max}$. The first graph shows the average of y_n and the second one the maximum over $n \in \{10^5 + 1, \ldots, 10^6\}$ of the relative error $\left|1 - \frac{y_n}{C_\alpha n^\alpha}\right|$, as α varies. The minimizing α is 0.9, corresponding to a uniform relative error of .0044. However we could not yet prove that such an intermediary regime happens.

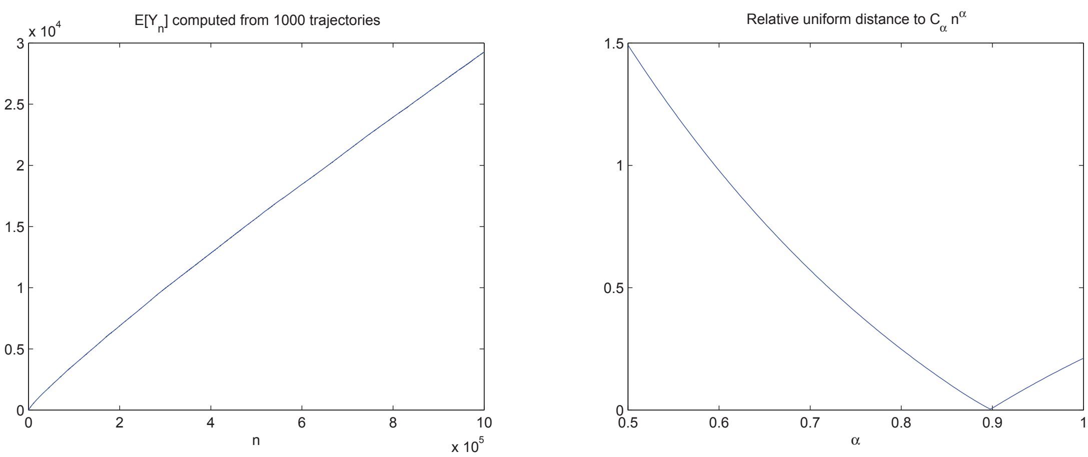

FIGURE 4.1. These plots refer to computer simulation: averages are taken over 10^3 trajectories up to time 10^6 (see section 4.5)

Appendix

Proof of Lemma 4.9. — For readibility and tractability reasons, we will prove this lemma in the case of only two Dirichlet random variables $(p_1, \ldots, p_{k+1})$ and $(p'_1, \ldots, p'_{l+1})$. The proof of the general case would go exactly along the same lines. The associativity property allows as well to reduce to the case when we normalize all components but one: $m_1 = k$ and $m_2 = l$ (by replacing the other marginals by their sum).

We set $\gamma := \alpha_{k+1}$ and $\gamma' := \alpha'_{l+1}$. Writing the index $(\cdot)_i$ instead of $(\cdot)_{1\le i\le k}$ and the same way with j and l, the left-hand side of the statement in this simplified setting equals

$$\int_{\{\sum_i x_i \le 1,\ \sum_j y_j \le 1\}} f\Big(\sum_i x_i + \sum_j y_j, \Big(\frac{x_i}{\sum_i x_i + \sum_j y_j}\Big)_i, \Big(\frac{y_j}{\sum_i x_i + \sum_j y_j}\Big)_j\Big)$$
$$c_0\Big(\prod_i x_i^{\alpha_i - 1}\Big)(1-\sum_i x_i)^{\gamma}\Big(\prod_j y_j^{\alpha'_j-1}\Big)(1-\sum_j y_j)^{\gamma'}\prod_i \mathrm{d}x_i \prod_j \mathrm{d}y_j,$$

for some positive c_0. We successively procede to the following changes of variable: $x_1 \mapsto u = \sum_i x_i + \sum_j y_j$, then $x_i \mapsto \widetilde{x}_i = \frac{x_i}{u}$ for every $i \neq 1$, and $y_j \mapsto \widetilde{y}_j = \frac{y_j}{u}$ for every j. The previous integral becomes

$$\int_{\{\sum_{i\neq 1}\widetilde{x}_i + \sum_j \widetilde{y}_j \le 1,\ 1-\frac{1}{u}\le \sum_j \widetilde{y}_j \le \frac{1}{u}\}} f\Big(u, 1-\sum_{i\neq 1}\widetilde{x}_i + \sum_j \widetilde{y}_j, (\widetilde{x}_i)_{i\neq 1}, (\widetilde{y}_j)\Big)\psi(u,(\widetilde{x}_i)_{i\neq 1},(\widetilde{y}_j)_j)$$
$$\mathrm{d}u \prod_{i\neq 1}\mathrm{d}\widetilde{x}_i \prod_j \mathrm{d}\widetilde{y}_j,$$

where

$$\psi(u,(x_i)_{i\neq 1},(y_j)_j) := c_0 u^{\sum_i \alpha_i + \sum_j \alpha'_j - 1}(1-\sum_{i\neq 1}\widetilde{x}_i - \sum_j \widetilde{y}_j)^{\alpha_1 - 1}$$
$$\cdot \prod_{i\neq 1}\widetilde{x}_i^{\alpha_i-1}(1-u(1-\sum_j \widetilde{y}_j))^{\beta}\prod_j \widetilde{y}_j^{\alpha'_j-1}(1-u\sum_j \widetilde{y}_j)^{\beta'}.$$

Bounding from above by 1 the last two factors of ψ where u appears, we get that the last quantity is less than

$$\int_{\{\sum_{i\neq 1}\widetilde{x}_i + \sum_j \widetilde{y}_j \le 1,\ u\le 2\}} f\Big(u, 1-\sum_{i\neq 1}\widetilde{x}_i + \sum_j \widetilde{y}_j, (\widetilde{x}_i)_{i\neq 1}, (\widetilde{y}_j)\Big)\theta(u,(\widetilde{x}_i)_{i\neq 1},(\widetilde{y}_j)_j)$$
$$\mathrm{d}u \prod_{i\neq 1}\mathrm{d}\widetilde{x}_i \prod_j \mathrm{d}\widetilde{y}_j,$$

where $\theta(u,(\widetilde{x}_i)_{i\neq 1},(\widetilde{y}_j)_j) := c_0\left(1-\sum_{i\neq 1}\widetilde{x}_i - \sum_j \widetilde{y}_j\right)^{\alpha_1-1}\prod_{i\neq 1}\widetilde{x}_i^{\alpha_i-1}\prod_j \widetilde{y}_j^{\alpha'_j-1}$.

This rewrites, for some positive c_1, as $c_1\widetilde{E}\left[f(\widetilde{\Sigma},\widetilde{p}_1,\ldots,\widetilde{p}_k,\widetilde{p'}_1,\ldots,\widetilde{p'}_l)\right]$, with the notations of the statement. We have here $\widetilde{P}(\widetilde{\Sigma}<\varepsilon) = c\int_0^{\varepsilon} u^{\sum_i \alpha_i + \sum_j \alpha'_j - 1}\,\mathrm{d}u = c'\varepsilon^{\sum_i \alpha_i + \sum_j \alpha'_j}$.

Proof of Proposition 4.12. — Without loss of generality, we may assume $\alpha_1 + \alpha_{-1} \ge 2\Sigma - 1$. As a consequence of Theorem 4.3, the time spent by the random walk inside the edge $(0, e_1)$ is non-integrable under P_0.

We prove that, for every $e \in \mathcal{V}$, P_0-a.s., $\frac{T_k^e}{k} \to_k +\infty$, where

$$T_k^e := \inf\{n \geq 0 : X_n \cdot e \geq 2k\}.$$

This implies that P_0-a.s., $\limsup_n \frac{X_n \cdot e}{n} \leq 0$ for all $e \in \mathcal{V}$, and thus the proposition. Let $e \in \mathcal{V}$. We introduce the exit times :

$$\tau_0 := \inf\{n \geq 0 : X_n \notin \{X_0, X_0 + e_1\}\}$$

(with a minus sign instead of the plus if $e = -e_1$) and, for $k \geq 1$,

$$\tau_k := \tau_0 \circ \Theta_{T_k^e},$$

with the convention that $\tau_k = \infty$ if $T_k^e = \infty$. The only dependence between the times τ_k, $k \in \mathbb{N}$, comes from the fact that $\tau_k = \infty$ implies $\tau_l = \infty$ for all $l \geq k$. The "2" in the definition of T_k^e causes indeed the τ_k's to depend on disjoint parts of the environment, namely slabs $\{x \in \mathbb{Z}^d : x \cdot e \in \{2k, 2k+1\}\}$. For $t_0, \ldots, t_k \in \mathbb{N}$, one has, using the Markov property at time T_k^e, the independence and the translation invariance of $\mathbb{P}$,

$$\begin{aligned} P_0(\tau_0 = t_0, \ldots, \tau_k = t_k) &= P_0(\tau_0 = t_0, \ldots, \tau_{k-1} = t_{k-1}, \tau_k = t_k, T_k^e < \infty) \\ &\leq P_0(\tau_0 = t_0, \ldots, \tau_{k-1} = t_{k-1}) P_0(\tau_0 = t_k) \\ &\leq \cdots \leq P_0(\tau_0 = t_0) \cdots P_0(\tau_0 = t_{k-1}) P_0(\tau_0 = t_k) \\ &= P(\widehat{\tau}_0 = t_0, \ldots, \widehat{\tau}_k = t_k), \end{aligned}$$

where, under P, the random variables $\widehat{\tau}_k$, $k \in \mathbb{N}$, are independent and have the same distribution as τ_0 (and hence are finite P-a.s.). From this we deduce that, for all $A \subset \mathbb{N}^{\mathbb{N}}$, $P_0((\tau_k)_k \in A) \leq P((\widehat{\tau}_k)_k \in A)$. In particular,

$$P_0\left(\liminf_k \frac{\tau_0 + \cdots + \tau_{k-1}}{k} < \infty\right) \leq P\left(\liminf_k \frac{\widehat{\tau}_0 + \cdots + \widehat{\tau}_{k-1}}{k} < \infty\right) = 0,$$

where the equality results from the law of large number for i.i.d. random variables (recall $E[\widehat{\tau}_k] = E_0[\tau_0] = \infty$). Finally, $T_k^e \geq \tau_0 + \cdots + \tau_{k-1}$, so that $\liminf_k \frac{T_k^e}{k} = \infty$ P_0-a.s., as wanted.

CHAPITRE 5

REVERSED DIRICHLET ENVIRONMENT AND DIRECTIONAL TRANSIENCE OF RANDOM WALKS IN DIRICHLET ENVIRONMENT

This article is co-authored with Christophe Sabot.

Abstract. We consider random walks in a random environment given by i.i.d. Dirichlet distributions at each vertex of $\mathbb{Z}^d$ or, equivalently, oriented edge reinforced random walks on $\mathbb{Z}^d$. The parameters of the distribution are a $2d$-uplet of positive real numbers indexed by the unit vectors of $\mathbb{Z}^d$. We prove that, as soon as these weights are nonsymmetric, the random walk is transient in a direction (i.e. it satisfies $X_n \cdot \ell \to_n +\infty$ for some ℓ) with positive probability. In dimension 2, this result is strenghened to an almost sure directional transience thanks to the 0-1 law from [**55**]. Our proof relies on the property of stability of Dirichlet environment by time reversal proved in [**40**]. In a first part of this paper, we also give a probabilistic proof of this property as an alternative to the change of variable computation used in [**40**].

Introduction

Random walks in a multidimensional random environment have attracted considerable interest in the last few years. Unlike the one-dimensional setting, this model remains however rather little understood. Recent advances focused especially in two directions: small perturbations of a deterministic environment (cf. for instance [**6**]) and ballisticity (cf. [**48**] for a survey), but few conditions are completely explicit or known to be sharp.

The interest in random Dirichlet environment stems from the desire to take advantage of the features of a *specific* multidimensional environment distribution that make a few computations explicitly possible, and hopefully provide through them a better intuition for the general situation. The choice of Dirichlet distribution appears as a natural one considering its close relationship with linearly reinforced random walk on oriented edges (cf. [**33**] and [**14**]). The opportunity of this choice was further confirmed by the derivation of a ballisticity criterion by Enriquez and Sabot [**15**] (later improved by Tournier [**51**]), and more especially by the recent proof by Sabot [**40**] that random walks in Dirichlet environment on $\mathbb{Z}^d$ are transient when $d \geq 3$.

The proof of the ballisticity criterion in **[14]** relies on an integration by part formula for Dirichlet distribution. This formula provides algebraic relations involving the Green function that allow to show that Kalikow's criterion applies. As for the proof of [**40**], one of its key tools is the striking property (Lemma 1 of [**40**]) that, provided the edge weights have null divergence, a reversed Dirichlet environment (on a finite graph) still is a Dirichlet environment.

In this paper we prove with this same latter property that random walks in Dirichlet environment on $\mathbb{Z}^d$ ($d \geq 1$) are directionally transient as soon as the weights are nonsymmetric. More precisely, under this condition, our result (Theorem 5.2) states that directional transience happens with positive probability: for some direction ℓ, $P_o(X_n \cdot \ell \to_n +\infty) > 0$. Combined with Kalikow's 0-1 law from [**24**], this proves that, almost surely, $|X_n \cdot \ell| \to_n +\infty$. Furthermore, in dimension 2, the 0-1 law proved by Zerner and Merkl (cf. [**55**]) enables to conclude that, almost surely, $X_n \cdot \ell \to_n +\infty$.

The above mentioned property of the reversed Dirichlet environment was derived in [**40**] by means of a complicated change of variable. In Section 5.2 of the present paper, we provide an alternative probabilistic proof of this important property.

5.1. Definitions and statement of the results

Let us precise the setting in this paper. Let $G = (V, E)$ be a directed graph, i.e. V is the set of vertices, and $E \subset V \times V$ is the set of edges. If $e = (x, y)$ is an edge, we respectively denote by $\underline{e} := x$ and $\overline{e} := y$ its tail and head. We suppose that each vertex has finite degree, i.e. that finitely many edges exit any vertex. G is said to be strongly connected if for any pair of vertices (x, y) there is a directed path from x to y in G. The set of environments on G is the set

$$\Delta := \left\{ (p_e)_{e\in E} \in (0,1)^E \;:\; \text{for all } x \in V, \sum_{e\in E, \underline{e}=x} p_e = 1 \right\}.$$

Let $(p_e)_{e\in E}$ be an environment. If $e = (x, y)$ is an edge, we shall write $p(x, y) := p_e$. Note that p extends naturally to a measure on the set of paths: if $\gamma = (x_1, x_2, \ldots, x_n) \in V^n$ is a path in G, i.e. if $(x_i, x_{i+1}) \in E$ for $i = 1, \ldots, n-1$, then we let

$$p(\{\gamma\}) = p(\gamma) := p(x_1, x_2)p(x_2, x_3)\cdots p(x_{n-1}, x_n).$$

To any environment $p = (p_e)_{e\in E}$ and any vertex x_0 we associate the distribution $P_{x_0,p}$ of the Markov chain $(X_n)_{n\geq 0}$ on V starting at x_0 with transition probabilities given by p. For every path $\gamma = (x_0, x_1, \ldots, x_n)$ in G starting at x_0, we have

$$P_{x_0,p}(X_0 = x_0, \ldots, X_n = x_n) = p(\gamma).$$

When necessary, we shall specify by a superscript $P^G_{x_0,p}$ which graph we consider.

Let $(\alpha_e)_{e\in E}$ be positive weights on the edges. For any vertex x, we let

$$\alpha_x := \sum_{e\in E, \underline{e}=x} \alpha_e$$

be the sum of the weights of the edges exiting from x. The Dirichlet distribution with parameter $(\alpha_e)_{e\in E}$ is the law $\mathbb{P}^{(\alpha)}$ of the random variable $p = (p_e)_{e\in E}$ in Δ such that the transition probabilities $(p_e)_{e\in E,\underline{e}=x}$ at each site x are independent Dirichlet random variables with parameter $(\alpha_e)_{e\in E,\underline{e}=x}$. Thus,

$$\mathrm{d}\mathbb{P}^{(\alpha)}(p) := \frac{\prod_{x\in V}\Gamma(\alpha_x)}{\prod_{e\in E}\Gamma(\alpha_e)}\prod_{e\in E} p_e^{\alpha_e-1}\prod_{e\in\widetilde{E}}\mathrm{d}p_e,$$

where $\widetilde{E}$ is obtained from E by removing arbitrarily, for each vertex x, one edge with origin x (the distribution does not depend on this choice). The corresponding expectation will be denoted by $\mathbb{E}^{(\alpha)}$. We may thus consider the probability measure $\mathbb{E}^{(\alpha)}[P_{x_0,p}(\cdot)]$ on random walks on G.

Time reversal. — Let us suppose that G is finite and strongly connected. Let $\check{G} = (V,\check{E})$ be the graph obtained from G by reversing all the edges, i.e. by swapping the head and tail of the edges. To any path $\gamma = (x_1, x_2, \ldots, x_n)$ in G we can associate the reversed path $\check{\gamma} := (x_n, \ldots, x_2, x_1)$ in $\check{G}$.

For any environment $p = (p_e)_{e\in E}$, the Markov chain with transition probabilities given by p is irreducible, hence (G being finite) we may define its unique invariant probability $(\pi_x)_{x\in V}$ on V and the environment $\check{p} = (\check{p}_e)_{e\in\check{E}}$ obtained by time reversal: for every $e \in E$,

$$\check{p}_{\check{e}} := \frac{\pi_{\underline{e}}}{\pi_{\overline{e}}} p_e.$$

For any family of weights $\alpha = (\alpha_e)_{e\in E}$, we define the reversed weights $\check{\alpha} = (\check{\alpha}_e)_{e\in\check{E}}$ simply by $\check{\alpha}_{\check{e}} := \alpha_e$ for any edge $e \in E$, and the divergence of α is given, for $x \in V$, by $\mathrm{div}(\alpha)(x) := \alpha_x - \check{\alpha}_x$.

The following proposition is Lemma 1 from [**40**].

Proposition 5.1. — *Suppose that the weights $\alpha = (\alpha_e)_{e\in E}$ have null divergence, i.e.*

$$\forall x \in V, \mathrm{div}(\alpha)(x) = 0 \text{ (or equivalently } \check{\alpha}_x = \alpha_x).$$

Then, under $\mathbb{P}^{(\alpha)}$, $\check{p}$ is a Dirichlet environment on $\check{G}$ with parameters $\check{\alpha} = (\check{\alpha}_e)_{e\in\check{E}}$.

In section 5.2, we give a neat probabilistic proof of this proposition. It also lies at the core of the proof of the directional transience.

Directional transience on $\mathbb{Z}^d$. — We consider the case $V = \mathbb{Z}^d$ ($d \geq 1$) with edges to the nearest neighbours. Let $(e_1, \ldots, e_d)$ be the canonical basis of $\mathbb{Z}^d$. An i.i.d. Dirichlet distribution on G is determined by a $2d$-vector $\vec{\alpha} = (\alpha_1, \beta_1, \ldots, \alpha_d, \beta_d)$ of positive weights. We define indeed the parameters α on E by $\alpha_{(x,x+e_i)} := \alpha_i$ and $\alpha_{(x,x-e_i)} := \beta_i$ for any vertex x and index $i \in \{1, \ldots, d\}$. In this context, we let $P_o := \mathbb{E}^{(\alpha)}[P_{o,p}(\cdot)]$ be the so-called annealed law.

Theorem 5.2. — *Assume $\alpha_1 > \beta_1$. Then*

$$P_o(X_n \cdot e_1 \underset{n}{\to} +\infty) > 0.$$

The 0-1 law proved by Kalikow in Lemma 1.1 of [**24**] and generalized to the non-uniformly elliptic case by Zerner and Merkl in Proposition 3 of [**55**], along with the 0-1 law proved by Zerner and Merkl in [**55**] (cf. also [**54**]) for the two-dimensional random walk allow to turn this theorem into almost sure results:

Corollary 5.3. — *Assume $\alpha_1 > \beta_1$.*

(i) *If $d \leq 2$, then*

$$P_o\text{-almost surely,} \qquad X_n \cdot e_1 \underset{n}{\to} +\infty.$$

(ii) *If $d \geq 3$, then*

$$P_o\text{-almost surely,} \qquad |X_n \cdot e_1| \underset{n}{\to} +\infty.$$

Remark. — *This theorem provides examples of non-ballistic random walks that are transient in a direction. Proposition 12 of* [**51**] *shows indeed that the condition $2\sum_j(\alpha_j + \beta_j) - \alpha_i - \beta_i \leq 1$ for a given i implies P_o-almost surely $\frac{X_n}{n} \to_n 0$ because of the existence of "finite traps", so that any choice of small unbalanced weights (less than $\frac{1}{4d}$ for instance) is such an example. Another simple example exhibiting this behaviour was communicated to us by Alexander Fribergh as well (cf. 2.3.2). It is however believed (cf.* [**48**] *p.227) that transience in a direction implies ballisticity if a uniform ellipticity assumption is made, i.e. if there exists $\kappa > 0$ such that, for every edge e, $\mathbb{P}$-almost surely $p_e \geq \kappa$.*

Under the hypothesis of Theorem 5.2 we conjecture that, for any $d \geq 1$,

$$P_o(X_n \cdot e_1 \underset{n}{\to} +\infty) = 1,$$

and that the following identity is true:

$$P_o(D = +\infty) = 1 - \frac{\beta_1}{\alpha_1},$$

where $D := \inf\{n \geq 0 : X_n \cdot e_1 < 0\}$. In this paper, only the "$\geq$" inequality is proved.

5.2. Reversed Dirichlet environment. Proof of Proposition 5.1

It suffices to prove that, for a given starting vertex o, the annealed distributions $\mathbb{E}^{(\alpha)}[P_{o,\check{p}}(\cdot)]$ and $\mathbb{E}^{(\check{\alpha})}[P_{o,p}(\cdot)]$ on walks on $\check{G}$ are the same.

Indeed, the annealed distribution $\mathbb{E}[P_{o,p}(\cdot)]$ characterizes the distribution $\mathbb{P}$ of the environment as soon as we assume that $\mathbb{P}$-almost every environment p is transitive recurrent. This results from considering the sample environment $p^{(n)}$ at time n:

$$\text{for every } e = (x, y) \in E, \quad p_e^{(n)} := \frac{|\{0 \leq i < n : (X_i, X_{i+1}) = e\}|}{|\{0 \leq i < n : X_i = x\}|}.$$

From the recurrence assumption and the law of large numbers we deduce that, for $\mathbb{P}$-almost every environment p, $p^{(n)}$ makes sense for large n and that, as $n \to \infty$, it converges almost surely under $P_{o,p}$, and thus under $\mathbb{E}[P_{o,p}(\cdot)]$, to a random variable $\widetilde{p}$ with distribution $\mathbb{P}$. This way we can recover the distribution of the environment from that of the annealed random walk.

We are thus reduced to proving that, for any finite path γ in G,

$$\mathbb{E}^{(\alpha)}[\check{p}(\check{\gamma})] = \mathbb{E}^{(\check{\alpha})}[p(\check{\gamma})]. \tag{5.2.1}$$

The only specific property of Dirichlet distribution to be used in the proof is the following "cycle reversal" property:

Lemma 5.4. — *For any cycle $\sigma = (x_1, x_2, \dots, x_n (= x_1))$ in G, $\mathbb{E}^{(\alpha)}[p(\sigma)] = \mathbb{E}^{(\check{\alpha})}[p(\check{\sigma})]$, i.e.*

$$\mathbb{E}^{(\alpha)}[p(x_1, x_2) \cdots p(x_{n-1}, x_n)] = \mathbb{E}^{(\check{\alpha})}[p(x_n, x_{n-1}) \cdots p(x_2, x_1)].$$

Proof. — Remembering that the annealed random walk in Dirichlet environment is an oriented edge linearly reinforced random walk where the initial weights on the edges are the parameters of the Dirichlet distribution (cf. [**14**]), the left-hand side of the previous equality writes

$$\mathbb{E}^{(\alpha)}[p(\sigma)] = \frac{\prod_{e\in E} \alpha_e(\alpha_e + 1) \cdots (\alpha_e + n_e(\sigma) - 1)}{\prod_{x\in V} \alpha_x(\alpha_x + 1) \cdots (\alpha_x + n_x(\sigma) - 1)},$$

where $n_e(\sigma)$ (resp. $n_x(\sigma)$) is the number of crossings of the oriented edge e (resp. the number of visits of the vertex x) in the path σ. The cyclicity gives $n_e(\sigma) = n_{\check{e}}(\check{\sigma})$ and $n_x(\sigma) = n_x(\check{\sigma})$ for all $e \in E, x \in V$. Furthermore, by assumption $\check{\alpha}_x = \alpha_x$ for every vertex x, and by definition $\alpha_e = \check{\alpha}_{\check{e}}$ for every edge e. This shows that the previous product matches the similar product with $\check{\alpha}$ and $\check{\sigma}$ instead of α and σ, hence the lemma. $\square$

If $\sigma = (x_1, x_2, \dots, x_n)$ is a cycle in G (thus $x_n = x_1$), the definition of the reversed environment gives

$$\begin{aligned}
p(\sigma) &= p(x_1, x_2)p(x_2, x_3) \cdots p(x_{n-1}, x_n)\\
&= \frac{\pi_{x_2}}{\pi_{x_1}}\check{p}(x_2, x_1)\frac{\pi_{x_3}}{\pi_{x_2}}\check{p}(x_3, x_2) \cdots \frac{\pi_{x_n}}{\pi_{x_{n-1}}}\check{p}(x_n, x_{n-1})\\
&= \check{p}(x_2, x_1)\check{p}(x_3, x_2) \cdots \check{p}(x_n, x_{n-1})\\
&= \check{p}(\check{\sigma}). \qquad (5.2.2)
\end{aligned}$$

Therefore, with the previous lemma: $\mathbb{E}^{(\alpha)}[\check{p}(\check{\sigma})] = \mathbb{E}^{(\check{\alpha})}[p(\check{\sigma})]$, which is Equation (5.2.1) for cycles.

Consider now a non-cycling path γ in G: $\gamma = (x = x_1, \dots, x_n = y)$ with $x \neq y$. The same computation as above shows that

$$\check{p}(\check{\gamma}) = \frac{\pi_x}{\pi_y} p(\gamma).$$

It is a well-known property of Markov chains that $\frac{\pi_x}{\pi_y} = E_{y,p}[N_x^y]$ where N_x^y is the number of visits to x before the next visit of y: $N_x^y := \sum_{i=0}^{H_y} \mathbf{1}_{\{X_i = x\}}$, with

$H_y := \inf\{n \geq 1 : X_n = y\}$. It can therefore be written, in a hopefully self-explanatory schematical way,

$$\frac{\pi_x}{\pi_y} = \sum_{k=0}^{\infty} P_{y,p}\left(N_x^y > k\right) = \sum_{k=0}^{\infty} p\,\Big(y \underset{\neq x,\, y}{\dashrightarrow} \underset{\geq k \text{ times}}{\underset{\circlearrowleft \neq x,\, y}{x}} \underset{\neq x,\, y}{\dashrightarrow} y\Big) = \sum_{k=0}^{\infty} p\,\Big(y \underset{\neq x,\, y}{\dashrightarrow} \underset{k \text{ times}}{\underset{\circlearrowleft \neq x,\, y}{x}}\Big),$$

where the subscripts "$\neq x, y$" mean that the paths sketched by the dashed arrows avoid x and y, and "$(\geq)k$ times" refers to the number of loops. Using the notation $\{x \xrightarrow{\gamma} y\}$ for the event where the walk follows the path γ (which goes from x to y), the Markov property at the time of k-th visit of x gives

$$\check{p}(\check{\gamma}) = \sum_{k=0}^{\infty} p\,\Big(y \underset{\neq x,\, y}{\dashrightarrow} \underset{k \text{ times}}{\underset{\circlearrowleft \neq x,\, y}{x}}\Big)\, p(\gamma) = \sum_{k=0}^{\infty} p\,\Big(y \underset{\neq x,\, y}{\dashrightarrow} \underset{k \text{ times}}{\underset{\circlearrowleft \neq x,\, y}{x}} \xrightarrow{\quad\gamma\quad} y\Big).$$

The paths in the probability on the right-hand side are *cycles*. Taking the expectation under $\mathbb{P}^{(\alpha)}$ of both sides, we can use the Lemma to reverse them. We get

$$\mathbb{E}^{(\alpha)}[\check{p}(\check{\gamma})] = \sum_{k=0}^{\infty} \mathbb{E}^{(\check{\alpha})}\Big[p\,\Big(y \xrightarrow{\quad\check{\gamma}\quad} \underset{k \text{ times}}{\underset{\circlearrowleft \neq x,\, y}{x}} \underset{\neq x,\, y}{\dashrightarrow} y\Big)\Big].$$

It only remains to notice that the summation over $k \in \mathbb{N}$ allows to drop the condition on the path after it has followed $\check{\gamma}$:

$$\mathbb{E}^{(\alpha)}[\check{p}(\check{\gamma})] = \mathbb{E}^{(\check{\alpha})}[p(\check{\gamma})].$$

This concludes the proof of the proposition.

5.3. Directional transience. Proof of Theorem 5.2

For any integer M, let us define the stopping times

$$T_M := \inf\{n \geq 0 : X_n \cdot e_1 \geq M\},$$
$$\widetilde{T}_M := \inf\{n \geq 0 : X_n \cdot e_1 \leq M\},$$

and in particular $D := \widetilde{T}_{-1}$. In addition, for any vertex x, we define $H_x := \inf\{n \geq 1 : X_n = x\}$.

Let $N, L \in \mathbb{N}^*$. Denoting $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$, we consider the finite and infinite "cylinders"

$$C_{N,L} := \{0, \ldots, L-1\} \times (\mathbb{Z}_N)^{d-1} \subset C_N := \mathbb{Z} \times (\mathbb{Z}_N)^{d-1},$$

endowed with an i.i.d. Dirichlet environment of parameter $\vec{\alpha}$. We are interested in the probability that the random walk starting at $o := (0, 0, \ldots, 0)$ exits $C_{N,L}$ to the right. Specifically, we shall prove

$$\mathbb{E}^{(\alpha)}\left[P_{o,p}^{C_N}(T_L < \widetilde{T}_{-1})\right] \geq 1 - \frac{\beta_1}{\alpha_1}. \tag{5.3.1}$$

To that aim, we introduce the finite weighted graph $G = G_{L,N}$ defined as follows (cf. also figure 5.1). The set of vertices of G is $V := C_{N,L+1} \cup \{\delta\}$. Let

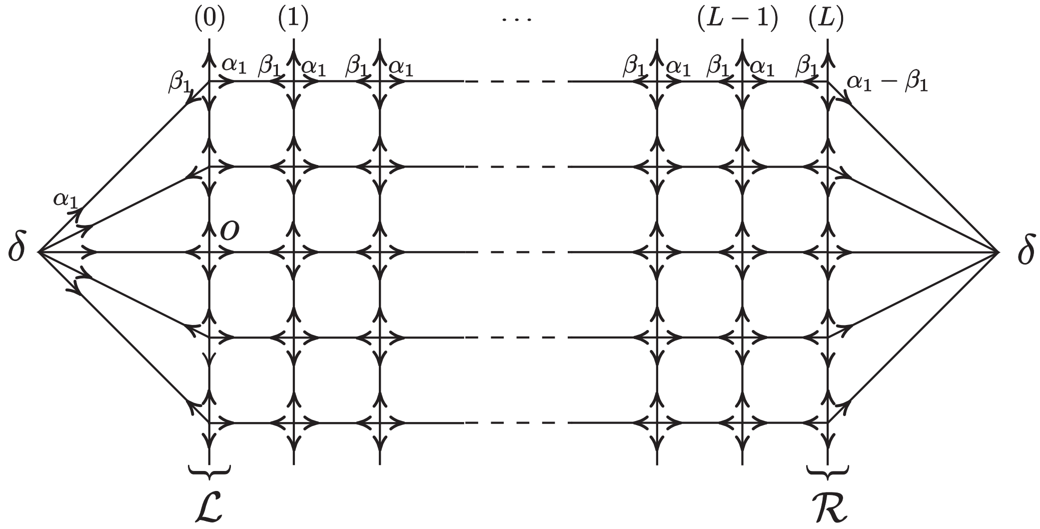

FIGURE 5.1. Definition of the weighted graph G. The top and bottom rows are identified (the graph is based on a cylinder) and the weights are the same on each line. Both vertices called δ are the same.

us denote by $\mathcal{L} := \{0\} \times (\mathbb{Z}_N)^{d-1}$ and $\mathcal{R} := \{L\} \times (\mathbb{Z}_N)^{d-1}$ the left and right ends of the cylinder. The edges of G inside $C_{N,L+1}$ go between nearest vertices for usual distance. In addition, "exiting" edges are introduced from the vertices in $\mathcal{L}$ and $\mathcal{R}$ into δ, and "entering" edges are introduced from δ into the vertices in $\mathcal{L}$. The edges in $C_{N,L+1}$ (including exiting edges) are endowed with weights naturally given by $\vec{\alpha}$ *except* that the edges exiting from $\mathcal{R}$ toward δ have weight $\alpha_1 - \beta_1 (> 0)$ instead of α_1. At last, the weight on the entering edges is α_1.

These weights, let us denote them by α', are chosen in such a way that $\operatorname{div} \alpha'(x) = 0$ for every vertex x (including δ, necessarily). We are thus in position to apply Proposition 5.1.

Using Equation (5.2.2) to reverse all the cycles in G that start at δ and get back to δ from $\mathcal{R}$ (for the first time), we get the following equality:

$$P^G_{\delta,p}\left(X_{H_\delta - 1} \in \mathcal{R}\right) = P^{\check{G}}_{\delta,\check{p}}\left(X_1 \in \mathcal{R}\right).$$

In the reversed graph $\check{G}$, there are N edges that exit δ to $\mathcal{L}$ (with reversed weight β_1), and N edges that exit δ to $\mathcal{R}$ (with reversed weight $\alpha_1 - \beta_1$). Combined with the Dirichlet distribution of $\check{p}$ under $\mathbb{P}$ given by Proposition 5.1 (notice that in fact we only reverse cycles, hence Lemma 5.4 suffices), the previous equality gives (5.3.2)

$$\mathbb{E}^{(\alpha')}\left[P^G_{\delta,p}\left(X_{H_\delta-1} \in \mathcal{R}\right)\right] = \mathbb{E}^{(\check{\alpha}')}\left[\sum_{e \in \check{E}, \underline{e} = \delta} p(e)\right] = \frac{N(\alpha_1 - \beta_1)}{N(\alpha_1 - \beta_1) + N\beta_1} = 1 - \frac{\beta_1}{\alpha_1}.$$

In the second equality we used the fact that under $\mathbb{P}^{(\check{\alpha}')}$ the marginal variable p_e follows a beta distribution with parameters $(\check{\alpha}'_e, (\sum_{e' \in \check{E}, \underline{e'} = \delta} \check{\alpha}'_{e'}) - \check{\alpha}'_e)$, and that the expected value of a Beta random variable with parameters α and β is $\frac{\alpha}{\alpha+\beta}$. Alternatively, this equality follows also from the distribution of the first step of an oriented edge reinforced random walk with initial weights given by α'.

Let us show how this equality implies the lower bound (5.3.1). First, because of the symmetry of the weighted graph, the distribution of X_1 under $\mathbb{E}^{(\alpha')}[P^G_{\delta,p}(\cdot)]$ is uniform on $\mathcal{L}$, hence, since $o \in \mathcal{L}$,

$$\mathbb{E}^{(\alpha')}\left[P^G_{\delta,p}\left(X_{H_\delta - 1} \in \mathcal{R}\right)\right] = \mathbb{E}^{(\alpha')}\left[P^G_{o,p}\left(X_{H_\delta - 1} \in \mathcal{R}\right)\right]. \tag{5.3.3}$$

Then we relate the exit probability out of $C_{N,L}$ endowed with parameters α with the exit probability out of $C_{N,L+1}$ endowed with modified parameters α'. These weights α and α' coincide in $C_{N,L}$; furthermore, a random walk on the cylinder C_N, from o, reaches the abcissa L before the abcissa $L+1$, so that (making a slight abuse of notation in using weights α' in C_N)

$$\mathbb{E}^{(\alpha)}\left[P^{C_N}_{o,p}(T_L < \widetilde{T}_{-1})\right] \geq \mathbb{E}^{(\alpha')}\left[P^{C_N}_{o,p}(T_{L+1} < \widetilde{T}_{-1})\right] = \mathbb{E}^{(\alpha')}\left[P^G_{o,p}\left(X_{H_\delta - 1} \in \mathcal{R}\right)\right]. \tag{5.3.4}$$

The equalities (5.3.2) and (5.3.3), along with (5.3.4), finally give (5.3.1).

To conclude the proof of the theorem, let us first take the limit as $N \to \infty$. Due to the ellipticity of Dirichlet environments, Lemma 4 of **[55]** shows that the random walk can't stay forever inside an infinite slab:

$$0 = \mathbb{E}^{(\alpha)}[P_{o,p}(T_L = \widetilde{T}_{-1} = \infty)] = \lim_{N\to\infty} \mathbb{E}^{(\alpha)}[P_{o,p}(T_L \wedge \widetilde{T}_{-1} > T^\perp_N)]$$

where $T^\perp_N := \inf\{n \geq 0 : |X_n - (X_n \cdot e_1)e_1| > N\}$ is the first time when the random walk is at distance greater than N from the axis $\mathbb{R}e_1$. Using this with the stopping time $T^\perp_{N/2}$, we can switch from the cylinder $C_{N,L}$ to an infinite slab:

$$\begin{aligned}
\mathbb{E}^{(\alpha)}[P^{C_N}_{o,p}(T_L < \widetilde{T}_{-1})] &= \mathbb{E}^{(\alpha)}[P^{C_N}_{o,p}(T_L < \widetilde{T}_{-1}, T^\perp_{N/2} > T_L)] \\
&\quad + \mathbb{E}^{(\alpha)}[P^{C_N}_{o,p}(T_L < \widetilde{T}_{-1}, T^\perp_{N/2} \leq T_L)] \\
&= \mathbb{E}^{(\alpha)}[P_{o,p}(T_L < \widetilde{T}_{-1}, T^\perp_{N/2} > T_L)] + o_N(1) \\
&= \mathbb{E}^{(\alpha)}[P_{o,p}(T_L < \widetilde{T}_{-1})] + o_N(1)',
\end{aligned}$$

hence, with (5.3.1),

$$\mathbb{E}^{(\alpha)}[P_{o,p}(T_L < \widetilde{T}_{-1})] \geq 1 - \frac{\beta_1}{\alpha_1}.$$

The limit when $L \to \infty$ simply involves decreasing events:

$$P_o(D = +\infty) = \mathbb{E}^{(\alpha)}[P_{o,p}(D = +\infty)] = \lim_{L\to\infty} \mathbb{E}^{(\alpha)}[P_{o,p}(T_L < \widetilde{T}_{-1})] \geq 1 - \frac{\beta_1}{\alpha_1} > 0.$$

Moreover, Lemma 4 from **[55]** shows more precisely that if a slab is visited infinitely often, then *both* half-spaces next to it are visited infinitely often as well. The event $\{D = \infty\} \cap \{\liminf_n X_n \cdot e_1 < M\}$ has therefore null P_o-probability for any $M > 0$, hence finally

$$P_o(X_n \cdot e_1 \underset{n}{\to} +\infty) \geq \lim_{M\to\infty} P_o(D = \infty, \liminf_n X_n \cdot e_1 \geq M) = P_o(D = \infty) > 0.$$

This concludes the proof of the theorem.

CHAPITRE 6

STABLE FLUCTUATIONS FOR BALLISTIC RANDOM WALKS IN RANDOM ENVIRONMENT ON $\mathbb{Z}$

This article is co-authored with Nathanaël Enriquez, Christophe Sabot and Olivier Zindy.

Abstract. We consider transient random walks in random environment on $\mathbb{Z}$ in the positive speed (ballistic) and critical zero speed regimes. A classical result of Kesten, Kozlov and Spitzer proves that the hitting time of level n, after proper centering and normalization, converges to a stable distribution, but does not describe its scale parameter. Following [**17**], where the (non-critical) zero speed case was dealt with, we give a new proof of this result in the subdiffusive case that provides a complete description of the limit law. Like in [**17**], the case of Dirichlet environment turns out to be remarkably explicit.

6.1. Introduction

Random walks in a one-dimensional random environment were first introduced in the late sixties as a toy model for DNA replication. The recent development of micromanipulation technics such as DNA unzipping has raised a renewed interest in this model in genetics and biophysics, cf. for instance [**4**] where it is involved in a DNA sequencing procedure. Its mathematical study was initiated by Solomon's 1975 article [**43**], characterizing the transient and recurrent regimes and proving a strong law of large numbers. A salient feature emerging from this work was the existence of an intermediary regime where the walk is transient with a zero asymptotic speed, in contrast with the case of simple random walks. Shortly afterward, Kesten, Kozlov and Spitzer [**27**] precised this result in giving limit laws in the transient regime. When suitably normalized, the (properly centered) hitting time of site n by the random walk was proved to converge toward a stable law as n tends to infinity, which implies a limit law for the random walk itself. In particular, this entailed that the ballistic case (i.e. with positive speed) further decomposes into a diffusive and a subdiffusive regimes.

The aim of this article is to fully characterize the limit law in the subdiffusive (non-Gaussian) regime. Our approach is based on the one used in the similar study of the zero speed regime [**17**] by three of the authors. The proof of [**27**] relied on the use of an embedded branching process in random environment (with immigration), which gives little insight into the localization of the random walk and no explicit parameters for the limit. Rather following Sinai's study [**42**] of the recurrent case and physicists' heuristics developed since then (cf. for instance [**7**]), we proceed to an analysis of the potential associated to the environment as a way to locate the "deep valleys" that are likely to slow down the walk the most. We thus prove that the fluctuations of the hitting time of n with respect to its expectation mainly come from the time spent at crossing a very small number of deep potential wells. Since these are well apart, this translates the situation to the study of an almost-i.i.d. sequence of exit times out of "deep valleys". The distribution of these exit times involves the expectation of some functional of a meander associated to the potential, which was shown in [**16**] to relate to Kesten's renewal series, making it possible to get explicit constants in the limit. The case of Beta distributions turns out to be fully explicit as a consequence of a result by Chamayou and Letac [**9**]. The proof also covers the non-ballistic regime, including the critical zero-speed case, which was not covered in [**17**].

Let us mention two other works relative to this setting. Mayer-Wolf, Roitershtein and Zeitouni [**30**] generalized the limit laws of [**27**] from i.i.d. to Markovian environment, still keeping with the branching process viewpoint. And Peterson [**36**] (following [**37**]), in the classical i.i.d. setting and using potential technics, proved that no quenched limit law (i.e. inside a fixed generic environment) exists in the ballistic subdiffusive regime.

The paper is organized as follows. Section 6.2 states the results. The notions of excursions and deep valleys are introduced in Section 6.3, which will enable us to give in Subsection 6.3.3 the sketch and organization of the proof that occupies the rest of the paper.

6.2. Notations and main results

Let $\omega := (\omega_i,\, i \in \mathbb{Z})$ be a family of i.i.d. random variables taking values in $(0,1)$ defined on Ω, which stands for the random environment. Denote by P the distribution of ω and by E the corresponding expectation. Conditioning on ω (i.e. choosing an environment), we define the random walk in random environment $X := (X_n,\, n \geq 0)$ starting from $x \in \mathbb{Z}$ as a nearest-neighbour random walk on $\mathbb{Z}$ with transition probabilities given by ω: if we denote by $P_{x,\omega}$ the law of the Markov chain $(X_n,\, n \geq 0)$ defined by $P_{x,\omega}(X_0 = x) = 1$ and

$$P_{x,\omega}(X_{n+1} = z \,|\, X_n = y) := \begin{cases} \omega_y, & \text{if } z = y+1, \\ 1-\omega_y, & \text{if } z = y-1, \\ 0, & \text{otherwise,} \end{cases}$$

then the joint law of (ω, X) is $\mathbb{P}_x(\mathrm{d}\omega, \mathrm{d}X) := P_{x,\omega}(\mathrm{d}X)P(\mathrm{d}\omega)$. For convenience, we let $\mathbb{P} := \mathbb{P}_0$. We refer to [**53**] for an overview of results on random walks in random environment. An important role is played by the sequence of variables

$$\rho_i := \frac{1-\omega_i}{\omega_i}, \qquad i \in \mathbb{Z}. \tag{6.2.1}$$

We will make the following assumptions in the rest of this paper.

Assumptions. —
(a) there exists $0 < \kappa < 2$ *for which* $E\left[\rho_0^\kappa\right] = 1$ *and* $E\left[\rho_0^\kappa \log^+ \rho_0\right] < \infty$*;*
(b) the distribution of $\log \rho_0$ *is non-lattice.*

We now introduce the hitting time $\tau(x)$ of site x for the random walk $(X_n,\, n \geq 0)$,

$$\tau(x) := \inf\{n \geq 1 : X_n = x\}, \qquad x \in \mathbb{Z}.$$

For $\alpha \in (1,2)$, let $\mathcal{S}_\alpha^{ca}$ be the completely asymmetric stable zero mean random variable of index α with characteristic function

$$E[\mathrm{e}^{it\mathcal{S}_\alpha^{ca}}] = \exp((-it)^\alpha) = \exp\left(|t|^\alpha \cos\frac{\pi\alpha}{2}\left(1 - i\,\mathrm{sgn}(t)\tan\frac{\pi\alpha}{2}\right)\right), \tag{6.2.2}$$

where we use the principal value of the logarithm to define $(-it)^\alpha (= \mathrm{e}^{\alpha\log(-it)})$ for real t, and $\mathrm{sgn}(t) := \mathbf{1}_{(0,+\infty)}(t) - \mathbf{1}_{(-\infty,0)}(t)$. Note that $\cos\frac{\pi\alpha}{2} < 0$.

For $\alpha = 1$, let $\mathcal{S}_1^{ca}$ be the completely asymmetric stable random variable of index 1 with characteristic function

$$E[\mathrm{e}^{it\mathcal{S}_1^{ca}}] = \exp(-\frac{\pi}{2}|t| - it\log|t|) = \exp\big(-\frac{\pi}{2}|t|(1 + i\frac{2}{\pi}\,\mathrm{sgn}(t)\log|t|)\big). \tag{6.2.3}$$

Moreover, let us introduce the constant C_K describing the tail of Kesten's renewal series $R := \sum_{k\geq 0}\rho_0\cdots\rho_k$, see [**26**]:

$$P(R > x) \sim C_K x^{-\kappa}, \qquad x \to \infty.$$

Note that several probabilistic representations are available to compute C_K numerically, which are equally efficient. The first one was obtained by Goldie [**21**], a second was conjectured by Siegmund [**41**], and a third one was obtained in [**16**].

The main result of the paper can be stated as follows. The symbol "$\xrightarrow{\text{(law)}}$" denotes the convergence in distribution.

Theorem 6.1. — *Under assumptions (a) and (b) we have, under* $\mathbb{P}$*, when* n *goes to infinity,*

– *if* $1 < \kappa < 2$*, letting* $v := \frac{1-E[\rho_0]}{1+E[\rho_0]}$*,*

$$\frac{\tau(n) - nv^{-1}}{n^{1/\kappa}} \xrightarrow{\text{(law)}} 2\left(-\frac{\pi\kappa^2}{\sin(\pi\kappa)}C_K^2 E[\rho_0^\kappa\log\rho_0]\right)^{1/\kappa}\mathcal{S}_\kappa^{ca} \tag{6.2.4}$$

and

$$\frac{X_n - nv}{n^{1/\kappa}} \xrightarrow{\text{(law)}} -2\left(-\frac{\pi\kappa^2}{\sin(\pi\kappa)}C_K^2 E[\rho_0^\kappa\log\rho_0]\right)^{1/\kappa} v^{1+\frac{1}{\kappa}}\mathcal{S}_\kappa^{ca}; \tag{6.2.5}$$

– *if* $\kappa = 1$*, for some deterministic sequences* $(u_n)_n, (v_n)_n$ *converging to 1,*

$$\frac{\tau(n) - u_n\frac{2}{E[\rho_0\log\rho_0]}n\log n}{n} \xrightarrow{\text{(law)}} \frac{2}{E[\rho_0\log\rho_0]}\mathcal{S}_1^{ca} \tag{6.2.6}$$

and

$$\frac{X_n - v_n \frac{E[\rho_0 \log \rho_0]}{2} \frac{n}{\log n}}{n/(\log n)^2} \xrightarrow{\text{(law)}} \frac{E[\rho_0 \log \rho_0]}{2} \mathcal{S}_1^{ca}. \tag{6.2.7}$$

In particular, for $\kappa = 1$, the following limits in probability hold:

$$\frac{\tau(n)}{n \log n} \xrightarrow{(p)} \frac{2}{E[\rho_0 \log \rho_0]} \quad \text{and} \quad \frac{X_n}{n/\log n} \xrightarrow{(p)} \frac{E[\rho_0 \log \rho_0]}{2}. \tag{6.2.8}$$

Remarks. —
- *The proof of the theorem will actually give an expression for the sequence $(u_n)_n$.*
- *The case $0 < \kappa < 1$, already settled in* [**17**], *also follows from (a subset of) the proof.*

This theorem takes a remarkably explicit form in the case of Dirichlet environment, i.e. when the law of ω_0 is $\mathrm{Beta}(\alpha, \beta) := \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbf{1}_{(0,1)}(x)\,\mathrm{d}x$, with $\alpha, \beta > 0$ and $B(\alpha,\beta) := \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$. An easy computation leads to $\kappa = \alpha - \beta$. Thanks to a very nice result of Chamayou and Letac [**9**] giving the explicit value of $C_K = \frac{1}{(\alpha-\beta)B(\alpha-\beta,\beta)}$ in this case, we obtain the following corollary.

Corollary 6.2. — *In the case where ω_0 has a distribution* $\mathrm{Beta}(\alpha,\beta)$, *with $1 \leq \alpha - \beta < 2$, Theorem 6.1 applies with $\kappa = \alpha - \beta$. Then we have, when n goes to infinity, if $1 < \alpha - \beta < 2$,*

$$\frac{\tau(n) - \frac{\alpha+\beta-1}{\alpha-\beta-1} n}{n^{\frac{1}{\alpha-\beta}}} \xrightarrow{\text{law}} 2 \left(-\frac{\pi}{\sin(\pi(\alpha-\beta))} \frac{\Psi(\alpha) - \Psi(\beta)}{B(\alpha-\beta,\beta)^2} \right)^{\frac{1}{\alpha-\beta}} \mathcal{S}_{\alpha-\beta}^{ca},$$

and

$$\frac{X_n - \frac{\alpha-\beta-1}{\alpha+\beta-1} n}{n^{\frac{1}{\alpha-\beta}}} \xrightarrow{\text{law}} -2 \left(-\frac{\pi}{\sin(\pi(\alpha-\beta))} \frac{\Psi(\alpha) - \Psi(\beta)}{B(\alpha-\beta,\beta)^2} \right)^{\frac{1}{\alpha-\beta}} \left(\frac{\alpha-\beta-1}{\alpha+\beta-1} \right)^{1+\frac{1}{\alpha-\beta}} \mathcal{S}_{\alpha-\beta}^{ca},$$

where Ψ denotes the classical digamma function, $\Psi(z) := (\log \Gamma)'(z) = \frac{\Gamma'(z)}{\Gamma(z)}$. Furthermore, if $\alpha - \beta = 1$, then we have

$$E[\rho_0 \log \rho_0] = \frac{B(\beta,\beta)}{2\beta}.$$

In the following, the constant C stands for a positive constant large enough, whose value can change from line to line. We henceforth assume that hypotheses (a) and (b) hold; in particular, wherever no further restriction is mentioned, we have $0 < \kappa < 2$.

6.3. Notion of valley – Proof sketch

Following Sinai [**42**] (in the recurrent case), and more recently the study of the case $0 < \kappa < 1$ in [**17**], we define notions of potential and valleys that enable to visualize where the random walk spends most of its time.

6.3.1. The potential. — The potential, denoted by $V = (V(x),\ x \in \mathbb{Z})$, is a function of the environment ω defined by $V(0) = 0$ and $\rho_x = \mathrm{e}^{V(x)-V(x-1)}$ for every $x \in \mathbb{Z}$, i.e.

$$V(x) := \begin{cases} \sum_{i=1}^{x} \log \rho_i & \text{if } x \geq 1, \\ 0 & \text{if } x = 0, \\ -\sum_{i=x+1}^{0} \log \rho_i & \text{if } x \leq -1, \end{cases}$$

where the ρ_i's are defined in (6.2.1). Under hypothesis (a), Jensen's inequality gives $E[\log \rho_0^\kappa] \leq \log E[\rho_0^\kappa] = 0$, and hypothesis (b) excludes the equality case $\rho_0 = 1$ a.s., hence $E[\log \rho_0] < 0$ and thus $V(x) \to \mp\infty$ a.s. when $x \to \pm\infty$.

Furthermore, we consider the weak descending ladder epochs of the potential, defined by $e_0 := 0$ and

$$e_{i+1} := \inf\{k > e_i :\ V(k) \leq V(e_i)\}, \qquad i \geq 0. \tag{6.3.1}$$

Observe that $(e_i - e_{i-1})_{i\geq 1}$ is a family of i.i.d. random variables. Moreover, hypothesis (a) of Theorem 6.1 implies that e_1 is exponentially integrable. Indeed, for all $n > 0$, for any $\lambda > 0$, $P(e_1 > n) \leq P(V(n) > 0) = P(\mathrm{e}^{\lambda V(n)} > 1) \leq E[\mathrm{e}^{\lambda V(n)}] = E[\rho_0^\lambda]^n$, and $E[\rho_0^\lambda] < 1$ for any $0 < \lambda < \kappa$ by convexity of $s \mapsto E[\rho_0^s]$.

It will be convenient to extend the sequence $(e_i)_{i\geq 0}$ to negative indices by letting

$$e_{i-1} := \sup\{k < e_i :\ \forall l < k, V(l) \geq V(k)\}, \qquad i \leq 0. \tag{6.3.2}$$

The structure of the sequence $(e_i)_{i\in\mathbb{Z}}$ will be better understood after Lemma 6.5.

Observe that the intervals $(e_i, e_{i+1}], i \in \mathbb{Z}$, stand for the excursions of the potential above its past minimum, provided $V(x) \geq 0$ when $x \leq 0$. Let us introduce H_i, the height of the excursion $(e_i, e_{i+1}]$, defined by

$$H_i := \max_{e_i \leq k \leq e_{i+1}} (V(k) - V(e_i)), \qquad i \in \mathbb{Z}.$$

Note that the random variables $(H_i)_{i\geq 0}$ are i.i.d. For notational convenience, we will write $H := H_0$.

In order to quantify what "high excursions" are, we need a key result of Iglehart [**23**] which gives the tail probability of H, namely

$$P(H > h) \sim C_I\, \mathrm{e}^{-\kappa h}, \qquad h \to \infty, \tag{6.3.3}$$

where

$$C_I := \frac{(1 - E[\mathrm{e}^{\kappa V(e_1)}])^2}{\kappa E[\rho_0^\kappa \log \rho_0] E[e_1]}.$$

This result comes from the following classical consequence of renewal theory (see [**19**]): if $S := \sup_{k\geq 0} V(k)$, then

$$P(S > h) \sim C_F\, \mathrm{e}^{-\kappa h}, \qquad h \to \infty, \tag{6.3.4}$$

where C_F satisfies $C_I = (1 - E[\mathrm{e}^{\kappa V(e_1)}])C_F$. For these two asymptotic estimates, refer also to Proposition 3.2.

6.3.2. The deep valleys. — The notion of deep valley is relative to the space scale. Let $n \geq 2$. To define the corresponding deep valleys, we extract from the excursions of the potential above its minimum these whose heights are greater than a critical height h_n, defined by

$$h_n := \frac{1}{\kappa}\log n - \log\log n.$$

Moreover, let q_n denote the probability that the height of such an excursion is larger than h_n. Due to (6.3.3), it satisfies

$$q_n := P(H > h_n) \sim C_I\,\mathrm{e}^{-\kappa h_n}, \qquad n \to \infty.$$

Then, let $(\sigma(i))_{i\geq 1}$ be the sequence of the indices of the successive excursions whose heights are greater than h_n. More precisely,

$$\begin{aligned}\sigma(1) &:= \inf\{j \geq 0 : H_j \geq h_n\},\\ \sigma(i+1) &:= \inf\{j > \sigma(i) : H_j \geq h_n\}, \qquad i \geq 1.\end{aligned}$$

We consider now some random variables depending only on the environment, which define the deep valleys.

Definition 2. — *For $i \geq 1$, let us introduce*

$$a_i := e_{\sigma(i)-D_n}, \qquad b_i := e_{\sigma(i)}, \qquad d_i := e_{\sigma(i)+1},$$

where

$$D_n := \left\lceil \frac{1+\gamma}{A\kappa}\log n \right\rceil, \tag{6.3.5}$$

with arbitrary $\gamma > 0$, and A equals $E[-V(e_1)]$ if this expectation is finite and is otherwise an arbitrary positive real number. For every $i \geq 1$, the piece of environment $(\omega_x)_{a_i<x\leq d_i}$ is called the i-th deep valley (with bottom at b_i).

Note that the definitions of a_i and d_i differ slightly from those in [**17**]. We shall denote by K_n the number of such deep valleys before e_n, i.e.

$$K_n := \#\{0 \leq i \leq n-1 : H_i \geq h_n\}.$$

Remark. — *In wider generality, our proof adapts easily if we choose h_n, D_n such that $n\mathrm{e}^{-\kappa h_n} \to +\infty$, $D_n \geq C h_n$ for a large C, and $n\mathrm{e}^{-2\kappa h_n} D_n \to 0$. These conditions ensure respectively that the first n deep valleys include the most significant ones, that they are wide enough (to the left) so as to make negligible the time spent on their left after the walk has reached their bottom, and that they are disjoint. A typical range for h_n is $\frac{1}{2\kappa}\log n + (1+\alpha)\log\log n \leq h_n \leq \frac{1}{\kappa}\log n - \varepsilon\log\log n$, where $\alpha, \varepsilon > 0$.*

6.3.3. Proof sketch. — The idea directing our proof of Theorem 6.1 is that the time $\tau(e_n)$ splits into (a) the time spent at crossing the numerous "small" excursions, which will give the first order nv^{-1} (or $n\log n$ if $\kappa=1$) and whose fluctuations are negligible on a scale of $n^{1/\kappa}$, and (b) the time spent inside deep valleys, which is on the order of $n^{1/\kappa}$, as well as its fluctuations, and will therefore provide the limit law after normalization. Moreover, with overwhelming probability, the deep valleys are disjoint and the times spent at crossing them may therefore be treated as independent random variables.

The proof divides into three parts: reducing the time spent in the deep valleys to an i.i.d. setting (Section 6.5); neglecting the fluctuations of the time spent in the shallow valleys (Section 6.6); and evaluating the tail probability of the time spent in *one* valley (Section 6.7). These elements shall indeed enable us to apply a classical theorem relative to i.i.d. heavy-tailed random variables (Section 6.8). Before that, a few preliminaries are necessary.

6.4. Preliminaries

This section divides into three independent parts. The first part recalls usual formulas about random walks in a one-dimensional potential. The second one adapts the main results from **[16]** in the present context. Finally the last part is devoted to the effect of conditioning the potential on $\mathbb{Z}_-$ (bearing in mind that this half of the environment has little influence on the random walk), which is a technical tool to provide stationarity for several sequences.

In the following, for any event A on the environments such that $P(A)>0$, we use the notations

$$P^A := P(\,\cdot\,|\,A) \quad \text{and} \quad \mathbb{P}^A := \mathbb{P}(\,\cdot\,|\,A) = P_\omega \times P^A(d\omega).$$

In addition, the specific notations

$$P^{\geq 0} := P(\,\cdot\,|\,\forall k\leq 0, V(k)\geq 0) \quad \text{and} \quad \mathbb{P}^{\geq 0} := P_\omega \times P^{\geq 0}(d\omega)$$

will prove themselves convenient.

6.4.1. Quenched formulas. — We recall here a few Markov chain formulas that are of repeated use throughout the paper.

Quenched exit probabilities. — For any $a\leq x\leq b$, (see **[53]**, formula (2.1.4))

$$P_{x,\omega}(\tau(b)<\tau(a)) = \frac{\sum_{a\leq k<x} \mathrm{e}^{V(k)}}{\sum_{a\leq k<b} \mathrm{e}^{V(k)}}. \tag{6.4.1}$$

In particular,

$$P_{x,\omega}(\tau(a)=\infty) = \frac{\sum_{a\leq k<x} \mathrm{e}^{V(k)}}{\sum_{k\geq a} \mathrm{e}^{V(k)}} \tag{6.4.2}$$

and

$$P_{a+1,\omega}(\tau(a)=\infty) = \left(\sum_{k\geq a} \mathrm{e}^{V(k)-V(a)}\right)^{-1}. \tag{6.4.3}$$

Thus $P_{0,\omega}(\tau(1)=\infty)=\left(\sum_{k\le 0}\mathrm{e}^{V(k)}\right)^{-1}=0$, P-a.s. because $V(k)\to+\infty$ a.s. when $k\to-\infty$, and $P_{1,\omega}(\tau(0)=\infty)=\left(\sum_{k\ge 0}\mathrm{e}^{V(k)}\right)^{-1}>0$, P-a.s. by the root test (using $E[\log\rho_0]<0$). This means that X is transient to $+\infty$ $\mathbb{P}$-a.s.

Quenched expectation. — For any $a<b$, P-a.s., (cf. [**53**])

$$\begin{aligned}E_{a,\omega}[\tau(b)] &= \sum_{a\le j<b}\sum_{i\le j}(1+\mathrm{e}^{V(i)-V(i-1)})\mathrm{e}^{V(j)-V(i)}\\ &= \sum_{a\le j<b}\sum_{i\le j}\alpha_{ij}\mathrm{e}^{V(j)-V(i)} \end{aligned}\tag{6.4.4}$$

where $\alpha_{ij}=2$ if $i<j$, and $\alpha_{jj}=1$. Thus, we have

$$E_{a,\omega}[\tau(b)]\le 2\sum_{a\le j<b}\sum_{i\le j}\mathrm{e}^{V(j)-V(i)}\tag{6.4.5}$$

and in particular

$$E_{a,\omega}[\tau(a+1)]=1+2\sum_{i<a}\mathrm{e}^{V(a)-V(i)}\le 2\sum_{i\le a}\mathrm{e}^{V(a)-V(i)}.\tag{6.4.6}$$

Quenched variance. — For any $a<b$, P-a.s., (cf. [**1**] or [**22**])

$$\begin{aligned}Var_{a,\omega}(\tau(b)) &= 4\sum_{a\le k<b}\sum_{j\le k}\mathrm{e}^{V(k)-V(j)}(1+\mathrm{e}^{V(j-1)-V(j)})\left(\sum_{l<j}\mathrm{e}^{V(j)-V(l)}\right)^2\\ &= 4\sum_{a\le k<b}\sum_{j<k}(\mathrm{e}^{V(k)+V(j)}+\mathrm{e}^{V(k)+V(j+1)})\left(\sum_{l\le j}\mathrm{e}^{-V(l)}\right)^2.\end{aligned}\tag{6.4.7}$$

Thus, we have

$$\begin{aligned}Var_{a,\omega}(\tau(b)) &\le 8\sum_{a\le k<b}\sum_{j\le k}\mathrm{e}^{V(k)+V(j)}\left(\sum_{l\le j}\mathrm{e}^{-V(l)}\right)^2\\ &\le 16\sum_{a\le k<b}\sum_{l'\le l\le j\le k}\mathrm{e}^{V(k)+V(j)-V(l)-V(l')}.\end{aligned}\tag{6.4.8}$$

6.4.2. Renewal estimates. — In this section we recall and adapt results from [**16**], which are very useful to bound finely the expectations of exponential functionals of the potential.

Let $R_- := \sum_{k\le 0}\mathrm{e}^{-V(k)}$. Then Lemma 3.3 in the introduction (or Lemma 3.2 from [**16**]) proves that

$$E^{\ge 0}[R_-]<\infty\tag{6.4.9}$$

and that more generally all the moments of R_- are finite under $P^{\ge 0}$.

Let us define

$$T_H := \min\{x\ge 0:\ V(x)=H\},$$

and

$$M_1 := \sum_{k<T_H}\mathrm{e}^{-V(k)},\qquad M_2 := \sum_{0\le k<e_1}\mathrm{e}^{V(k)-H}.$$

Let $Z := M_1 M_2 \mathrm{e}^H$. Theorem 2.2 (together with Remark 7.1) of [**16**] proves that $P^{\geq 0}(Z > t, H = S) \sim C_U t^{-\kappa}$ as $t \to \infty$, where

$$C_U := C_I \left(\frac{C_K}{C_F}\right)^2 = \kappa E[\rho_0^\kappa \log \rho_0] E[e_1] (C_K)^2. \tag{6.4.10}$$

The next lemma shows that the condition $\{H = S\}$ can be dropped.

Lemma 6.3. — *We have*

$$P^{\geq 0}(Z > t) \sim \frac{C_U}{t^\kappa}, \qquad t \to \infty.$$

Proof. — All moments of $M_1 M_2$ are finite under $P^{\geq 0}$. Indeed, $M_2 \leq e_1$, $M_1 \leq e_1 + \sum_{k \leq 0} \mathrm{e}^{-V(k)}$ and the random variables e_1 and $\sum_{k \leq 0} \mathrm{e}^{-V(k)}$ have all moments finite under $P^{\geq 0}$ (cf. after (6.3.1) and (6.4.9)). For any $\ell_t > 0$,

$$P^{\geq 0}(Z > t, H < \ell_t) \leq P^{\geq 0}(M_1 M_2 > t\mathrm{e}^{-\ell_t}) \leq \frac{E^{\geq 0}[(M_1 M_2)^2]}{(t\mathrm{e}^{-\ell_t})^2}.$$

Since $\kappa < 2$, we may choose ℓ_t such that $\ell_t \to \infty$ and $t^\kappa = o(t^2 \mathrm{e}^{-2\ell_t})$ as $t \to \infty$, hence $P^{\geq 0}(Z > t, H < \ell_t) = o(t^{-\kappa})$. On the other hand, Z is independent of $S' := \sup_{x \geq e_1} V(x) - V(e_1) \geq S$, hence

$$\begin{aligned} P^{\geq 0}(Z > t, H > \ell_t, S > H) &\leq P^{\geq 0}(Z > t, H > \ell_t) P^{\geq 0}(S' > \ell_t) \\ &= P^{\geq 0}(Z > t, H > \ell_t) o(1) \end{aligned}$$

as $t \to \infty$, so that

$$\begin{aligned} P^{\geq 0}(Z > t, H > \ell_t) &\sim P^{\geq 0}(Z > t, H > \ell_t, H = S) \\ &= P^{\geq 0}(Z > t, H = S) + o(t^{-\kappa}) \sim C_U t^{-\kappa}. \end{aligned}$$

Thus we finally have

$$P^{\geq 0}(Z > t) = P^{\geq 0}(Z > t, H > \ell_t) + P^{\geq 0}(Z > t, H < \ell_t) = C_U t^{-\kappa} + o(t^{-\kappa}). \quad \square$$

We will actually need moments involving

$$M_1' := \sum_{k < e_1} \mathrm{e}^{-V(k)}$$

instead of $M_1 (\leq M_1')$. The next result is an adaptation of Lemma 4.1 from [**16**], i.e. our Lemma 3.4, to the present situation, together with (6.3.3), with a novelty coming from the difference between M_1' and M_1.

Lemma 6.4. — *For any $\alpha, \beta, \gamma \geq 0$, there is a constant C such that, for large $h > 0$,*

$$E^{\geq 0}[(M_1')^\alpha (M_2)^\beta \mathrm{e}^{\gamma H} | H < h] \leq \begin{cases} C & \text{if } \gamma < \kappa, \\ Ch & \text{if } \gamma = \kappa, \\ C\mathrm{e}^{(\gamma - \kappa)h} & \text{if } \gamma > \kappa \end{cases} \tag{6.4.11}$$

and, if $\gamma < \kappa$,

$$E^{\geq 0}[(M_1')^\alpha (M_2)^\beta \mathrm{e}^{\gamma H} | H \geq h] \leq C\mathrm{e}^{\gamma h}. \tag{6.4.12}$$

Proof. — Let $M := (M_1')^\alpha (M_2)^\beta$. Note first that $M_1' \le R_- + e_1$ and $M_2 \le e_1$, so that all moments of M_1' and M_2 are finite under $P^{\ge 0}$ (cf. (6.4.9) and after (6.3.1)). Hölder inequality then gives $E^{\ge 0}[M] < \infty$ and, since e^H has moments up to order κ (excluded) by (6.3.3), $E^{\ge 0}[(M_1')^\alpha (M_2)^\beta \mathrm{e}^{\gamma H}] < \infty$ for $\gamma < \kappa$, which proves the very first bound.

Let us now prove in the other cases that we may insert the condition $\{S = H\}$ in the expectations. Let $\ell = \ell(h) := \frac{1}{\gamma}\log h$. We have

$$E^{\ge 0}[M\mathrm{e}^{\gamma H}\mathbf{1}_{\{H<h\}}] \le E^{\ge 0}[M]h + E^{\ge 0}[M\mathrm{e}^{\gamma H}\mathbf{1}_{\{H<h, H>\ell\}}] \tag{6.4.13}$$

Since M and H are independent of $S' := \sup_{x\ge e_1} V(x) - V(e_1)$, and $\{S > H > \ell\} \subset \{S' > \ell\}$,

$$E^{\ge 0}[M\mathrm{e}^{\gamma H}\mathbf{1}_{\{H<h,H>\ell\}}\mathbf{1}_{\{S>H\}}] \le E^{\ge 0}[M\mathrm{e}^{\gamma H}\mathbf{1}_{\{H<h\}}]P(S' > \ell).$$

Then $P(S' > \ell) \to 0$ when $h \to \infty$, hence adding this left-hand side to (6.4.13) gives

$$E^{\ge 0}[M\mathrm{e}^{\gamma H}\mathbf{1}_{\{H<h\}}](1 + o(1)) \le E^{\ge 0}[M]h + E^{\ge 0}[M\mathrm{e}^{\gamma H}\mathbf{1}_{\{H<h\}}\mathbf{1}_{\{H=S\}}].$$

Given that $P(H = S) > 0$, and $h \le \mathrm{e}^{(\gamma-\kappa)h}$ for large h when $\gamma > \kappa$, it thus suffices to prove the last two bounds of (6.4.11) with $E^{\ge 0}[M\mathrm{e}^{\gamma H}|H < h, H = S]$ as the left-hand side. As for (6.4.12), the introduction of ℓ is useless to prove similarly (skipping (6.4.13)) that we may condition by $\{H = S\}$.

For any $r > 0$, by Lemma 4.1 of **[16]** (which is also Lemma 3.4 page 32),

$$E^{\ge 0}[(M_2)^r | \lfloor H \rfloor, H = S] \le C_r. \tag{6.4.14}$$

On the other hand, $M_1' = M_1 + \sum_{T_H<k<e_1} \mathrm{e}^{-V(k)}$. Let $r > 0$. We have, by Lemma 4.1 of **[16]**, $E^{\ge 0}[(M_1)^r | \lfloor H \rfloor, H = S] \le C_r$. As for the other term, it results from Lemma 3.4 of **[16]** that $(H, \sum_{T_H\le k<e_1} \mathrm{e}^{-V(k)})$ has same distribution under $P^{\ge 0}(\cdot|H = S)$ as $(H, \sum_{T_H^-<k\le 0} \mathrm{e}^{V(k)-H})$ where $T_H^- := \sup\{k \le 0 | V(k) > H\}$, and we claim that there is $C_r' > 0$ such that, for all $N \in \mathbb{N}$,

$$E\left[\left(\sum_{T_N^-<k\le 0} \mathrm{e}^{V(k)}\right)^r\right] \le C_r'\mathrm{e}^{rN}. \tag{6.4.15}$$

Before we prove this inequality, let us use it to conclude that

$$\begin{aligned} E^{\ge 0}[(M_1')^r | \lfloor H \rfloor, H = S] &\le 2^r \left(E^{\ge 0}[(M_1)^r | \lfloor H \rfloor, H = S] + \mathrm{e}^{-r\lfloor H \rfloor} C \mathrm{e}^{r(\lfloor H \rfloor + 1)}\right) \\ &\le C'. \end{aligned} \tag{6.4.16}$$

For readibility reasons, we write the proof of (6.4.15) when $r = 2$, the case of higher integer values being exactly similar and implying the general case (if $0 < r < s$, $E[X^r] \le E[X^s]^{r/s}$ for any positive X). We have

$$E\left[\left(\sum_{T_N^-<k\le 0} \mathrm{e}^{V(k)}\right)^2\right] \le \sum_{0\le m,n<N} \mathrm{e}^{n+1}\mathrm{e}^{m+1} E[\nu([n, n+1))\nu([m, m+1))] \tag{6.4.17}$$

where $\nu(A) := \#\{k \le 0 : V(k) \in A\}$ for all $A \subset \mathbb{R}$. For any $n \in \mathbb{N}$, Markov property at time $\sup\{k \le 0 : V(k) \in [n, n+1)\}$ implies that $E[\nu([n, n+1))^2] \le$

$E[\nu([-1,1))^2]$. This latter expectation is finite because $V(1)$ has a negative mean and is exponentially integrable; more precisely, $\nu([-1,1))$ is exponentially integrable as well: for $\lambda > 0$, for all $k \geq 0$, $P(V(-k) < 1) \leq e^{\lambda} E[e^{\lambda V(1)}]^k = e^{\lambda} E[\rho^{\lambda}]^k$ hence, choosing $\lambda > 0$ small enough so that $E[\rho^{\lambda}] < 1$ (cf. assumption (a)), we have, for all $p \geq 0$,

$$\begin{aligned} P(\nu([-1,1)) > p) &\leq P(\exists k \geq p \text{ s.t. } V(-k) < 1) \\ &\leq \sum_{k \geq p} P(V(-k) < 1) \leq e^{\lambda}(1 - E[\rho^{\lambda}])^{-1} E[\rho^{\lambda}]^p. \end{aligned}$$

Thus, using Cauchy-Schwarz inequality (or $ab \leq \frac{1}{2}(a^2 + b^2)$) to bound the expectations uniformly, the right-hand side of (6.4.17) is less than Ce^{2N} for some constant C. This proves (6.4.15).

Finally, assembling (6.4.14) and (6.4.16) leads to

$$E^{\geq 0}[M | \lfloor H \rfloor, H = S] \leq E^{\geq 0}[(M_1')^{2\alpha} | \lfloor H \rfloor, H = S]^{1/2} E^{\geq 0}[(M_2)^{2\beta} | \lfloor H \rfloor, H = S]^{1/2} \leq C$$

hence, conditioning by $\lfloor H \rfloor$,

$$E^{\geq 0}[M e^{\gamma H} \mathbf{1}_{\{H < h\}} | H = S] \leq C' E[e^{\gamma(\lfloor H \rfloor + 1)} \mathbf{1}_{\{\lfloor H \rfloor < h\}}] \leq C'' E[e^{\gamma H} \mathbf{1}_{\{H < h+1\}}],$$

and similarly

$$E^{\geq 0}[M e^{\gamma H} \mathbf{1}_{\{H > h\}} | H = S] \leq C'' E[e^{\gamma H} \mathbf{1}_{\{H > h-1\}}].$$

The conclusion of the lemma is then a simple consequence of the tail estimate (6.3.3) and the usual formulas

$$E[e^{\gamma H} \mathbf{1}_{\{H > h\}}] = e^{\gamma h} P(H > h) + \int_h^{\infty} \gamma e^{\gamma u} P(H > u)\, du$$

and

$$E[e^{\gamma H} \mathbf{1}_{\{H < h\}}] = 1 - e^{\gamma h} P(H > h) + \int_0^h \gamma e^{\gamma u} P(H > u)\, du. \qquad \square$$

6.4.3. Environment on the left of 0. — By definition, the distribution of the environment is translation invariant. However, the distribution of the "environment seen from e_k", i.e. of $(\omega_{e_k+p})_{p \in \mathbb{Z}}$, depends on k. When suitably conditioning the environment on $\mathbb{Z}_-$, this problem vanishes.

Recall we defined both e_i for $i \geq 0$ and $i \leq 0$, cf. (6.3.2).

Lemma 6.5. — *Under $P^{\geq 0}$, the sequence $(e_{i+1} - e_i)_{i \in \mathbb{Z}}$ is i.i.d., and more precisely the sequence of the excursions $(V(e_i + l) - V(e_i))_{0 \leq l \leq e_{i+1} - e_i}$, $i \in \mathbb{Z}$, is i.i.d..*

Proof. — Let us denote $\mathcal{L} := \{\forall l < 0,\ V(l) \geq 0\}$. Let Φ, Ψ be positive measurable functions respectively defined on finite paths indexed by $\{0, \ldots, k\}$ for any k and on infinite paths indexed by $\mathbb{Z}_-$). We have

$$\begin{aligned} &E^{\geq 0}[\Psi((V(e_{-1} + l) - V(e_{-1}))_{l \leq 0}) \Phi((V(e_{-1} + l) - V(e_{-1}))_{0 \leq l \leq -e_{-1}})] \\ &\quad = \sum_{k=-\infty}^{0} E[\Psi((V(k + l) - V(k))_{l \leq 0}) \Phi((V(k + l) - V(k))_{0 \leq l \leq -k}) \mathbf{1}_{A_k}] P(\mathcal{L})^{-1}, \end{aligned}$$

where $A_k := \{e_{-1} = k\} \cap \mathcal{L} = \{\forall l < k, V(l) \geq V(k), V(k) \geq V(0), \forall k < l < 0, V(l) > V(k)\}$. Using the fact that $(V(k+l) - V(k))_{l\in\mathbb{Z}}$ has same distribution as $(V(l))_{l\in\mathbb{Z}}$, this becomes

$$\sum_{k=-\infty}^{0} E[\Psi((V(l))_{l\leq 0})\mathbf{1}_{(\forall l<0, V(l)\geq 0)}\Phi((V(l))_{0\leq l\leq -k})\mathbf{1}_{(V(-k)\leq 0, \forall 0<l<-k, V(l)>0)}]P(\mathcal{L})^{-1}.$$

Finally, the independence between $(V(l))_{l\leq 0}$ and $(V(l))_{l\geq 0}$ shows that the previous expression equals

$$\begin{aligned}\sum_{k=-\infty}^{0} & E^{\geq 0}[\Psi((V(l))_{l\leq 0})]E[\Phi((V(l))_{0\leq l\leq -k})\mathbf{1}_{(V(0)\geq V(-k), \forall 0<l<-k, V(l)>V(0))}] \\ &= E^{\geq 0}[\Psi((V(l))_{l\leq 0})]E[\Phi((V(l))_{0\leq l\leq e_1})],\end{aligned}$$

hence

$$\begin{aligned}E^{\geq 0}&[\Psi((V(e_{-1}+l) - V(e_{-1}))_{l\leq 0})\Phi((V(e_{-1}+l) - V(e_{-1}))_{0\leq l\leq -e_{-1}})] \\ &= E^{\geq 0}[\Psi((V(l))_{l\leq 0})]E[\Phi((V(l))_{0\leq l\leq e_1})].\end{aligned}$$

By induction we deduce that, under $P^{\geq 0}$, the excursions to the left are independent and distributed like the first excursion to the right. In addition, $(V(l))_{l\geq 0}$ and $(V(l))_{l\leq 0}$ are independent and, due to Markov property, the excursions to the right are i.i.d.. This concludes the proof: all the excursions, to the left or to the right, are independent and have same distribution under $P^{\geq 0}$. □

6.5. Independence of the deep valleys

The independence between deep valleys goes through imposing these valleys to be disjoint (i.e. $a_i > d_{i-1}$ for all i) and neglecting the time spent on the left of a valley while it is being crossed (i.e. the time spent on the left of a_i before a_{i+1} is reached).

NB. — All the results and proofs from this section hold for any parameter $\kappa > 0$.

For any integers x, y, z, let us define the time

$$\widetilde{\tau}^{(z)}(x, y) := \#\{\tau(x) \leq k \leq \tau(y) : \; X_k \leq z\}$$

spent on the left of z between the first visit to x and the first visit to y, and the total time

$$\widetilde{\tau}^{(z)} := \#\{k \geq \tau(z) : \; X_k \leq z\}$$

spent on the left of z after the first visit of z. Of course, $\widetilde{\tau}^{(z)}(x, y) \leq \widetilde{\tau}^{(z)}$ if $z \leq x$.

We consider the event

$$NO(n) := \{0 < a_1\} \cap \bigcap_{i=1}^{K_n-1} \{d_i < a_{i+1}\},$$

which means that the large valleys before e_n lie entirely on $\mathbb{Z}_+$ and don't overlap.

The following two propositions will enable us to reduce to i.i.d. deep valleys.

Proposition 6.6. — *We have*

$$P(NO(n)) \xrightarrow[n]{} 1.$$

Proof of Proposition 6.6. — Choose $\varepsilon > 0$ and define the event

$$A_K(n) := \{K_n \le (1+\varepsilon)C_I(\log n)^\kappa\}.$$

Since K_n is a binomial random variable of mean $nq_n \sim_n C_I(\log n)^\kappa$, it follows from the law of large numbers that $P(A_K(n))$ converges to 1 as $n \to \infty$. On the other hand, if the event $NO(n)^c$ occurs, then there exists $1 \le i \le K_n$ such that there is at least one high excursion among the first D_n excursions to the right of d_{i-1} (with $d_0 = 0$). Thus,

$$\begin{aligned} P(NO(n)^c) &\le P(A_K(n)^c) + (1+\varepsilon)C_I(\log n)^\kappa(1-(1-q_n)^{D_n}) \\ &\le o(1) + (1+\varepsilon)C_I(\log n)^\kappa q_n D_n = o(1) \end{aligned}$$

Indeed, for any $0 < x < 1$ and $\alpha > 0$, we have $1-(1-x)^\alpha \le \alpha x$ by concavity of $x \mapsto 1-(1-x)^\alpha$. □

For $x \ge 0$, define

$$a(x) := \max\{e_k :\ k \in \mathbb{Z},\ e_{k+D_n} \le x\}. \tag{6.5.1}$$

In particular, $a(b_i) = a_i$ for all $i \ge 1$, and $a(0) = e_{-D_n}$.

Proposition 6.7. — *Under* $\mathbb{P}$,

$$\frac{1}{n^{1/\kappa}}\sum_{i=1}^{K_n} \widetilde{\tau}^{(a_i)}(b_i, d_i) = \frac{1}{n^{1/\kappa}}\sum_{k=0}^{n-1} \widetilde{\tau}^{(a(e_k))}(e_k, e_{k+1})\mathbf{1}_{\{H_k \ge h_n\}} \xrightarrow[n]{(p)} 0.$$

Proof. — The equality is trivial from the definitions. The second expression has the advantage that, under $\mathbb{P}^{\ge 0}$, all the terms have same distribution because of Lemma 6.5. To overcome the fact that $\widetilde{\tau}^{(a(0))}(0, e_1)\mathbf{1}_{\{H \ge h_n\}}$ is not integrable for $0 < \kappa \le 1$, we introduce the event

$$\begin{aligned} A_n &:= \{\text{for } i = 1, \ldots, K_n,\ H_{\sigma(i)} \le V(a_i) - V(b_i)\} \\ &= \bigcap_{k=0}^{n-1} \{H_k < h_n\} \cup \{h_n \le H_k \le V(e_{k-D_n}) - V(e_k)\}. \end{aligned}$$

Let us prove that $P^{\ge 0}((A_n)^c) = o_n(1)$. By Lemma 6.5,

$$P^{\ge 0}((A_n)^c) \le nP^{\ge 0}(H \ge h_n,\ H > V(e_{-D_n})).$$

Choose $0 < \gamma' < \gamma'' < \gamma$ (cf. (6.3.5)) and define $l_n := \frac{1+\gamma'}{\kappa}\log n$. We get

$$P^{\ge 0}((A_n)^c) \le n\big(P(H \ge l_n) + P(H \ge h_n)P^{\ge 0}(V(e_{-D_n}) < l_n)\big). \tag{6.5.2}$$

Equation (6.3.3) gives $P(H \ge l_n) \sim_n C_I \mathrm{e}^{-\kappa l_n} = C_I n^{-(1+\gamma')}$ and $P(H \ge h_n) \sim_n C_I n^{-1}(\log n)^\kappa$. Under $P^{\ge 0}$, $V(e_{-D_n})$ is the sum of D_n i.i.d. random variables distributed like $-V(e_1)$. Therefore, for any $\lambda > 0$,

$$P^{\ge 0}(V(e_{-D_n}) < l_n) \le \mathrm{e}^{\lambda l_n} E[\mathrm{e}^{-\lambda(-V(e_1))}]^{D_n}.$$

Since $\frac{1}{\lambda}\log E[\mathrm{e}^{-\lambda(-V(e_1))}] \to -E[-V(e_1)] \in [-\infty, 0)$ as $\lambda \to 0^+$, we can choose $\lambda > 0$ such that $\log E[\mathrm{e}^{-\lambda(-V(e_1))}] < -\lambda A\frac{1+\gamma''}{1+\gamma}$ (where A was defined after (6.3.5)), hence $E[\mathrm{e}^{-\lambda(-V(e_1))}]^{D_n} \le n^{-\lambda\frac{1+\gamma''}{\kappa}}$. Thus, $P^{\ge 0}(V(e_{-D_n}) < l_n) \le n^{-\lambda\frac{\gamma''-\gamma'}{\kappa}}$. Using these estimates in (6.5.2) concludes the proof that $P^{\ge 0}((A_n)^c) = o_n(1)$.

Let us now prove the Proposition itself. By Markov inequality, for all $\delta > 0$,

$$\mathbb{P}^{\ge 0}\left(\frac{1}{n^{1/\kappa}}\sum_{k=0}^{n-1}\widetilde{\tau}^{(a(e_k))}(e_k, e_{k+1})\mathbf{1}_{\{H\ge h_n\}} > \delta\right)$$
$$\le P^{\ge 0}((A_n)^c) + \frac{1}{\delta n^{1/\kappa}}\mathbb{E}^{\ge 0}\left[\sum_{k=0}^{n-1}\widetilde{\tau}^{(a(e_k))}(e_k, e_{k+1})\mathbf{1}_{\{H\ge h_n\}}\mathbf{1}_{A_n}\right]$$
$$\le o_n(1) + \frac{n}{\delta n^{1/\kappa}}E^{\ge 0}\left[E_\omega[\widetilde{\tau}^{(e_{-D_n})}(0, e_1)]\mathbf{1}_{\{H > h_n,\ H < V(e_{-D_n})\}}\right]. \tag{6.5.3}$$

Note that we have $E_\omega[\widetilde{\tau}^{(e_{-D_n})}(0, e_1)] = E_\omega[N]E_\omega[T_1]$, where N is the number of crossings from $e_{-D_n}+1$ to e_{-D_n} before the first visit at e_1, and T_1 is the time for the random walk to go from e_{-D_n} to $e_{-D_n}+1$ (for the first time, for instance); furthermore, these two terms are independent under $P^{\ge 0}$. Using (6.4.1), we have

$$E_\omega[N] = \frac{P_{0,\omega}(\tau(e_{-D_n}) < \tau(e_1))}{P_{e_{-D_n}+1,\omega}(\tau(e_1) < \tau(e_{-D_n}))} = \sum_{0\le x<e_1}\mathrm{e}^{V(x)-V(e_{-D_n})} = M_2\mathrm{e}^{H-V(e_{-D_n})}$$

hence, on the event $\{H < V(e_{-D_n})\}$, $E_\omega[N] \le M_2$.

The length of an excursion to the left of e_{-D_n} is computed as follows, due to (6.4.6):

$$E_\omega[T_1] = E_{e_{-D_n},\omega}[\tau(e_{-D_n}+1)] \le 2\sum_{x\le e_{-D_n}}\mathrm{e}^{-(V(x)-V(e_{-D_n}))}.$$

The law of $(V(x) - V(e_{-D_n}))_{x\le e_{-D_n}}$ under $P^{\ge 0}$ is $P^{\ge 0}$ because of Lemma 6.5. Therefore,

$$E^{\ge 0}[E_\omega[T_1]] \le 2E^{\ge 0}\left[\sum_{x\le 0}\mathrm{e}^{-V(x)}\right] = 2E^{\ge 0}[R_-] < \infty$$

with (6.4.9). We conclude that the right-hand side of (6.5.3) is less than

$$o_n(1) + \frac{n}{\delta n^{1/\kappa}}2E^{\ge 0}[R_-]E[M_2\mathbf{1}_{\{H>h_n\}}].$$

Since Lemma 6.4 gives $E[M_2\mathbf{1}_{\{H>h_n\}}] \le CP(H > h_n) \sim_n C'\mathrm{e}^{-\kappa h_n} = C'n^{-1}(\log n)^\kappa$, this whole expression converges to 0, which concludes. $\square$

6.6. Fluctuation of interarrival times

For any $x \le y$, we define the inter-arrival time $\tau(x,y)$ between sites x and y by

$$\tau(x,y) := \inf\{n \ge 0 :\ X_{\tau(x)+n} = y\}, \qquad x, y \in \mathbb{Z}.$$

Let

$$\tau_{\mathrm{IA}} := \sum_{i=0}^{K_n}\tau(d_i, b_{i+1}\wedge e_n) = \sum_{k=0}^{n-1}\tau(e_k, e_{k+1})\mathbf{1}_{\{H_k<h_n\}}$$

(with $d_0 = 0$) be the time spent at crossing small excursions before $\tau(e_n)$. The aim of this section is the following bound on the fluctuations of τ_{IA}.

Proposition 6.8. — *For any $1 \le \kappa < 2$, under $\mathbb{P}^{\ge 0}$,*

$$\frac{1}{n^{1/\kappa}}(\tau_{\mathrm{IA}} - \mathbb{E}^{\ge 0}[\tau_{\mathrm{IA}}]) \xrightarrow[n]{(p)} 0. \tag{6.6.1}$$

Remark. — *This limit also holds for $0 < \kappa < 1$ in a simple way: we have, in this case, using* (6.4.5) *and Lemma 6.4,*

$$\begin{aligned}\mathbb{E}^{\ge 0}[\tau_{\mathrm{IA}}] = nE^{\ge 0}[E_\omega[\tau(e_1)]\mathbf{1}_{\{H \le h_n\}}] &\le nE[2M_1' M_2 \mathrm{e}^{H} \mathbf{1}_{\{H \le h_n\}}] \\ &\le Cn\mathrm{e}^{(1-\kappa)h_n} = o(n^{1/\kappa}),\end{aligned}$$

hence both $n^{-1/\kappa}\tau_{\mathrm{IA}}$ and its expectation converge to 0 in probability.

By Chebychev inequality, this Proposition will come as a direct consequence of Lemma 6.10 bounding the variance of τ_{IA}. However, a specific caution is necessary in the case $\kappa = 1$; indeed, the variance is infinite in this case, because of the rare but significant fluctuations originating from the time spent by the walk when it backtracks into deep valleys. Our proof in this case consists in proving first that we may neglect in probability (using a first-moment method) the time spent backtracking into these deep valleys; and then that this brings us to the computation of the variance of τ_{IA} in an environment where small excursions have been substituted for the high ones (thus removing the non-integrability problem).

Subsection 6.6.1 is dedicated to this reduction to an integrable setting, which is only involved in the case $\kappa = 1$ of Proposition 6.8 and of the main theorem (but holds in greater generality), while Subsection 6.6.2 states and proves the bounds on the variance, implying Proposition 6.8.

6.6.1. Reduction to small excursions (required for the case $\kappa = 1$). — Let $h > 0$. Let us denote by d_- the right end of the first excursion on the left of 0 that is higher than h:

$$d_- := \max\{e_k :\ k \le 0, H_{k-1} \ge h\}.$$

Remember $\widetilde{\tau}^{(d_-)}(0, e_1)$ is the time spent on the left of d_- before the walk reaches e_1.

Lemma 6.9. — *There exists $C > 0$, independent of h, such that*

$$\mathbb{E}^{\ge 0}[\widetilde{\tau}^{(d_-)}(0, e_1)\mathbf{1}_{\{H \le h\}}] \le C \begin{cases} \mathrm{e}^{-(2\kappa-1)h} & \text{if } \kappa < 1 \\ h\mathrm{e}^{-h} & \text{if } \kappa = 1 \\ \mathrm{e}^{-\kappa h} & \text{if } \kappa > 1. \end{cases} \tag{6.6.2}$$

Proof. — Let us decompose $\widetilde{\tau}^{(d_-)}(0, e_1)$ into the successive excursions to the left of d_-:

$$\widetilde{\tau}^{(d_-)}(0, e_1) = \sum_{m=1}^{N} T_m,$$

where N is the number of crossings from $d_- + 1$ to d_- before $\tau(e_1)$, and T_m is the time for the walk to go from d_- to $d_- + 1$ on the m-th time. Under P_ω, the times

T_m, $m \geq 1$, are i.i.d. and independent of N (i.e., more properly, the sequence $(T_m)_{1\leq m\leq N}$ can be prolonged to an infinite sequence with these properties). We have, using Markov property and then (6.4.1),

$$E_\omega[N] = \frac{P_{0,\omega}(\tau(d_-) < \tau(e_1))}{P_{d_-+1,\omega}(\tau(e_1) < \tau(d_-))} = \sum_{0\leq x<e_1} \mathrm{e}^{V(x)-V(d_-)}$$

and, from (6.4.6),

$$E_\omega[T_1] = E_{d_-,\omega}[\tau(d_-+1)] \leq 2\sum_{x\leq d_-} \mathrm{e}^{-(V(x)-V(d_-))}.$$

Therefore, by Wald identity and Lemma 6.5,

$$\begin{aligned}\mathbb{E}^{\geq 0}[\widetilde{\tau}^{(d_-)}(0,e_1)\mathbf{1}_{\{H\leq h\}}] &= E^{\geq 0}[E_\omega[N]E_\omega[T_1]\mathbf{1}_{\{H\leq h\}}] \\ &\leq 2E\left[\sum_{0\leq x<e_1} \mathrm{e}^{V(x)}\mathbf{1}_{\{H\leq h\}}\right]E^{\geq 0}[\mathrm{e}^{-V(d_-)}]E^{\Lambda(h)}\left[\sum_{x\leq 0}\mathrm{e}^{-V(x)}\right],\end{aligned} \tag{6.6.3}$$

where $\Lambda(h) := \{\forall k\leq 0,\ V(k)\geq 0\}\cap\{H_{-1}\geq h\}$. The first expectation can be written as $E[M_2\mathrm{e}^H\mathbf{1}_{\{H\leq h\}}]$. For the second one, note that $d_- = e_{-W}$, where W is a geometric random variable of parameter $q := P(H\geq h)$; and, conditional on $\{W=n\}$, the distribution of $(V(k))_{e_{-W}\leq k\leq 0}$ under $P^{\geq 0}$ is the same as that of $(V(k))_{e_{-n}\leq k\leq 0}$ under $P^{\geq 0}(\cdot|\text{for } k=0,\dots,n-1, H_{-k}<h)$. Therefore,

$$E^{\geq 0}[\mathrm{e}^{-V(d_-)}] = E^{\geq 0}\big[E[\mathrm{e}^{-V(e_1)}|H<h]^W\big] = \frac{q}{1-(1-q)E[\mathrm{e}^{V(e_1)}|H<h]},$$

and $(1-q)E[\mathrm{e}^{V(e_1)}|H<h]$ converges to $E[\mathrm{e}^{V(e_1)}]<1$ when $h\to\infty$ (the inequality comes from assumption (b)), hence this quantity is uniformly bounded from above by $c<1$ for large h. In addition, (6.3.3) gives $q\sim C_I\mathrm{e}^{-\kappa h}$ when $h\to\infty$, hence

$$E^{\geq 0}[\mathrm{e}^{-V(d_-)}] \leq C\mathrm{e}^{-\kappa h},$$

where C is independent of h. Finally, let us consider the last term of (6.6.3). We have

$$\begin{aligned}&E^{\Lambda(h)}\left[\sum_{x\leq 0}\mathrm{e}^{-V(x)}\right] \\ &\quad = E\left[\sum_{e_{-1}<x\leq 0}\mathrm{e}^{-V(x)}\middle| H_{-1}\geq h\right] + E^{\geq 0}\left[\sum_{x\leq e_{-1}}\mathrm{e}^{-(V(x)-V(e_{-1}))}\right]E[\mathrm{e}^{-V(e_{-1})}] \\ &\quad \leq E[M_1'|H\geq h] + E^{\geq 0}[R_-]E[\mathrm{e}^{V(e_1)}]\end{aligned}$$

hence, using Lemma 6.4, (6.4.9) and $V(e_1)\leq 0$, this term is bounded by a constant. The statement of the lemma then follows from the application of Lemma 6.4 to $E[M_2\mathrm{e}^H\mathbf{1}_{\{H\leq h\}}]$. □

The part of the inter-arrival time τ_{IA} spent at backtracking in high excursions can be written as follows:

$$\begin{aligned}\widetilde{\tau_{\mathrm{IA}}} &:= \widetilde{\tau}^{(d_-)}(0, b_1 \wedge e_n) + \sum_{i=1}^{K_n} \widetilde{\tau}^{(d_i)}(d_i, b_{i+1} \wedge e_n) \\ &= \sum_{k=0}^{n-1} \widetilde{\tau}^{(d(e_k))}(e_k, e_{k+1}) \mathbf{1}_{\{H_k < h_n\}},\end{aligned}$$

where, for $x \in \mathbb{Z}$,

$$d(x) := \max\{e_k : \ k \in \mathbb{Z}, \ e_k \leq x, \ H_{k-1} \geq h_n\}.$$

In particular, $d(0) = d_-$ in the previous notation with $h = h_n$.

Note that, under $\mathbb{P}^{\geq 0}$, because of Lemma 6.5, the terms of the above sum have same distribution as $\widetilde{\tau}^{(d(0))}(0, e_1)\mathbf{1}_{\{H<h_n\}}$, hence

$$\mathbb{E}^{\geq 0}[\widetilde{\tau_{\mathrm{IA}}}] = n\mathbb{E}^{\geq 0}[\widetilde{\tau}^{(d(0))}(0, e_1)\mathbf{1}_{\{H<h_n\}}].$$

Thus, for $\mathbb{E}^{\geq 0}[\widetilde{\tau_{\mathrm{IA}}}]$ to be negligible with respect to $n^{1/\kappa}$, it suffices that the expectation on the right-hand side be negligible with respect to $n^{1/\kappa - 1}$. In particular, for $\kappa = 1$, it suffices that it converges to 0, which is readily seen from (6.6.2). Thus, for $\kappa = 1$,

$$\frac{1}{n^{1/\kappa}} \mathbb{E}^{\geq 0}[\widetilde{\tau_{\mathrm{IA}}}] \underset{n}{\longrightarrow} 0, \tag{6.6.4}$$

hence in particular $n^{-1/\kappa}\widetilde{\tau_{\mathrm{IA}}} \to 0$ in probability under $\mathbb{P}^{\geq 0}$. Note that (6.6.4) actually holds for any $\kappa \geq 1$.

Let us introduce the modified environment, where independent small excursions are substituted for the high excursions. In order to avoid obfuscating the redaction, we will only introduce little notation regarding this new environment.

Let us enlarge the probability space in order to accommodate for a new family of independent excursions indexed by $\mathbb{N}^* \times \mathbb{Z}$ such that the excursion indexed by (n, k) has same distribution as $(V(x))_{0 \leq x \leq e_1}$ under $P(\cdot | H \leq h_n)$. Thus we are given, for every $n \in \mathbb{N}^*$, a countable family of independent excursions lower than h_n. For every n, we define the modified environment of height less than h_n by replacing all the excursions of V that are higher than h_n by new independent ones that are lower than h_n. Because of Lemma 6.5, this construction is especially natural under $P^{\geq 0}$, where it has stationarity properties.

In the following, we will denote by P' the law of the modified environment relative to the height h_n guessed from the context (hence also a definition of $(P^{\geq 0})'$, for instance).

Remark. — *Repeating the proof done under $P^{\geq 0}$ for $(P^{\geq 0})'$, we see that R_- still has all finite moments in the modified environment, and that these moments are bounded uniformly in n. In particular, the bound for $E^{\geq 0}[(M_1')^\alpha (M_2)^\beta \mathrm{e}^{\gamma H} \mathbf{1}_{\{H \leq h_n\}}]$ given in Lemma 6.4 is unchanged for $(E^{\geq 0})'$ (writing $M_1' = R_- + \sum_{0 \leq k < e_1} \mathrm{e}^{-V(x)}$*

and using $(a+b)^\alpha \le 2^\alpha(a^\alpha+b^\alpha)$). On the other hand,

$$\begin{aligned} E'[R] &= \sum_{i=0}^{\infty} E'[\mathrm{e}^{V(e_i)}]E'\left[\sum_{e_i\le k<e_{i+1}} \mathrm{e}^{V(k)-V(e_i)}\right] \\ &= \sum_{i=0}^{\infty} E[\mathrm{e}^{V(e_1)}|H\le h_n]^i E[M_2\mathrm{e}^H|H\le h_n], \end{aligned}$$

and $E[\mathrm{e}^{V(e_1)}|H\le h_n]\le c$ for some $c<1$ independent of n because this expectation is smaller than 1 for all n and it converges toward $E[\mathrm{e}^{V(e_1)}]<1$ as $n\to\infty$. Hence, by Lemma 6.4,

$$\text{(6.6.5)} \qquad \textit{if } \kappa=1, \qquad E'[R]\le Ch_n$$

This is the only difference that will appear in the following computations.

Assuming that $d(0)$ keeps being defined with respect to the usual heights, (6.6.2) (with $h=h_n$) is still true for the walk in the modified environment. Indeed, the change only affects the environment on the left of $d(0)$, hence the only difference in the proof involves the times T_m: in (6.6.3), one should substitute $(E^{\ge 0})'$ for $E^{\Lambda(h)}$, and this factor is uniformly bounded in both cases because of the above remark about R_-.

We deduce that the time $\widetilde{\tau_{\mathrm{IA}}}'$, defined similarly to $\widetilde{\tau_{\mathrm{IA}}}$ except that the excursions on the left of the points $d(e_i)$ (i.e. the times similar to T_m in the previous proof) are performed in the modified environment, still satisfies, for $\kappa=1$,

$$\text{(6.6.6)} \qquad \frac{1}{n^{1/\kappa}}\mathbb{E}^{\ge 0}[\widetilde{\tau_{\mathrm{IA}}}'] \xrightarrow[n]{} 0.$$

Note now that

$$\tau_{\mathrm{IA}}' := \tau_{\mathrm{IA}} - \widetilde{\tau_{\mathrm{IA}}} + \widetilde{\tau_{\mathrm{IA}}}'$$

is the time spent at crossing the (original) small excursions, in the environment where the high excursions have been replaced by new independent small excursions. Indeed, the high excursions are only involved in τ_{IA} during the backtracking of the walk to the left of $d(e_i)$ for some $0\le i<n$. Assembling (6.6.4) and (6.6.6), it is equivalent (for $\kappa=1$) to prove (6.6.1) or

$$\frac{1}{n^{1/\kappa}}(\tau_{\mathrm{IA}}' - \mathbb{E}^{\ge 0}[\tau_{\mathrm{IA}}']) \xrightarrow[n]{(p)} 0,$$

and it is thus sufficient to prove

$$\mathbb{V}\mathrm{ar}^{\ge 0}(\tau_{\mathrm{IA}}') = o_n(n^{2/\kappa}).$$

6.6.2. Bounding the variance of τ_{IA}. — Because of the previous subsection, the proof of Proposition 6.8 will follow from the following Lemma.

Lemma 6.10. — *We have, for $1<\kappa<2$,*

$$\mathbb{V}\mathrm{ar}^{\ge 0}(\tau_{\mathrm{IA}}) = o_n(n^{2/\kappa})$$

and, for $1\le\kappa<2$,

$$\mathbb{V}\mathrm{ar}^{\ge 0}(\tau_{\mathrm{IA}}') = o_n(n^{2/\kappa}).$$

We recall that the second bound is only introduced to settle the case $\kappa = 1$; it would suffice for $1 < \kappa < 2$ as well, but introduces unnecessary complication. The computations being very close for τ_{IA} and τ'_{IA}, we will write below the proof for τ_{IA} and indicate line by line where changes happen for τ'_{IA}. Let us stress that, when dealing with τ'_{IA}, all the indicator functions $\mathbf{1}_{\{H_\cdot \le h_n\}}$ (which define the small valleys) would refer to the original heights, while all the potentials $V(\cdot)$ appearing along the computation (which come from quenched expectations of times spent by the walk) would refer to the modified environment.

Since we have

$$\mathbb{V}\mathrm{ar}^{\ge 0}(\tau_{\mathrm{IA}}) = E^{\ge 0}[\,Var_\omega(\tau_{\mathrm{IA}})] + Var^{\ge 0}(E_\omega[\tau_{\mathrm{IA}}]),$$

it suffices to prove the following two results:

$$E^{\ge 0}[\,Var_\omega(\tau_{\mathrm{IA}})] = o_n(n^{2/\kappa}) \tag{6.6.7}$$

$$Var^{\ge 0}(E_\omega[\tau_{\mathrm{IA}}]) = o_n(n^{2/\kappa}). \tag{6.6.8}$$

Proof of (6.6.7). — We have

$$\tau_{IA} = \sum_{p=0}^{n-1} \tau(e_p, e_{p+1}) \mathbf{1}_{\{H_p < h_n\}} \tag{6.6.9}$$

and by Markov property, the above times are independent under $P_{o,\omega}$. Hence

$$Var_\omega(\tau_{\mathrm{IA}}) = \sum_{p=0}^{n-1} Var_\omega(\tau(e_p, e_{p+1})) \mathbf{1}_{\{H_p < h_n\}}.$$

Under $P^{\ge 0}$, the distribution of the environment seen from e_p does not depend on p, hence

$$E^{\ge 0}[\,Var_\omega(\tau_{\mathrm{IA}})] = nE^{\ge 0}[\,Var_\omega(\tau(e_1))\mathbf{1}_{\{H < h_n\}}]. \tag{6.6.10}$$

We use Formula (6.4.8):

$$Var_\omega(\tau(e_1))\mathbf{1}_{\{H<h_n\}} \le 16 \sum_{l' \le l \le j \le k \le e_1,\ 0 \le k} \mathrm{e}^{V(k)+V(j)-V(l)-V(l')} \mathbf{1}_{\{H<h_n\}}. \tag{6.6.11}$$

Let us first consider the part of the sum where $j \ge 0$. By noting that the indices satisfy $l' \le j$ and $l \le k$, this part is seen to be less than $(M'_1 M_2 \mathrm{e}^H)^2 \mathbf{1}_{\{H<h_n\}}$. Lemma 6.4 shows that its expectation is less than $C\mathrm{e}^{(2-\kappa)h_n}$. *For τ'_{IA}: The same holds, because of the remark p. 85.*

It remains to deal with the indices $j < 0$. This part rewrites as

$$\sum_{l', l \le j < 0} \mathrm{e}^{V(j)-V(l)-V(l')} \cdot \sum_{0 \le k < e_1} \mathrm{e}^{V(k)} \mathbf{1}_{\{H<h_n\}}. \tag{6.6.12}$$

Since $V_{|\mathbb{Z}_+}$ and $V_{|\mathbb{Z}_-}$ are independent under P, so are the two above factors. The second one equals $\mathrm{e}^H M_2 \mathbf{1}_{\{H<h_n\}}$. Let us split the first one according to the excursion $[e_{u-1}, e_u)$ containing j; it becomes

$$\sum_{u \le 0} \mathrm{e}^{-V(e_{u-1})} \sum_{e_{u-1} \le j < e_u} \mathrm{e}^{V(j)-V(e_{u-1})} \left(\sum_{l \le j} \mathrm{e}^{-(V(l)-V(e_{u-1}))} \right)^2. \tag{6.6.13}$$

We have $V(e_{u-1}) \geq V(e_u)$ and, under $P^{\geq 0}$, $V(e_u)$ is independent of $(V(e_u + k) - V(e_u))_{k\leq 0}$ and thus of $(V(e_{u-1}+k) - V(e_{u-1}))_{k\leq e_u - e_{u-1}}$, which has same distribution as $(V(k))_{k\leq e_1}$. Therefore, the expectation of (6.6.13) with respect to $P^{\geq 0}$ is less than

$$\sum_{u\leq 0} E^{\geq 0}[\mathrm{e}^{-V(e_u)}] E^{\geq 0}\left[\sum_{0\leq j<e_1} \mathrm{e}^{V(j)}\left(\sum_{l\leq j}\mathrm{e}^{-V(l)}\right)^2\right]$$
$$\leq (1 - E[\mathrm{e}^{V(e_1)}])^{-1} E^{\geq 0}[\mathrm{e}^H (M_1')^2 M_2].$$

Thus the expectation of (6.6.12) with respect to $P^{\geq 0}$ is bounded by

$$(1 - E[\mathrm{e}^{V(e_1)}])^{-1} E^{\geq 0}[\mathrm{e}^H (M_1')^2 M_2] E^{\geq 0}[\mathrm{e}^H M_2 \mathbf{1}_{\{H<h_n\}}].$$

From Lemma 6.4, we conclude that this term is less than a constant if $\kappa > 1$. The part corresponding to $j \geq 0$ therefore dominates; this finishes the proof of (6.6.7). *For τ'_{IA}: The first factor is $(1 - E[\mathrm{e}^{V(e_1)}|H < h_n])^{-1}$, which is uniformly bounded because it converges to $(1 - E[\mathrm{e}^{V(e_1)}])^{-1} < \infty$ and, using Lemma 6.4, the two other factors are each bounded by a constant if $\kappa > 1$ and by Ch_n if $\kappa = 1$ (cf. again the remark p. 85). Thus, the part corresponding to $j \geq 0$ still dominates in this case.*

We have proved $E^{\geq 0}[\mathit{Var}_\omega(\tau_{\mathrm{IA}}))] \leq Cn\mathrm{e}^{(2-\kappa)h_n}$. Since $n\mathrm{e}^{(2-\kappa)h_n} = \frac{n^{2/\kappa}}{(\log n)^{2-\kappa}}$, this concludes.

Proof of (6.6.8). — From equation (6.6.9) we deduce

$$E_\omega[\tau_{\mathrm{IA}}] = \sum_{p=0}^{n-1} E_\omega[\tau(e_p, e_{p+1})]\mathbf{1}_{\{H_p<h_n\}}$$

hence, using (6.4.4),

$$\mathit{Var}^{\geq 0}(E_\omega[\tau_{\mathrm{IA}}]) = \sum_{i\leq j,\ k\leq l} (E^{\geq 0}[A_{ij}A_{kl}] - E^{\geq 0}[A_{ij}]E^{\geq 0}[A_{kl}]), \tag{6.6.14}$$

where $A_{ij} := \alpha_{ij}\mathrm{e}^{V(j)-V(i)}\mathbf{1}_{\{0\leq j<e_n,\ H(j)<h_n\}}$ for any indices $i < j$, and $H(j) := H_q$ when $e_q \leq j < e_{q+1}$.

Let us split this sum according to the relative order of i, j, k, l and bound each term separately. Note that, up to multiplication by 2, we may assume $j \leq l$, hence we only have to consider $i \leq j \leq k \leq l$ and $i, k \leq j \leq l$ (either $i \leq k$ or $k \leq i$).

• $i \leq j \leq k \leq l$. Let us split again according to the excursion containing j. The summand of (6.6.14) equals

$$\sum_{q=0}^{n-1}\Big(E^{\geq 0}[A_{ij}\mathbf{1}_{\{e_q\leq j<e_{q+1}\}}A_{kl}] - E^{\geq 0}[A_{ij}\mathbf{1}_{\{e_q\leq j<e_{q+1}\}}]E^{\geq 0}[A_{kl}]\Big). \tag{6.6.15}$$

In addition, we write $A_{kl} = A_{kl}\mathbf{1}_{\{k\geq e_{q+1}\}} + A_{kl}\mathbf{1}_{\{e_q\leq k<e_{q+1}\}}$ in both terms in order to split according to whether j and k lie in the same excursion.

Let us consider the case when j and k are in different excursions. Because of Markov property at time e_{q+1} and of the stationarity of the distribution of the environment, we have, for any $i \le j \le k \le l$,

$$E^{\ge 0}[A_{ij}\mathbf{1}_{\{e_q \le j < e_{q+1}\}}A_{kl}\mathbf{1}_{\{e_{q+1}\le k\}}] = E^{\ge 0}[A_{ij}\mathbf{1}_{\{e_q\le j<e_{q+1}\}}]E^{\ge 0}[A_{kl}\mathbf{1}_{\{e_{q+1}\le k\}}],$$

hence these terms do not contribute to the sum (6.6.15). *The same holds for* τ'_{IA}.

For the remainder of the sum, we will only need to bound the "expectation of the square" part of the variance, i.e. the terms coming from $E^{\ge 0}[A_{ij}A_{kl}]$.

Let us turn to the case when j and k lie in the same excursion $[e_q, e_{q+1})$. We have (remember $\alpha_{ij} \le 2$)

$$\begin{aligned}\sum_{i\le j\le k\le l}\sum_{q=0}^{n-1} & E^{\ge 0}\left[A_{ij}A_{kl}\mathbf{1}_{\{e_q\le j\le k<e_{q+1}\}}\right] \\ & \le 4\sum_{q=0}^{n-1}E^{\ge 0}\left[\sum_{i\le j\le k\le l}\mathrm{e}^{V(j)-V(i)}\mathbf{1}_{\{e_q\le j\le k<e_{q+1}\}}\mathrm{e}^{V(l)-V(k)}\mathbf{1}_{\{H_q<h_n\}}\right].\end{aligned}$$

Because of Lemma 6.5, the last expectation, which involves a function of $(V(e_q+l)-V(e_q))_{l\in\mathbb{Z}}$, does not depend on q. Thus it equals

$$\begin{aligned}4n\sum_{i\le j\le k\le l} & E^{\ge 0}[\mathrm{e}^{V(j)-V(i)}\mathbf{1}_{\{0\le j\le k<e_1\}}\mathrm{e}^{V(l)-V(k)}\mathbf{1}_{\{H<h_n\}}] \\ & \le 4nE^{\ge 0}\left[\sum_{i\le j\le e_1,\ 0\le j}\mathrm{e}^{V(j)-V(i)}\sum_{0\le k\le l,\ k<e_1}\mathrm{e}^{V(l)-V(k)}\mathbf{1}_{\{H<h_n\}}\right].\end{aligned}$$

Splitting according to whether $l < e_1$ or $l \ge e_1$, the variable in the last expectation is bounded by $(M'_1M_2\mathrm{e}^H)^2\mathbf{1}_{\{H<h_n\}} + (M'_1M_2\mathrm{e}^H)(M'_1\sum_{l\ge e_1}\mathrm{e}^{V(l)})\mathbf{1}_{\{H<h_n\}}$. Note that $\sum_{l\ge e_1}\mathrm{e}^{V(l)} \le \sum_{l\ge e_1}\mathrm{e}^{V(l)-V(e_1)}$, which has same distribution as R and is independent of M'_1, M_2, H. The above bound thus becomes

$$4n\Big(E^{\ge 0}[(M'_1)^2(M_2)^2\mathrm{e}^{2H}\mathbf{1}_{\{H<h_n\}}] + E^{\ge 0}[(M'_1)^2M_2\mathrm{e}^H\mathbf{1}_{\{H<h_n\}}]E[R]\Big).$$

From Lemma 6.4, this is less than $4n(C\mathrm{e}^{(2-\kappa)h_n}+C) \le C'n\mathrm{e}^{(2-\kappa)h_n}$. *For* τ'_{IA}, *this is unchanged when* $\kappa > 1$; *and, if* $\kappa = 1$, *the above expression is bounded by* $4n(C\mathrm{e}^{(2-\kappa)h_n}+C(h_n)^2)$, *cf.* (6.6.5), *hence the bound remains the same.*

- $i, k \le j \le l$ (either $i \le k$ or $k \le i$). We have

$$\begin{aligned}\sum_{i,k\le j\le l} & E^{\ge 0}[A_{ij}A_{kl}] \\ & \le 8\sum_{p=0}^{n-1}E^{\ge 0}\left[\sum_{i\le k\le j\le l}\mathrm{e}^{V(l)-V(k)+V(j)-V(i)}\mathbf{1}_{\{e_p\le l<e_{p+1}\}}\mathbf{1}_{\{H_p<h_n\}}\right].\end{aligned}$$

Using Lemma 6.5 we see that the above expectation, which involves a function of $(V(e_p+l)-V(e_p))_{l\in\mathbb{Z}}$, does not depend on p. Therefore, it equals

$$8nE^{\ge 0}\left[\sum_{i\le k\le j\le l\le e_1,\ l\ge 0}\mathrm{e}^{V(l)+V(j)-V(k)-V(i)}\mathbf{1}_{\{H<h_n\}}\right].$$

The quantity in the expectation matches exactly the formula in (6.6.11) that was used as a bound for $Var_\omega(\tau(e_1))\mathbf{1}_{\{H<h_n\}}$ (with different names for the indices:

(i,k,j,l) becomes (l',l,j,k)). Thus, it follows from the proof of (6.6.7) that

$$\sum_{i,k\le j\le l} E^{\ge 0}[A_{ij}A_{kl}] \le Cn\mathrm{e}^{(2-\kappa)h_n} = o_n(n^{2/\kappa}).$$

We have obtained the expected upper bound for each of the orderings, hence the lemma.

6.6.3. A subsequent Lemma. — The previous proofs of (6.6.7) and (6.6.8) entail the following bound for the crossing time of one low excursion:

Lemma 6.11. — *We have, for* $1<\kappa<2$,

$$E^{\ge 0}[E_\omega[\tau(e_1)^2]\mathbf{1}_{\{H<h\}}] \le C\mathrm{e}^{(2-\kappa)h},$$

and similarly for $(E^{\ge 0})'$ *when* $1\le\kappa<2$.

Proof. — We have $E_\omega[\tau(e_1)^2] = \mathit{Var}_\omega(\tau(e_1)) + E_\omega[\tau(e_1)]^2$. Equation (6.6.10) and the remainder of the proof of (6.6.7) give:

$$E^{\ge 0}[\mathit{Var}_\omega(\tau(e_1))\mathbf{1}_{\{H<h\}}] \le C\mathrm{e}^{(2-\kappa)h}.$$

In order to see that the proof of (6.6.8) implies the remainding bound:

$$E^{\ge 0}[E_\omega[\tau(e_1)]^2\mathbf{1}_{\{H<h\}}] \le C\mathrm{e}^{(2-\kappa)h},$$

it suffices to take $n=1$ in the proof (except of course in "h_n") and to notice that, although our proof gave a bound for the *variance* of $E_\omega[\tau(e_1)]$, we actually only needed to substract the "squared expectation"-terms (cf. (6.6.14)) corresponding to indices lying in different excursions... a situation which doesn't occur when $n=1$. Thus our proof in fact gives (in this case only) a bound for the "expectation of the square" of $E_\omega[\tau(e_1)]$. □

6.7. A general estimate for the occupation time of a deep valley

In this section we establish a precise annealed estimate for the tail distribution of the time spent by the particle to cross the first positive excursion of the potential above its past infimum. Since we shall use this result to estimate the occupation time of deep valleys previously introduced, it is relevant to condition the potential to be nonnegative on $\mathbb{Z}_-$. The main result of this section is the following.

Proposition 6.12. — *The tail distribution of the hitting time of the first negative record* e_1 *satisfies*

$$t^\kappa\,\mathbb{P}^{\ge 0}\left(\tau(e_1)\ge t\right) \longrightarrow C_T, \qquad t\to\infty,$$

where the constant C_T *is given by*

$$C_T := 2^\kappa\,\Gamma(\kappa+1)C_U. \tag{6.7.1}$$

The idea of the proof is the following. We show first that the the height of the first excursion has to be larger than a function h_t (of order $\log t$). Secondly, we prove that conditional on $H \geq h_t$ the environment has locally "good" properties. Finally, we decompose the passage from 0 to e_1 into the sum of a random geometrically distributed number of unsuccessful attempts to reach e_1 from 0 (i.e. excursions of the particle from 0 to 0 which do not hit e_1), followed by a successful attempt. This enables us to prove that $\tau(e_1)$ behaves as an exponentially distributed random variable with mean $2Z$ where Z is defined by $Z := M_1 M_2 \,\mathrm{e}^H$ and whose tail distribution is studied in **[16]** and recalled in Lemma 6.3.

In this proof, we denote $\tau(e_1)$ by τ.

6.7.1. The height of the first excursion has to be large.

— Let the critical height h_t be a function of t defined by

$$h_t := \log t - \log\log t, \qquad t \geq \mathrm{e}^{\mathrm{e}}. \tag{6.7.2}$$

Lemma 6.13. — *We have*

$$\mathbb{P}^{\geq 0}(\tau(e_1) > t,\, H \leq h_t) = o(t^{-\kappa}), \qquad t \to \infty.$$

Proof. — Let us first assume that $0 < \kappa < 1$. Then, by Markov inequality, we get

$$\begin{aligned}\mathbb{P}^{\geq 0}(\tau > t, H \leq h_t) = E^{\geq 0}[P_\omega(\tau > t)\mathbf{1}_{\{H \leq h_t\}}] &\leq \frac{1}{t} E^{\geq 0}[E_\omega[\tau]\mathbf{1}_{\{H \leq h_t\}}] \\ &\leq \frac{1}{t} E^{\geq 0}[2M_1' M_2 \mathrm{e}^H \mathbf{1}_{\{H \leq h_t\}}] \leq \frac{1}{t} C\mathrm{e}^{(1-\kappa)h_t},\end{aligned}$$

where the last inequality follows from Lemma 6.4. Since $t^{-1}\mathrm{e}^{(1-\kappa)h_t} = t^{-\kappa}(\log t)^{-(1-\kappa)}$, this settles this case.

Let us now assume $1 < \kappa < 2$. By Markov inequality, we get

$$\mathbb{P}^{\geq 0}(\tau > t, H \leq h_t) \leq \frac{1}{t^2} E^{\geq 0}[E_\omega[\tau^2]\mathbf{1}_{\{H \leq h_t\}}].$$

Applying Lemma 6.11 yields $\mathbb{P}^{\geq 0}(\tau > t, H \leq h_t) \leq Ct^{-2}\mathrm{e}^{(2-\kappa)h_t}$, which concludes the proof of Lemma 6.13 when $\kappa \neq 1$.

For $\kappa = 1$, neither of the above techniques works: the first one is too rough, and $Var_\omega(\tau)$ is not integrable. We shall modify τ so as to make $Var_\omega(\tau)$ integrable. To this end, let us refer to Subsection 6.6.1 and denote by d_- the right end of the first excursion on the left of 0 that is higher than h_t, and by $\widetilde{\tau} := \widetilde{\tau}^{(d_-)}(0, e_1)$ the time spent on the left of d_- before reaching e_1. By Lemma 6.9 we have $\mathbb{E}^{\geq 0}[\widetilde{\tau}\mathbf{1}_{\{H<h_t\}}] \leq Ch_t \mathrm{e}^{-h_t} \leq C(\log t)^2 t^{-1}$. Let us also introduce $\widetilde{\tau}'$, which is defined like $\widetilde{\tau}$ but in the modified environment, i.e. by replacing the high excursions (on

the left of d_-) by small ones (cf. after Lemma 6.9). Then we have

$$\begin{aligned}\mathbb{P}^{\geq 0}(\tau > t, H < h_t) &\leq \mathbb{P}^{\geq 0}(\widetilde{\tau} > (\log t)^3, H < h_t) + \mathbb{P}^{\geq 0}(\tau - \widetilde{\tau} > t - (\log t)^3, H < h_t)\\ &\leq \frac{1}{(\log t)^3}\mathbb{E}^{\geq 0}[\widetilde{\tau}\mathbf{1}_{\{H<h_t\}}] + \mathbb{P}^{\geq 0}(\tau - \widetilde{\tau} + \widetilde{\tau}' > t - (\log t)^3, H < h_t)\\ &= o(t^{-1}) + (\mathbb{P}^{\geq 0})'(\tau > t - (\log t)^3, H < h_t)\\ &\leq o(t^{-1}) + \frac{1}{(t-(\log t)^3)^2}(E^{\geq 0})'[E_\omega[\tau^2]\mathbf{1}_{\{H<h_t\}}],\end{aligned}$$

and Lemma 6.11 allows us to conclude just like in the case $1 < \kappa < 2$.

Remark. — *An alternative proof for* $\kappa = 1$*, avoiding the use of a modified environment, would consist in bounding the heights of all excursions on the left of 0 by increasing quantities so as to give this event overwhelming probability; this method is used after* (6.9.5).

□

6.7.2. "Good" environments. — Let us introduce the following events

$$\begin{aligned}\Omega_t^{(1)} &:= \{e_1 \leq C\log t\},\\ \Omega_t^{(2)} &:= \left\{\max\{-V^{\downarrow}(0,T_H),\, V^{\uparrow}(T_H,e_1)\} \leq \alpha\log t\right\},\\ \Omega_t^{(3)} &:= \left\{R^- \leq (\log t)^4 t^\alpha\right\},\end{aligned}$$

where $\max(0, 1-\kappa) < \alpha < \min(1, 2-\kappa)$ is arbitrary, and R^- will be introduced in Subsection 6.7.3. Then, we define the set of "good" environments at time t by

$$\Omega_t := \Omega_t^{(1)} \cap \Omega_t^{(2)} \cap \Omega_t^{(3)}.$$

The following result tells that "good" environments are asymptotically typical.

Lemma 6.14. — *The event* Ω_t *satisfies*

$$P(\Omega_t^c,\, H \geq h_t) = o(t^{-\kappa}), \qquad t \to \infty.$$

The proof of this result is easy but technical and postponed to the Appendix page 103.

6.7.3. Preliminary results: two h-processes. — In order to estimate finely the time spent in a deep valley, we decompose the passage from 0 to e_1 into the sum of a random geometrically distributed number, denoted by N, of unsuccessful attempts to reach e_1 from 0 (i.e. excursions of the particle from 0 to 0 which do not hit e_1), followed by a successful attempt. More precisely, N is a geometrically distributed random variable with parameter $1-p$ satisfying

$$1 - p = \frac{\omega_0}{\sum_{x=0}^{e_1-1} \mathrm{e}^{V(x)}} = \frac{\omega_0}{M_2\mathrm{e}^H}, \tag{6.7.3}$$

and we can write $\tau(e_1) = \sum_{i=1}^N F_i + G$, where the F_i's are the durations of the successive i.i.d. failures and G that of the first success. The accurate estimation of the time spent by each (successful and unsuccessful) attempt leads us to consider two h-processes where the random walker evolves in two modified potentials, one corresponding to the conditioning on a failure (see the potential $\widehat{V}$) and the other to the conditioning on a success (see the potential $\bar{V}$). Note that this approach was first introduced in **[17]** to estimate the quenched Laplace transform of the occupation time of a deep valley in the case $0 < \kappa < 1$.

The failure case: the h-potential $\widehat{V}$. — Let us fix a realization of ω. To introduce the h-potential $\widehat{V}$, we define $h(x) := P_{x,\omega}(\tau(0) < \tau(e_1))$. We introduce, for $0 < x < e_1$, $\widehat{\omega}_x := \omega_x \frac{h(x+1)}{h(x)}$ and, for $x \leq 0$, $\widehat{\omega}_x := \omega_x$. Then classically, under $P_{0,\omega}(\cdot|\tau(0) < \tau(e_1))$, $(X_n)_{0\leq n\leq\tau(0)}$ has same law as under $P_{0,\widehat{\omega}}$. In particular, the length F of the first failed attempt at reaching e_1 from 0 (conditional on its existence) satisfies

$$E_{0,\omega}[F] = E_{0,\omega}[\tau(0)|\tau(0) < \tau(e_1)] = E_{0,\widehat{\omega}}[\tau(0)].$$

Since h is a harmonic function, we have $1 - \widehat{\omega}_x = (1-\omega_x)\frac{h(x-1)}{h(x)}$. Note that $h(x)$ satisfies, see (6.4.1),

$$h(x) = \sum_{k=x}^{e_1-1} \mathrm{e}^{V(k)} \Bigg(\sum_{k=0}^{e_1-1} \mathrm{e}^{V(k)}\Bigg)^{-1}, \qquad 0 < x < e_1. \tag{6.7.4}$$

Now, $\widehat{V}$ can be defined for $x \geq 0$ by $\widehat{V}(x) := \sum_{i=1}^{x} \log \frac{1-\widehat{\omega}_i}{\widehat{\omega}_i}$. We obtain for any $0 \leq x < y < e_1$,

$$\widehat{V}(y) - \widehat{V}(x) = (V(y) - V(x)) + \log\left(\frac{h(x)\,h(x+1)}{h(y)\,h(y+1)}\right). \tag{6.7.5}$$

Since $h(x)$ is a decreasing function of x (by definition), we get for any $0 \leq x < y \leq e_1$,

$$\widehat{V}(y) - \widehat{V}(x) \geq V(y) - V(x). \tag{6.7.6}$$

From **[17]** (see Lemma 12) or the above preliminaries, we recall the following explicit computations for the first and second moments of F. For any environment ω, we have

$$E_\omega[F] = 2\,\omega_0\Bigg(\sum_{i=-\infty}^{-1} \mathrm{e}^{-V(i)} + \sum_{i=0}^{e_1-1} \mathrm{e}^{-\widehat{V}(i)}\Bigg) =: 2\,\omega_0\,\widehat{M}_1,$$

and

$$E_\omega\left[F^2\right] = 4\omega_0\,R^+ + 4(1-\omega_0)\,R^-, \tag{6.7.7}$$

where R^+ and R^- are defined by

$$R^+ := \sum_{i=1}^{e_1-1}\Bigg(1 + 2\sum_{j=0}^{i-2} \mathrm{e}^{\widehat{V}(j)-\widehat{V}(i-1)}\Bigg)\Bigg(\mathrm{e}^{-\widehat{V}(i-1)} + 2\sum_{j=i+1}^{e_1-1} \mathrm{e}^{-\widehat{V}(j-1)}\Bigg),$$

$$R^- := \sum_{i=-\infty}^{-1}\Bigg(1 + 2\sum_{j=i+2}^{0} \mathrm{e}^{V(j)-V(i+1)}\Bigg)\Bigg(\mathrm{e}^{-V(i+1)} + 2\sum_{j=-\infty}^{i-1} \mathrm{e}^{-V(j+1)}\Bigg).$$

Moreover, we can prove the following useful properties.

Lemma 6.15. — *For all* $t \geq 1$, *we have on* Ω_t

$$Var_\omega(F) \leq C(\log t)^4 t^\alpha, \tag{6.7.8}$$

$$M_2 \leq C \log t, \tag{6.7.9}$$

$$|\widehat{M}_1 - M_1| \leq o(t^{-\delta}) M_1, \tag{6.7.10}$$

with $\delta \in (0, 1-\alpha)$.

The proof of this result is postponed to the appendix.

The success case: the h-potential $\bar{V}$. — In a similar way, we introduce the h-potential $\bar{V}$ by defining $g(x) := P_{x,\omega}(\tau(e_1) < \tau(0)) = 1-h(x)$. For any $0 < x < e_1$, we introduce $\bar{\omega}_x := \omega_x \frac{g(x+1)}{g(x)}$, and $\bar{\omega}_0 = 1$. Then classically, under $P_{0,\omega}(\cdot|\tau(e_1) < \tau(0))$, $(X_n)_{0 \leq n \leq \tau(e_1)}$ has same law as under $P_{0,\bar{\omega}}$. In particular, the length G of the first successful attempt at reaching e_1 from 0 satisfies

$$E_{0,\omega}[G] = E_{0,\omega}[\tau(e_1)|\tau(e_1) < \tau(0)] = E_{0,\bar{\omega}}[\tau(e_1)].$$

Since g is a harmonic function, we have $1 - \bar{\omega}_x = (1-\omega_x)\frac{g(x-1)}{g(x)}$. Note that $g(x)$ satisfies, see (6.4.1),

$$g(x) = \sum_{k=0}^{x-1} \mathrm{e}^{V(k)} \Big(\sum_{k=0}^{e_1-1} \mathrm{e}^{V(k)} \Big)^{-1}, \qquad 0 < x < e_1. \tag{6.7.11}$$

Then, $\bar{V}$ can be defined for $x \geq 0$ by

$$\bar{V}(x) := \sum_{i=1}^{x} \log \frac{1-\bar{\omega}_i}{\bar{\omega}_i}.$$

Moreover, for any $0 < x < y \leq e_1$, we have

$$\bar{V}(y) - \bar{V}(x) = (V(y) - V(x)) + \log \left(\frac{g(x)\, g(x+1)}{g(y) g(y+1)} \right). \tag{6.7.12}$$

Since $g(x)$ is a increasing function of x, we get for any $0 \leq x < y \leq e_1$,

$$\bar{V}(y) - \bar{V}(x) \leq V(y) - V(x). \tag{6.7.13}$$

Moreover, we have for any environment ω (see (6.4.5)),

$$E_\omega[G] \leq 2 \sum_{0 \leq i \leq j < n} \mathrm{e}^{\bar{V}(j) - \bar{V}(i)}. \tag{6.7.14}$$

Using this expression, we can use the "good" properties of the environment to obtain the following bound.

Lemma 6.16. — *For all* $t \geq 1$, *we have on* Ω_t

$$E_\omega[G] \leq C(\log t)^4 t^\alpha.$$

The proof of this result is again postponed to the appendix.

6.7.4. Proof of Proposition 6.12. — Recalling Lemma 6.13 and Lemma 6.14, the proof of Proposition 6.12 boils down to showing that

$$t^{\kappa} P(H \geq h_t)\, \mathbb{P}^{\overline{\Omega}_t}(\tau \geq t) \longrightarrow C_T, \qquad t \to \infty,$$

where

$$\overline{\Omega}_t := \Omega_t \cap \{H \geq h_t\} \cap \{\forall x \leq 0,\ V(x) \leq 0\}.$$

Using the notations introduced in the previous subsections we can first write

$$\mathbb{P}^{\overline{\Omega}_t}(\tau \geq t) = E^{\overline{\Omega}_t}\left[P_\omega(\tau \geq t)\right] = E^{\overline{\Omega}_t}\left[\sum_{k \geq 0}(1-p)p^k P_\omega\Big(\sum_{i=1}^{k} F_i + G \geq t\Big)\right]. \tag{6.7.15}$$

Moreover, note that we will use ϵ_t in this subsection to denote a function which tends to 0 when t tends to infinity but whose value can change from line to line.

Proof of the lower bound. — Let us introduce $\xi_t := (\log t)^{-1}$ for $t \geq \mathrm{e}$ and

$$K_+ := \frac{\mathrm{e}^{\xi_t} t}{E_\omega[F]} = \frac{\mathrm{e}^{\xi_t} t}{2\omega_0 \widehat{M}_1}. \tag{6.7.16}$$

Since the random variable G is nonnegative, the sum in (6.7.15) is larger than

$$\begin{aligned} &\sum_{k \geq 0}(1-p)p^k P_\omega\Big(\sum_{i=1}^{k} F_i \geq t,\ k \geq K_+\Big) \\ &\geq p^{K_+} - (1-p)\sum_{k \geq K_+} p^k P_\omega\Big(\sum_{i=1}^{k} F_i \leq t,\ k \geq K_+\Big). \end{aligned} \tag{6.7.17}$$

Now for any $k \geq K_+$, the probability term in (6.7.17) is less than

$$\begin{aligned} P_\omega\Big(\sum_{i=1}^{k} F_i \leq \mathrm{e}^{-\xi_t} k E_\omega[F]\Big) &\leq \frac{Var_\omega(F)}{k E_\omega[F]^2 (1-\mathrm{e}^{-\xi_t})^2} \\ &\leq \frac{Var_\omega(F)}{K_+ E_\omega[F]^2 (1-\mathrm{e}^{-\xi_t})^2} \leq \frac{Var_\omega(F)}{t(1-\mathrm{e}^{-\xi_t})^2}, \end{aligned} \tag{6.7.18}$$

the last inequality being a consequence of the definition of K_+ given by (6.7.16) together with the fact that $\mathrm{e}^{\xi_t} E_\omega[F] \geq 1$. Therefore, assembling (6.7.17) and (6.7.18) yields

$$\mathbb{P}^{\overline{\Omega}_t}(\tau \geq t) \geq E^{\overline{\Omega}_t}\left[\left(1 - \frac{Var_\omega(F)}{t(1-\mathrm{e}^{-\xi_t})^2}\right) p^{K_+}\right].$$

Since $\alpha < 2-\kappa < 1$, Lemma 6.15 implies

$$\mathbb{P}^{\overline{\Omega}_t}(\tau \geq t) \geq (1-\epsilon_t)\, E^{\overline{\Omega}_t}\left[p^{K_+}\right]. \tag{6.7.19}$$

Furthermore, recalling (6.7.3) and (6.7.16) yields

$$E^{\overline{\Omega}_t}\left[p^{K_+}\right] = E^{\overline{\Omega}_t}\left[\Big(1 - \frac{\omega_0}{M_2 \mathrm{e}^H}\Big)^{\frac{\mathrm{e}^{\xi_t} t}{2\omega_0 \widehat{M}_1}}\right] \geq E^{\overline{\Omega}_t}\left[\Big(1 - \frac{\omega_0}{M_2 \mathrm{e}^H}\Big)^{\frac{\mathrm{e}^{\xi'_t} t}{2\omega_0 M_1}}\right],$$

where the inequality is a consequence of Lemma 6.15 and $\xi_t' := \xi_t - \log(1 - o(t^{-\delta})) = \xi_t + o(t^{-\delta})$. Then, observe that $\omega_0/M_2\mathrm{e}^H \le \mathrm{e}^{-h_t}$ and recall that $\log(1-x) \ge -x(1+x)$, for x small enough, such that we obtain

$$E^{\overline{\Omega}_t}\left[p^{K_+}\right] \ge E^{\overline{\Omega}_t}\left[\exp\left\{-\frac{\mathrm{e}^{\xi_t'}t}{2Z}(1+\frac{\omega_0}{M_2\mathrm{e}^H})\right\}\right] \ge E^{\overline{\Omega}_t}\left[\mathrm{e}^{-\frac{\mathrm{e}^{\xi_t''}t}{2Z}}\right],$$

where we recall that $Z = M_1M_2\,\mathrm{e}^H$ and $\xi_t'' := \xi_t' + \log(1+\mathrm{e}^{-h_t}) = \xi_t + o(t^{-\delta})$. Moreover Lemma 6.14 implies

$$E^{\overline{\Omega}_t}\left[\mathrm{e}^{-\frac{\mathrm{e}^{\xi_t''}t}{2Z}}\right] \ge (1-\epsilon_t)E^{\Omega_t^*}\left[\mathrm{e}^{-\frac{\mathrm{e}^{\xi_t''}t}{2Z}}\right] - \epsilon_t\frac{t^{-\kappa}}{P(H\ge h_t)},$$

where

$$\Omega_t^* := \{H \ge h_t\} \cap \{V(x) \ge 0,\ \forall x \le 0\}.$$

Now, we would like to integrate with respect to Z. To this goal, let us introduce the notation $F_Z^{(t)}(z) := P^{\ge 0}(Z > z \,|\, H \ge h_t)$. An integration by part yields

$$\begin{aligned} E^{\Omega_t^*}\left[\mathrm{e}^{-\frac{\mathrm{e}^{\xi_t''}t}{2Z}}\right] &= \int_{\mathrm{e}^{h_t}}^{\infty}\mathrm{e}^{-\frac{\mathrm{e}^{\xi_t''}t}{2z}}\,\mathrm{d}F_Z^{(t)}(z) \\ &= -\mathrm{e}^{-\frac{\mathrm{e}^{\xi_t''}t}{2\mathrm{e}^{h_t}}}F_Z^{(t)}(\mathrm{e}^{h_t}) + \int_{\mathrm{e}^{h_t}}^{\infty}\frac{\mathrm{e}^{\xi_t''}t}{2z^2}\mathrm{e}^{-\frac{\mathrm{e}^{\xi_t''}t}{2z}}F_Z^{(t)}(z)\,\mathrm{d}z. \qquad (6.7.20) \end{aligned}$$

Then, let us make the crucial observation that

$$F_Z^{(t)}(z) = \frac{P^{\ge 0}(Z>z)}{P(H\ge h_t)} - \frac{P^{\ge 0}(Z>z,\,H<h_t)}{P(H\ge h_t)}.$$

Therefore, denoting by I the integral in (6.7.20), we can write $I = I_1 - I_2$, where I_1 and I_2 are given by

$$\begin{aligned} I_1 &:= \frac{1}{P(H\ge h_t)}\int_{\mathrm{e}^{h_t}}^{\infty}\frac{\mathrm{e}^{\xi_t''}t}{2z^2}\mathrm{e}^{-\frac{\mathrm{e}^{\xi_t''}t}{2z}}P^{\ge 0}(Z>z)\,\mathrm{d}z, \\ I_2 &:= \frac{1}{P(H\ge h_t)}\int_{\mathrm{e}^{h_t}}^{\infty}\frac{\mathrm{e}^{\xi_t''}t}{2z^2}\mathrm{e}^{-\frac{\mathrm{e}^{\xi_t''}t}{2z}}P^{\ge 0}(Z>z,\,H<h_t)\,\mathrm{d}z. \end{aligned}$$

To treat I_1, let us recall that Lemma 6.3 gives the tail behaviour of Z under $P^{\ge 0}$:

$$(1-\epsilon_t)C_Uz^{-\kappa} \le P^{\ge 0}(Z>z) \le (1+\epsilon_t)C_Uz^{-\kappa}, \qquad (6.7.21)$$

for all $z \ge \mathrm{e}^{h_t}$. Hence, we are led to compute the integral

$$\int_{\mathrm{e}^{h_t}}^{\infty}\frac{\mathrm{e}^{\xi_t''}t}{2z^2}\mathrm{e}^{-\frac{\mathrm{e}^{\xi_t''}t}{2z}}z^{-\kappa}\,\mathrm{d}z = \mathrm{e}^{-\kappa\xi_t''}2^{\kappa}\left(\int_0^{\frac{\mathrm{e}^{\xi_t''}t}{2}}\mathrm{e}^{-y}y^{\kappa}\,\mathrm{d}y\right)t^{-\kappa}, \qquad (6.7.22)$$

by making the change of variables given by $y = \mathrm{e}^{\xi_t''}t/2z$. Observe that the integral in (6.7.22) is close to $\Gamma(\kappa+1)$ when t tends to infinity (indeed $\xi_t'' \to 0$). Therefore, recalling (6.3.3) and that $C_T = C_U2^{\kappa}\Gamma(\kappa+1)$, we obtain

$$(1-\epsilon_t)C_Tt^{-\kappa} \le I_1\,P(H\ge h_t) \le (1+\epsilon_t)C_Tt^{-\kappa}. \qquad (6.7.23)$$

We turn now to I_2. Repeating the proof of Corollary 4.2 in **[16]** yields

$$P^{\ge 0}(Z>z,\,H<h_t) \le Cz^{-\eta}\mathrm{e}^{(\eta-\kappa)h_t},$$

for any $\eta > \kappa$ and all $z \geq \mathrm{e}^{h_t}$. Therefore, repeating the previous computation, we get

(6.7.24)

$$I_2\, P(H \geq h_t) \leq C2^\eta \left(\int_0^{\frac{\mathrm{e}^{\xi_t''} t}{2}} \mathrm{e}^{-y} y^\eta \,\mathrm{d}y \right) \mathrm{e}^{(\eta-\kappa)h_t} t^{-\eta} \leq C2^\eta \Gamma(\eta+1) \mathrm{e}^{(\eta-\kappa)h_t} t^{-\eta},$$

which yields $I_2\, P(H \geq h_t) \leq \epsilon_t t^{-\kappa}$, by choosing η larger than κ and recalling (6.7.2).

Then assembling (6.7.23) and (6.7.24) implies $I\, P(H \geq h_t) \geq (1-\epsilon_t) C_T t^{-\kappa}$ and coming back to (6.7.19)–(6.7.20), we obtain

$$P(H \geq h_t)\, \mathbb{P}^{\overline{\Omega}_t}(\tau \geq t) \geq -\mathrm{e}^{-\frac{\mathrm{e}^{\xi_t''} t}{2\mathrm{e}^{h_t}}} P(H \geq h_t) + (1-\epsilon_t) C_T t^{-\kappa},$$

which concludes the proof of the lower bound since $\exp\{-\frac{\mathrm{e}^{\xi_t''} t}{2\mathrm{e}^{h_t}}\} P(H \geq h_t) = o(t^{-\kappa})$ when t tends to infinity; indeed $y^\kappa \mathrm{e}^{-cy} \to 0$ when $y \to \infty$ and $t^{-1}\mathrm{e}^{h_t} \to 0$ when $t \to \infty$, see (6.7.2).

Proof of the upper bound. — Using still the notations $\xi_t := (\log t)^{-1}$ for $t \geq \mathrm{e}$, let us now introduce

$$K_- := \frac{\mathrm{e}^{-\xi_t} t}{E_\omega[F]} = \frac{\mathrm{e}^{-\xi_t} t}{2\omega_0 \widehat{M}_1}.$$

Let also $\eta_t := \xi_t - \frac{1}{2}\xi_t^2$, so that $0 < \eta_t < 1 - \mathrm{e}^{-\xi_t}$. The sum in (6.7.15) is smaller than

$$p^{K_-} + (1-p) \sum_{k \leq K_-} p^k P_\omega\Big(\sum_{i=1}^k F_i + G \geq t\Big)$$

$$\leq p^{K_-} + \frac{E_\omega[G]}{\eta_t t} + (1-p) \sum_{k \leq K_-} p^k P_\omega\Big(\sum_{i=1}^k F_i \geq t(1-\eta_t)\Big), \tag{6.7.25}$$

the inequality being a consequence of Chebychev inequality. Furthermore, observe that $k \leq K_-$ implies $t \geq k e^{\xi_t} E_\omega[F]$ hence the probability term in (6.7.25) is less than

$$P_\omega\Big(\sum_{i=1}^k F_i - kE_\omega[F] \geq k(\mathrm{e}^{\xi_t}(1-\eta_t)-1)E_\omega[F]\Big) \leq \frac{Var_\omega(F)}{k(\mathrm{e}^{\xi_t}(1-\eta_t)-1)^2 E_\omega[F]^2}$$

(remembering $1-\eta_t > \mathrm{e}^{-\xi t}$). Therefore,

$$P_\omega(\tau \geq t) \leq p^{K_-} + \frac{E_\omega[G]}{\eta_t t} + \frac{Var_\omega(F)}{(\mathrm{e}^{\xi_t}(1-\eta_t)-1)^2 E_\omega[F]^2} \sum_{k \leq K_-} \frac{(1-p)p^k}{k}.$$

The last sum is less than $(1-p)\log\frac{1}{1-p} = \frac{\omega_0}{M_2 \mathrm{e}^H} \log \frac{M_2 \mathrm{e}^H}{\omega_0}$. On the event $\overline{\Omega}_t$, we have $\mathrm{e}^H \geq \mathrm{e}^{h_t}$, $E_\omega[F] \geq 1$, $M_2 \geq 1$, $\frac{1}{2} \leq \omega_0 \leq 1$, and (6.7.8), hence

$$\mathbb{P}^{\overline{\Omega}_t}(\tau \geq t) \leq E^{\overline{\Omega}_t}\left[p^{K_-}\right] + E^{\overline{\Omega}_t}\left[\frac{E_\omega[G]}{\eta_t t}\right] + \frac{C(\log t)^4 t^\alpha}{(\mathrm{e}^{\xi_t}(1-\eta_t)-1)^2} \frac{1}{\mathrm{e}^{h_t}} E^{\overline{\Omega}_t}\left[\log(2M_2 \mathrm{e}^H)\right].$$

Let us now bound the three terms in the right-hand side of the previous equation. Consider the last one. Using Lemma 6.4 and (6.3.3), we have

$$E[\log(M_2 \mathrm{e}^H)|H \geq h_t] = E[\log M_2|H \geq h_t] + E[H|H \geq h_t] \leq C + h_t$$

for some constant C. When $t \to \infty$, $\mathrm{e}^{\xi_t}(1-\eta_t) - 1 \sim \frac{\xi_t^3}{6} = \frac{1}{6\log^3 t}$. Since $\mathrm{e}^{h_t} = \frac{t}{\log t}$ and $\alpha < 1$, the whole term is seen to converge polynomially to zero. In particular,

$$\frac{C(\log t)^4 t^\alpha}{(\mathrm{e}^{\xi_t}(1-\eta_t)-1)^2} \frac{1}{\mathrm{e}^{h_t}} E^{\overline{\Omega}_t}\left[\log(2M_2\mathrm{e}^H)\right] \leq \epsilon_t \frac{t^{-\kappa}}{P(H \geq h_t)}. \tag{6.7.26}$$

For the second term, Lemma 6.16 implies

$$E^{\overline{\Omega}_t}\left[\frac{E_\omega[G]}{\xi_t t}\right] \leq C\frac{(\log t)^4 t^\alpha}{\xi_t t} \leq \epsilon_t \frac{t^{-\kappa}}{P(H \geq h_t)}, \tag{6.7.27}$$

since $\alpha < 1$. Finally, for the first expectation, we repeat the arguments of the proof of the upper bound obtained for I. More precisely, recalling (6.7.3) and (6.7.16), we get

$$E^{\overline{\Omega}_t}\left[p^{K_-}\right] \leq E^{\overline{\Omega}_t}\left[(1 - \frac{\omega_0}{M_2 \mathrm{e}^H})^{\frac{\mathrm{e}^{-\xi'_t}t}{2\omega_0 M_1}}\right] \leq E^{\overline{\Omega}_t}\left[\mathrm{e}^{-\frac{\mathrm{e}^{-\xi'_t}t}{2Z}}\right],$$

where the first inequality is a consequence of Lemma 6.15 and $\xi'_t := \xi_t - \log(1 + o(t^{-\delta})) = \xi_t + o(t^{-\delta})$, while the second inequality is a consequence of $\log(1-x) \leq -x$ for $0 < x < 1$. Then, an integration by part yields

$$E^{\overline{\Omega}_t}\left[p^{K_-}\right] \leq \frac{1+\epsilon_t}{P(H \geq h_t)} \int_{\mathrm{e}^{h_t}}^{\infty} \frac{\mathrm{e}^{-\xi'_t}t}{2z^2} \mathrm{e}^{-\frac{\mathrm{e}^{-\xi'_t}t}{2z}} P^{\geq 0}(Z > z)\,\mathrm{d}z.$$

Making the change of variables given by $y = \mathrm{e}^{-\xi'_t}t/2z$ and recalling (6.7.21) imply

$$P(H \geq h_t)\, E^{\overline{\Omega}_t}\left[p^{K_-}\right] \leq (1+\epsilon_t) C_T t^{-\kappa}. \tag{6.7.28}$$

Now, assembling (6.7.26), (6.7.27) and (6.7.28) concludes the proof of the upper bound.

6.8. Proof of Theorem 6.1

The results from Sections 6.5 and 6.6 enable us to reduce the proof of Theorem 6.1 to an equivalent i.i.d. setting and thus to apply a classic limit theorem.

NB: we first prove the theorem under $\mathbb{P}^{\geq 0}$, and the statement under $\mathbb{P}$ will follow.

6.8.1. Reduction to i.i.d. random variables. — For all $i \geq 0$, let $Z_i := \tau(e_i, e_{i+1})$, so that $(Z_i)_{i\geq 0}$ is a stationary sequence under $\mathbb{P}^{\geq 0}$ (cf. Lemma 6.5) and

$$\tau(e_n) = Z_0 + \cdots + Z_{n-1}.$$

Let us also enlarge the probability space $(\Omega \times \mathbb{Z}^{\mathbb{N}}, \mathcal{B}, \mathbb{P}^{\geq 0})$ in order to introduce an i.i.d. sequence $(\omega^{(i)}, (X_n^{(i)})_{n\geq 0})_{i\geq 0}$ of environments and random walks distributed according to $\mathbb{P}^{\geq 0}$. Since the excursions of V are independent, it is possible to couple ω and $(\omega^{(i)})_{i\geq 0}$ in such a way that, for all $i \geq 0$, $H^{(i)} = H_i$, or more generally that the first excursion of $\omega^{(i)}$ and the $(i+1)$-th excursion of ω are the same. It suffices indeed to build $\omega^{(i)}$ from the excursion $(\omega_{e_i+x})_{1\le x\le e_{i+1}-e_i}$ of ω and from independent environments with law $P^{\ge 0}$ on both sides of it.

For all integers $i\ge 0$, we may now introduce

$$\widehat{Z}_i := \tau^{(i)}(e_1^{(i)})$$

which is defined like $Z_1(=\tau(e_1))$ but relatively to $(\omega^{(i)},X^{(i)})$ instead of (ω,X). By construction, $(\widehat{Z}_i)_{i\ge 0}$ is a sequence of i.i.d. random variables distributed like Z_1 under $\mathbb{P}^{\ge 0}$.

For $1<\kappa<2$. — We have the decomposition (where indices i range from 0 to $n-1$)

$$\begin{aligned}\tau(e_n)-\mathbb{E}^{\ge 0}[\tau(e_n)] &= \Bigg(\sum_{H_i<h_n} Z_i - \mathbb{E}^{\ge 0}\Big[\sum_{H_i<h_n} Z_i\Big]\Bigg)\\ &\quad+\Bigg(\sum_{H_i\ge h_n} Z_i\Bigg)\mathbf{1}_{NO(n)^c}+\Bigg(\sum_{H_i\ge h_n}\widetilde{\tau}_i\Bigg)\mathbf{1}_{NO(n)}\\ &\quad+\Bigg(\sum_{H_i\ge h_n} Z_i^*\Bigg)\mathbf{1}_{NO(n)}-\mathbb{E}^{\ge 0}\Big[\sum_{H_i\ge h_n} Z_i\Big],\end{aligned}\tag{6.8.1}$$

where, if $H_i\ge h_n$ and j is such that $\sigma(j)=i$ (i.e. $e_i=b_j$), $\widetilde{\tau}_i=\widetilde{\tau}^{(a_j)}(e_i,e_{i+1})$ is the time spent on the left of a_j after the first visit of e_i and before reaching e_{i+1}, and $Z_i^*=Z_i-\widetilde{\tau}_i$.

Due to Propositions 6.8, 6.6 and 6.7 respectively, the first three terms are negligible in $\mathbb{P}^{\ge 0}$-probability with respect to $n^{1/\kappa}$, hence

$$\frac{\tau(e_n)-\mathbb{E}^{\ge 0}[\tau(e_n)]}{n^{1/\kappa}}=\frac{1}{n^{1/\kappa}}\Bigg(\Bigg(\sum_{H_i\ge h_n} Z_i^*\Bigg)\mathbf{1}_{NO(n)}-\mathbb{E}^{\ge 0}\Big[\sum_{H_i\ge h_n} Z_i\Big]\Bigg)+o(1),$$

where $o(1)$ is a random variable converging to 0 in $\mathbb{P}^{\ge 0}$-probability.

For $\kappa=1$. — Let

$$a_n := \inf\big\{t>0:\ \mathbb{P}^{\ge 0}(\tau(e_1)>t)\le n^{-1}\big\}.$$

(Note that $a_n\sim_n C_T n$ by Proposition 6.12). With the same definitions as above, we decompose

$$\begin{aligned}\tau(e_n)-n\mathbb{E}^{\ge 0}[\tau(e_1)\mathbf{1}_{\{\tau(e_1)<a_n\}}] &= \Bigg(\sum_{H_i<h_n} Z_i - \mathbb{E}^{\ge 0}\Big[\sum_{H_i<h_n} Z_i\Big]\Bigg)\\ &\quad+\Bigg(\sum_{H_i\ge h_n} Z_i\Bigg)\mathbf{1}_{NO(n)^c}+\Bigg(\sum_{H_i\ge h_n}\widetilde{\tau}_i\Bigg)\mathbf{1}_{NO(n)}\\ &\quad+\Bigg(\sum_{H_i\ge h_n} Z_i^*\Bigg)\mathbf{1}_{NO(n)}-n\mathbb{E}^{\ge 0}[Z_1(\mathbf{1}_{\{Z_1<a_n,H\ge h_n\}}-\mathbf{1}_{\{Z_1\ge a_n,H<h_n\}})].\end{aligned}$$

Note that the last term accounts for the difference between the restriction according to the value of $\tau(e_1)$, used on the left-hand side and that we need for applying the limit theorem, and the restriction according to the height, used in the right-hand side decomposition and throughout the paper.

Again, the first three terms are negligible with respect to n, hence $n^{-1}(\tau(e_n)-n\mathbb{E}^{\geq 0}[\tau(e_1)\mathbf{1}_{\{\tau(e_1)<a_n\}}])$ equals

(6.8.2)
$$\frac{1}{n}\left(\left(\sum_{H_i\geq h_n} Z_i^*\right)\mathbf{1}_{NO(n)} - n\mathbb{E}^{\geq 0}[Z_1(\mathbf{1}_{\{Z_1<a_n, H\geq h_n\}} - \mathbf{1}_{\{Z_1\geq a_n, H<h_n\}})]\right) + o(1).$$

Let us resume to the general case $1\leq\kappa<2$. Observe that $Z^*_{\sigma(j)}$ is the time to go from b_j to d_j for a random walk reflected at a_j, hence it depends only on the environment between a_j+1 and d_j. On the other hand, under $P(\cdot|K_n = m, NO(n))$, the pieces $(\omega_{b_j+x})_{a_j<b_j+x\leq d_j}$ of the environment, for $j=1,\ldots,m$, are i.i.d. with same distribution as $(\omega_x)_{e_{-D_n}<x\leq e_1}$ under

$$P^{\geq 0}(\cdot|H\geq h_n, H_{-k}<h_n \text{ for } k=1,\ldots,D_n).$$

Remember indeed that $a_1>0$ on $NO(n)$; and due to our definition of deep valleys, conditioning by the value of K_n only affects the number of deep valleys and not their individual distributions, while conditioning by $NO(n)$ implies the independence and imposes the excursions between a_j and b_j to be small, for $j=1,\ldots,K_n$.

As a consequence, the term $\left(\sum_{H_i\geq h_n} Z_i^*\right)\mathbf{1}_{NO(n)}$ has same distribution under $\mathbb{P}^{\geq 0}$ as $\left(\sum_{H_i\geq h_n}\widehat{Z}_i^*\right)\mathbf{1}_{NO(n)}$ under $\mathbb{P}(\cdot|\widehat{NO}(n))$, where $\widehat{Z}_i^*\mathbf{1}_{\{H_i\geq h_n\}}$ is defined like $Z_1^*\mathbf{1}_{\{H\geq h_n\}}$ but relative to $(\omega^{(i)},X^{(i)})$, and

$$\widehat{NO}(n) := \left\{\text{for } j=1,\ldots,K_n,\ H_{-1}^{(\sigma(j))}<h_n,\ldots,H_{-D_n}^{(\sigma(j))}<h_n\right\}$$

is the event that D_n small excursions precede the high excursions in the i.i.d. framework.

We deduce that, for $1<\kappa<2$, the characteristic function satisfies

$$\mathbb{E}^{\geq 0}\left[\mathrm{e}^{i\lambda n^{-1/\kappa}(\tau(e_n)-\mathbb{E}^{\geq 0}[\tau(e_n)])}\right]$$
$$= \mathbb{E}^{\geq 0}\left[\exp\left(i\lambda n^{-1/\kappa}(\left(\sum_{H_i\geq h_n}\widehat{Z}_i^*\right)\mathbf{1}_{NO(n)} - \mathbb{E}^{\geq 0}\left[\sum_{H_i\geq h_n} Z_i\right])\right)\middle|\widehat{NO}(n)\right] + o_n(1)$$

(6.8.3)
$$= \mathbb{E}^{\geq 0}\left[\exp\left(i\lambda n^{-1/\kappa}(\left(\sum_{H_i\geq h_n}\widehat{Z}_i^*\right)\mathbf{1}_{NO(n)} - \mathbb{E}^{\geq 0}\left[\sum_{H_i\geq h_n} Z_i\right])\right)\right] + o_n(1)'.$$

The last equality comes from $P(\widehat{NO}(n))\to_n 1$, cf. Lemma 6.17 below, and from the fact that the term in the expectation is bounded by 1. We have of course similar equalities for $\kappa=1$ from (6.8.2).

The following lemma will enable us to put the neglected terms back in the sum, now with $\widehat{Z}_i$ instead of Z_i, and thus complete the reduction to i.i.d. random variables. For $i\geq 0$, let $\widehat{\widehat{\tau}}_i$ be the time spent by $X^{(i)}$ on the left of e_{-D_n} ($=a_1$ if $H>h_n$) before e_1 is reached, hence $\widehat{Z}_i=\widehat{Z}_i^*+\widehat{\widehat{\tau}}_i$.

Lemma 6.17. — *We have*

$$P(\widehat{NO}(n)) \xrightarrow[n]{} 1,$$

$$\frac{1}{n^{1/\kappa}} \sum_{i=0}^{n-1} \widehat{\widehat{\tau}}_i \mathbf{1}_{\{H^{(i)} \geq h_n\}} \xrightarrow[n]{(p)} 0,$$

$$\text{(6.8.4)} \qquad \frac{1}{n^{1/\kappa}} \left(\sum_{i=0}^{n-1} \widehat{Z}_i \mathbf{1}_{\{H^{(i)} < h_n\}} - E\left[\sum_{i=0}^{n-1} \widehat{Z}_i \mathbf{1}_{\{H^{(i)} < h_n\}} \right] \right) \xrightarrow[n]{(p)} 0.$$

Proof. — These results follow respectively from the proofs of Propositions 6.6, 6.7 and 6.8, made easier by the independence of the random variables $\widehat{Z}_0, \dots, \widehat{Z}_{n-1}$. More precisely, the proofs of Propositions 6.6 and 6.7 hold in this i.i.d. context almost without a change. And since the random variables $\widehat{Z}_i \mathbf{1}_{\{H^{(i)} < h_n\}}$, $i \geq 0$, are independent, the proof of (6.8.4) for $1 < \kappa < 2$ would follow from

$$n \mathbb{V}\mathrm{ar}^{\geq 0}(\tau(e_1) \mathbf{1}_{\{H < h_n\}}) = o(n^{2/\kappa}),$$

and thus from $n \mathbb{E}^{\geq 0}[\tau(e_1)^2 \mathbf{1}_{\{H < h_n\}}] = o(n^{2/\kappa})$, which is given by Lemma 6.11. For $\kappa = 1$, the same modification of the environment as in Subsection 6.6.1 adapts immediately. □

From this lemma and (6.8.3), recomposing (6.8.1) with variables $\widehat{Z}_i$ (and using $NO(n)$ again, not $\widehat{NO}(n)$), we finally have, for $1 < \kappa < 2$,

(6.8.5)
$$\mathbb{E}^{\geq 0} \left[\mathrm{e}^{i\lambda n^{-1/\kappa}(\tau(e_n) - \mathbb{E}^{\geq 0}[\tau(e_n)])} \right] = \mathbb{E} \left[\mathrm{e}^{i\lambda n^{-1/\kappa}(\widehat{Z}_0 + \dots + \widehat{Z}_{n-1} - \mathbb{E}[\widehat{Z}_0 + \dots + \widehat{Z}_{n-1}])} \right] + o_n(1).$$

Note that we used the equality $\mathbb{E}^{\geq 0}[\sum_{H_i > h_n} Z_i] = \mathbb{E}[\sum_{H^{(i)} > h_n} \widehat{Z}_i]$, which results from the equality in distribution of $Z_i \mathbf{1}_{\{H_i \geq h_n\}}$ and $\widehat{Z}_i \mathbf{1}_{\{H^{(i)} \geq h_n\}}$ under $\mathbb{P}^{\geq 0}$.

As a conclusion, this shows that, for $1 < \kappa < 2$, $\frac{\tau(e_n) - \mathbb{E}^{\geq 0}[\tau(e_n)]}{n^{1/\kappa}}$ has same limit in law under $\mathbb{P}^{\geq 0}$ (if any) as $\frac{\widehat{Z}_0 + \dots + \widehat{Z}_{n-1} - n\mathbb{E}^{\geq 0}[\widehat{Z}_0]}{n^{1/\kappa}}$, where the random variables $\widehat{Z}_i$, $i \geq 0$, are i.i.d. with same distribution as $\tau(e_1)$ under $\mathbb{P}^{\geq 0}$.

For $\kappa = 1$, the same procedure shows that $\frac{\tau(e_n) - n\mathbb{E}^{\geq 0}[\tau(e_1) \mathbf{1}_{\{\tau(e_1) < a_n\}}]}{n}$ has same limit in law under $\mathbb{P}^{\geq 0}$, if any, as $\frac{\widehat{Z}_0 + \dots + \widehat{Z}_{n-1} - n\mathbb{E}^{\geq 0}[\widehat{Z}_0 \mathbf{1}_{\{\widehat{Z}_0 < a_n\}}]}{n}$.

6.8.2. Conclusion of the proof. — Let us quote (a particular case of) Theorem 2.7.7 from [**13**]:

Theorem 6.18. — *Suppose $X_1, X_2, \dots$ are i.i.d. nonnegative random variables with a distribution that satisfies*

$$\mathbb{P}(X_1 > x) = x^{-\alpha} L(x)$$

where $1 \leq \alpha < 2$ and L is slowly varying. Let $S_n := X_1 + \dots + X_n$,

$$a_n := \inf\left\{x : \mathbb{P}(X_1 > x) \leq n^{-1}\right\} \quad \textit{and} \quad b_n := n\mathbb{E}[X_1 \mathbf{1}_{\{X_1 < a_n\}}].$$

Then, if $1<\alpha<2$,

$$\frac{S_n - nE[X_1]}{a_n} \xrightarrow[n]{\text{(law)}} (-\Gamma(1-\alpha))^{1/\alpha}\mathcal{S}_\alpha^{ca},$$

where $\mathcal{S}_\alpha^{ca}$ *is a centered completely asymmetric stable random variable of index* α*, defined in* (6.2.2).

And if $\alpha=1$,

$$\frac{S_n - b_n}{a_n} \xrightarrow[n]{\text{(law)}} c + \mathcal{S}_1^{ca},$$

where $c = 1-\gamma$ *(*$\gamma \simeq 0.577$ *being Euler's constant), and* $\mathcal{S}_1^{ca}$ *was defined in* (6.2.3).

Remarks. —

– *Durrett* [**13**] *actually gives a different parametrization of the limit law. The above parameters are obtained by comparing the real and imaginary parts of expressions (7.11) and (7.13) (where there is a sign error) of* [**13**]*, using the following identities:* $\int_0^\infty \frac{1-\cos x}{x^{\alpha+1}}\,\mathrm{d}x = \cos\left(\frac{\pi\alpha}{2}\right)\Gamma(1-\alpha)$ *(for any* $0<\alpha<2$*), and* $\int_0^1 \frac{\sin u - u}{u^2}\,\mathrm{d}u + \int_1^\infty \frac{\sin u}{u}\,\mathrm{d}u = 1-\gamma$*. The value of* c *is however unimportant in the following.*
– *If* $\mathbb{P}(X_1 > x) \sim_{x\to\infty} \frac{C}{x^\alpha}$*, then we have* $a_n \sim_n C^{1/\alpha} n^{1/\alpha}$*.*

Thanks to Proposition 6.12 and to the previous reduction (6.8.5) to an i.i.d. framework, Theorem 6.18 gives that, for $1<\kappa<2$,

$$\text{under } \mathbb{P}^{\geq 0},\ \frac{\tau(e_n) - \mathbb{E}^{\geq 0}[\tau(e_n)]}{n^{1/\kappa}} \xrightarrow[n]{\text{(law)}} (-\Gamma(1-\alpha)C_T)^{1/\kappa}\mathcal{S}_\kappa^{ca}.$$

The random walk is almost-surely transient to $+\infty$ under both $\mathbb{P}$ and $\mathbb{P}^{\geq 0}$ (cf. after (6.4.3)), hence the total time spent on $\mathbb{Z}_-$ is finite in both cases and thus trivially negligible with respect to $n^{1/\kappa}$. Since random walks under distributions $\mathbb{P}$ and $\mathbb{P}^{\geq 0}$ can simply be coupled so that they coincide after erasure of the time spent on $\mathbb{Z}_-$, we conclude that the above limit (with same centering) holds under $\mathbb{P}$ as well.

We deduce, using the law of large numbers and the central limit theorem for $(e_n)_n$ that (cf. the conclusion of [**27**])

$$\text{under } \mathbb{P},\ \frac{\tau(n) - nv^{-1}}{n^{1/\kappa}} \xrightarrow[n]{\text{(law)}} (-\Gamma(1-\alpha)E[e_1]^{-1}C_T)^{1/\kappa}\mathcal{S}_\kappa^{ca}, \tag{6.8.6}$$

where $v^{-1} := \frac{1}{E[e_1]}\mathbb{E}^{\geq 0}[\tau(e_1)]$. Since (6.8.6) yields $\frac{\tau(n)}{n} \to_n v^{-1}$ in probability, comparison with Solomon [**43**] gives the value $v^{-1} = \mathbb{E}[\tau(1)] = \frac{1+E[\rho_0]}{1-E[\rho_0]}$. By (6.7.1),

$$(-\Gamma(1-\kappa)E[e_1]^{-1}C_T)^{1/\kappa} = \left(-\Gamma(1-\kappa)2^\kappa\Gamma(1+\kappa)E[e_1]^{-1}C_U\right)^{1/\kappa},$$

and Euler's reflection formula $\Gamma(1+\kappa)\Gamma(1-\kappa) = \frac{\pi\kappa}{\sin\pi\kappa}$, together with the expression of C_U recalled in (6.4.10) leads to the value of Equation (6.2.4).

Finally, the limit law for X_n results using transience to $+\infty$, cf. [**27**], pp.167–168.

For $\kappa = 1$, we get

$$\text{under } \mathbb{P}^{\geq 0}, \quad \frac{\tau(e_n) - n\mathbb{E}^{\geq 0}[\tau(e_1)\mathbf{1}_{\{\tau(e_1)<a_n\}}]}{n^{1/\kappa}} \xrightarrow[n]{\text{(law)}} C_T(1-\gamma) + C_T\mathcal{S}_1^{ca}.$$

Furthermore, using Proposition 6.12, when $n \to \infty$,

$$\mathbb{E}^{\geq 0}[\tau(e_1)\mathbf{1}_{\{\tau(e_1)<a_n\}}] = \int_0^{a_n} \mathbb{P}^{\geq 0}(\tau(e_1) > t)\,\mathrm{d}t \sim C_T \log(a_n) \sim C_T \log n.$$

Like in the previous case, we may substitute $\mathbb{P}$ for $\mathbb{P}^{\geq 0}$ (letting the centering term unchanged). Goldie **[21]** proved that, when $\kappa = 1$, $C_K = \frac{1}{E[\rho_0 \log \rho_0]}$, hence $C_T = \frac{2E[e_1]}{E[\rho_0 \log \rho_0]}$. This concludes the proof of Theorem 6.1 (cf. **[27]** again for the inversion argument).

NB: for additional details relating to the conclusion (passage from $\tau(e_n)$ to $\tau(n)$ and inversion argument), the reader may refer to Appendix A.

6.9. Appendix

6.9.1. Proof of Lemma 6.14. — Recalling the definition of Ω_t, the proof of Lemma 6.14 boils down to showing that for $i = 1, 2, 3$,

$$P((\Omega_t^{(i)})^c,\, H \geq h_t) = o(t^{-\kappa}), \qquad t \to \infty. \tag{6.9.1}$$

The case $i = 1$ is trivial. Indeed, the fact that e_1 has some finite exponential moments (see after (6.3.1)) implies that $P((\Omega_t^{(1)})^c) = o(t^{-\kappa})$ when t tends to infinity.

Furthermore, this result implies that the case $i = 2$ is a consequence of

$$P((\Omega_t^{(2)})^c,\, \Omega_t^{(1)},\, H \geq h_t) = o(t^{-\kappa}), \qquad t \to \infty.$$

Then, let us observe that $V^{\uparrow}(T_H, e_1)$ is less than $V^{\uparrow}(T_{h_t}, e_1)$ which is bounded by $V^{\uparrow}(T_{h_t}, T_{h_t} + \lceil C \log t \rceil)$ on $\Omega_t^{(1)}$. Applying the strong Markov property at time T_{h_t}, we get that $P(V^{\uparrow}(T_H, e_1) \geq \alpha \log t,\, \Omega_t^{(1)},\, H \geq h_t)$ is bounded by

$$\begin{aligned} P(H \geq h_t)\, P(V^{\uparrow}(0, \lceil C \log t \rceil) \geq \alpha \log t) &\leq P(H \geq h_t)\, P\big(\max_{1 \leq k \leq \lceil C \log t \rceil} H_k \geq \alpha \log t\big) \\ &\leq C(\log t)\, P(H \geq h_t)\, P(H \geq \alpha \log t). \end{aligned}$$

Recalling that $h_t = \log t - \log\log t$, that $\alpha > 0$ together with Iglehart's result yields

$$P(V^{\uparrow}(T_H, e_1) \geq \alpha \log t,\, \Omega_t^{(1)},\, H \geq h_t) = o(t^{-\kappa}), \qquad t \to \infty.$$

Then to prove (6.9.1) for $i = 2$, it remains to show that

$$P(V^{\downarrow}(0, T_H) \leq -\alpha \log t,\, H \geq h_t) = o(t^{-\kappa}), \qquad t \to \infty. \tag{6.9.2}$$

Observing that $V^{\downarrow}(0, T_H) = \min\{V^{\downarrow}(0, T_{h_t}),\, V^{\downarrow}(T_{h_t}, T_H)\}$, we will treat each term separately. From the trivial inclusion

$$\big\{V^{\downarrow}(0, T_{h_t}) \leq -\alpha \log t,\, H \geq h_t\big\} \subset \big\{T^{\downarrow}(\alpha \log t) < T_{h_t} < T_{(-\infty,0]}\big\},$$

it follows that $P(V^{\downarrow}(0,T_{h_t}) \leq -\alpha \log t,\, H \geq h_t)$ is less than

$$\sum_{p=\lfloor \alpha \log t \rfloor}^{\lfloor h_t \rfloor} P(M_\alpha \in [p,p+1),\, T^{\downarrow}(\alpha \log t) < T_{h_t} < T_{(-\infty,0]}),$$

where $M_\alpha := \max\{V(k) : 0 \leq k \leq T^{\downarrow}(\alpha \log t)\}$. Applying the strong Markov property at time $T^{\downarrow}(\alpha \log t)$, we bound the term of the previous sum by $P(S \geq p)\, P(S \geq h_t - (p+1-\alpha \log t))$. Then recalling that there exists C such that $P(S \geq p) \leq C\mathrm{e}^{-\kappa p}$ for all $p \geq 0$ (see (6.3.4)), we obtain the uniform bound $C\mathrm{e}^{-\kappa(h_t+\alpha \log t)}$ for the summand, which yields

$$P(V^{\downarrow}(0,T_{h_t}) \leq -\alpha \log t,\, H \geq h_t) \leq C h_t \mathrm{e}^{-\kappa(h_t+\alpha \log t)} = o(t^{-\kappa}), \qquad t \to \infty, \tag{6.9.3}$$

since $h_t = \log t - \log\log t$ and $\alpha > 0$. Furthermore, applying again the strong Markov property at T_{h_t}, we obtain

$$P(V^{\downarrow}(T_{h_t},T_H) \leq -\alpha \log t,\, H \geq h_t) \leq P(H \geq h_t) P(V^{\downarrow}(0,T_S) \leq -\alpha \log t).$$

Then, applying the strong Markov property at $T^{\downarrow}(\alpha \log t)$, we get that $P(V^{\downarrow}(0,T_S) \leq -\alpha \log t)$ is less than $P(S > \alpha \log t)$, which yields

$$P(V^{\downarrow}(T_{h_t},T_H) \leq -\alpha \log t,\, H \geq h_t) \leq C \mathrm{e}^{-\kappa(h_t+\alpha \log t)} = o(t^{-\kappa}), \qquad t \to \infty. \tag{6.9.4}$$

Now assembling (6.9.3) and (6.9.4) implies (6.9.2) and concludes the proof of the case $i = 2$.

Let us consider the last case $i = 3$. Since R^- depends only on $\{V(x),\, x \leq 0\}$, and $P(H > h_t) \sim C_I t^{-\kappa} (\log t)^{\kappa}$ when $t \to \infty$, it suffices to prove $P^{\geq 0}(R^- > (\log t)^4 t^{\alpha}) = o((\log t)^{-\kappa})$. This would follow (for any $\alpha > 0$) from Markov property if $E^{\geq 0}[R^-] < \infty$. We have (changing indices and incorporating the single terms into the sums):

$$\begin{aligned} R^- &= \sum_{i \leq 0} \left(1 + 2 \sum_{i \leq j \leq 0} \mathrm{e}^{V(j)-V(i)}\right) \left(\mathrm{e}^{-V(i)} + 2 \sum_{k \leq i-1} \mathrm{e}^{-V(k)}\right) \\ &\leq 4 \sum_{k \leq i \leq j \leq 0} \mathrm{e}^{V(j)-V(i)-V(k)}, \end{aligned} \tag{6.9.5}$$

and this latter quantity was already seen to be integrable under $P^{\geq 0}$, after (6.6.12), when $1 < \kappa < 2$. In order to deal with the case $0 < \kappa \leq 1$, let us introduce the event

$$A_t = \bigcap_{k=1}^{\infty} \{H_{-k} < \frac{1}{\kappa} \log k^2 + \log t + \log\log t\}.$$

On one hand, by (6.3.3), $P((A_t)^c) \leq \sum_{k=1}^{\infty} \frac{C}{k^2 (t \log t)^{\kappa}} = \left(\sum_{k=1}^{\infty} \frac{C}{k^2}\right) \frac{t^{-\kappa}}{(\log t)^{\kappa}} = o(t^{-\kappa})$. On the other hand, proceding like after (6.6.12),

$$E^{\geq 0}[R^- \mathbf{1}_{A_t}] \leq 4 \sum_{u \leq 0} E^{\geq 0}[\mathrm{e}^{-V(e_u)}] E^{\geq 0}[(M_1')^2 M_2 \mathrm{e}^H \mathbf{1}_{\{H < \frac{1}{\kappa} \log u^2 + \log t + \log\log t\}}]$$

and $E^{\geq 0}[\mathrm{e}^{-V(e_u)}] = E[\mathrm{e}^{V(e_1)}]^u$ hence, using Lemma 6.4, when $0 < \kappa < 1$,

$$E^{\geq 0}[R^- \mathbf{1}_{A_t}] \leq 4 \left(\sum_{u \leq 0} E[\mathrm{e}^{V(e_1)}]^u \frac{1}{u^{2(1-\kappa)/\kappa}} \right) (t \log t)^{1-\kappa} = C(t \log t)^{1-\kappa},$$

and when $\kappa = 1$,

$$E^{\geq 0}[R^- \mathbf{1}_{A_t}] \leq 4 \sum_{u \leq 0} E[\mathrm{e}^{V(e_1)}]^u (\frac{1}{\kappa} \log u^2 + \log t + \log \log t) \leq C \log t.$$

Finally, by Markov inequality,

$$P^{\geq 0}(R^- > t^\alpha (\log t)^4) \leq P^{\geq 0}((A_t)^c) + \frac{1}{t^\alpha (\log t)^4} E^{\geq 0}[R^- \mathbf{1}_{A_t}]$$

is negligible with respect to $(\log t)^{-\kappa}$ for any $\alpha \geq 1 - \kappa$ when $0 < \kappa < 1$, and for any $\alpha > 0$ when $\kappa = 1$.

6.9.2. Proof of Lemma 6.15. — The proof of (6.7.9) is a direct consequence of the definitions of M_2 and Ω_t. Then, we shall first prove (6.7.8). Since $Var_\omega(F) \leq E_\omega [F^2]$, we shall bound $E_\omega [F^2]$. Recalling (6.7.7) implies

$$R^+ \leq C(\log t)^3 \mathrm{e}^{-\widehat{V}^{\downarrow}(0,e_1)} \max_{0 \leq j \leq e_1} \mathrm{e}^{-\widehat{V}(j)},$$

on Ω_t. To bound $\widehat{V}^{\downarrow}(0, e_1)$ by below, observe first that (6.7.6) yields $\widehat{V}^{\downarrow}(0, T_H) \geq V^{\downarrow}(0, T_H) \geq -\alpha \log t$ on Ω_t. Moreover, (6.7.4) together with (6.7.5) imply that $\widehat{V}(y) - \widehat{V}(x)$ is greater on Ω_t than

$$[V(y) - \max_{y \leq j \leq e_1 - 1} V(j)] - [V(x) - \max_{x \leq j \leq e_1 - 1} V(j)] - \log \log t - O(1),$$

for any $T_H \leq x \leq y \leq e_1$, which yields $\widehat{V}^{\downarrow}(T_H, e_1) \geq -\alpha \log t - \log \log t - O(1)$ on Ω_t. Furthermore, since (6.7.4) and (6.7.5) imply that $\widehat{V}(T_H)$ is larger than $\max_{0 \leq j \leq T_H} \widehat{V}(j) - \log \log t - O(1)$, assembling $\widehat{V}^{\downarrow}(0, T_H) \geq -\alpha \log t$ with $\widehat{V}^{\downarrow}(T_H, e_1) \geq -\alpha \log t - \log \log t - O(1)$ yields

$$\widehat{V}^{\downarrow}(0, e_1) \geq -\alpha \log t - \log \log t - O(1). \tag{6.9.6}$$

Then, coming back to (6.9.2), we have to bound $\max_{0 \leq j \leq e_1} \exp\{-\widehat{V}(j)\}$. Recalling (6.7.6), we have $\min_{0 \leq j \leq T_H} \widehat{V}(j) \geq \min_{0 \leq j \leq T_H} V(j) \geq 0$, by definition of the deep valleys. Moreover, it follows from (6.9.6) that, for any $T_H \leq j \leq e_1$,

$$\begin{aligned} \min_{T_H \leq j \leq e_1} \widehat{V}(j) &= \min_{T_H \leq j \leq e_1} (\widehat{V}(j) - \widehat{V}(T_H)) + \widehat{V}(T_H) \\ &\geq \widehat{V}^{\downarrow}(T_H, e_1) + h_t \geq h_t - \alpha \log t - \log \log t - O(1), \end{aligned}$$

which is greater than 0 for t large enough. Therefore, recalling (6.9.6) and (6.9.2), we get $R^+ \leq C(\log t)^4 t^\alpha$ on Ω_t. This result together with the fact that $R^- \leq C(\log t)^4 t^\alpha$ on Ω_t concludes the proof of (6.7.8).

In a second step, we prove (6.7.10). To this aim, observe first that $-S_1 \leq \widehat{M_1} - M_1 \leq S_2$, where $S_1 := \sum_{i=0}^{T_H - 1} |\mathrm{e}^{-V(i)} - \mathrm{e}^{-\widehat{V}(i)}|$ and $S_2 := \sum_{i=T_H}^{e_1 - 1} \mathrm{e}^{-\widehat{V}(i)}$. By definition of Ω_t and since $T_H \geq T_{h_t}$ we get $S_2 \leq C(\log t)\, \mathrm{e}^{-h_t - \widehat{V}^{\downarrow}(0,e_1)}$ which yields

$S_2 = o(t^{-\delta})$, when $t \to \infty$ by recalling (6.9.6). To bound S_1, the definition of $h(\cdot)$ (from the h-process) given in Subsection 6.7.3 implies

$$S_1 \leq \sum_{i=0}^{T_H-1} \mathrm{e}^{-V(i)}(1-h(i)) \leq C(\log t)\mathrm{e}^{-h_t} \sum_{i=0}^{T_H-1} \mathrm{e}^{\max_{0\leq j\leq i} V(j)-V(i)},$$

on Ω_t. Since $T_H \leq e_1 \leq C\log t$ on Ω_t, we obtain $S_1 \leq C(\log t)^2\mathrm{e}^{-h_t-V^{\downarrow}(0,T_H)}$. This concludes the proof of (6.7.10) by recalling that $V^{\downarrow}(0,T_H)$ is larger than $-\alpha\log t$.

6.9.3. Proof of Lemma 6.16. — Recalling (6.7.14), we get $E_\omega[G] \leq C(\log t)^2\mathrm{e}^{\bar{V}^{\uparrow}(0,e_1)}$ on Ω_t. Therefore the proof of Lemma 6.16 boils down to finding an upper bound for the largest rise $\bar{V}^{\uparrow}(0,e_1)$ of $\bar{V}$ inside the interval $[0,e_1]$. Observe first that (6.7.13) allows to bound the largest rise $\bar{V}^{\uparrow}(T_H,e_1)$ of $\bar{V}$ on the interval $[T_H,e_1]$ by the largest rise of V on this interval, which is less than $\alpha\log t$ on Ω_t. Concerning the largest rise of $\bar{V}$ on the interval $[0,T_H]$, we notice, taking into account the small size of the fluctuations of V controlled by Ω_t, that (6.7.11) and (6.7.12) imply that the difference $\bar{V}(y)-\bar{V}(x)$ is less or equal than

$$[V(y) - \max_{0\leq j\leq y} V(j)] - [V(x) - \max_{0\leq j\leq x} V(j)] + \log\log t + O(1),$$

which yields $\bar{V}^{\uparrow}(0,T_H) \leq \alpha\log t + \log\log t + O(1)$ on Ω_t. Furthermore, (6.7.13) and the fact that $\max_{T_H\leq y\leq e_1} V(y) \leq V(T_H)$ yields $\max_{T_H\leq y\leq e_1}\bar{V}(y) \leq \bar{V}(T_H)$ imply

$$\bar{V}^{\uparrow}(0,e_1) \leq \max\left\{\bar{V}^{\uparrow}(0,T_H), \bar{V}^{\uparrow}(T_H,e_1)\right\},$$

from which we conclude the proof of Lemma 6.16.

APPENDICE A

CONCLUSION OF THE PROOF OF THEOREM 6.1

The proof of chapter 6 gives the limit in distribution of the (centered and normalized) hitting times $\tau(e_n)$, refering to **[27]** for the derivation of the limit of the process $(X_t)_{t\geq 0}$ itself. In order to give a more complete account of the proof, we hereby present the main details of these arguments.

We first consider the case $1<\kappa<2$ (see A.3 for $\kappa\leq 1$). Let us thus assume

$$\frac{\tau(e_n)-n\mathbb{E}^{\geq 0}[\tau(e_1)]}{n^{1/\kappa}} \xrightarrow[n]{\text{(law)}} \mathcal{S} \tag{A.0.1}$$

for some random variable $\mathcal{S}$.

A.1. From $\tau(e_n)$ to $\tau(k)$

Let us choose $\frac{1}{2}<\alpha<\min(1,\frac{1}{\kappa})$ (this is possible since $0<\kappa<2$). Let $E:=E[e_1]$. The event

$$A_k := \{e_{\lfloor\frac{k-k^\alpha}{E}\rfloor} < k < e_{\lfloor\frac{k+k^\alpha}{E}\rfloor}\}, \qquad k\geq 1, \tag{A.1.1}$$

satisfies $P(A_k)\to_k 1$ by application of the central limit theorem to the sequence $(e_n)_n$ and because $\alpha>\frac{1}{2}$. The sequence $(\tau(k))_{k\geq 0}$ being increasing, we have, on A_k,

$$\tau(e_{\lfloor\frac{k-k^\alpha}{E}\rfloor}) \leq \tau(k) \leq \tau(e_{\lfloor\frac{k+k^\alpha}{E}\rfloor}), \tag{A.1.2}$$

hence, for all $t\in\mathbb{R}$,

$$\mathbb{P}\left(\frac{\tau(k)-\frac{k}{E}\mathbb{E}^{\geq 0}[\tau(e_1)]}{(\frac{k}{E})^{1/\kappa}}<t\right) \leq P((A_k)^c)+\mathbb{P}\left(\frac{\tau(e_{\lfloor\frac{k-k^\alpha}{E}\rfloor})-\frac{k}{E}\mathbb{E}^{\geq 0}[\tau(e_1)]}{(\frac{k}{E})^{1/\kappa}}<t\right)$$

and

$$\mathbb{P}\left(\frac{\tau(k)-\frac{k}{E}\mathbb{E}^{\geq 0}[\tau(e_1)]}{(\frac{k}{E})^{1/\kappa}}<t\right) \geq \mathbb{P}\left(\frac{\tau(e_{\lfloor\frac{k+k^\alpha}{E}\rfloor})-\frac{k}{E}\mathbb{E}^{\geq 0}[\tau(e_1)]}{(\frac{k}{E})^{1/\kappa}}<t\right)-P((A_k)^c).$$

Since $\alpha<\min(1,1/\kappa)$ we have, as $k\to\infty$, $k\sim k\pm k^\alpha$ (for the numerator) and $\frac{k^\alpha}{k^{1/\kappa}}\to 0$ (for the centering), hence

$$\frac{\tau(e_{\lfloor\frac{k-k^\alpha}{E}\rfloor})-\frac{k}{E}\mathbb{E}^{\geq 0}[\tau(e_1)]}{(\frac{k}{E})^{1/\kappa}} \xrightarrow[k]{\text{(law)}} \mathcal{S},$$

and similarly for $e_{\lfloor \frac{k+k^\alpha}{E} \rfloor}$. This, with $P(A_k) \to 1$, completes the proof of

$$\frac{\tau(k) - k\frac{\mathbb{E}^{\geq 0}[\tau(e_1)]}{E[e_1]}}{k^{1/\kappa}} \xrightarrow[k]{\text{(law)}} \frac{1}{E[e_1]^{1/\kappa}}\mathcal{S}. \tag{A.1.3}$$

Note that $\frac{\mathbb{E}^{\geq 0}[\tau(e_1)]}{E[e_1]} = \mathbb{E}[\tau(1)]$. This follows for instance from Solomon's Theorem [**43**] and the fact that (A.1.3) implies $\frac{\tau(k)}{k} \to_k \frac{\mathbb{E}^{\geq 0}[\tau(e_1)]}{E[e_1]}$ in law.

A.2. From $\tau(k)$ to X_t

For any integers ζ, η and t, we have

(A.2.1)
$$\{\tau(\zeta) \geq t\} \subset \{X_t \leq \zeta\} \subset \{\tau(\zeta+\eta) \geq t\} \cup \left\{ \inf_{s \geq \tau(\zeta+\eta)} X_s - (\zeta+\eta) \leq -\eta \right\}.$$

Markov property under P_ω and the stationarity of the environment under P imply that the probability of the latter event equals $P(\inf_{s\geq 0} X_s \leq -\eta)$. In particular, since X is almost surely transient to $+\infty$, this probability converges to 0 as $\eta \to +\infty$, uniformly with respect to ζ and n.

Let $x \in \mathbb{R}$. For all $t \in \mathbb{N}$, we define

$$\zeta(t) := \lfloor vt + xt^{1/\kappa} \rfloor,$$

so that

$$\{X_t \leq \zeta(t)\} = \left\{ \frac{X_t - vt}{t^{1/\kappa}} \leq x \right\}. \tag{A.2.2}$$

As $t \to \infty$, we have $\zeta(t) \sim vt$, hence $\zeta(t) = vt + x(\frac{1}{v}\zeta(t))^{1/\kappa}(1+o(1))$ and thus $t = v^{-1}\zeta(t) - \frac{x}{v^{1+1/\kappa}}\zeta(t)^{1/\kappa}(1+o(1))$. This leads to

$$\mathbb{P}(\tau(\zeta(t)) \geq t) = \mathbb{P}\left(\frac{\tau(\zeta(t)) - v^{-1}\zeta(t)}{\zeta(t)^{1/\kappa}} \geq -\frac{x}{v^{1+1/\kappa}} + o_t(1) \right),$$

where $o_t(1)$ is a deterministic function converging to 0 as $t \to \infty$. By (A.1.3), this probability tends to $1 - L(-v^{-(1+1/\kappa)}x)$ where L is the probability distribution function of $\frac{1}{E[e_1]^{1/\kappa}}\mathcal{S}$.

By (A.2.1) we have,for all integers $t, \eta > 0$,

$$\mathbb{P}(\tau(\zeta(t)) \geq t) \leq \mathbb{P}(X_t \leq \zeta(t)) \leq \mathbb{P}(\tau(\zeta(t)+\eta) \geq t) + \mathbb{P}(\inf_{s\geq 0} X_s \leq -\eta).$$

First taking the limit as $t \to \infty$ for the extreme sides, and then as $\eta \to \infty$, we get

$$\mathbb{P}(X_t \leq \zeta(t)) \xrightarrow[t]{} 1 - L(-v^{1+1/\kappa}x) = \mathbb{P}\Big(-\frac{v^{1+\frac{1}{\kappa}}}{E[e_1]}\mathcal{S} \leq x \Big),$$

which, in view of (A.2.2), gives

$$\frac{X_t - vt}{t^{1/\kappa}} \xrightarrow[t]{\text{(law)}} -\frac{v^{1+\frac{1}{\kappa}}}{E[e_1]}\mathcal{S}.$$

A.3. Zero speed case

The first part holds without change if $\kappa \leq 1$.

For $0 < \kappa < 1$, the inversion part works the same with $\zeta(t) = \lfloor xt^\kappa \rfloor$ (hence $t = x^{1/\kappa}\zeta(t)^{1/\kappa}(1 + o(1))$.

For $\kappa = 1$, the computation is more delicate because $b(n) := \frac{1}{C}u_n n \log n$ (where $C := \frac{1}{2}E[\rho_0 \log \rho_0]$) is not explicitly invertible. If $u_n = 1$, we would asymptotically have $b^{-1}(t) \sim C\frac{t}{\log t}$. In fact, it is possible to find $v_t \to 1$ so that $b(Cv_t\frac{t}{\log t}) = t + o(1)$ and if we then define, for an real x and a integer $t \geq 2$,

$$\zeta(t) := \left\lfloor Cv_t \frac{t}{\log t} + \frac{t}{(\log t)^2}x \right\rfloor,$$

then we may get $t = b(\zeta(t)) - \frac{x}{C^2}\zeta(t)(1 + o(1))$. From this point on, the above proof adapts seamlessly.

APPENDICE B

A NOTE ON EDGE-REINFORCED RANDOM WALKS

Ce qui suit est une preuve simple d'un résultat initiallement dû à Merkl et Rolles [**31**] sur les marches renforcées par arêtes (non-orientées). Cette partie n'est pas directement liée au reste de la thèse.

Abstract.Although the question of the recurrence of linearly edge reinforced random walks (ERRW) on infinite graphs has known important breakthroughs in the recent years (cf. notably [**32**]), it seems that the only known proof that one almost-sure return implies the recurrence of the walk is based on the difficult fact that ERRWs on infinite graphs are mixtures of Markov chains (cf. [**31**]). We provide in this note a short and simple proof of that property, with the finite case as the only tool.

Let $G = (V, E)$ be a locally finite undirected graph, and $\alpha = (\alpha_e)_{e\in E}$ be a family of positive real numbers. The *linearly edge reinforced random walk on G with initial weights α starting at $o \in V$* is the nearest-neighbour random walk $(X_k)_{k\geq 0}$ on V defined as follows: $X_0 = o$; then, at each step, the walk crosses a neighbouring edge chosen with a probability proportional to its weight; and the weight of an edge is increased by 1 after it is traversed.

The only property to be used in this note is the following consequence of De Finetti's theorem for Markov chains (cf. [**12**], and [**25**] for instance): if G is finite, then there exists a probability measure μ on transition matrices on G such that the law of the ERRW X is $\int P_\omega(\cdot)d\mu(\omega)$ where P_ω is the law of the Markov chain on V with transition ω starting at o.

Here is the statement of the (main) part of Theorem 2.1 in [**31**] (cf. remark after the proof).

Theorem B.1. — *For the linearly edge-reinforced random walk (ERRW) on any locally finite weighted graph, the following two statements are equivalent:*

(i) *the ERRW returns to its starting point with probability 1;*

(ii) *the ERRW returns to its starting point infinitely often with probability 1.*

Proof. — On finite graphs, this result follows from a Borel-Cantelli argument (cf. [**25**] and the remark after the proof). Let us therefore denote by $\mathbb{P}$ the law of the ERRW on an *infinite* locally finite weighted graph G starting at o. Assume that condition (i) holds.

For any $n \in \mathbb{N}$, we introduce the finite graph G_n defined from the ball $B(n+1)$ of center o and radius $n+1$ in G by identifying the points at distance $n+1$ from o to a new point δ_n. The law of the ERRW on G_n (with same weights as in G) starting at o is denoted by $\mathbb{P}_{G_n}$.

Let us also define the successive return times $\tau^{(1)}, \tau^{(2)}, \dots$ of the ERRW at o, the exit time T_n from $B(n)$, and the hitting time τ_{δ_n} of δ_n in G_n. Note that the laws $\mathbb{P}$ and $\mathbb{P}_{G_n}$ may be naturally coupled in such a way that the trajectories coincide up to time $T_n = \tau_{\delta_n}$.

We have, for all $k \geq 1$,

$$\mathbb{P}(\tau^{(k)} < \infty) = \mathbb{P}(\tau^{(k)} < \infty, \tau^{(k)} < T_n) + \mathbb{P}(T_n < \tau^{(k)} < \infty),$$

and the second term converges to 0 when $n \to \infty$ since $T_n \geq n \underset{n}{\longrightarrow} \infty$. Therefore,

$$\begin{aligned} \mathbb{P}(\tau^{(k)} < \infty) &= \mathbb{P}(\tau^{(k)} < \infty, \tau^{(k)} < T_n) + o_n(1) \\ &= \mathbb{P}_{G_n}(\tau^{(k)} < \infty, \tau^{(k)} < \tau_{\delta_n}) + o_n(1). \end{aligned} \tag{B.0.1}$$

(NB: the condition $\tau^{(k)} < \infty$ on last line could be dropped since (ii) is true for ERRW on finite graphs). In particular, assumption (i) gives:

$$\lim_n \mathbb{P}_{G_n}(\tau^{(1)} < \infty, \tau^{(1)} < \tau_{\delta_n}) = \mathbb{P}(\tau^{(1)} < \infty) = 1. \tag{B.0.2}$$

Since G_n is finite, we may write $\mathbb{P}_{G_n}$ as a mixture of Markov chains: $\mathbb{P}_{G_n}(\cdot) = \int P_{G_n,\omega}(\cdot) d\mu_n(\omega)$. Thus we have, according to (B.0.2),

$$\lim_n \int P_{G_n,\omega}(\tau^{(1)} < \infty, \tau^{(1)} < \tau_{\delta_n}) d\mu_n = 1$$

and, for all $k \geq 1$, according to (B.0.1) and Markov property (applied $k-1$ times),

$$\begin{aligned} \mathbb{P}(\tau^{(k)} < \infty) &= \lim_n \int P_{G_n,\omega}(\tau^{(k)} < \infty, \tau^{(k)} < \tau_{\delta_n})\, d\mu_n \\ &= \lim_n \int P_{G_n,\omega}(\tau^{(1)} < \infty, \tau^{(1)} < \tau_{\delta_n})^k\, d\mu_n. \end{aligned}$$

We may conclude that the last limit equals 1 thanks to the following very simple Lemma:

Lemma B.2. — *If $(f_n)_n, (\mu_n)_n$ are respectively a sequences of measurable functions and probability measures such that, for all n, $0 \leq f_n \leq 1$, and $\int f_n d\mu_n \underset{n}{\longrightarrow} 1$, then:*

$$\text{for every integer } k \geq 1, \ \int (f_n)^k d\mu_n \underset{n}{\longrightarrow} 1.$$

Proof of the lemma. — By Jensen's inequality, $\int (f_n)^k d\mu_n \geq \left(\int f_n d\mu_n\right)^k$. Furthermore, $\int (f_n)^k d\mu_n \leq 1$, hence the limit. □

As a conclusion, $\mathbb{P}(\tau^{(k)} < \infty) = 1$ for all $k \geq 1$, hence $\mathbb{P}(\forall k, \tau^{(k)} < \infty) = 1$, which is (ii). □

Remark. — *Condition* (ii) *implies that the ERRW visits every edge in the connected component of the starting point infinitely often in both directions, by means of the conditional Borel-Cantelli lemma, cf. the end of the proof of Theorem 1.1 in* [**32**] *or Proposition 1 of* [**25**] *for a direct proof.*

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