

N°d'ordre : 4141

THÈSE

PRÉSENTÉE À

L'UNIVERSITÉ BORDEAUX I

ÉCOLE DOCTORALE DE MATHÉMATIQUES ET D'INFORMATIQUE

Par **Yong liang XIONG**

POUR OBTENIR LE GRADE DE

DOCTEUR

SPÉCIALITÉ : **MATHEMATIQUES APPLIQUEES ET CALCUL
SCIENTIFIQUE**

**Analyse par simulation numérique de la réduction de la
traînée et des caractéristiques d'écoulements bidimensionnels
par l'ajout de polymères en solution**

Soutenue le 9 Decembre 2010 à l'Institut de Mathématiques de Bordeaux

Devant la commission d'examen formée de :

M. Charles Henri BRUNEAU, PROF. , Université Bordeaux I	Directeur
M. Guillaume CHIAVASSA , MdC., Ecole Centrale de Marseille	Rapporteur
M. Michel DEVILLE , PROF., Ecole Polytechnique Fédérale de Lausanne	Président
M. Angelo IOLLO , PROF. , Université Bordeaux I	Examineur
M. Hamid KELLAY , PROF. , Université Bordeaux I	Examineur
M. Laurent THAIS , MdC., Ecole Polytechnique Universitaire de Lille	Rapporteur

Institut de Mathématiques de Bordeaux

Université Bordeaux 1

351 cours de la Libération,

33405 Talence Cedex (FRANCE)

Acknowledgments

I would like to express my gratitude to all those who helped me in the past three years.

First of all, I would like to extend my sincere gratitude to my supervisor, Professor Charles henri BRUNEAU, even though there is no words can express my appreciation for his great help. He helped me at all aspects from academic question to daily life. Every time when I face difficulty, I always get his effective helps or advices. I do appreciate his patience, encouragement, and professional instructions during last three years. He also shared his abundant numerical experiences with me. He has spent much time correcting my thesis. Without his patient help and illuminating instruction, the completion of this thesis would not have been possible.

I am also deeply indebted to Professor Hamid KELLAY. He kindly gave me many good advices and helps from suggesting some interesting problem in this thesis to recommending literatures. I benefited a lot from his profound knowledge in turbulent flow. He also made experiments and share the photos with us. He made great effects and contributions to understand the numerical results. Without his help, this thesis could not have reached its present form.

I would like to thank Mr. Patrick FRISHER shared their wavelet tools with me, and thank our never met colleagues for effective email communications.

Special thanks should go to Mr. Guillaume CHIAVASSA and Mr. Laurent THAIS, they reviewed this thesis and give some useful comments and advices to ameliorate the thesis.

I would like to thank all the members of the jury.

I deeply appreciate all my friends in Bordeaux.

I should finally like to express my gratitude to my family. Although my parents live far away in China and my wife live in Norway, their unconditional love and support are felt everyday.

Numerical Investigation on Drag Reduction and Two-Dimensional Turbulence in Diluted Polymer Solutions

Abstract: Polymer plays an important role on the drag reduction and modification of the structure of flow. In this thesis, Oldroyd B model is employed to study the effect of viscoelasticity for polymer solutions diluted by two-dimensional direct numerical simulation. The obstacles are taken into account by penalization method. The artificial boundary condition is imposed without any reflection on the channel outlet. Flow past a cylinder is investigated in detailed by present numerical methods. The numerical codes are valid for predicting the drag force and capturing the important character of viscoelastic flow by comparing with experimental and other numerical results. The drag map of the cylinder is obtained at a wide range of Reynolds number and Weissenberg number space. The detailed characteristics of viscoelastic flow are reported in the thesis. The effects of polymer on two-dimensional turbulent flow are also discussed by grid turbulent flow.

Keywords: Numerical simulation, drag reduction, two dimensional turbulent flow, Oldroyd-B, polymer solutions, flow past a cylinder, grid turbulent flow, penalization method, artificial boundary condition, multigrid method.

Analyse par simulation numérique de la réduction de la traînée et des caractéristiques d'écoulements bidimensionnels par l'ajout de polymères en solution

Résumé: Les polymères jouent un rôle important sur la réduction de la traînée et la modification de la structure des écoulements. Nous avons utilisé le modèle Oldroyd-B pour étudier l'effet de la viscoélasticité de solutions de polymères dilués sur des écoulements en deux dimensions autour d'obstacles dans un canal. Les obstacles sont pris en compte par la méthode de pénalisation volumique et des conditions aux limites artificielles sans réflexion sont imposées à la sortie du canal. La discrétisation est effectuée par des schémas performants en différences finies et la résolution par une méthode multigrille. Les simulations numériques sont effectuées pour une large gamme de nombres de Reynolds et de nombres de Weissenberg. Les caractéristiques détaillées des écoulements viscoélastiques sont analysées et comparées entre elles et celles de l'écoulement du fluide sans polymère. En particulier les jeux de paramètres conduisant à une augmentation ou à une baisse de la traînée. Enfin les effets du polymère sur des écoulements turbulents sont aussi analysés.

Mots clefs: Simulation numérique, la réduction de la traînée, écoulements bidimensionnels, Oldroyd-B model, polymères, méthode de pénalisation, condition à la limite artificielle, méthode multigrille.

Contents

1	Introduction	1
2	Background knowledge and literature review	5
2.1	Newtonian turbulence	5
2.1.1	Energy balance and transfer	6
2.1.2	Kolmogorov's hypotheses and theory	8
2.1.3	Two-dimensional turbulence	11
2.2	Polymer-turbulence interaction	15
2.2.1	Polymer solutions	15
2.2.2	Oldroyd-B model	16
2.2.3	The way to discover the drag reduction mechanism	19
2.3	Flow past a circular cylinder	23
2.3.1	Newtonian flow	23
2.3.2	Non-Newtonian flow	24
3	Numerical methods and Validation	29
3.1	Governing equations	29
3.1.1	Dimensionless governing equations	29
3.1.2	Artificial stress diffusion term	30
3.1.3	Penalization Method	31
3.1.4	Final equations and Drag coefficient	33
3.2	Initial datum and boundary conditions	35
3.2.1	Initial datum	35

3.2.2	Inlet boundary condition	35
3.2.3	Wall boundary condition	38
3.2.4	Outlet condition	38
3.3	Discretization	42
3.3.1	Time discretization	42
3.3.2	Space discretization	43
3.4	Multigrid solver	48
3.5	Numerical validation	50
3.5.1	Comparison for confined cylinder in a channel	50
3.5.2	Comparison with three-dimensional experimental results	53
4	Flows past a circular cylinder	57
4.1	The effects of polymer viscosity	58
4.2	The drag behavior with polymer	62
4.2.1	Drag enhancement and drag reduction	62
4.2.2	The evolution of the drag and lift coefficient with polymer	69
4.3	The flow modification by polymer	73
4.3.1	Stokes flow	74
4.3.2	Laminar flow with steady wake	78
4.3.3	Laminar flow with unsteady wake	83
4.3.4	Transitional flow	90
4.3.5	Turbulent flow	97
4.4	High Weissenberg number instability	102
4.4.1	Stokes flow instability	102
4.4.2	The up-down symmetry broken by polymer	105
4.4.3	The instability at high Reynolds numbers	105
5	Two dimensional grid turbulence	111
5.1	The flow fields and the modification by polymer	112
5.2	The effects of the polymer to turbulent structures and the spectra	117

6	Conclusions and future works	131
6.1	Conclusions	131
6.2	Future works	132

Chapter 1

Introduction

Drag reduction and flow structure modification by polymer additives has received considerable attention since Toms's discovery in 1949 [142]. It has been known that the addition of a small amount of polymer into a turbulent Newtonian solvent (parts per million by weight) can reduce skin friction drag by up to 80%. This technology has been used to increase flow rate at the same pumping cost for oil pipelines. Polymer also has another important application in reducing the drag of ship and submarine hulls to achieve a higher speed and reduced energy cost. For the first kind of application, the polymer solution is surrounded by solid boundaries. Quite a number of experimental works were implemented to compare the difference of turbulent channel flow between Newtonian and diluted polymers in solution. For the second kind of application, inhomogeneous polymer solution flows outside of the boundary; it was investigated as skin friction reduction in external turbulent boundary with slot-injected polymer by Fontaine [54].

However, for the applied consideration, pressure difference drag plays an important part of the whole drag, i.e. an optimal scheme for skin friction reduction of turbulent boundary layer may be valueless for the reason of increasing of pressure difference. Hence, it is reasonable to classify the problem as drag reduction for flow past a bluff body question, which is also widely investigated for Newtonian fluid but received less attention to polymeric solution.

A polymeric molecule consists in a long chain composed by the repetition of a large number of monomers. For a typical drag reduction polymeric molecule, the number of monomers is as much as 10^6 . The variety of available molecular structures makes polymers have a broad range of mechanical characteristic, especially for the elastic behavior. Upon addition of small amounts of polymer to a Newtonian fluid the resulting solution becomes viscoelastic. The stress is proportional to the deformation rate under

the effect of viscosity and is proportional to the deformation itself under the effect of elasticity. A viscoelastic solution can be thought of as a mixture of both characteristics. The viscoelastic fluid has therefore a different behavior from the Newtonian fluid. As an example, the Newtonian fluid is pushed away from the rotating rod by the centrifugal force, on the other hand the viscoelastic fluid move toward the rod and climb up the rod, which is well known as Weissenberg effect. The viscoelasticity of polymer solutions can dramatically modify the flow behavior such as changing the stability of laminar motion or transition to turbulence, modifying the structure of vortices or vortex street and can influence the transport of mass, momentum and scalar.

Due to the important applications of drag reduction, and due to the large number of strange flow behaviors and the complex mechanism relating to turbulence, a large number of studies have been devoted to the understanding of the behavior of viscoelastic fluids. One can know the abundant history before 1995 on this topic by a huge bibliography [104]. In the past, a lot of theoretical models have been put forward in order to understand the mechanism and the effect of polymeric solution. However, it is still far to say we understand viscoelastic flow well, so that some important hydrodynamic behavior of polymer solutions was discovered only recently even at small Reynolds number [62, 63, 55], and the drag modification mechanism in turbulent flows is still ill-understood.

It is believed that recent advances in direct numerical simulation (DNS), which avoids the confusion induced by the use of turbulent models, have enabled the study of non-Newtonian effects on turbulent drag reduction to achieve a complete understanding of this physical phenomenon. Considering three-dimensional DNS is still too numerically expensive to get sufficient solution in multi-dimensional parameter space, two-dimensional DNS facilitates the implementation of this technology.

Among the vast literature on drag reduction, wall bounded flows have received much more attention than other flows, mainly due to its practical significance and early experimental and theoretic works. However, we believe that studying viscoelastic flows past circular cylinders will evoke the second application for drag reduction by polymers directly, and may give useful insights to understand the drag reduction mechanism.

Many studies have been reported on the flow past a circular cylinder for Newtonian fluids and on channel flow for Non-Newtonian fluids. On the other hand, relatively few studies have been conducted on the flow past a circular cylinder for non-Newtonian fluids. Especially there is no numerical research on the flow about a circular cylinder at high Reynolds number to the author's knowledge. One of the reasons is that DNS is still computationally expensive for three-dimensional turbulence, the computational cost of DNS for external flow is much more than the DNS on internal flow such as channel flow

at the same Reynolds number for geometry reasons, and the introduction of many new unknowns in viscoelastic models makes the three-dimensional DNS unfeasible. On the other hand, the results of simulations by turbulent closure models are not convincing for viscoelastic flows past a bluff body. However, inspired by the latest investigations on two-dimensional turbulence of polymer solutions [2, 82], an investigation of two-dimensional viscoelastic flows using DNS may shed light on the viscoelastic flow around a bluff body. It is worth emphasizing that the existing mechanism of drag reduction does not exclude two-dimensional turbulence.

The experiments of the polymer solutions flow past a cylinder at high Reynold number are very few, and they have not given a clear conclusion on the drag behavior because the polymer solutions they used are either severe shear thinning or degrade under shear [37, 124]. Until now, there are both drag reduction and drag enhancement reports for the non-Newtonian fluid flow past a circular cylinder at low Reynold number [109, 155, 55], and there is not a conclusive review on these disagreement. This is the part of motivation for this dissertation, which will investigate the flow behavior at a broad range of both Reynolds number and Weissenberg number by DNS to understand the effect of viscoelasticity on the flow and get the drag reduction map. Specially, the flows past an array of circular cylinders will generate isotropic turbulence, which is the footstone of some drag reduction theory [139, 135] but experiment and DNS of such flows are few [4]. Hence, the isotropic two-dimensional turbulence of dilute polymer solution is also the scope of this dissertation. In this dissertation, comparisons of two-dimensional Newtonian flow and non-Newtonian flow will be made with recent two-dimensional soap film experiment [80, 2, 82].

The dissertation is organized as follows: chapter 2 is a review of background knowledge and relevant literature. In chapter 3, a detailed numerical implementation method was given for both Newtonian and Oldroyd-B constitutive equations based non-Newtonian fluid, whereafter the numerical validation will be presented at the end of the chapter. In chapter 4, we first show the effects of concentration by polymer, then we perform DNS for flow past a circular cylinder in a wide range of Reynolds number and Weissenberg number parameter space. In chapter 5, effects on grid turbulence are investigated. Conclusions and future perspectives are given in chapter 6.

Chapter 2

Background knowledge and literature review

The investigation of turbulence-polymer interactions involved in polymer drag reduction includes two independent disciplines: fluid dynamics and polymer science. Each of them is an active research field involving a large research community, and has important applications itself. A complete review on these is difficult due to the overwhelming amount of past research. This chapter only presents a brief introduction to turbulence theory, the relevant literature on polymeric turbulent flow drag reduction and an important topic: flow past a bluff body.

2.1 Newtonian turbulence

Turbulence is difficult to define precisely even though a lot of scientists had tried to define it. It is identified by its characteristic: highly random unsteady with chaotic variations of pressure and velocity in space and in time, its effective diffusivity on mass, heat and momentum, and its dissipation. The Reynolds number suggests the tendency of flow transit from laminar to turbulent flow, it is defined as:

$$Re \equiv \frac{\rho UL}{\mu} = \frac{UL}{\nu}. \quad (2.1)$$

Where ρ is the density, μ is the viscosity and $\nu \equiv \mu/\rho$ is the kinematic viscosity. U and L are respectively the characteristic velocity and length scales. The governing equations for an incompressible Newtonian fluid are continuity and the Navier-Stokes(N-S) equation

$$\nabla \cdot \mathbf{u} = 0 \quad (2.2)$$

$$\frac{D\mathbf{u}}{Dt} \equiv \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f} \quad (2.3)$$

where $\mathbf{u}(\mathbf{x}, t)$ is the velocity vector, $p(\mathbf{x}, t)$ is the local pressure and $\mathbf{f}$ represents the external forces per unit mass. $\mathbf{u} \cdot \nabla \mathbf{u}$ is called inertial or non-linear term, which is responsible for the transfer of kinetic energy in the turbulent cascade, $\nu \nabla^2 \mathbf{u}$ is the dissipative viscous term. In fact, the Reynolds number gives the relative weight between the inertial term and the viscous term. When it is small, the viscous term is the dominant that gives a laminar flow.

The N-S equations can be solved only for laminar flow with simple initial and boundary conditions. For high Reynolds number flow or complex initial and boundary conditions, the solution can only be obtained through numerical simulation. Moreover, a typical feature of turbulence is chaos, i.e. the N-S equations are sensitive to the initial condition, which drastically reduces the interest of the exact solutions. Hence, turbulence theory aims at the prediction of the statistical properties of the flow but not to find its analytic solution.

2.1.1 Energy balance and transfer

Energy balance

In absence of external forcing for N-S equation, one multiplies by u_i and uses continuity equation to obtain:

$$\frac{\partial u_i u_i}{2 \partial t} + \frac{\partial u_j u_i u_i}{2 \partial x_j} = -\frac{u_i}{\rho} \frac{\partial p}{\partial x_i} + \nu u_i \nabla^2 u_i. \quad (2.4)$$

Assuming periodic boundary condition on a cubic volume V of size L

$$\mathbf{u}(x + nL, y + mL, z + qL) = \mathbf{u}(x, y, z) \quad \forall x, y, z \in \mathbb{R} \quad \forall n, m, q \in \mathbb{Z} \quad (2.5)$$

or vanishing boundary condition on the volume V

$$\mathbf{u} \big|_{\partial V} = 0. \quad (2.6)$$

Then the mean kinetic energy of the fluid is

$$E = \frac{1}{V} \int_V |\mathbf{u}|^2 d\mathbf{r} = \langle \frac{1}{2} |\mathbf{u}|^2 \rangle \quad (2.7)$$

where angular brackets are used to denote averages over the volume V . In equation (2.4), the integral of second and third term vanishes considering continuity condition. For the last term considering

$$\langle \mathbf{u} \cdot \nabla^2 \mathbf{u} \rangle = -\langle (\nabla \times \mathbf{u})^2 \rangle \quad \text{if} \quad \nabla \cdot \mathbf{u} = 0. \quad (2.8)$$

Then the energy balance reads

$$\frac{dE}{dt} = -\nu \langle |w|^2 \rangle. \quad (2.9)$$

Where we have introduced the *vorticity* of the fluid $w = \nabla \times \mathbf{u}$. Defining the mean *enstrophy* as

$$Z = \langle \frac{1}{2} |w|^2 \rangle \quad (2.10)$$

the energy balance reads:

$$\frac{dE}{dt} = -2\nu Z. \quad (2.11)$$

Which shows that in absence of external forcing and for $\nu = 0$, the kinetic energy is conserved. However, at the high Reynolds number corresponding to $\nu \rightarrow 0$, the energy dissipation rate does not vanish, but approximately reaches a constant value [126] :

$$\epsilon \equiv -\frac{dE}{dt}. \quad (2.12)$$

Where ϵ is the energy dissipation, this is known as *dissipative anomaly* and implies in the limit $\nu = 0$ that the enstrophy must grow in order to compensate the decreasing viscosity, which was known as *inviscid enstrophy catastrophe*. For Euler equation (when $\nu = 0$ in N-S equation), one can not assume the velocity and pressure fields are smooth sufficiently to get the relation above, hence the conclusion about the kinetic energy is conservation may break down [56].

Energy Transfer

It is well known that the non-linear term in N-S equation plays a fundamental role in turbulence; however, as shown above by the energy balance, this term does not play any effective role. Its importance in turbulence is to take in charge the energy transfer among the various scales of motion which is the origin of the turbulent cascade. Following the idea of Obukhov [106], if one considers the velocity $\mathbf{u}(\mathbf{x})$ as a L -periodic function, it can be decomposed as Fourier series

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k}} \hat{\mathbf{u}}(\mathbf{k}) e^{\frac{2i\pi}{L} \mathbf{k} \cdot \mathbf{x}} = \mathbf{u}_K^<(\mathbf{x}) + \mathbf{u}_K^>(\mathbf{x}), \quad \mathbf{k} \in \mathbb{Z}^3 \quad (2.13)$$

Where low-pass filtered velocity $\mathbf{u}_K^<(\mathbf{x})$ identified as eddies of scale larger than l ($= K^{-1}$) and high-pass filtered velocity $\mathbf{u}_K^>(\mathbf{x})$ corresponding to eddies of scale less than l are defined as follow

$$\mathbf{u}_K^<(\mathbf{x}) \equiv \sum_{|\mathbf{k}| < K} \hat{\mathbf{u}}(\mathbf{k}) e^{\frac{2i\pi}{L} \mathbf{k} \cdot \mathbf{x}}, \quad \mathbf{k} \in \mathbb{Z}^3 \quad (2.14)$$

$$\mathbf{u}_K^>(\mathbf{x}) \equiv \sum_{|\mathbf{k}| > K} \hat{\mathbf{u}}(\mathbf{k}) e^{\frac{2i\pi}{L} \mathbf{k} \cdot \mathbf{x}}, \quad \mathbf{k} \in \mathbb{Z}^3. \quad (2.15)$$

After using the low-pass filtering operator to N-S equations, a scale-by-scale energy budget equation can be obtained as follow [56]

$$\partial_t \mathcal{E}(K) + \Pi_E(K) = \mathcal{D}_E(K) + \mathcal{F}_E(K) \quad (2.16)$$

where

$$\mathcal{E}(K) = \frac{1}{2} \langle |\mathbf{u}_K^<|^2 \rangle \quad (2.17)$$

is the cumulative energy between wave number 0 and K , $\Pi_E(K)$ is the energy flux due to the nonlinear terms through wave number K , $\mathcal{D}_E(K)$ denotes the energy dissipation, and $\mathcal{F}_E(K)$ represents the cumulative energy injection by the force. Then energy spectrum is defined by

$$E(k) = \frac{d\mathcal{E}(k)}{dk}. \quad (2.18)$$

The equation (2.16) can be interpreted as the rate of change of energy scales smaller than $l = K^{-1}$ equal to the energy injected at such scale by the force minus the energy dissipation at this scale minus the flux of energy to smaller scales by non linear interactions. For some special flow, energy injection and dissipation happen at a fixed given scale.

2.1.2 Kolmogorov's hypotheses and theory

Kolmogorov's hypotheses

Richardson proposed the conceptual picture of turbulent cascade by using a simple dimensional analysis of N-S equations. It shows that the large scale of turbulence is broken down into smaller and smaller scales until the smallest scale. At the smallest scale, viscosity becomes effective and will dissipate the kinetic energy into heat energy. This idea was made into a quantitative theory by Kolmogorov (K41 theory). Before the formation of the theory, three hypotheses were postulated by him as follow:

- **The local isotropy hypothesis:** *At sufficiently high Reynolds number, the small scale turbulent motions, i.e., scales much smaller than the integral length (L), are statistically isotropic.*
- **The first similarity hypothesis:** *In every turbulent flow at sufficiently high Reynolds number, the statistics of the small scale motions have a universal form that is uniquely determined by the viscosity ν and dissipation rate ϵ .*

- **The second similarity hypothesis:** *In every turbulent flow at sufficiently high Reynolds number, the statistics of the motions at scales that are much smaller than the integral scale and much larger than the smallest scale have a universal form that is uniquely determined by the dissipation rate ϵ and the characteristic length of the scale of interest.*

From the viewpoint above, one can identify three different ranges in turbulence:

- **Injective range:** Injective range corresponds to the large scales, which are sustained by the energy injecting of external force.
- **Inertial range:** Large scale eddies in injective range are deformed and stretched by the fluid dynamics, until they become smaller and smaller scale eddies keeping their whole energy. These small scale eddies become smaller and smaller scale eddies taking their whole energy. The process is repeated in such a way that energy is transferred to smaller and smaller structures. These small scale and smaller scale eddies correspond to the inertial range, the whole energy is transferred among these scales without loss.
- **Dissipative range:** this range corresponds to the smallest structures of the flow, the energy is dissipated by viscosity, the cascade is stopped.

One can utilize dimensional analysis to estimate the time required for transferring energy from an eddy with size l to a smaller eddies as $\tau_l \sim l/u_l$, where u_l denotes the root mean square (RMS) velocity fluctuation on the scale l . The same scale eddy dissipating its energy requires much more time as $\tau_l^{dis} \sim l^2/\nu$. The ratio of dissipating time to transferring time corresponds to the Reynolds number for the eddy. For statistically steady state turbulence, energy flux in inertial range l should keep a constant ϵ .

$$\epsilon = \Pi(l) \sim \frac{E(l)}{\tau_l} \sim \frac{u_l^3}{l}. \quad (2.19)$$

Then the velocity and characteristic times can scale as:

$$u_l \sim \epsilon^{1/3} l^{1/3} \quad (2.20)$$

$$\tau_l \sim \epsilon^{-1/3} l^{2/3}. \quad (2.21)$$

Setting $\tau_l = \tau_l^{dis}$, one gets the Kolmogorov scale η to distinguish the inertial and dissipative range, the dissipative velocity, and dissipative characteristic time.

$$\eta \sim \epsilon^{-1/4} \nu^{3/4} \quad (2.22)$$

$$u_d \sim (\epsilon \nu)^{1/4} \quad (2.23)$$

$$\tau_d \sim (\nu/\epsilon)^{1/2} \quad (2.24)$$

Structure function

From the similarity hypothesis, the N-S equations in the inviscid limit have scale invariance:

$$t, \mathbf{r}, \mathbf{u} \Leftrightarrow \lambda^{1-h} t, \lambda \mathbf{r}, \lambda^h \mathbf{u} \quad \lambda \in \mathbb{R}_+, h \in \mathbb{R} \quad (2.25)$$

for a statistical turbulent velocity field. In inertial range, the velocity fluctuations on scale l are supposed to be self-similar

$$\delta u(\lambda l) \sim \lambda^h \delta u(l) \quad (2.26)$$

where

$$\delta u(l) \equiv (\mathbf{u}(\mathbf{x} + l) - \mathbf{u}(\mathbf{x})) \cdot l/l. \quad (2.27)$$

This shows the probability distribution of velocity increment is rescaled according to the scaling exponent h . Hence, in the limit of infinite Reynolds number, n -th order structure functions are expected to display a power law behavior

$$S_p(l) \equiv \langle (\delta u_l)^p \rangle \sim l^{hp}. \quad (2.28)$$

In 1941, Kolmogorov derived an exact relation for the third order longitudinal structure function. This relation is the famous four-fifths law.

- **Four-fifths law:** *In the limit of infinite Reynolds number the third order (longitudinal) structure function of homogeneous isotropic turbulence, evaluated for increments l small compared to the integral scale, is given in terms of the mean energy dissipation per unit mass ϵ by*

$$S_3(l) \equiv \langle (\delta u_l)^3 \rangle = -\frac{4}{5} \epsilon l. \quad (2.29)$$

This law fixes the value of exponent $h = 1/3$, and gives an exact constant of structure. In association with scaling hypothesis for the structure function, Kolmogorov scaling law reads

$$S_p(l) = C_p \epsilon^{p/3} l^{p/3}. \quad (2.30)$$

Especially, second order structure function with the same dimension to energy. One can get the energy spectrum relation

$$E(k) \sim \epsilon^{2/3} k^{-5/3} \quad (2.31)$$

by a simple dimensional analysis from two-thirds law. $k^{-5/3}$ energy spectrum has been observed in many different physical experiments. For low order structure functions, experimental results are in agreement with this law. With increasing p , experimental results show that the scaling exponents are systematically smaller than $p/3$ [67]. The phenomenon is thought to be associated to intermittency of turbulence, which is an important topic in modern turbulence research and many studies have been devoted to this subject [56].

Furthermore, ergodic hypothesis and Taylor frozen turbulence hypothesis are usually used to interpret observations and will be used in this thesis. Under the hypothesis of ergodicity, ensemble averaging is replaced by time or space averaging. For flows with mean velocity U , when its fluctuation velocity $u' = u - U$ is relatively small with respect to the mean flow, the fluctuating velocity field for a short time delay τ is frozen and simply advected by the mean velocity U . Hence, velocity in time series at one fix point can transfer to velocity structure in space by Taylor's frozen turbulence assumption using the formula:

$$u'_x(x, t + \tau) = u'_x(x - U\tau, t). \quad (2.32)$$

2.1.3 Two-dimensional turbulence

For the intermediate scale flows in the atmosphere and in the ocean, the thickness of fluid and the fluctuations are small compared to other directions that the flow can be described as two-dimensional. Also in plasma physics, turbulent motions in the plane perpendicular to magnetic are violently confined and again can be analyzed in two dimensions.

Following the pioneering theoretical work by Kraichnan, Batchelor and Leith (KBL)[84, 81, 9, 87, 88], two-dimensional turbulence is far from a simplification of three-dimensional turbulence, but includes many particular phenomena which are validated by experiments using two-dimensional soap films and electromagnetically driven thin layers [79].

Conservation in two-dimensional turbulence

By taking the curl of the N-S equation, one gets an equation for the vorticity as follows:

$$\frac{Dw}{Dt} \equiv \frac{\partial w}{\partial t} + \mathbf{u} \cdot \nabla w = (w \cdot \nabla)\mathbf{u} + \nu \nabla^2 w + \mathbf{g} \quad (2.33)$$

where $\mathbf{g}$ is the curl of the forcing, it vanishes for a constant force. In two-dimensional flows, vorticity is always perpendicular to velocity plane so that vortex stretching term $(w \cdot \nabla)\mathbf{u}$ disappears. This is the ultimate reason resulting in the difference between

two and three dimensional turbulence. In the limit $Re \rightarrow \infty$, to keep energy balance, enstrophy increases infinitely by vortex stretching. However, in two dimension situation, stretching of vortex is impossible, vorticity is conserved along the fluid particle trajectories in the limit $Re \rightarrow \infty$. In fact, enstrophy is a conserved quantity in two-dimensional turbulence similarly to energy in three-dimensional turbulence. In the absence of external forcing $f = 0$, there is an additional balance equation for the enstrophy in two-dimensional turbulence:

$$\epsilon_2 \equiv \frac{dZ}{dt} = -2\nu P \quad P \equiv \langle \frac{1}{2} |\nabla \times w|^2 \rangle. \quad (2.34)$$

The quantity P is called the *palinstrophy*, ϵ_2 denotes enstrophy dissipation as its three dimensional counterpart: energy dissipation ϵ . Therefore, in the limit $\nu \rightarrow 0$, at variance with three dimensional turbulent flows, the viscous dissipation of energy vanishes in the two dimensional case.

In the same way as the decomposition of the velocity by Obukhov, the vorticity can be split into high-pass filtered vorticity and low-pass filtered vorticity. Likewise, one gets a scale-by-scale enstrophy budget equation by using vorticity splitting to equation 2.33:

$$\partial_t \mathcal{Z}(K) + \Pi_Z(K) = \mathcal{D}_Z(K) + \mathcal{F}_Z(K) \quad (2.35)$$

where

$$\mathcal{Z}(K) = \frac{1}{2} \langle |w_K^\leq|^2 \rangle \quad (2.36)$$

is the cumulative enstrophy between wave number 0 and K , $\Pi_Z(K)$ is the enstrophy flux due to the nonlinear terms through wave number K , $\mathcal{D}_Z(K)$ denotes the enstrophy dissipation, and $\mathcal{F}_Z(K)$ represents the cumulative enstrophy injection by the force. Then enstrophy spectrum is defined by

$$Z(k) = \frac{d\mathcal{Z}(k)}{dk}. \quad (2.37)$$

In Fourier space, the energy and enstrophy spectra having the relation:

$$Z(k) \sim k^2 E(k). \quad (2.38)$$

Tung and Gkioulekas [146] wrote this relation in a general form using the same notation

$$Z(k) = \Lambda(k) E(k). \quad (2.39)$$

And the fluxes have also a similar relation

$$\frac{\partial \Pi_Z(k)}{\partial t} = \Lambda(k) \frac{\partial \Pi_E(k)}{\partial t} \quad (2.40)$$

In fully developed turbulence, there are two kinds of inviscid invariants, the energy and the enstrophy. Moreover, the vanishing of the energy dissipation in the limit $Re \rightarrow \infty$ makes it impossible to have a constant flux cascade of energy toward small scales, thus the turbulent cascade picture is modified. Kraichnan showed that there are two kinds of cascades in two dimensional turbulence: *direct enstrophy cascade* where enstrophy is mainly transferred toward small scales until dissipated by viscosity; and *inverse energy cascade* where energy is transported to large scale.

Direct enstrophy cascade

Kraichnan and Batchelor proposed that enstrophy cascade toward small scales until dissipation occurs by viscosity like energy cascade in three-dimensional case. In the cascade range, the smallest scale is associated with viscosity ν and enstrophy dissipation ϵ_2 . From a dimensional argument, one gets the smallest scale in two-dimensional turbulence

$$l_d \sim \epsilon_2^{-1/6} \nu^{1/2}. \quad (2.41)$$

This shows that the number of degrees of freedom in two dimensional turbulence is much less than in the three-dimensional case. One benefits the reduction of the computational cost both in the reducing of dimension and decreasing of the degree of freedom at one dimension. One can also get the characteristic velocity u_l and time τ at scale l by dimensional argument as follow:

$$u_l \sim \epsilon_2^{1/3} l \quad (2.42)$$

$$\tau \sim \epsilon_2^{-1/3}. \quad (2.43)$$

Noticing the life of eddies is independent of the scale l , the smaller scales can not empty out space for the large eddies breaking up in time. Therefore, Kraichnan showed that this may result in nonlocality of cascade which was interpreted by Tabling [141] that large scales are directly coupled to the small scales. In enstrophy cascade, the energy and enstrophy spectra read:

$$E(k) = C' \epsilon_2^{2/3} k^{-3} \quad (2.44)$$

$$Z(k) = C' \epsilon_2^{2/3} k^{-1} \quad (2.45)$$

where C' is the Kraichnan Batchelor constant. However in the limit $Re \rightarrow \infty$, the integral of enstrophy suffers from “infrared divergences” as the injection wave-number tends to zero. A subsequently correction was given by Kraichnan as

$$E(k) = C' \epsilon_2^{2/3} k^{-3} (Ln(k/k_0))^{-1/3} \quad (2.46)$$

where k_0 is the injection wave-number. This correction successfully avoids the divergence on the enstrophy transfer rate. However, in the limit $Re \rightarrow \infty$, corrected

enstrophy is still divergent at $k \rightarrow \infty$. In fact, enstrophy is always divergent unless the exponent is smaller than -3, and this can result in conflict in dimensional argument. A variety of energy spectrum slopes, from -3 to -6, were observed in direct numerical simulation (DNS) and experiments [127, 2, 79]. In enstrophy range, its structure function supposes to depend on l and ϵ_2 , from dimensional argument, $S_2(l) \sim \epsilon_2^{2/3} l^2$. A logarithmic correction factor following the energy spectrum correction by Kraichnan is permissive under the frame of dimensional analysis. In fact, vorticity and ϵ_2 exclude dimension in length, vorticity structure function defined by $W_p(l) = \langle (w(\mathbf{x} + \mathbf{l}) - w(\mathbf{x}))^p \rangle$ is used to expect having a zero exponent for distance of two points l . For the following structure function:

$$W_p(l) = l^{\xi_p} \quad (2.47)$$

Eyink [50] showed that the scaling exponents ξ_p are not larger than zero for $p \geq 3$. Falkovich and Lebedev showed $W_p(l)$ have logarithmic behavior, and there isn't intermittency in the enstrophy range [51].

Inverse energy cascade

The inverse energy cascade in two-dimensional turbulence transfers energy from the injection scale towards large scales with a constant flux of energy, i.e. $\Pi(l) = -\epsilon$. Characteristic times, velocities and energy spectrum have the same form as in three dimensions as follow:

$$\tau_l \sim \epsilon^{-1/3} l^{2/3} \quad l \geq l_{inj} \quad (2.48)$$

$$u_l \sim \epsilon^{1/3} l^{1/3} \quad l \geq l_{inj} \quad (2.49)$$

$$E(k) = C \epsilon^{2/3} k^{-5/3} \quad k \leq k_{inj}. \quad (2.50)$$

Evidence of the inverse energy cascade and the scaling of the energy spectrum comes from numerical simulations [57, 127]. However, the slight deviation from Kolmogorov scaling was paid attention to recently. Intermittency effect in the inverse energy cascade is taken for the reason of the deviations [15, 113].

In two-dimensional turbulence, it is possible to repeat the derivation of the Kolmogorov's *four-fifths law*. Then one gets the *three-half law* for the third order structure function in the limit $Re \rightarrow \infty$ as follow:

$$S_3(l) = \frac{3}{2} \epsilon l \quad l \geq l_{inj}. \quad (2.51)$$

2.2 Polymer-turbulence interaction

2.2.1 Polymer solutions

As a result of the thermal agitation, a polymer molecule is assumed to be a spherical coil in a homogeneous flow. The radius of gyration of polymeric molecule, which is the statistically average size, is related to the number of monomers N as:

$$R_0 = N^{3/5}a \quad (2.52)$$

where a denotes the length of the single monomer according to Flory [53]. Typical values of R_0 are of order $0.1\mu m$, which is generally smaller than Kolmogorov dissipation scale in turbulence. On the contrary, the molecule is stretched into an elongated shape in an inhomogeneous flow, which can be characterized by its end-to-end distance R . An elongated molecule at the same time has the trend to relax to its equilibrium configuration as a result of Brownian effect. After the shear is excluded, the characteristic time for an elongated molecule recovering its equilibrium configuration is called the relaxation time, which is an important index for viscoelastic solutions. According to Zimm [166], a convenient measure of the relaxation time for the linear chain is given by:

$$\tau = \frac{\mu R_0^3}{k_B T} \quad (2.53)$$

where μ is the solvent viscosity, k_B is the Boltzmann constant, T is the solution temperature. The ratio of the relaxation time of the polymer τ to the characteristic time of flow L/u is an important index to describe the viscoelasticity of the flow, known as Weissenberg number that reads:

$$Wi = \tau u / L \quad (2.54)$$

There is an elastic component which is proportional to the polymer deformation tensor for a viscoelastic solution. The elastic stresses can be dominated by the elongation of polymer molecule. The polymer elongation can also be limited by their reaction on the flow; polymers can reduce the stretching rates in flow. This results in a dynamical equilibrium state characterized by a constant average elongation, which is decided by the flow structure, the polymer concentration and the molecular character .

The molecular viscosity of the fluid is the result of chaotic movements and collisions among the fluid molecule. There is a significant increase of the molecular viscosity when a small amount long-chained polymer was put into the solution. In contrast to Newtonian fluids, the viscosity of polymer solution may or may not vary with shear rate, but most polymer solutions are inherently shear thinning (i.e., the viscosity decreases

with the shear rate). This is because the alignment of a segment of a chain with the flow produces no drag and therefore no contribution to the shear stress. The stretching and alignment increases with shear rate so that a polymer solution is more or less shear thinning [73]. However, when a polymer solution is so dilute, the variation of the viscosity with the shear rate can be ignored. In this thesis, only a Boger fluid which is a liquid that is elastic and has a constant viscosity is modeled.

2.2.2 Oldroyd-B model

In order to model the effect of a polymer on Newtonian fluid, it is necessary to set up a model which can express the interaction between the polymer and the flow. The Oldroyd-B model is a well-known linear viscoelastic model for diluted polymer solutions based on the dumbbell model. The dumbbell model is one of the simplest models to describe the behavior of a molecule of polymer [108].

Taking two spherical beads and connecting them together with a linear spring, the spring is intended to represent the entropic tendency of the polymer coil to end up in a random relaxed state. If the coil is stretched out, it will tend to recoil. Except the spring force acting on each bead, there are also Brownian motion and a drag force if the bead is moving relative to the fluid around it.

Let us assume that the bead with radius a is instantaneously at a position $\mathbf{x}$, and suppose the other bead is instantaneously at the position $\mathbf{x} + \mathbf{r}$. Because the bead is very small, it has no inertia, and Newton's second law becomes a force balance: total force on bead is zero. The spring force pulling it towards the other bead is $\lambda \mathbf{r}$, λ is a spring constant. Because it comes from thermal forces, its size is $\lambda = \frac{3kT}{a^2}$. If the bead at $\mathbf{x}$ is moving with the velocity $\mathbf{U}$, And the fluid around it has velocity $\mathbf{u}(\mathbf{x})$, the drag force on the bead will be $6\pi\mu a[\mathbf{U} - \mathbf{u}(\mathbf{x})]$ in Stokes regime. So the force balance on the bead reads:

$$\frac{3kT}{a^2}\mathbf{r} + 6\pi\mu a[\mathbf{U} - \mathbf{u}(\mathbf{x})] = 0. \quad (2.55)$$

Then the velocity of the bead is:

$$\mathbf{U} = \frac{kT}{2\pi\mu a^3}\mathbf{r} + \mathbf{u}(\mathbf{x}). \quad (2.56)$$

In the same way, the other bead is positioned at $\mathbf{x} + \mathbf{r}$ and the spring force on it is the same. So the velocity is :

$$\mathbf{U}_2 = -\frac{kT}{2\pi\mu a^3}\mathbf{r} + \mathbf{u}(\mathbf{x} + \mathbf{r}). \quad (2.57)$$

In the absence of Brownian motion, these two velocities show how the bead positions evolve:

$$\frac{d\mathbf{x}}{dt} = \frac{kT}{2\pi\mu a^3} \mathbf{r} + \mathbf{u}(\mathbf{x}) \frac{d(\mathbf{x} + \mathbf{r})}{dt} = \frac{kT}{2\pi\mu a^3} \mathbf{x} + \mathbf{u}(\mathbf{x} + \mathbf{r}). \quad (2.58)$$

Finally we subtract the two to get the evolution of $\mathbf{x}$:

$$\frac{d\mathbf{x}}{dt} = -\frac{kT}{\pi\mu a^3} \mathbf{x} + \mathbf{u}(\mathbf{x} + \mathbf{r}) - \mathbf{u}(\mathbf{r}). \quad (2.59)$$

Since $\mathbf{r}$ is microscopically small relative to the scale over which $\mathbf{u}$ changes, we can use a Taylor series for $\mathbf{u}(\mathbf{x} + \mathbf{r})$

$$\frac{d\mathbf{x}}{dt} = -\frac{kT}{\pi\mu a^3} \mathbf{x} + \mathbf{x} \cdot \nabla \mathbf{u}(\mathbf{x}). \quad (2.60)$$

The constant $2kT/\pi\mu a^3$ is called $1/\tau$; it represents the inverse relaxation time of the dumbbell following distortion caused by the flow:

$$\frac{d\mathbf{r}}{dt} = -\frac{1}{2\tau} \mathbf{r} + \mathbf{r} \cdot \nabla \mathbf{u}(\mathbf{x}). \quad (2.61)$$

Polymer extra stress

Now suppose there is a suspension containing many of these dumbbells. We look at the tiny surface element with area δS and unit normal $\mathbf{n}$. The force associated with a single dumbbell that crosses the surface is $\lambda \mathbf{r}$; the stress also depends on how many dumbbells will cross the surface. Clearly a long dumbbell is more likely to cross the surface than a short one; and a dumbbell which is lined up with $\mathbf{n}$ is more likely to cross the surface than one which is perpendicular to it. If there are m dumbbells per unit volume, then we expect the number crossing the surface element to be $m \mathbf{r} \cdot \mathbf{n} \delta S$, then the extra stress exerted by the dumbbells is

$$\boldsymbol{\sigma}_p \cdot \mathbf{n} \delta S = \lambda \mathbf{r} \cdot (m \mathbf{r} \cdot \mathbf{n} \delta S) \quad (2.62)$$

so

$$\boldsymbol{\sigma}_p = m \lambda \mathbf{r} \mathbf{r}. \quad (2.63)$$

Adding Brownian motion

The way a dumbbell will evolve has been worked out when its motion is deterministic, and so the stress it will cause, but to complete the model, one need to add Brownian motion of the two beads. After adding a standard 3D Brownian motion to the evolution of the vector $\mathbf{r}$ $d\mathbf{r} = (-\frac{1}{2\tau} \mathbf{r} + \mathbf{r} \cdot \nabla \mathbf{u}(\mathbf{x}))dt + \tau^{-1/2} dB_t$.

The coefficient $\tau^{-1/2}$ has been chosen for two reasons: it has the correct dimension; but also because this leads to a dumbbell with no flow having a natural length of 1. Because one eventually wants to work out the extra stresses, one defines the conformation tensor. Considering the definition, it is a symmetric tensor as follows:

$$\boldsymbol{\sigma}(\mathbf{r}, t) = \mathbb{E}[\mathbf{r}(\mathbf{x}, t)\mathbf{r}(\mathbf{x}, t)]. \quad (2.64)$$

as we take a time-step dt , the position $\mathbf{x}$ of dumbbell will move to

$$\mathbf{x} + d\mathbf{x} = \mathbf{x} + \mathbf{u}(\mathbf{x})dt. \quad (2.65)$$

Then after one time-step we have

$$\begin{aligned} \boldsymbol{\sigma}(\mathbf{x} + \mathbf{u}(\mathbf{x})dt, t + dt) &= \mathbb{E}[(\mathbf{r} + d\mathbf{r})(\mathbf{r} + d\mathbf{r})] \\ &= \mathbb{E}[\mathbf{r}\mathbf{r} + \mathbf{r}d\mathbf{r} + d\mathbf{r}\mathbf{r} + d\mathbf{r}d\mathbf{r}] \\ &= \mathbb{E}[\mathbf{r}\mathbf{r}] + \mathbb{E}[\mathbf{r}d\mathbf{r}] + \mathbb{E}[d\mathbf{r}\mathbf{r}] + \mathbb{E}[d\mathbf{r}d\mathbf{r}]. \end{aligned} \quad (2.66)$$

Looking at these four terms separately we have the following equations:

$$\begin{aligned} \mathbb{E}[\mathbf{r}\mathbf{r}] &= \boldsymbol{\sigma}(\mathbf{x}, t) \\ \mathbb{E}[\mathbf{r}d\mathbf{r}] &= \mathbb{E}[\mathbf{r}((-\frac{1}{2\tau}\mathbf{r} + \mathbf{r} \cdot \nabla \mathbf{u}(\mathbf{x}))dt + \tau^{-1/2}dB_t)] \\ &= \mathbb{E}[\mathbf{r}(-\frac{1}{2\tau}\mathbf{r} + \mathbf{r} \cdot \nabla \mathbf{u}(\mathbf{x}))dt] \\ &= -\frac{1}{2\tau}\boldsymbol{\sigma}dt + \boldsymbol{\sigma} \cdot \nabla \mathbf{u}dt \\ \mathbb{E}[d\mathbf{r}\mathbf{r}] &= \mathbb{E}[(\frac{1}{2\tau}\mathbf{r} + \mathbf{r} \cdot \nabla \mathbf{u}(\mathbf{x}))dt + \tau^{-1/2}dB_t]\mathbf{r}] \\ &= \mathbb{E}[(\frac{1}{2\tau}\mathbf{r} + \mathbf{r} \cdot \nabla \mathbf{u}(\mathbf{x}))dt]\mathbf{r}] \\ &= \frac{1}{2\tau}\boldsymbol{\sigma}dt + (\nabla \mathbf{u})^T \cdot \boldsymbol{\sigma}dt \\ \mathbb{E}[d\mathbf{r}d\mathbf{r}] &= \mathbb{E}[(\frac{1}{2\tau}\mathbf{r} + \mathbf{r} \cdot \nabla \mathbf{u}(\mathbf{x}))dt + \tau^{-1/2}dB_t](\frac{1}{2\tau}\mathbf{r} + \mathbf{r} \cdot \nabla \mathbf{u}(\mathbf{x}))dt + \tau^{-1/2}dB_t)] \\ &= \mathbb{E}[(\tau^{-\frac{1}{2}}dB_t)(\tau^{-\frac{1}{2}}dB_t)] \\ &= \tau^{-1}Idt. \end{aligned} \quad (2.67)$$

we also need to look at the left-hand side term

$$\begin{aligned} \boldsymbol{\sigma}(\mathbf{x} + \mathbf{u}(\mathbf{x})dt, t + dt) &= \boldsymbol{\sigma}(\mathbf{x} + \mathbf{u}(\mathbf{x})dt, t + dt) - \boldsymbol{\sigma}(\mathbf{x}, t + dt) + \boldsymbol{\sigma}(\mathbf{x}, t + dt) \\ &= (\mathbf{u}dt \cdot \nabla)\boldsymbol{\sigma} + \boldsymbol{\sigma}(\mathbf{x}, t + dt) - \boldsymbol{\sigma}(\mathbf{x}, t) + \boldsymbol{\sigma}(\mathbf{x}, t) \\ &= (\mathbf{u}dt \cdot \nabla)\boldsymbol{\sigma} + \frac{\partial \boldsymbol{\sigma}}{\partial t}dt + \boldsymbol{\sigma}(\mathbf{x}, t). \end{aligned} \quad (2.68)$$

Putting these all together gives the Oldroyd-B model as follow:

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \nabla \mathbf{u} - (\nabla \mathbf{u})^T \cdot \boldsymbol{\sigma} = -\frac{1}{\tau}(\boldsymbol{\sigma} - I). \quad (2.69)$$

and the polymer extra stress is expressed by:

$$\boldsymbol{\sigma}_p = m\lambda\boldsymbol{\sigma} = \frac{\epsilon\eta}{\tau}\boldsymbol{\sigma} \quad (2.70)$$

where $\epsilon = \frac{m\lambda\tau}{\eta}$ is the zero-shear contribution of polymers to the total polymer solution viscosity η . For an incompressible fluid ($\nabla \cdot \mathbf{u} = 0$) with constant density ρ , the momentum conservation law with polymer reads:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + (1 - \epsilon)\nu \nabla^2 \mathbf{u} + \frac{\epsilon\nu}{\tau} \nabla \cdot \boldsymbol{\sigma} + \mathbf{f} \quad (2.71)$$

where ν denotes the kinematic viscosity of the polymer solution.

2.2.3 The way to discover the drag reduction mechanism

Experimental studies

Experiment was the most important source of information to investigate the drag reduction. After the pioneering experimental works of Toms [142], Virk [151, 153] was the first to perform a comprehensive and exhaustive experimental investigation and to review the phenomenon in pipe flows. He compiled many available experimental data on pipe flows with smooth surfaces by that time and made several important observations. He identified three regimes for the behavior of pipe flow with polymer additives. The first one occurs in the laminar regime, there is no drag reduction and the velocity profiles obey Poiseuille flow. The second one was called the polymeric regime. In this regime, the flow is turbulent and drag reduction can be observed. The onset of drag reduction seems to occur at a universal characteristic wall shear stress, the polymer properties such as the polymer molecular weight and concentration affect the amount of drag reduction. The last one is maximum drag reduction (MDR). He was the first to demonstrate the existence of a maximum drag reduction asymptote which was found to be independent of the polymer parameters. The mean velocity profile of a drag reduction pipe flow in the boundary layer can be also summarized by three zones: the viscous sublayer, an elastic sublayer characteristic of drag reduction and a log layer which is similar to that of Newtonian turbulence but shifted upwards. Moreover, Virk [152] also indicated that the roughness of the pipe decreases the drag reduction. Before the discussion of the drag reduction mechanism, Lumley [97] made a review on the relation between polymer solutions and drag reduction from the foregone experimental works. From his observations, the linear structure molecules rather than molecules with many branches are effective on drag reduction; the number of monomer

units in the main chain rather than molecular weight, is the important factor; the effectiveness of drag reduction is monotone in flexibility of polymer molecules; the increase of main-chain length decreases the relative concentration requirement.

Recognizing the importance of turbulent boundary layer for drag reduction, some studies have focused on studying the effect of polymer additives on the near wall turbulent structure. Achia and Thompson [1] found that polymers suppress the formation of streaks and the eruption of bursts in the turbulent pipe flows. Luchik and Tiederman [94] used Laser Doppler Velocimetry (LDV) to measure the velocity components in a drag reduced channel flow. They observed a dampening of the velocity fluctuations normal to the wall in the buffer region, an increase in the streak spacing and a decreased frequency between bursts. Harder and Tiederman [66] observed a reduction of the Reynolds stress but the presence of the added polymer stress could only be accounted for at high concentrations of the polymer additive. However, Warholic et al. [158, 159] later observed that there is reduced Reynolds stress and a non-zero polymer stress over the whole cross section of the channel.

In order to consider the practical application involving external flow, many studies are taken up with external turbulent boundary layer by injection of polymer solution. Fontaine et al. [54] used LDV and laser induced fluorescence to study the velocity field in a turbulent boundary layer. An immediate deceleration of the flow near the wall was observed after the injection. They also observed the suppression of the wall-normal velocity fluctuations and of the Reynolds stress by the action of polymer. With the increase of the streamwise distance from the injection slot, the inhomogeneous polymer solution behaves like bulk polymer solutions. Petrie and Fontaine [117] compared the effects of drag reduction by a polymer solution to slot injected polymer solutions in a boundary layer. The homogeneous solutions change the entire boundary layer, while the effects of the injected solutions are restricted to the near wall region. However, the modifications of the near wall region and the amount of drag reduction by the homogeneous solutions are equivalent to slot injected polymer solutions. Petrie et al. [116] studied the effect of surface roughness on drag reduction between homogeneous solutions and injected solutions. They observed that higher concentrations are required as roughness increases in homogeneous polymer flows. However, the drag reductions obtained on rough surfaces are larger than the smooth one in the case of slot injections. White et al. [163] used particle imaging velocimetry (PIV) to investigate a turbulent boundary layer with slot injected polymer solutions. They observed significant modifications of the near wall turbulent structure with a coarsening of the low speed streaks and suppression on the intensity of the near wall vertical structures. Soomandepalli et al. [134] indicated that the polymers tend to accumulate around the low speed streaks

in the near wall region.

Other examples of important experimental work on turbulent friction drag reduction by additives can be found in Virk et al. [154], Ptasinski et al. [120], Hou et al. [68] and Paschkewitz et al. [114] Experimental works on flow past a bluff body in polymer solution are addressed in the next section.

Analytical and computational studies

Lumley [97], who is among the first to shed some light on the mechanism for polymer drag reduction, concluded that the stretching of polymers begins at a threshold characteristic time scale instead of a length scale. The polymer molecules are stretched by the flow when the relaxation time of polymer is larger than the characteristic time of turbulence. He described that an increase of the effective viscosity, which comes from the stretching of the polymer molecules in the inertial layer of the boundary layer, induces the drag reduction mechanism. He showed that the enhanced viscosity outside the viscous sublayer would increase the thickness of the buffer layer on turbulent boundary layer, and would result in the increase of flow rate for a given pressure drop. He also showed that the drag reduction saturated at high concentrations and is proportional to the concentration for lower concentrations. Lumley theory is qualitatively plausible to explain the change of turbulent boundary layers by polymers near the wall. For the effect of polymer on turbulence outside the boundary layer, this theory does not provide all the necessary clues.

In contrast with lumley's viscous model, Tabor and De Gennes [139] proposed an elastic energy model to emphasize the link between turbulence and polymers. They argued that the polymers in turbulent flows exhibit elastic properties even at very low concentrations. Hence, they can store up some of the cascading energy of turbulence and this energy is not dissipated by viscosity thus giving rise to an effective drag reduction. This theory also predicts the existence of a critical concentration below which drag reduction can not be observed. The original De Gennes theory deals with isotropic turbulence, it was extended to pipe flows to deal with the onset of drag reduction and the maximum drag reduction asymptote by Sreenivasan and White [135]. Considering that the elastic energy of a polymer is related to the number of monomers per unit mass, the molecular weight of the polymer and the number of monomers in the chain is not important in this theoretical framework. Furthermore, this theory says that the polymers only need to be stretched partially to affect the flow, which is a much less stringent requirement than Lumley's theory.

There are many other theories showing a plausible mechanism for the drag reduction.

Based on the velocity profile of the turbulent boundary layer in experiments, Virk [153] proposed an elastic sublayer model. Ryskin [123] introduced the so-called yo-yo model which shows that the elongation of a polymer chain in an extensional flow does follow the elongation of the fluid element if the strain rate exceeds a critical value. However, the polymer chain does not deform affinely with the fluid but unravels, whereby the central portion of the chain is straightened out first and then remains taut and grows in length at the expense of the two coiled portions at the ends. The taut central portion of the polymer chain produces large additional stresses by a dissipative mechanism. By analyzing vorticity disturbances in axisymmetric elongational flow, Rabin and Zielinska [121] showed that the enhancement of large scale vorticity is a consequence of elastic energy storage by the polymer molecules, thus inhibiting the energy cascade toward the dissipative scales. By simplifying the FENE-P model (Finitely Extensible Nonlinear Elastic model which the closure was proposed by Peterlin), Benzi et al. [11] proposed a shell model which can interpret maximum drag reduction (MDR) mechanism well. L'vov et al. [95] proposed an elucidation on the mechanism of drag reduction by polymers in turbulent wall-bounded flows arguing that polymer stretching results in the suppression of momentum flux to the wall.

Numerical simulations have become increasingly important due to recent advances in polymer models, computational ability and numerical methods, and the more comprehensive information they supplied relative to experiments. Recently the FENE-P model was widely used in polymeric DNS to study turbulence drag reduction of polymer solutions [48, 49, 42, 47, 90, 162]. Sureshkumar and Beris [129] were the first to attempt the DNS of a turbulent channel flow with polymer drag reduction at high Reynolds numbers using Oldroyd-B model. They put forward an artificial stress diffusion method to improve the stability at high Weissenberg number, which was widely popular in recent DNS. Ptasinski et al [120] were the first to achieve the high drag reduction regime where a change in the slope of the log-law in the mean velocity profile is observed. Baron and Sibilla [8] demonstrated that a finite difference method can reproduce all the main features of polymer drag reduced flows. They suggested that elongated polymers inhibit turbulence regeneration by opposing pressure redistribution from streamwise to cross-flow velocity fluctuations [125]. Stone et al. [136] investigated a polymer solution in plane Couette flow containing exact coherent states (ECS) in order to avoid the complications due to the randomness of turbulence. The ECS has the ability to capture the main structure of the buffer layer. They believed that this kind of simpler flow can reproduce the main structures of polymer drag reduction [137].

De Angelis et al. [3, 4] investigated the different modes of energy transfer in their spectral simulations. They found that the special profile of the most energetic modes

is hardly changed between viscoelastic and Newtonian flows. They indicated that the drag reduction is seen in the energy containing modes rather than dissipative modes. L'vov et al. [95] discovered high quality hydroelastic wave. This is the result of energy transfer between turbulent energy and polymer potential energy. The interaction between polymers and turbulence was investigated by numerical experiments of Dubief et al. [46]. They found that the polymers reduce drag by damping near wall vortices and sustain turbulence by injecting energy into the streamwise velocity component in the very near wall region [47].

On the other hand, Reynolds averaged Navier-Stokes (RANS) approaches which are more computationally efficient than DNS are attempted. Considering many new non-linear terms in the case of viscoelastic flows, it is more difficult to fabricate new closure models than traditional Newtonian flows. Leighton et al. [86] proposed a RANS model by expressing the unclosed mean stretching term as a combination of all available tensors and assuming a Perterlin-type closure for the mean restoring term. Although the model predicts the mean velocity profiles well, several parameters need to be calibrated, and the model can not reproduce accurately the anisotropy of the turbulent normal stresses in the near wall region. The performance of this model compared with channel flow DNS is not satisfactory. Cruz and Pinho [39, 40] proposed a new $k - \epsilon$ model for viscoelastic flows. However, the model showed large over-prediction of the drag reduction for pipe flows compared with DNS results. L'vov et al. [96] proposed another closure in the limit of small polymer stretching using the Hookean force-extension law. It is not completely satisfactory either by comparison with channel flow DNS. These considerations suggest that it is far to say the understanding of viscoelastic flow is adequate so that it is still very difficult to build a practical viscoelastic turbulent model.

2.3 Flow past a circular cylinder

2.3.1 Newtonian flow

External flows past obstacles, such as a circular cylinder, have been studied extensively because of their many practical applications. For example, tubes in tubular and in pin-type heat exchangers which are used extensively in the cooling of electronic-components and in food and polymer processing applications. On the other hand, the flow over a circular cylinder includes abundant flow phenomena such as: boundary layer flow separation, wake flow, vortex shedding, vortex-induced oscillations, aerodynamic loading, momentum balance, and the lift and drag forces, shear layer, boundary transition and so on. Hence, to study the flow around a circular cylinder is fundamental to a com-

prehensive understanding of other complex flows over bluff body or rod bundles. The flow over a circular cylinder has been used both as a benchmark and as a legitimate research case.

The flow past a circular cylinder is only related to the Reynolds number. When $Re \leq 1$, the inertial force takes the secondary role in flow, the flow is dominated by viscous force, the upstream and downstream streamlines keep symmetry, the drag coefficient is approximately inversely proportional to Re , the flow is known as Stokes flow. The symmetry between upstream and downstream streamlines is lost upon the increase of Re up to about 4, and the fluid particles first flow along the surface of cylinder and separate from the solid surface at the side culmination (90 degrees). The separated fluid elements form a pair of stationary symmetric vortices known as attached vortices. The pair of attached vortices is elongated with the increase of Reynolds number until the shedding happens at $Re > 49$. The downstream side of the cylinder is no longer steady. At both sides of the downstream of the cylinder, there is vortex shedding periodically to form a regular arrangement of the vortex array, which is called Karman vortex street. When the Reynolds number reaches 300, the vortex street loses its regularity and periodicity gradually, but the separation point (about 82 degrees) before the cylinder remains laminar boundary layer near the wall, the wake is still a laminar wake until a transition happens in the wake. When $Re = 200000 \sim 500000$, the laminar boundary layer may be transformed at any time to a turbulent boundary layer, the drag coefficient decreases dramatically as a result of the separation point moving to the rear of the cylinder. After this, the separation point moves back to the front of the cylinder, which corresponds to the increase of the drag coefficient, along with the increase of the Reynolds number. It is believed that the drag coefficient is independent of Reynolds number at $Re > 3 \times 10^6$.

For more detailed reviews and important works please refer to Williamson [161], Karniadakis and Triantafyllou [76], Unal et al. [148, 149], Hammache and Gharib [65] and Prasad and Williamson [119].

2.3.2 Non-Newtonian flow

A majority of studies on the flow past a cylinder focus on creeping flow at low Reynolds number. The two most important topics on the flow are the modification of velocity field by elasticity and drag reduction. The first theoretical analysis of viscoelastic flow past a cylinder was carried out by Ulmann and Denn [147], they linearized the momentum equation by Oseen approximation and limited the upper convected Maxwell (UCM) model at small deformation rates to a linear constitutive equation. They showed

that the drag coefficient decreases and streamline shifts upstream for polymer solution compared with a Newtonian fluid due to elastic effects. Their approach has been criticized since near the stagnation point of the cylinder surface small deformation rate precondition is extremely severe even at low Reynolds number, furthermore their approximation is not uniformly valid in the domain for any non-zero Weissenberg number, which results in an unrealistic boundary condition [102, 164].

Mena and co-workers have performed several investigations on creeping viscoelastic flow past spheres and cylinders. Mena and Caswell [102] executed a strict asymptotic expansion in the limit of Reynolds and Weissenberg number small enough for Oldroyd-B constitutive model. In direct contrast to the result of Ulmann, they found that elasticity shifts the streamlines downstream and decreases the drag for flow past a cylinder. Their analysis results were confirmed partly by experiments of flows over cylinders and spheres, for which the drag reduction was observed to be quadratic in Wi at low Weissenberg number [18] and a downstream shift of the streamlines was observed at $Wi < 1$ [98].

The elastic instability is now becoming a more and more important topic for the flow of polymer solution [62, 63]. The coupling of a perturbation for radial extensional flow to the base-state stress gradients leading to a perturbation in the non-Newtonian hoop stress is thought as the mechanism of instability of viscoelastic flow with curved streamline [85]. McKinley et al. [101] demonstrated that the instability is generated in the wake near the rear of stagnation point then convected downstream. They found that the steady planar stagnation flow in the downstream wake becomes a three-dimensional cellular structure when the Weissenberg number exceeds a critical value. Shiang et al. [133] performing experiments for viscoelastic flow past a confined cylinder showed similar behavior in which the viscoelastic flow bifurcates from steady two-dimensional flow to steady three-dimensional flow for values of Weissenberg number greater than a critical value. In two-dimensional numerical simulations, a periodical bubble at the rear of the cylinder was found, this also induces a periodical drag force for the viscoelastic creeping flow past a cylinder in these simulation results [109]. On the flow instability side, Cadot and Kumar [29] suggested that viscoelasticity delayed the development of two-dimensional instability by injecting polymer solutions into a cylinder wake. Sahin and Owens [130] also indicated the critical Reynolds number for the onset of vortex shedding in the viscoelastic fluid is higher than in the Newtonian case by DNS and linear stability analysis. This conclusion was observed in the succedent experiment works of Pipe and Monkewitz [118]. They found the critical Reynolds number increased for the low concentrations of PEO solutions. Their LDA measurements have shown that the lateral velocity fluctuations on the wake centreline are similar to the Newtonian

case at low concentrations although a reduction in strength is noted at their highest concentration. They suggest that the interaction of normal stress differences and a shear rate dependent elongational viscosity on the development of the instability. We have not found the work on the elastic instability and flow instability at higher Reynolds number.

The flow past a confined cylinder has no geometrical singularities and includes both elongational and shear deformation. Thus it is suitable to serve as a benchmark both for numerical and experimental investigation on effects of viscoelasticity. A confined cylinder in channel flow with blockage ratio 0.5 had served as benchmark since the proposal by Brown and McKinley [19] and a large number of numerical and experimental works are related to this configuration [6, 33, 31, 45, 52, 69, 71, 70, 93, 101, 103, 111, 109, 138]. The drag behavior can refer to the table 2.1. Their results show consistently drag first slightly decreases then increases with Wi , just like drag behavior on flow past a sphere [73]. On the other hand, the experiment has not validated the same drag behavior for this configuration, they found drag enhancement for the polymer solutions. It should be noticed that the blockage ratio 0.5 is too large to separate the influence of the constraining wall and the extensional flow near the stagnation point, and may cause severe three-dimensional effects as suggested by the experimental results [155].

Author	method	model	drag behavior
Baaijens et al. (1997)	DG	Giesekus/PTT	Reduction
Liu et al. (1998)	FEA	Giesekus/FENE-P/FENE-CR	Reduction
Dou et al. (1999)	FVM	Oldroyd-B	Reduction
Fan et al. (1999)	FEA	Oldroyd-B	Reduction
Owens et al. (2002)	SEM	Oldroyd-B	Reduction
Verhelst et al. (2004)	Experiment	—	Enhancement
Oliveira (2005)	FVM	FENE-P	Reduction
Hulsen et al. (2005)	FEA	Oldroyd-B	Reduction when $Wi < 2.0$

FVM, finite volume method; FEM, finite element method; SEM, spectral element method; DG, Discontinuous Galerkin.

Table 2.1: Literature review on drag behavior for the viscoelastic fluid past a cylinder with $B = 0.5$ at Stokes regime.

Numerical simulations of unbounded creeping flow of the viscoelastic flow past cylinders and spheres using a suspension of dumbbells model with finite extensibility were performed by Chilcott and Rallison [33]. The drag coefficient variation with Weissenberg number for the flow past a sphere first dips slightly, and then rises for all model parameter. For the flow past a cylinder, the drag increases with Weissenberg number, a first

decreasing process which exists for the flow about a sphere has not been found. This may be due to the blockage ratio [69]. In their numerical investigation, there is no drag reduction at small blockage ratio (β) when $\beta < 0.1$, however the trend is completely reversed for flow with stronger blockage. Recent experiment using polyethylene oxide (PEO) with molecular weight 4 million to study the flow around an optical fiber cylinder in the Stokes regime was performed by Francois et al. [55]. The drag enhancement above a critical Weissenberg number for different polymer concentration solutions was observed. There is no drag reduction in their subsequent pipe flow experiment where the drag was measured for a specially designed pipe with large roughness by small cylinders on the surface of the wall. Their experimental investigations shed some light on the drag reduction obtained with polymer solutions in rough pipes reported in early experiment [152].

James and Acosta [74] measured the heat transfer and drag for the flow of diluted polymer solutions past cylinder at $Re < 50$. The concentration increase results in the reduction of heat transfer and enhancement of drag force at a fixed Reynolds number. They suggested the importance of Re/Wi as an index to describe the effect on elasticity. They also used a dye-injection to visualize the flow fluid which showed an increase of effective wake size, and streamline asymmetry for the increasing polymer concentration.

Kato and Mizuno [77] measured the drag force in the Re range of 1 to 5×10^4 for the flow over a cylinder with and without polymer. They found the drag forces in the Re range of 1 to 100 are barely affected by the elastic effect. Important drag reduction was found in the Re range of 10^3 to 5×10^4 for the polymer solution, and the drag reduction is less affected by the Reynolds number in the range of $Re > 8 \times 10^3$. They also found drag reduction can be reduced at high Weissenberg number for high Reynolds number flow. From the measurement of pressure distribution around a cylinder, they found the polymer solution rises the back pressure and delays the separation for the high Reynolds number flow. It was thought the direct reason of drag reduction.

In the numerical simulations of viscoelastic flows by Oliveira [110], the drag reduction was found in the Re range of 50 to 150 for the FENE-CR fluid. He also pointed out the vortex shedding frequency is attenuated by elasticity, and the formation length of wake is increased by the polymer. These conclusions can also be found in the experimental works of Cadot and coworkers [30, 29]. Moreover, his simulation results show that the rms lateral velocity are suppressed by the polymer, which is in agreement with the two-dimensional soap film experiment results.

Even though the drag reduction is listed as one of the most important topic in non-Newtonian fluid dynamics [41], we found it is still difficult to clarify the drag behavior for the flow past a cylinder from the literature review above. These results are somewhat

Author	Method	Re range	drag
James et al. (1970)	Experiment	1~50	Enhancement
Kato et al. (1983)	Experiment	1000~50000	Reduction
Chilcott et al. (1988)	FD	<1	Enhancement
Huang et al. (1995)	FEM	0.1~10	Enhancement
Oliveira (2001)	FVM	50~150	Reduction
Cressman et al. (2001)	Experiment	O(100)	—
Francois et al. (2008)	Experiment	<< 50	Enhancement

Table 2.2: Literature review on drag behavior for the viscoelastic fluid past a cylinder.

contradictory. We will return to this topic by DNS in chapter 4 after introducing our numerical method in chapter 3.

Chapter 3

Numerical methods and Validation

3.1 Governing equations

3.1.1 Dimensionless governing equations

The governing equations of an unsteady incompressible flow for a Newtonian fluid without body force are the evolution Navier-Stokes equations written in primitive variables as:

$$\begin{cases} \nabla \cdot \mathbf{u} = 0 \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} \end{cases} \quad (3.1)$$

where $\mathbf{u}(\mathbf{x}, t) = (u(\mathbf{x}, t), v(\mathbf{x}, t))$ is the velocity vector, $p(\mathbf{x}, t)$ is the local pressure, ρ is the density of the fluid, ν denotes the kinematic viscosity of the fluid and $\mathbf{x} = (x, y)$ represents the space vector in Cartesian coordinates.

The governing equations of an unsteady incompressible flow for Oldroyd-B model without body force are rewritten in primitive variables as:

$$\begin{cases} \nabla \cdot \mathbf{u} = 0 \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + (1 - \epsilon) \nu \nabla^2 \mathbf{u} + \frac{\epsilon \nu}{\tau} \nabla \cdot \boldsymbol{\sigma} \\ \partial_t \boldsymbol{\sigma} + (\mathbf{u} \cdot \nabla) \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \nabla \mathbf{u} - (\nabla \mathbf{u})^T \cdot \boldsymbol{\sigma} = -\frac{1}{\tau} (\boldsymbol{\sigma} - I) \end{cases} \quad (3.2)$$

where $\boldsymbol{\sigma}$ is a symmetrical conformation tensor that gives three additional unknowns in two dimensions $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$, ϵ is the zero-shear contribution of polymers to the total polymer solution viscosity and τ denotes the relaxation time of the polymer solution. Setting $\epsilon = 0$ restore the Navier-Stokes equations for a Newtonian fluid; therefore we will only pay attention to the equations of the Oldroyd-B fluids.

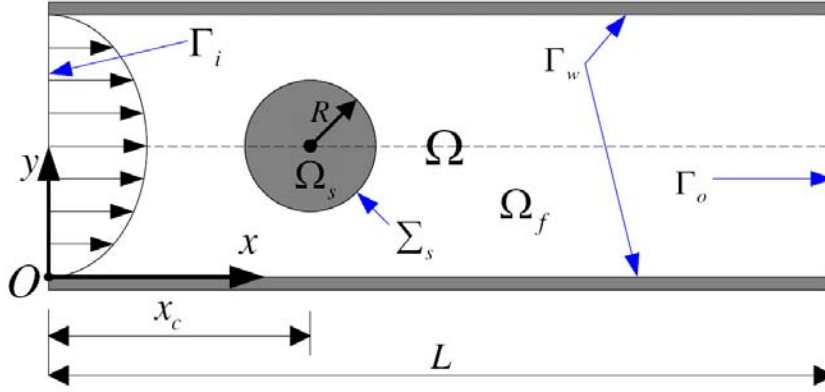


Figure 3.1: Sketch of the computational domain

In this thesis, we are interested in computing the flow behind obstacles in a channel (figure 3.1). Let Ω be a connected bounded domain in $\mathbb{R}^2$ in the first quadrant of the Cartesian coordinates. The fluid flows in the domain from inlet boundary Γ_i , then flows past the bluff body Ω_s in the channel delimited by the wall boundaries Γ_w ; and finally goes out through the outlet boundary Γ_o .

Considering non-dimensional equations is convenient for comparison analysis, so we study non-dimensionalization equations of the above governing equations. If the space vector $\mathbf{x}$ is undimensioned by the channel height H , the velocity vector $\mathbf{u}$ is non-dimensionalized by the mean velocity of inlet boundary U and the pressure p is non-dimensionalized by $\frac{1}{2}\rho U^2$, we can get the following non-dimensional governing equations:

$$\begin{cases} \nabla \cdot \mathbf{u} = 0 \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{Re} [(1 - \epsilon) \Delta \mathbf{u} + \frac{\epsilon}{Wi} \nabla \cdot \boldsymbol{\sigma}] \\ \partial_t \boldsymbol{\sigma} + (\mathbf{u} \cdot \nabla) \boldsymbol{\sigma} + \frac{\boldsymbol{\sigma} - \mathbf{I}}{Wi} = (\nabla \mathbf{u}) \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot (\nabla \mathbf{u})^T \end{cases} \quad (3.3)$$

where p , $\mathbf{u}$ and $\mathbf{x}$ are, respectively, the non-dimensional pressure, velocity vector and space vector; $Re = \frac{UH}{\nu}$, $Wi = \frac{\tau U}{H}$ are the Reynolds number and the Weissenberg number for the reference velocity U and reference scale H .

3.1.2 Artificial stress diffusion term

Performing a direct numerical simulation of turbulence coupled to the Oldroyd-B model is a real challenge due to the loss of the numerical stability. Sureshkumar and Beris

[129] overcame these instabilities by introducing a stress diffusion term. This is the method used in our simulation. Adding this stress diffusion term, the new governing equations read:

$$\begin{cases} \nabla \cdot \mathbf{u} = 0 \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{Re} [(1 - \epsilon) \Delta \mathbf{u} + \frac{\epsilon}{Wi} \nabla \cdot \boldsymbol{\sigma}] \\ \partial_t \boldsymbol{\sigma} + (\mathbf{u} \cdot \nabla) \boldsymbol{\sigma} + \frac{\boldsymbol{\sigma} - \mathbf{I}}{Wi} = (\nabla \mathbf{u}) \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot (\nabla \mathbf{u})^T + \kappa \Delta \boldsymbol{\sigma} \end{cases} \quad (3.4)$$

where κ is a coefficient to choose. Sureshkumar and Beris had investigated the effect of the addition of this term on the stability and accuracy systematically. If the magnitude of this term is reasonably small, convergence is established as $\delta h \sim \delta t \rightarrow 0$, where δh is a characteristic mesh size and δt is the time step. They proposed the reasonable small value of κ should be the same order as $\delta h^2 / \delta t$. In this thesis κ is no more than 10^{-3} to restrict the artificial term as small as possible. For the Newtonian fluid, this term is absent, we will see their results are consistent with the viscoelastic fluid at small Weissenberg number. It suggests that the small artificial term doesn't affect the flow behavior.

3.1.3 Penalization Method

Instead of solving the governing equations in the fluid domain Ω_f , the penalization method is used to compute the incompressible Navier-Stokes equations around obstacles on the whole domain Ω . In this method, a penalization term is added on the velocity defined on the volume of the solid body considered as a porous body. This corresponds to a Brinkman type model with variable permeability where the fluid domain has a very large permeability and the porous medium has a very low permeability. Angot et al. [5] have performed numerical tests to confirm the efficiency and accuracy of the method for the Newtonian fluid. In this thesis, we will use this method to describe the solid obstacles Ω_s in the channel domain Ω . For the L^2 penalization, the first idea to force the velocity to be small in Ω_s is to solve:

$$\begin{cases} \nabla \cdot \mathbf{u}_K = 0 \\ \partial_t \mathbf{u}_K + (\mathbf{u}_K \cdot \nabla) \mathbf{u}_K + \frac{1}{K} \mathbf{1}_{\Omega_s} \mathbf{u}_K = -\nabla p_K + \frac{1}{Re} [(1 - \epsilon) \Delta \mathbf{u}_K + \frac{\epsilon}{Wi} \nabla \cdot \boldsymbol{\sigma}_K] \\ \partial_t \boldsymbol{\sigma}_K + (\mathbf{u}_K \cdot \nabla) \boldsymbol{\sigma}_K + \frac{\boldsymbol{\sigma}_K - \mathbf{I}}{Wi} + \frac{1}{K} \mathbf{1}_{\Omega_s} (\boldsymbol{\sigma}_K - \boldsymbol{\sigma}_s) = (\nabla \mathbf{u}_K) \cdot \boldsymbol{\sigma}_K + \boldsymbol{\sigma}_K \cdot (\nabla \mathbf{u}_K)^T + \kappa \Delta \boldsymbol{\sigma}_K \end{cases} \quad (3.5)$$

in the whole domain Ω , where K is the non-dimensional permeability coefficient used in the penalization term. We denote $\mathbf{1}_{\Omega_s}$ the characteristic function:

$$\begin{cases} \mathbf{1}_{\Omega_s}(x) = 1 & \text{if } x \in \Omega_s \\ \mathbf{1}_{\Omega_s}(x) = 0 & \text{if } x \in \Omega_f \end{cases} \quad (3.6)$$

Let's set $\mathbf{u}_K = \mathbf{u} + K\tilde{\mathbf{u}}$, $p_K = p + K\tilde{p}$ and $\boldsymbol{\sigma}_K = \boldsymbol{\sigma} + K\tilde{\boldsymbol{\sigma}}$ to derive formally the equations satisfied by $\mathbf{u}$, p , $\boldsymbol{\sigma}$ and $\tilde{\mathbf{u}}$, $\tilde{p}$, $\tilde{\boldsymbol{\sigma}}$. Then we get from the momentum equation and constitutive equation by identifying the terms of same order:

$$\mathbf{1}_{\Omega_s} \mathbf{u} = 0 \quad (3.7)$$

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \mathbf{1}_{\Omega_s} \tilde{\mathbf{u}} = -\nabla p + \frac{1}{Re} [(1 - \epsilon) \Delta \mathbf{u} + \frac{\epsilon}{Wi} \nabla \cdot \boldsymbol{\sigma}] \quad (3.8)$$

$$\partial_t \tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \tilde{\mathbf{u}} = -\nabla \tilde{p} + \frac{1}{Re} [(1 - \epsilon) \Delta \tilde{\mathbf{u}} + \frac{\epsilon}{Wi} \nabla \cdot \tilde{\boldsymbol{\sigma}}] \quad (3.9)$$

and

$$\mathbf{1}_{\Omega_s} (\boldsymbol{\sigma} - \boldsymbol{\sigma}_s) = 0 \quad (3.10)$$

$$\partial_t \boldsymbol{\sigma} + (\mathbf{u} \cdot \nabla) \boldsymbol{\sigma} + \frac{\boldsymbol{\sigma} - \mathbf{I}}{Wi} + \mathbf{1}_{\Omega_s} \tilde{\boldsymbol{\sigma}} = (\nabla \mathbf{u}) \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot (\nabla \mathbf{u})^T + \kappa \Delta \boldsymbol{\sigma} \quad (3.11)$$

$$\partial_t \tilde{\boldsymbol{\sigma}} + (\tilde{\mathbf{u}} \cdot \nabla) \boldsymbol{\sigma} + (\mathbf{u} \cdot \nabla) \tilde{\boldsymbol{\sigma}} + \frac{\tilde{\boldsymbol{\sigma}}}{Wi} = (\nabla \tilde{\mathbf{u}}) \cdot \boldsymbol{\sigma} + (\nabla \mathbf{u}) \cdot \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma} \cdot (\nabla \tilde{\mathbf{u}})^T + \tilde{\boldsymbol{\sigma}} \cdot (\nabla \mathbf{u})^T + \kappa \Delta \tilde{\boldsymbol{\sigma}} \quad (3.12)$$

Thus by (3.7) and (3.10), $\mathbf{u}$ and $\boldsymbol{\sigma} - \boldsymbol{\sigma}_s$ vanish in Ω_s . Consequently the final governing equations of Oldroyd-B fluid read:

$$\begin{cases} \nabla \cdot \mathbf{u} = 0 & \text{in }]0, T[\times \Omega \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{\mathbf{u}}{K} = -\nabla p + \frac{1}{Re} [(1 - \epsilon) \Delta \mathbf{u} + \frac{\epsilon}{Wi} \nabla \cdot \boldsymbol{\sigma}] & \text{in }]0, T[\times \Omega \\ \partial_t \boldsymbol{\sigma} + (\mathbf{u} \cdot \nabla) \boldsymbol{\sigma} + \frac{\boldsymbol{\sigma} - \boldsymbol{\sigma}_s}{K} + \frac{\boldsymbol{\sigma} - \mathbf{I}}{Wi} = (\nabla \mathbf{u}) \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot (\nabla \mathbf{u})^T + \kappa \Delta \boldsymbol{\sigma} & \text{in }]0, T[\times \Omega \end{cases} \quad (3.13)$$

where $\boldsymbol{\sigma}_s$ is the value of conformation tensor in the solid. The parameter K is set to 10^{16} in the fluid zone Ω_f to recover the genuine equations and to 10^{-7} in the solid zone to enforce $\mathbf{u}$ to vanish and $\boldsymbol{\sigma}$ to be equal to $\boldsymbol{\sigma}_s$, the numerical validation shows this parameter value are reasonable to get the genuine solutions of NS equations [5]. It is also possible to remove the penalization term of the tensor, however our numerical experiments show it is numerically unstable.

For the value of $\boldsymbol{\sigma}_s$, there are two kinds of considerations. The first one considers there is no polymer in the solid, so $\boldsymbol{\sigma}_s$ is set to zero in the obstacles. The second way is to solve the governing equation with the zero velocity conditions in the solid zone and on the solid boundary and then $\boldsymbol{\sigma}_s$ can be approximated by the unit tensor (as shown in subsection 3.2.3). However the simulation results show these two values of $\boldsymbol{\sigma}_s$ give about the same results both for the force and for the flow fields as shown in table 3.1. In the present thesis, we have adopted the first way to study the effect of polymer.

Re	Wi	Cd_a	Cd_b	$\ E\ _a$	$\ E\ _b$	$\ Z\ _a$	$\ Z\ _b$
1.0	1.0	27.81	27.82	4.765	4.765	68.54	68.54
1000	1.0	2.497	2.497	5.094	5.094	525.7	525.8
5000	1.0	2.361	2.329	5.031	5.026	685.4	680.6
2000	2.0	1.778	1.784	4.850	4.856	455.4	458.5
1000	6.0	4.018	4.028	4.622	4.622	433.6	434.5
1.0	10.0	44.75	44.78	4.697	4.697	71.54	71.55

Table 3.1: Drag coefficient, energy and enstrophy for several cases with two kinds of σ_s , wherein the subscript a denotes the result setting $\sigma_s = 0$ and b denotes the result setting $\sigma_s = \mathbf{I}$.

3.1.4 Final equations and Drag coefficient

The final equations can be described as follow:

$$\left\{ \begin{array}{ll} \nabla \cdot \mathbf{u} = 0 & \text{in }]0, T[\times \Omega \\ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{\mathbf{u}}{K} = -\nabla p + \frac{1}{Re} [(1 - \epsilon) \Delta \mathbf{u} + \frac{\epsilon}{Wi} \nabla \cdot \sigma] & \text{in }]0, T[\times \Omega \\ \frac{\partial \sigma}{\partial t} + (\mathbf{u} \cdot \nabla) \sigma + \frac{\sigma}{K} + \frac{\sigma - \mathbf{I}}{Wi} = (\nabla \mathbf{u}) \cdot \sigma + \sigma \cdot (\nabla \mathbf{u})^T + \kappa \Delta \sigma & \text{in }]0, T[\times \Omega \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}(\mathbf{x}), \quad \sigma(\mathbf{x}, 0) = \sigma(\mathbf{x}) & \text{in } \Omega \\ \mathbf{u}(\cdot, t) = 0 \quad \text{and wall boundary conditions for } \sigma & \text{on }]0, T[\times \Gamma_w \\ \mathbf{u}(\cdot, t) = \mathbf{u}_{inlet}(t), \quad \sigma(\cdot, t) = \sigma_{inlet}(t) & \text{on }]0, T[\times \Gamma_i \\ \text{and an artificial condition} & \text{on }]0, T[\times \Gamma_o \end{array} \right. \quad (3.14)$$

The force acting on the obstacles are composed by the pressure difference, viscous surface friction and elastic force of the polymer molecules, which is absent for the Newtonian fluids. The traditional method to calculate the drag force by integrating the traction of these three components along the surface is given by:

$$F_x = \int_{\Sigma_s} \left(-p + \frac{2(1 - \epsilon)}{Re} \frac{\partial u}{\partial x} + \frac{\epsilon \sigma_{xx}}{Re \cdot Wi} \right) n_x \cdot dS + \left(\frac{1 - \epsilon}{Re} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \frac{\epsilon \sigma_{xy}}{Re \cdot Wi} \right) n_y \cdot dS \quad (3.15)$$

where n_x and n_y are, respectively, projections on x or y axis of the unit normal vector of the solid surface; Σ_s denotes the surface of obstacle and dS is the differential scale of the solid surface.

In the same way, the lift force of the obstacle is calculated as follows:

$$F_y = \int_{\Sigma_s} \left(-p + \frac{2(1 - \epsilon)}{Re} \frac{\partial v}{\partial y} + \frac{\epsilon \sigma_{yy}}{Re \cdot Wi} \right) n_y \cdot dS + \left(\frac{1 - \epsilon}{Re} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + \frac{\epsilon \sigma_{xy}}{Re \cdot Wi} \right) n_x \cdot dS \quad (3.16)$$

Here, the penalization method used in this thesis allows us to compute the drag and lift forces by integrating the penalization term inside the solid body. Indeed using the Green's theorem, the force acting on the obstacles read

$$\mathbf{F} = \iint_{\Omega_s} \left(-\nabla p + \frac{1}{Re} [(1 - \epsilon)\Delta \mathbf{u} + \frac{\epsilon}{Wi} \nabla \cdot \boldsymbol{\sigma}] \right) dxdy \quad (3.17)$$

Instead the force on the right hand side of momentum equation in 3.14 by the left hand side, we have

$$\mathbf{F} = \iint_{\Omega_s} \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{\mathbf{u}}{K} \right) dxdy \quad (3.18)$$

Because the unsteady term and convection term are small inside the obstacles, the drag and lift forces can be approximated by

$$F_x \approx \iint_{\Omega_s} \frac{u}{K} dxdy \quad (3.19)$$

$$F_y \approx \iint_{\Omega_s} \frac{v}{K} dxdy \quad (3.20)$$

It should be noted that the equations and the unknowns used above have been non-dimensionalized, therefore the drag and lift force above have no dimensions. They are transformed to compare to other results in the literature. As an illustration, figure 3.2 is the evolutionary drag of a cylinder, it shows that the drags obtained by these two methods are very close.

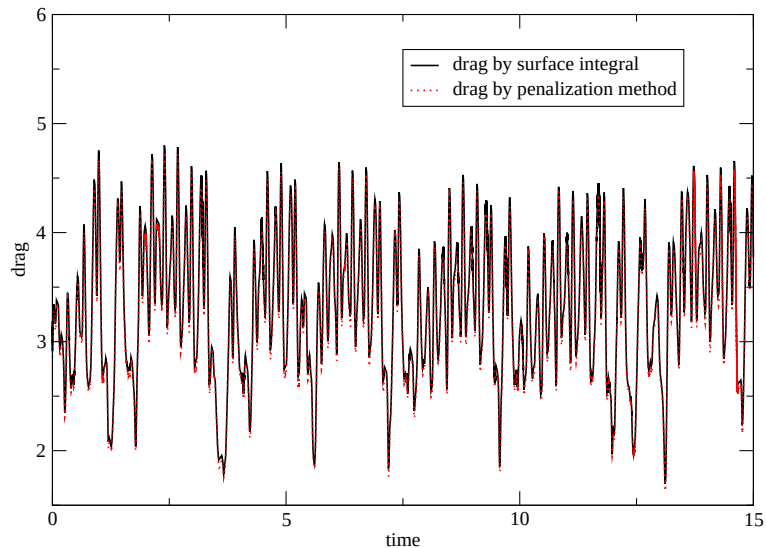


Figure 3.2: Comparison of the drag coefficient given by two different methods: direct surface integral method and penalization method

3.2 Initial datum and boundary conditions

3.2.1 Initial datum

It is necessary to supply an initial condition for starting the calculation even though the results of interest are those that reached a statistical equilibrium. A typical initial datum used in our code is given as follows:

$$\left\{ \begin{array}{lll} u(\mathbf{x}, 0) = 6(1 - y)y & in & \Omega_f \\ u(\mathbf{x}, 0) = 0 & in & \Omega_s \\ v(\mathbf{x}, 0) = 0 & in & \Omega \\ \sigma_{xx}(\mathbf{x}, 0) = 1 & in & \Omega_f \\ \sigma_{xx}(\mathbf{x}, 0) = 0 & in & \Omega_s \\ \sigma_{xy}(\mathbf{x}, 0) = 0 & in & \Omega \\ \sigma_{yy}(\mathbf{x}, 0) = 1 & in & \Omega_f \\ \sigma_{yy}(\mathbf{x}, 0) = 0 & in & \Omega_s \end{array} \right. \quad (3.21)$$

Because we are not interested in the results before having reached a statistical equilibrium, we use coarser grids consuming less cpu time to get an approximate result, then this result is employed to serve as initial datum on the fine grid.

3.2.2 Inlet boundary condition

On the inlet boundary, a two-dimensional planar Poiseuille flow is used for the inlet velocity $\mathbf{u}$. For a steady Poiseuille flow, the Navier-Stokes equations can be simplified. Namely we have the mass conservation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (3.22)$$

and the two momentum equations:

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \quad (3.23)$$

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} \quad (3.24)$$

where the stress is defined by

$$\tau_{xx} = 2\eta \frac{\partial u}{\partial x} \quad (3.25)$$

$$\tau_{xy} = \eta \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \quad (3.26)$$

$$\tau_{yy} = 2\eta \frac{\partial v}{\partial y} \quad (3.27)$$

The no-slip boundary condition on the solid walls reads

$$u = v = 0 \quad \text{on} \quad \Gamma_w \quad (3.28)$$

The Poiseuille flow is steady in the x-direction. So we can write:

$$u = U(y), v = 0, \tau_{xx} = \tau_{yy} = 0, \tau_{xy} = \eta \frac{\partial u}{\partial y}. \quad (3.29)$$

So the two momentum equations become:

$$\frac{\partial p}{\partial x} = \frac{\partial \tau_{xy}}{\partial y} \quad (3.30)$$

$$\frac{\partial p}{\partial y} = \frac{\partial \tau_{xy}}{\partial x} \quad (3.31)$$

As τ_{xy} depends on y , the pressure p can only depend on x . From the equation 3.30, $\partial p / \partial x$ can only depend on y , so it must be a constant A , then we get:

$$p = p_0 - Ax \quad (3.32)$$

$$-A = \frac{\partial p}{\partial x} = \eta \frac{d^2 U(y)}{dy^2} \quad (3.33)$$

From the no-slip boundary condition on the wall, we get the solution

$$U(y) = -\frac{A}{2\eta}(y^2 - Hy) = \frac{AH^2}{2\eta} \left(\frac{y}{H} - \frac{y^2}{H^2} \right) \quad (3.34)$$

It is a parabolic velocity profile with the maximum speed u_m at the centreline $y = H/2$.

Using the flow rate q , we have:

$$\int_0^H u dy = q \quad (3.35)$$

and we get:

$$A = \frac{12\eta q}{H^3} \quad (3.36)$$

So the velocity profile can be written as:

$$u = \frac{6q}{H} \left(\frac{y}{H} - \frac{y^2}{H^2} \right) \quad (3.37)$$

That is the Poiseuille flow we consider for our dilute polymers in solution on the entrance section. Considering $q = 1$ and $H = 1$ after the non-dimensionalization, the velocity boundary condition at the inlet reads

$$u(0, y, t) = 6y(1 - y); \quad v(0, y, t) = 0, \quad \text{on} \quad \Gamma_i \times (0, T) \quad (3.38)$$

Writting the governing equations of Oldroyd-B model in two dimensions:

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\ Re[\frac{\partial u}{\partial t} + (u \cdot \frac{\partial u}{\partial x} + v \cdot \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x}) + \frac{u}{K} - (1 - \epsilon)(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) - \frac{\epsilon}{We}(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y}) = 0 \\ Re[\frac{\partial v}{\partial t} + (u \cdot \frac{\partial v}{\partial x} + v \cdot \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y}) + \frac{v}{K} - (1 - \epsilon)(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) - \frac{\epsilon}{We}(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y}) = 0 \\ \sigma_{xx} - 1 + Wi(\frac{\partial \sigma_{xx}}{\partial t} + u \cdot \frac{\partial \sigma_{xx}}{\partial x} + v \cdot \frac{\partial \sigma_{xx}}{\partial y} + \frac{\sigma_{xx}}{K} - 2\sigma_{xx}\frac{\partial u}{\partial x} - 2\sigma_{xy}\frac{\partial u}{\partial y} - \kappa(\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{xx}}{\partial y^2})) = 0 \\ \sigma_{xy} + Wi(\frac{\partial \sigma_{xy}}{\partial t} + u \cdot \frac{\partial \sigma_{xy}}{\partial x} + v \cdot \frac{\partial \sigma_{xy}}{\partial y} + \frac{\sigma_{xy}}{K} - \sigma_{xx}\frac{\partial v}{\partial x} - \sigma_{yy}\frac{\partial u}{\partial y} - \kappa(\frac{\partial^2 \sigma_{xy}}{\partial x^2} + \frac{\partial^2 \sigma_{xy}}{\partial y^2})) = 0 \\ \sigma_{yy} - 1 + Wi(\frac{\partial \sigma_{yy}}{\partial t} + u \cdot \frac{\partial \sigma_{yy}}{\partial x} + v \cdot \frac{\partial \sigma_{yy}}{\partial y} + \frac{\sigma_{yy}}{K} - 2\sigma_{yy}\frac{\partial v}{\partial y} - 2\sigma_{xy}\frac{\partial v}{\partial x} - \kappa(\frac{\partial^2 \sigma_{yy}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2})) = 0 \end{array} \right. \quad (3.39)$$

and considering the last term is small and was added to ensure the numerical stability, we drop it and remove the terms to zero on the inlet section to get the following equations for the tensor:

$$\left\{ \begin{array}{ll} \sigma_{xx} - 1 + Wi(-2\sigma_{xy}\frac{\partial u}{\partial y}) = 0 & on \ \Gamma_i \times (0, T) \\ \sigma_{xy} + Wi(-\sigma_{yy}\frac{\partial u}{\partial y}) = 0 & on \ \Gamma_i \times (0, T) \\ \sigma_{yy} - 1 = 0 & on \ \Gamma_i \times (0, T) \end{array} \right. \quad (3.40)$$

Thus we can write the boundary condition for the tensor:

$$\left\{ \begin{array}{ll} \sigma_{xx} = 1 + 2Wi(\sigma_{xy}\frac{\partial u}{\partial y}) & on \ \Gamma_i \times (0, T) \\ \sigma_{xy} = Wi\frac{\partial u}{\partial y} & on \ \Gamma_i \times (0, T) \\ \sigma_{yy} = 1 & on \ \Gamma_i \times (0, T) \end{array} \right. \quad (3.41)$$

Putting the above results into the full constitutive equations, including the numerical stability term which we ignore above, we can get the boundary condition:

$$\left\{ \begin{array}{ll} \sigma_{xx} = 1 + 2Wi\sigma_{xy}\frac{\partial u}{\partial y} + Wi \cdot \kappa \frac{\partial^2(1+2Wi\sigma_{xy}\frac{\partial u}{\partial y})}{\partial y^2} & on \ \Gamma_i \times (0, T) \\ \sigma_{xy} = Wi(\sigma_{yy}\frac{\partial u}{\partial y}) + Wi \cdot \kappa \frac{\partial^2(Wi\frac{\partial u}{\partial y})}{\partial y^2} & on \ \Gamma_i \times (0, T) \\ \sigma_{yy} = 1 & on \ \Gamma_i \times (0, T) \end{array} \right. \quad (3.42)$$

Using the velocity profile on the inlet boundary, we can rewrite the boundary condition for both $\mathbf{u}$ and $\boldsymbol{\sigma}$ as:

$$\left\{ \begin{array}{ll} u(., t) = 6(1 - y)y & on \ \Gamma_i \times]0, T[\\ v(., t) = 0 & on \ \Gamma_i \times]0, T[\\ \sigma_{xx}(., t) = 1 + 72Wi^2(1 - 2y)^2 + 288\kappa Wi^3 & on \ \Gamma_i \times]0, T[\\ \sigma_{xy}(., t) = 6Wi(1 - 2y) & on \ \Gamma_i \times]0, T[\\ \sigma_{yy}(., t) = 1 & on \ \Gamma_i \times]0, T[\end{array} \right. \quad (3.43)$$

3.2.3 Wall boundary condition

The no-slip wall boundary condition is used setting $\mathbf{u} = 0$ on the wall. If we put this condition into the constitutive equations, we can get as for the inlet boundary condition, the condition for the tensor at the wall:

$$\begin{cases} \sigma_{xx} = 1 + 2Wi\sigma_{xy}\frac{\partial u}{\partial y} + Wi \cdot \kappa \frac{\partial^2(1+2Wi\sigma_{xy}\frac{\partial u}{\partial y})}{\partial y^2} & \text{on } \Gamma_w \\ \sigma_{xy} = Wi(\sigma_{yy}\frac{\partial u}{\partial y}) + Wi \cdot \kappa \frac{\partial^2(Wi\frac{\partial u}{\partial y})}{\partial y^2} & \text{on } \Gamma_w \\ \sigma_{yy} = 1 & \text{on } \Gamma_w \end{cases} \quad (3.44)$$

However, in contrast with the inlet boundary, the term $\partial u/\partial y$ is not known at the wall. The elastic stress on the wall can be expressed in terms of the velocity gradient at last time step through solving the momentum and the constitutive equations of Oldroyd-B fluid. For an evolution simulation in time, the boundary condition at $t = n\delta t$ is updated by the flow field of $t = (n-1)\delta t$. So the mathematical description for the wall boundary condition is given by:

$$\begin{cases} u(., t) = 0 & \text{on } \Gamma_w \times]0, T[\\ v(., t) = 0 & \text{on } \Gamma_w \times]0, T[\\ \sigma_{xx}(., n\delta t) = [1 + 2Wi\sigma_{xy}\frac{\partial u}{\partial y} + Wi \cdot \kappa \frac{\partial^2(1+2Wi\sigma_{xy}\frac{\partial u}{\partial y})}{\partial y^2}] \big|_{t=(n-1)\delta t} & \text{on } \Gamma_w \times]0, T[\\ \sigma_{xy}(., n\delta t) = [Wi(\sigma_{yy}\frac{\partial u}{\partial y}) + Wi \cdot \kappa \frac{\partial^2(Wi\frac{\partial u}{\partial y})}{\partial y^2}] \big|_{t=(n-1)\delta t} & \text{on } \Gamma_w \times]0, T[\\ \sigma_{yy}(., t) = 1 & \text{on } \Gamma_w \times]0, T[\end{cases}$$

3.2.4 Outlet condition

In many numerical simulation of the flow past a cylinder in a confined channel, a very long channel extending 30 times its height is used to be sure a Poiseuille flow is recovered at the end of the channel. Then the outlet boundary condition can be the same as at the inlet section [45]. To reduce the computational resources, an efficient artificial non-reflecting boundary condition is used at the exit section.

The aim of this section is to derive a properly artificial non-reflecting boundary condition on the outlet boundary. For the full governing equations as in Bruneau and Fabrie [21]:

$$\begin{cases} \nabla \cdot \mathbf{u} = 0 & \text{in }]0, T[\times \Omega \\ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{\mathbf{u}}{K} = -\nabla p + \frac{1}{Re} [(1-\epsilon)\Delta \mathbf{u} + \frac{\epsilon}{Wi} \nabla \cdot \boldsymbol{\sigma}] & \text{in }]0, T[\times \Omega \\ \frac{\partial \boldsymbol{\sigma}}{\partial t} + (\mathbf{u} \cdot \nabla) \boldsymbol{\sigma} + \frac{\boldsymbol{\sigma}}{K} + \frac{\boldsymbol{\sigma} - \mathbf{I}}{Wi} = (\nabla \mathbf{u}) \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot (\nabla \mathbf{u})^T + \kappa \Delta \boldsymbol{\sigma} & \text{in }]0, T[\times \Omega \end{cases} \quad (3.45)$$

Denoting $(\mathbf{u}_0, p_0, \boldsymbol{\sigma}_0)$ the solution of the steady linear problem, we present natural boundary conditions using a weak formulation in velocity-pressure-tensor involving the stress tensor $\delta(\mathbf{u}, p, \boldsymbol{\sigma}) = \frac{1}{Re}[(1 - \epsilon)(\nabla \mathbf{u} + \nabla \mathbf{u}^T) + \epsilon \frac{\boldsymbol{\sigma} - I}{Wi}] - p\mathbf{I}$ that is $\delta(\mathbf{u}, p, \boldsymbol{\sigma}) = \frac{2(1-\epsilon)}{Re}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2 + \frac{\epsilon}{Re} \frac{\boldsymbol{\sigma} - I}{Wi} - p\mathbf{I}$ by setting $\mathbf{w} = \mathbf{u} - \mathbf{u}_0, q = p - p_0, \boldsymbol{\chi} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_0$, the momentum and constitutive equations above can be transferred to:

$$\left\{ \begin{array}{l} \frac{\partial \mathbf{w}}{\partial t} + ((\mathbf{w} + \mathbf{u}_0) \cdot \nabla)(\mathbf{w} + \mathbf{u}_0) + \frac{(\mathbf{w} + \mathbf{u}_0)}{K} + \text{div} \delta(\mathbf{w}, q, \boldsymbol{\chi}) = 0 \quad \text{in }]0, T[\times \Omega \\ \\ \partial_t \boldsymbol{\chi} + (\mathbf{w} + \mathbf{u}_0) \cdot \nabla (\boldsymbol{\sigma}_0 + \boldsymbol{\chi}) - \nabla (\mathbf{w} + \mathbf{u}_0) (\boldsymbol{\sigma}_0 + \boldsymbol{\chi}) - \\ (\boldsymbol{\sigma}_0 + \boldsymbol{\chi}) (\nabla (\mathbf{w} + \mathbf{u}_0))^T + \frac{(\boldsymbol{\sigma}_0 + \boldsymbol{\chi})}{K} + \frac{1}{Wi} ((\boldsymbol{\sigma}_0 + \boldsymbol{\chi}) - I) - \kappa \text{div} \Delta (\boldsymbol{\sigma}_0 + \boldsymbol{\chi}) = 0 \quad \text{in }]0, T[\times \Omega \end{array} \right.$$

We get the weak formulation of the momentum equation using a test function ϕ as follows:

$$\begin{aligned} \int_{\Omega} \partial_t \mathbf{w} \cdot \phi dx + \int_{\Omega} ((\mathbf{w} + \mathbf{u}_0) \cdot \nabla) \mathbf{w} \cdot \phi dx + \int_{\Omega} ((\mathbf{w} + \mathbf{u}_0) \cdot \nabla) \mathbf{u}_0 \cdot \phi dx - \\ \int_{\Omega} \text{div} \delta(\mathbf{w}, q, \boldsymbol{\chi}) \cdot \phi dx + \int_{\Omega} \frac{\mathbf{w} + \mathbf{u}_0}{K} \cdot \phi dx = 0 \end{aligned}$$

Then by the Green and Stokes formulas:

$$\begin{aligned} \int_{\Omega} \partial_t \mathbf{w} \cdot \phi dx + \frac{1}{2} \int_{\Omega} (((\mathbf{w} + \mathbf{u}_0) \cdot \nabla) \mathbf{w} \cdot \phi - ((\mathbf{w} + \mathbf{u}_0) \cdot \nabla) \phi \cdot \mathbf{w}) dx + \\ \frac{1}{2} \int_{\Gamma_o} (\mathbf{w} + \mathbf{u}_0) \cdot \mathbf{n} \mathbf{w} \cdot \phi d\gamma + \int_{\Omega} ((\mathbf{w} + \mathbf{u}_0) \cdot \nabla) \mathbf{u}_0 \cdot \phi dx - \int_{\Omega} \text{div} \delta(\mathbf{w}, q, \boldsymbol{\chi}) : \nabla \phi dx + \\ \int_{\Omega} \frac{\mathbf{w} + \mathbf{u}_0}{K} \cdot \phi dx - \int_{\Gamma_o} \delta(\mathbf{w}, q, \boldsymbol{\chi}) \cdot \mathbf{n} \cdot \phi d\gamma = 0 \end{aligned}$$

Setting the negative part of the first boundary term and the second boundary term in the equation above to zero, we can get the boundary condition as follows:

$$\delta(\mathbf{w}, q, \boldsymbol{\chi}) \cdot \mathbf{n} + \frac{1}{2} ((\mathbf{w} + \mathbf{u}_0) \cdot \mathbf{n})^- \mathbf{w} = 0 \quad (3.46)$$

and using the expression of $\mathbf{u}$ we get:

$$\delta(\mathbf{u}, p, \boldsymbol{\sigma}) \cdot \mathbf{n} + \frac{1}{2} (\mathbf{u} \cdot \mathbf{n})^- (\mathbf{u} - \mathbf{u}_{ref}) = \delta(\mathbf{u}^{ref}, p^{ref}, \boldsymbol{\sigma}^{ref}) \cdot \mathbf{n} \quad (3.47)$$

It can be written as follow:

$$\begin{pmatrix} \left(\frac{2(1-\epsilon)}{Re} \frac{\partial u}{\partial x} + \frac{\epsilon(\sigma_{xx}-1)}{Re \cdot Wi} - p \right) & \frac{1-\epsilon}{Re} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \frac{\epsilon \sigma_{xy}}{Re \cdot Wi} \\ \frac{1-\epsilon}{Re} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \frac{\epsilon \sigma_{xy}}{Re \cdot Wi} & \frac{2(1-\epsilon)}{Re} \frac{\partial v}{\partial y} + \frac{\epsilon(\sigma_{yy}-1)}{Re \cdot Wi} - p \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2} \min(0, u) \begin{pmatrix} u - u^{ref} \\ v - v^{ref} \end{pmatrix} \\ = \begin{pmatrix} \left(\frac{2(1-\epsilon)}{Re} \frac{\partial u^{ref}}{\partial x} + \frac{\epsilon(\sigma_{xx}^{ref}-1)}{Re \cdot Wi} - p^{ref} \right) & \frac{1-\epsilon}{Re} \left(\frac{\partial u^{ref}}{\partial y} + \frac{\partial v^{ref}}{\partial x} \right) + \frac{\epsilon \sigma_{xy}^{ref}}{Re \cdot Wi} \\ \frac{1-\epsilon}{Re} \left(\frac{\partial u^{ref}}{\partial y} + \frac{\partial v^{ref}}{\partial x} \right) + \frac{\epsilon \sigma_{xy}^{ref}}{Re \cdot Wi} & \frac{2(1-\epsilon)}{Re} \frac{\partial v^{ref}}{\partial y} + \frac{\epsilon(\sigma_{yy}^{ref}-1)}{Re \cdot Wi} - p^{ref} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so we get the following equation:

$$\begin{pmatrix} \left(\frac{2(1-\epsilon)}{Re} \frac{\partial u}{\partial x} + \frac{\epsilon(\sigma_{xx}-1)}{Re \cdot Wi} - p \right) \\ \frac{1-\epsilon}{Re} \left(\frac{\partial u^{ref}}{\partial y} + \frac{\partial v^{ref}}{\partial x} \right) + \frac{\epsilon \sigma_{xy}^{ref}}{Re \cdot Wi} \end{pmatrix} - \frac{1}{2} \min(0, u) \begin{pmatrix} u - u^{ref} \\ v - v^{ref} \end{pmatrix} \\ = \begin{pmatrix} \left(\frac{2(1-\epsilon)}{Re} \frac{\partial u^{ref}}{\partial x} + \frac{\epsilon(\sigma_{xx}^{ref}-1)}{Re \cdot Wi} - p^{ref} \right) \\ \frac{1-\epsilon}{Re} \left(\frac{\partial u^{ref}}{\partial y} + \frac{\partial v^{ref}}{\partial x} \right) + \frac{\epsilon \sigma_{xy}^{ref}}{Re \cdot Wi} \end{pmatrix} \quad (3.48)$$

then we have

$$p = p^{ref} + \frac{2(1-\epsilon)}{Re} \frac{\partial u}{\partial x} + \frac{\epsilon \cdot (\sigma_{xx} - 1)}{Re \cdot We} \\ - \frac{\min(0, u)}{2} (u - u^{ref}) - \frac{2(1-\epsilon)}{Re} \frac{\partial u^{ref}}{\partial x} + \frac{\epsilon \cdot (\sigma_{xx}^{ref} - 1)}{Re \cdot Wi}$$

and

$$\frac{1-\epsilon}{Re} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \frac{\epsilon \cdot \sigma_{xy}}{Re \cdot Wi} - \frac{\min(0, u)}{2} \cdot (v - v^{ref}) \\ = \frac{1-\epsilon}{Re} \left(\frac{\partial u^{ref}}{\partial y} + \frac{\partial v^{ref}}{\partial x} \right) + \frac{\epsilon \cdot \sigma_{xy}^{ref}}{Re \cdot Wi}$$

If we take the solution near the outlet boundary as reference flow, so at the boundary of outlet, we have the following equations embedded in our code:

$$p(i+1, j) = p(i, j) + \frac{2(1-\epsilon) \cdot \left(\frac{u(i+2, j) - u(i+1, j)}{dx} - \frac{u(i+1, j) - u(i, j)}{dx} \right)}{Re} \\ + \frac{\epsilon(\sigma_{xx}(i+1, j) - \sigma_{xx}(i, j))}{Re \cdot Wi} - \frac{\min(0, u(i+1, j))}{2} (u(i+1, j) - u(i, j))$$

and

$$\begin{aligned} & \frac{v(i+1, j) - v(i, j)}{dx} + \frac{u(i+1, j) - u(i+1, j-1)}{dy} + \frac{\epsilon \cdot (\sigma_{xy}(i+1, j+1) - \sigma_{xy}(i, j+1))}{Wi \cdot (1 - \epsilon)} \\ &= \frac{Re * \min(0, \frac{(u(i+1, j) + u(i+1, j-1))}{2})}{2(1 - \epsilon)} \frac{v(i+1, j) + v(i-1, j)}{2} \\ & \quad + \frac{v(i, j) - v(i-1, j)}{dx} + \frac{u(i, j) - u(i, j-1)}{dy} \end{aligned}$$

In the same way , we get formally the weak formulation for the tensor:

$$\begin{aligned} & \int_{\Omega} \partial_t \boldsymbol{\chi} \cdot \phi dx + \int_{\Omega} ((\mathbf{w} + \mathbf{u}_0) \cdot \nabla) \boldsymbol{\chi} \cdot \phi dx + \int_{\Omega} ((\mathbf{w} + \mathbf{u}_0) \cdot \nabla) \boldsymbol{\sigma}_0 \cdot \phi dx \\ & - \int_{\Omega} ((\boldsymbol{\chi} + \boldsymbol{\sigma}_0) \cdot \nabla (\mathbf{w} + \mathbf{u}_0)) \cdot \phi dx - \int_{\Omega} (\nabla (\mathbf{w} + \mathbf{u}_0)^T \cdot (\boldsymbol{\chi} + \boldsymbol{\sigma}_0)) \cdot \phi dx + \int_{\Omega} \frac{1}{Wi} (\boldsymbol{\chi} - \mathbf{I}) \cdot \phi dx \\ & - \int_{\Omega} \kappa \operatorname{div} \cdot \nabla \boldsymbol{\chi} \cdot \phi dx + \int_{\Omega} \frac{\boldsymbol{\chi} + \boldsymbol{\sigma}_0}{K} \cdot \phi dx \end{aligned}$$

Then by Green and Stokes formulas:

$$\begin{aligned} & \int_{\Omega} \partial_t \boldsymbol{\chi} \cdot \phi dx + \frac{1}{2} \int_{\Omega} [((\mathbf{w} + \mathbf{u}_0) \cdot \nabla) \boldsymbol{\chi} \cdot \phi - ((\mathbf{w} + \mathbf{u}_0) \cdot \nabla) \phi \cdot \boldsymbol{\chi}] dx \\ & + \frac{1}{2} \int_{\Gamma_0} ((\mathbf{w} + \mathbf{u}_0) \cdot \mathbf{n}) \boldsymbol{\chi} \cdot \phi d\gamma + \frac{1}{2} \int_{\Omega} [((\mathbf{w} + \mathbf{u}_0) \cdot \nabla) \boldsymbol{\sigma}_0 \cdot \phi - ((\mathbf{w} + \mathbf{u}_0) \cdot \nabla) \phi \cdot \boldsymbol{\sigma}_0] dx \\ & + \frac{1}{2} \int_{\Gamma_0} ((\mathbf{w} + \mathbf{u}_0) \cdot \mathbf{n}) \boldsymbol{\sigma}_0 \cdot \phi d\gamma - \int_{\Omega} ((\boldsymbol{\chi} + \boldsymbol{\sigma}_0) \cdot \nabla (\mathbf{w} + \mathbf{u}_0)) \cdot \phi dx \\ & - \int_{\Omega} (\nabla (\mathbf{w} + \mathbf{u}_0)^T \cdot (\boldsymbol{\chi} + \boldsymbol{\sigma}_0)) \cdot \phi dx + \int_{\Omega} \frac{1}{Wi} (\boldsymbol{\chi} - \mathbf{I}) \cdot \phi dx + \int_{\Omega} \kappa \nabla \boldsymbol{\chi} : \nabla \phi dx \\ & - \int_{\Gamma_0} \nabla \boldsymbol{\chi} \cdot \mathbf{n} \phi d\gamma + \int_{\Omega} \frac{\boldsymbol{\chi} + \boldsymbol{\sigma}_0}{K} \cdot \phi dx = 0 \end{aligned}$$

If we do the same work as with the momentum equation, we can get a similar results as follows:

$$\frac{1}{2} [(\mathbf{w} + \mathbf{u}_0) \cdot \mathbf{n}]^- \boldsymbol{\chi} + \frac{1}{2} [(\mathbf{w} + \mathbf{u}_0) \cdot \mathbf{n}]^- \boldsymbol{\sigma}_0 + \kappa \cdot \nabla \boldsymbol{\chi} \cdot \mathbf{n} = 0 \quad (3.49)$$

So we can get the following boundary condition:

$$\kappa \partial_{\mathbf{n}} \boldsymbol{\sigma} + \frac{1}{2} (\mathbf{u} \cdot \mathbf{n})^- (\boldsymbol{\sigma} - \boldsymbol{\sigma}_{ref}) = \kappa \partial_{\mathbf{n}} \boldsymbol{\sigma}_{ref} \quad (3.50)$$

Hence the artificial outlet boundary condition in our code can be written as:

$$\left\{ \begin{array}{ll} \delta(\mathbf{u}, p, \boldsymbol{\sigma}) \cdot \mathbf{n} + \frac{1}{2}(\mathbf{u} \cdot \mathbf{n})^-(\mathbf{u} - \mathbf{u}^{ref}) = \delta(\mathbf{u}^{ref}, p^{ref}, \boldsymbol{\sigma}^{ref}) \cdot \mathbf{n} & \text{on } \Gamma_o \times]0, T[\\ \kappa \partial_{\mathbf{n}} \boldsymbol{\sigma} + \frac{1}{2}(\mathbf{u} \cdot \mathbf{n})^-(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{ref}) = \kappa \partial_{\mathbf{n}} \boldsymbol{\sigma}^{ref} & \text{on } \Gamma_o \times]0, T[\end{array} \right. \quad (3.51)$$

The final problem can be written as follow:

$$\left\{ \begin{array}{ll} \nabla \cdot \mathbf{u} = 0 & \text{in }]0, T[\times \Omega \\ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{\mathbf{u}}{K} = -\nabla p + \frac{1}{Re} [(1 - \epsilon) \Delta \mathbf{u} + \frac{\epsilon}{Wi} \nabla \cdot \boldsymbol{\sigma}] & \text{in }]0, T[\times \Omega \\ \frac{\partial \boldsymbol{\sigma}}{\partial t} + (\mathbf{u} \cdot \nabla) \boldsymbol{\sigma} + \frac{\boldsymbol{\sigma}}{K} + \frac{\boldsymbol{\sigma} - \mathbf{I}}{Wi} = (\nabla \mathbf{u}) \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot (\nabla \mathbf{u})^T + \kappa \Delta \boldsymbol{\sigma} & \text{in }]0, T[\times \Omega \\ u(., t) = 6(1 - y)y & \text{on } \Gamma_i \times]0, T[\\ v(., t) = 0 & \text{on } \Gamma_i \times]0, T[\\ \sigma_{xx}(., t) = 1 + 72Wi^2(1 - 2y)^2 + 288\kappa Wi^3 & \text{on } \Gamma_i \times]0, T[\\ \sigma_{xy}(., t) = 6Wi(1 - 2y) & \text{on } \Gamma_i \times]0, T[\\ \sigma_{yy}(., t) = 1 & \text{on } \Gamma_i \times]0, T[\\ u(., t) = 0 & \text{on } \Gamma_w \times]0, T[\\ v(., t) = 0 & \text{on } \Gamma_w \times]0, T[\\ \sigma_{xx}(., n\delta t) = [1 + 2Wi\sigma_{xy} \frac{\partial u}{\partial y} + Wi \cdot \kappa \frac{\partial^2(1 + 2Wi\sigma_{xy} \frac{\partial u}{\partial y})}{\partial y^2}] \big|_{t=(n-1)\delta t} & \text{on } \Gamma_w \times]0, T[\\ \sigma_{xy}(., n\delta t) = [Wi(\sigma_{yy} \frac{\partial u}{\partial y}) + Wi \cdot \kappa \frac{\partial^2(Wi \frac{\partial u}{\partial y})}{\partial y^2}] \big|_{t=(n-1)\delta t} & \text{on } \Gamma_w \times]0, T[\\ \sigma_{yy}(., t) = 1 & \text{on } \Gamma_w \times]0, T[\\ \delta(\mathbf{u}, p, \boldsymbol{\sigma}) \cdot \mathbf{n} + \frac{1}{2}(\mathbf{u} \cdot \mathbf{n})^-(\mathbf{u} - \mathbf{u}^{ref}) = \delta(\mathbf{u}^{ref}, p^{ref}, \boldsymbol{\sigma}^{ref}) \cdot \mathbf{n} & \text{on } \Gamma_o \times]0, T[\\ \kappa \partial_{\mathbf{n}} \boldsymbol{\sigma} + \frac{1}{2}(\mathbf{u} \cdot \mathbf{n})^-(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{ref}) = \kappa \partial_{\mathbf{n}} \boldsymbol{\sigma}^{ref} & \text{on } \Gamma_o \times]0, T[\\ \text{and an initial field} & \text{in } \Omega \end{array} \right.$$

3.3 Discretization

For the problem of the previous section, the approximation is performed by means of finite differences for the velocity, pressure and for the elastic stress tensor. In this method, the spatial domain is discretized with an uniform Cartesian mesh and the time interval is split into uniform time steps δt .

3.3.1 Time discretization

A second-order Gear scheme is used to discretize the unsteady term in our code. The use of a second-order scheme is crucial as the first-order Euler scheme for instance does not describe properly the evolution of the solution unless a very low CFL number is

used. The first-order scheme can be used for the first time steps to get very fast a coarse initial field. The linear terms are treated implicitly, while on the contrary the convection terms are treated explicitly. We denote $\mathbf{u}^n$ for instance the approximation of $\mathbf{u}$ at time $t_n = n\delta t$, δt is the time step. The semi-discretized second-order Gear scheme governing equations read:

$$\left\{ \begin{array}{l} \nabla \cdot \mathbf{u}^n = 0 \\ \frac{3\mathbf{u}^n}{2\delta t} - \frac{(1-\epsilon)}{4\mathbf{u}^{n-1} - \mathbf{u}^{n-2}} \Delta \mathbf{u}^n + \frac{\mathbf{u}^n}{K} + \nabla p^n = \\ \frac{2\delta t}{4\mathbf{u}^{n-1} - \mathbf{u}^{n-2}} - 2(\mathbf{u}^{n-1} \cdot \nabla) \mathbf{u}^{n-1} + (\mathbf{u}^{n-2} \cdot \nabla) \mathbf{u}^{n-2} + \frac{\epsilon}{ReWi} (2\nabla \cdot \boldsymbol{\sigma}^{n-1} - \nabla \cdot \boldsymbol{\sigma}^{n-2}) \\ \frac{3\boldsymbol{\sigma}^n}{2\delta t} + \frac{\boldsymbol{\sigma}^n - \mathbf{I}}{Wi} + \frac{\boldsymbol{\sigma}^n}{K} - \kappa \Delta \boldsymbol{\sigma}^n = \frac{4\boldsymbol{\sigma}^{n-1} - \boldsymbol{\sigma}^{n-2}}{2\delta t} - 2(\mathbf{u}^{n-1} \cdot \nabla) \boldsymbol{\sigma}^{n-1} + (\mathbf{u}^{n-2} \cdot \nabla) \boldsymbol{\sigma}^{n-2} + \\ 2\nabla \mathbf{u}^{n-1} \boldsymbol{\sigma}^{n-1} + 2\boldsymbol{\sigma}^{n-1} (\nabla \mathbf{u}^T)^{n-1} - \nabla \mathbf{u}^{n-2} \boldsymbol{\sigma}^{n-2} - \boldsymbol{\sigma}^{n-2} (\nabla \mathbf{u}^T)^{n-2} \end{array} \right. \quad (3.52)$$

3.3.2 Space discretization

The system of equations is discretized in space by finite differences on an uniform staggered grid G_h the mesh size of which is $h = \delta x = \delta y$, where δx and δy are the step discretizations in each direction. The discrete values of the pressure p and the diagonal elements of the conformation tensor σ_{xx} , σ_{yy} are located at the center of each cell, the discrete values of the other element σ_{xy} are located at the corner of the cell and those of the velocity field are located at the middle of the sides as shown on figure 3.3.

Our governing equations are composed of the first derivative term $\partial\phi/\partial x$ and the second derivative term $\partial^2\phi/\partial x^2$ of a generic variable ϕ . The Taylor series expansion is a useful approach to approximate the derivatives. The first derivative term, the pressure gradient term $\partial p/\partial x$ as an example, is approximated by the central difference scheme (CDS) at the point $(i - \frac{1}{2}, j)$ by

$$\left(\frac{\partial p}{\partial x}\right)_{i-\frac{1}{2},j} \approx \frac{p_{i+1,j} - p_{i,j}}{\delta x}. \quad (3.53)$$

In the same way, the second derivative term for instance $\frac{\partial^2 u}{\partial y^2}$ is approximated by the central difference scheme (CDS) at the point $(i - \frac{1}{2}, j)$ by

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{i-\frac{1}{2},j} \approx \frac{u_{i-\frac{1}{2},j+1} - 2u_{i-\frac{1}{2},j} + u_{i-\frac{1}{2},j-1}}{(\delta y)^2}. \quad (3.54)$$

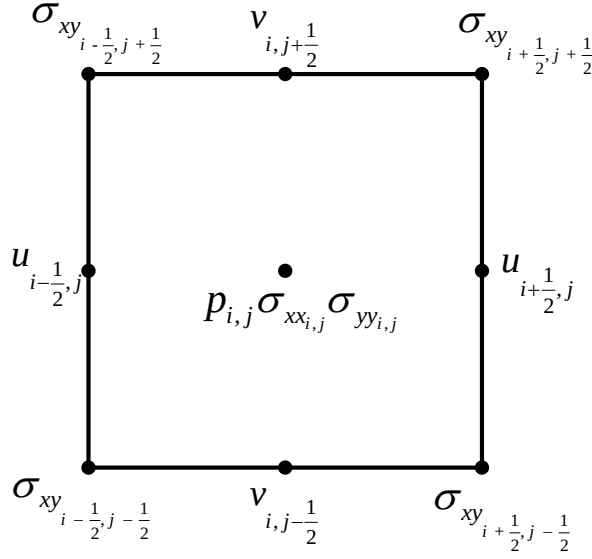


Figure 3.3: A staggered cell

It is noted that the discretization of this term at the boundary yields modified formulas due to the use of staggered grids. For example, on the bottom wall we have

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{i-\frac{1}{2},1} \approx \frac{\frac{8}{3}u_{i-\frac{1}{2},\frac{1}{2}} - 4u_{i-\frac{1}{2},1} + \frac{4}{3}u_{i-\frac{1}{2},2}}{(\delta y)^2} \quad (3.55)$$

to better take into account the no-slip boundary condition. The convection terms are approximated by a third-order Murman-Like scheme. For instance $-u\partial_x u - v\partial_y u$ is approximated at point $(i - \frac{1}{2}, j)$ by

$$\begin{aligned} & -\frac{u_{i,j}}{3} \frac{\Delta_{i,j}u}{\delta x} - \frac{5u_{i-1,j}}{6} \frac{\Delta_{i-1,j}u}{\delta x} + \frac{u_{i-2,j}}{6} \frac{\Delta_{i-2,j}u}{\delta x} & \text{if } u_{i-1,j} > 0 \\ & -\frac{u_{i-1,j}}{3} \frac{\Delta_{i-1,j}u}{\delta x} - \frac{5u_{i,j}}{6} \frac{\Delta_{i,j}u}{\delta x} + \frac{u_{i+1,j}}{6} \frac{\Delta_{i+1,j}u}{\delta x} & \text{if } u_{i,j} < 0 \\ & -\frac{v_{i-\frac{1}{2},j+\frac{1}{2}}}{3} \frac{\Delta_{i-\frac{1}{2},j+\frac{1}{2}}u}{\delta y} - \frac{5v_{i-\frac{1}{2},j-\frac{1}{2}}}{6} \frac{\Delta_{i-\frac{1}{2},j-\frac{1}{2}}u}{\delta y} + \frac{v_{i-\frac{1}{2},j-\frac{3}{2}}}{6} \frac{\Delta_{i-\frac{3}{2},j-\frac{3}{2}}u}{\delta y} & \text{if } v_{i-\frac{1}{2},j-\frac{1}{2}} > 0 \\ & -\frac{v_{i-\frac{1}{2},j-\frac{1}{2}}}{3} \frac{\Delta_{i-\frac{1}{2},j-\frac{1}{2}}u}{\delta y} - \frac{5v_{i-\frac{1}{2},j+\frac{1}{2}}}{6} \frac{\Delta_{i-\frac{1}{2},j+\frac{1}{2}}u}{\delta y} + \frac{v_{i-\frac{1}{2},j+\frac{3}{2}}}{6} \frac{\Delta_{i-\frac{3}{2},j+\frac{3}{2}}u}{\delta y} & \text{if } v_{i-\frac{1}{2},j+\frac{1}{2}} < 0 \end{aligned}$$

where $u_{i,j} = \frac{1}{2}(u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j})$ and $v_{i-\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2}(v_{i,j+\frac{1}{2}} + v_{i-1,j+\frac{1}{2}})$ are obtained by linear interpolation, $\Delta_{i,j}u = (u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j})$ and $\Delta_{i-\frac{1}{2},j+\frac{1}{2}}u = (u_{i-\frac{1}{2},j+1} - u_{i-\frac{1}{2},j})$.

The corresponding term for the vertical component of the velocity v is discretized in

the same way. The construction of this scheme refers to [27] where a comparison of various third order schemes is presented. The detail discretizations of the governing equations are written as follow:

The mass conservation equation at the pressure point (i, j) is discretized by:

$$\frac{u_{i+\frac{1}{2},j}^n - u_{i-\frac{1}{2},j}^n}{\delta x} + \frac{v_{i,j+\frac{1}{2}}^n - v_{i,j-\frac{1}{2}}^n}{\delta y} = 0 \quad (3.56)$$

The first component of momentum equation is approximated at point $(i - \frac{1}{2}, j)$ by:

$$\begin{aligned} & \frac{3u_{i-\frac{1}{2},j}^n}{2\delta t} - \frac{(1-\epsilon)}{Re} \left(\frac{u_{i+\frac{1}{2},j}^n - 2u_{i-\frac{1}{2},j}^n + u_{i-\frac{3}{2},j}^n}{\delta x^2} + \frac{u_{i-\frac{1}{2},j+1}^n - 2u_{i-\frac{1}{2},j}^n + u_{i-\frac{1}{2},j-1}^n}{\delta y^2} \right) + \frac{p_{i,j}^n - p_{i-1,j}^n}{\delta x} + \frac{u_{i-\frac{1}{2},j}^n}{K} = \\ & \frac{2u_{i-\frac{1}{2},j}^{n-1}}{\delta t} - \frac{u_{i-\frac{1}{2},j}^{n-2}}{2\delta t} - 2 \cdot ct^{n-1} + ct^{n-2} + 2 \frac{\epsilon}{Wi \cdot Re} \left(\frac{\sigma_{xx,i,j}^{n-1} - \sigma_{xx,i-1,j}^{n-1}}{\delta x} + \frac{\sigma_{xy,i-\frac{1}{2},j+\frac{1}{2}}^{n-1} - \sigma_{xy,i-\frac{1}{2},j-\frac{1}{2}}^{n-1}}{\delta y} \right) - \\ & \frac{\epsilon}{Wi \cdot Re} \left(\frac{\sigma_{xx,i,j}^{n-2} - \sigma_{xx,i-1,j}^{n-2}}{\delta x} + \frac{\sigma_{xy,i-\frac{1}{2},j+\frac{1}{2}}^{n-2} - \sigma_{xy,i-\frac{1}{2},j-\frac{1}{2}}^{n-2}}{\delta y} \right) \end{aligned} \quad (3.57)$$

Where ct denotes the convection term. At any time step, we can approximate it at point $(i - \frac{1}{2}, j)$ by:

$$\begin{aligned} ct_{i-\frac{1}{2},j} &= \frac{u_{i,j}}{3} \frac{\Delta_{i,j}u}{\delta x} + \frac{5u_{i-1,j}}{6} \frac{\Delta_{i-1,j}u}{\delta x} - \frac{u_{i-2,j}}{6} \frac{\Delta_{i-2,j}u}{\delta x} && \text{when } u_{i-1,j} > 0 \\ &+ \frac{u_{i-1,j}}{3} \frac{\Delta_{i-1,j}u}{\delta x} + \frac{5u_{i,j}}{6} \frac{\Delta_{i,j}u}{\delta x} - \frac{u_{i+1,j}}{6} \frac{\Delta_{i+1,j}u}{\delta x} && \text{when } u_{i,j} < 0 \\ &+ \frac{v_{i-\frac{1}{2},j+\frac{1}{2}}}{3} \frac{\Delta_{i-\frac{1}{2},j+\frac{1}{2}}u}{\delta y} + \frac{5v_{i-\frac{1}{2},j-\frac{1}{2}}}{6} \frac{\Delta_{i-\frac{1}{2},j-\frac{1}{2}}u}{\delta y} - \frac{v_{i-\frac{1}{2},j-\frac{3}{2}}}{6} \frac{\Delta_{i-\frac{1}{2},j-\frac{3}{2}}u}{\delta y} && \text{when } v_{i-\frac{1}{2},j-\frac{1}{2}} > 0 \\ &+ \frac{v_{i-\frac{1}{2},j-\frac{1}{2}}}{3} \frac{\Delta_{i-\frac{1}{2},j-\frac{1}{2}}u}{\delta y} + \frac{5v_{i-\frac{1}{2},j+\frac{1}{2}}}{6} \frac{\Delta_{i-\frac{1}{2},j+\frac{1}{2}}u}{\delta y} - \frac{v_{i-\frac{1}{2},j+\frac{3}{2}}}{6} \frac{\Delta_{i-\frac{1}{2},j+\frac{3}{2}}u}{\delta y} && \text{when } v_{i-\frac{1}{2},j+\frac{1}{2}} < 0 \end{aligned} \quad (3.58)$$

In the same way, the second component of momentum equation at $(i, j - \frac{1}{2})$ can be approximated by:

$$\begin{aligned} & \frac{3v_{i,j-\frac{1}{2}}^n}{2\delta t} - \frac{(1-\epsilon)}{Re} \left(\frac{v_{i+\frac{1}{2},j-\frac{1}{2}}^n - 2v_{i,j-\frac{1}{2}}^n + v_{i-\frac{1}{2},j-\frac{1}{2}}^n}{\delta x^2} + \frac{v_{i,j+\frac{1}{2}}^n - 2v_{i,j-\frac{1}{2}}^n + v_{i,j-\frac{3}{2}}^n}{\delta y^2} \right) + \frac{p_{i,j}^n - p_{i,j-1}^n}{\delta y} + \frac{v_{i,j-\frac{1}{2}}^n}{K} = \\ & \frac{2v_{i,j-\frac{1}{2}}^{n-1}}{\delta t} - \frac{v_{i,j-\frac{1}{2}}^{n-2}}{2\delta t} - 2 \cdot ct^{n-1} + ct^{n-2} + 2 \frac{\epsilon}{Wi \cdot Re} \left(\frac{\sigma_{yy,i,j}^{n-1} - \sigma_{yy,i,j-1}^{n-1}}{\delta y} + \frac{\sigma_{xy,i+\frac{1}{2},j-\frac{1}{2}}^{n-1} - \sigma_{xy,i-\frac{1}{2},j-\frac{1}{2}}^{n-1}}{\delta x} \right) - \\ & - \frac{\epsilon}{Wi \cdot Re} \left(\frac{\sigma_{yy,i,j}^{n-2} - \sigma_{yy,i,j-1}^{n-2}}{\delta y} + \frac{\sigma_{xy,i+\frac{1}{2},j-\frac{1}{2}}^{n-2} - \sigma_{xy,i-\frac{1}{2},j-\frac{1}{2}}^{n-2}}{\delta x} \right) \end{aligned} \quad (3.59)$$

we can approximate ct at point $(i, j - \frac{1}{2})$ by:

$$\begin{aligned}
 ct_{i,j-\frac{1}{2}} &= \frac{u_{i-\frac{1}{2},j-\frac{1}{2}}}{3} \frac{\Delta_{i-\frac{1}{2},j+\frac{1}{2}} v}{\delta x} + \frac{5u_{i-\frac{1}{2},j-\frac{1}{2}}}{6} \frac{\Delta_{i-\frac{1}{2},j-\frac{1}{2}} v}{\delta x} - \frac{u_{i-\frac{3}{2},j-\frac{1}{2}}}{6} \frac{\Delta_{i-\frac{3}{2},j-\frac{1}{2}} v}{\delta x} \quad \text{when } u_{i-\frac{1}{2},j-\frac{1}{2}} > 0 \\
 &+ \frac{u_{i-\frac{1}{2},j-\frac{1}{2}}}{3} \frac{\Delta_{i-\frac{1}{2},j-\frac{1}{2}} v}{\delta x} + \frac{5u_{i+\frac{1}{2},j-\frac{1}{2}}}{6} \frac{\Delta_{i+\frac{1}{2},j-\frac{1}{2}} v}{\delta x} - \frac{u_{i+\frac{3}{2},j-\frac{1}{2}}}{6} \frac{\Delta_{i+\frac{3}{2},j-\frac{1}{2}} v}{\delta x} \quad \text{when } u_{i+\frac{1}{2},j-\frac{1}{2}} < 0 \\
 &+ \frac{v_{i,j}}{3} \frac{\Delta_{i,j} v}{\delta y} + \frac{5v_{i,j-1}}{6} \frac{\Delta_{i,j-1} v}{\delta y} - \frac{v_{i,j-2}}{6} \frac{\Delta_{i,j-2} v}{\delta y} \quad \text{when } v_{i,j-1} > 0 \\
 &+ \frac{v_{i-1,j}}{3} \frac{\Delta_{i-1,j} v}{\delta y} + \frac{5v_{i,j}}{6} \frac{\Delta_{i,j} v}{\delta y} - \frac{v_{i+1,j}}{6} \frac{\Delta_{i+1,j} v}{\delta y} \quad \text{when } v_{i,j} < 0
 \end{aligned} \tag{3.60}$$

Where $v_{i,j} = \frac{1}{2}(v_{i,j+\frac{1}{2}} + v_{i,j-\frac{1}{2}})$ and $u_{i+\frac{1}{2},j-\frac{1}{2}} = \frac{1}{2}(u_{i+\frac{1}{2},j} + u_{i+\frac{1}{2},j-1})$ are obtained by linear interpolation, $\Delta_{i,j} v = v_{i,j+\frac{1}{2}} - v_{i,j-\frac{1}{2}}$ and $\Delta_{i+\frac{1}{2},j-\frac{1}{2}} v = v_{i+1,j-\frac{1}{2}} - v_{i,j-\frac{1}{2}}$.

As seen in figure 3.3, we put the conformation tensor components σ_{xx} and σ_{yy} at the center of the cell, and σ_{xy} at the corner of the cell, then we can discretize the first component of tensor at point (i, j) as:

$$\begin{aligned}
 &\frac{3\sigma_{xx,i,j}^n}{2\delta t} + \frac{1}{W_i}(\sigma_{xx,i,j}^n - 1) + \frac{\sigma_{xx,i,j}^n}{K} - \kappa \left(\frac{\sigma_{xx,i+1,j}^n - 2\sigma_{xx,i,j}^n + \sigma_{xx,i-1,j}^n}{\delta x^2} + \frac{\sigma_{xx,i,j+1}^n - 2\sigma_{xx,i,j}^n + \sigma_{xx,i,j-1}^n}{\delta y^2} \right) = \frac{2\sigma_{xx,i,j}^{n-1}}{\delta t} \\
 &- \frac{\sigma_{xx,i,j}^{n-2}}{2\delta t} - 2tt^{n-1} + tt^{n-2} + 4(\sigma_{xx,i,j}^{n-1} \left(\frac{u_{i+\frac{1}{2},j}^{n-1} - u_{i-\frac{1}{2},j}^{n-1}}{\delta x} \right) + \frac{\sigma_{xy,i+\frac{1}{2},j+\frac{1}{2}}^{n-1} + \sigma_{xy,i+\frac{1}{2},j-\frac{1}{2}}^{n-1} + \sigma_{xy,i-\frac{1}{2},j-\frac{1}{2}}^{n-1} + \sigma_{xy,i-\frac{1}{2},j+\frac{1}{2}}^{n-1}}{4} \\
 &\quad \frac{(u_{i+\frac{1}{2},j+1}^{n-1} + u_{i-\frac{1}{2},j+1}^{n-1}) - (u_{i+\frac{1}{2},j-1}^{n-1} + u_{i-\frac{1}{2},j-1}^{n-1})}{4\delta y}) - 2(\sigma_{xx,i,j}^{n-2} \left(\frac{u_{i+\frac{1}{2},j}^{n-2} - u_{i-\frac{1}{2},j}^{n-2}}{\delta x} \right) + \\
 &\quad \frac{\sigma_{xy,i+\frac{1}{2},j+\frac{1}{2}}^{n-2} + \sigma_{xy,i+\frac{1}{2},j-\frac{1}{2}}^{n-2} + \sigma_{xy,i-\frac{1}{2},j-\frac{1}{2}}^{n-2} + \sigma_{xy,i-\frac{1}{2},j+\frac{1}{2}}^{n-2}}{4} \cdot \frac{(u_{i+\frac{1}{2},j+1}^{n-2} + u_{i-\frac{1}{2},j+1}^{n-2}) - (u_{i+\frac{1}{2},j-1}^{n-2} + u_{i-\frac{1}{2},j-1}^{n-2})}{4\delta y})
 \end{aligned} \tag{3.61}$$

where tt denotes the transport term. Similarly, the transport term of the tensor $u \cdot \partial_x \sigma_{xx} + v \cdot \partial_y \sigma_{xx}$ can be approximated at point (i, j) by:

$$\begin{aligned}
 &\frac{u_{i+\frac{1}{2},j}}{3} \frac{\Delta_{i+\frac{1}{2},j} \sigma_{xx}}{\delta x} + \frac{5u_{i-\frac{1}{2},j}}{6} \frac{\Delta_{i-\frac{1}{2},j} \sigma_{xx}}{\delta x} - \frac{u_{i+\frac{3}{2},j}}{6} \frac{\Delta_{i+\frac{3}{2},j} \sigma_{xx}}{\delta x} + \quad \text{when } u_{i-\frac{1}{2},j} > 0 \\
 &\frac{u_{i-\frac{1}{2},j}}{3} \frac{\Delta_{i-\frac{1}{2},j} \sigma_{xx}}{\delta x} + \frac{5u_{i+\frac{1}{2},j}}{6} \frac{\Delta_{i+\frac{1}{2},j} \sigma_{xx}}{\delta x} - \frac{u_{i+\frac{3}{2},j}}{6} \frac{\Delta_{i+\frac{3}{2},j} \sigma_{xx}}{\delta x} + \quad \text{when } u_{i+\frac{1}{2},j} < 0 \\
 &\frac{v_{i,j+\frac{1}{2}}}{3} \frac{\Delta_{i,j+\frac{1}{2}} \sigma_{xx}}{\delta y} + \frac{5v_{i,j-\frac{1}{2}}}{6} \frac{\Delta_{i,j-\frac{1}{2}} \sigma_{xx}}{\delta y} - \frac{v_{i,j-\frac{3}{2}}}{6} \frac{\Delta_{i,j-\frac{3}{2}} \sigma_{xx}}{\delta y} + \quad \text{when } v_{i,j-\frac{1}{2}} > 0 \\
 &\frac{v_{i,j-\frac{1}{2}}}{3} \frac{\Delta_{i,j-\frac{1}{2}} \sigma_{xx}}{\delta y} + \frac{5v_{i,j+\frac{1}{2}}}{6} \frac{\Delta_{i,j+\frac{1}{2}} \sigma_{xx}}{\delta y} - \frac{v_{i,j+\frac{3}{2}}}{6} \frac{\Delta_{i,j+\frac{3}{2}} \sigma_{xx}}{\delta y} \quad \text{when } v_{i,j+\frac{1}{2}} < 0
 \end{aligned} \tag{3.62}$$

where $\Delta_{i+\frac{1}{2},j}\sigma_{xx} = (\sigma_{xx_{i+1,j}} - \sigma_{xx_{i,j}})$ and $\Delta_{i,j+\frac{1}{2}}\sigma_{xx} = (\sigma_{xx_{i,j+1}} - \sigma_{xx_{i,j}})$.

The second component of the tensor at point $(i - \frac{1}{2}, j - \frac{1}{2})$ is approximated as:

$$\begin{aligned}
 & \frac{3\sigma_{xy_{i-\frac{1}{2},j-\frac{1}{2}}}^n}{2\delta t} + \frac{1}{We}\sigma_{xy_{i-\frac{1}{2},j-\frac{1}{2}}}^n + \frac{\sigma_{xy_{i-\frac{1}{2},j-\frac{1}{2}}}^n}{K} - \kappa \left(\frac{\sigma_{xy_{i-\frac{3}{2},j-\frac{1}{2}}}^n - 2\sigma_{xy_{i-\frac{1}{2},j-\frac{1}{2}}}^n + \sigma_{xy_{i+\frac{1}{2},j-\frac{1}{2}}}^n}{\delta x^2} + \right. \\
 & \left. \frac{\sigma_{xy_{i-\frac{1}{2},j-\frac{3}{2}}}^n - 2\sigma_{xy_{i-\frac{1}{2},j-\frac{1}{2}}}^n + \sigma_{xy_{i-\frac{1}{2},j+\frac{1}{2}}}^n}{\delta y^2} \right) = \frac{2\sigma_{xy_{i-\frac{1}{2},j-\frac{1}{2}}}^{n-1}}{\delta t} - \frac{\sigma_{xy_{i-\frac{1}{2},j-\frac{1}{2}}}^{n-2}}{2\delta t} - 2tt^{n-1} + tt^{n-2} + \\
 & \frac{\sigma_{xx_{i,j}}^{n-1} + \sigma_{xx_{i-1,j}}^{n-1} + \sigma_{xx_{i,j-1}}^{n-1} + \sigma_{xx_{i-1,j-1}}^{n-1}}{2} \cdot \frac{v_{i,j-\frac{1}{2}}^{n-1} - v_{i-1,j-\frac{1}{2}}^{n-1}}{\delta x} + \\
 & \frac{\sigma_{yy_{i,j}}^{n-1} + \sigma_{yy_{i-1,j}}^{n-1} + \sigma_{yy_{i,j-1}}^{n-1} + \sigma_{yy_{i-1,j-1}}^{n-1}}{2} \cdot \frac{v_{i-\frac{1}{2},j}^{n-1} - v_{i-\frac{1}{2},j-1}^{n-1}}{\delta y} - \\
 & \frac{\sigma_{xx_{i,j}}^{n-2} + \sigma_{xx_{i-1,j}}^{n-2} + \sigma_{xx_{i,j-1}}^{n-2} + \sigma_{xx_{i-1,j-1}}^{n-2}}{4} \cdot \frac{v_{i,j-\frac{1}{2}}^{n-2} - v_{i-1,j-\frac{1}{2}}^{n-2}}{\delta x} - \\
 & \frac{\sigma_{yy_{i,j}}^{n-2} + \sigma_{yy_{i-1,j}}^{n-2} + \sigma_{yy_{i,j-1}}^{n-2} + \sigma_{yy_{i-1,j-1}}^{n-2}}{4} \cdot \frac{v_{i-\frac{1}{2},j}^{n-2} - v_{i-\frac{1}{2},j-1}^{n-2}}{\delta y}
 \end{aligned} \tag{3.63}$$

Similarly, the transport terms of tensor $u \cdot \partial_x \sigma_{xy} + v \cdot \partial_y \sigma_{xy}$ can be approximated at point $(i - \frac{1}{2}, j - \frac{1}{2})$ by:

$$\begin{aligned}
 & \frac{u_{i,j-\frac{1}{2}}}{3} \frac{\Delta_{i,j-\frac{1}{2}}\sigma_{xy}}{\delta x} + \frac{5u_{i-1,j-\frac{1}{2}}}{6} \frac{\Delta_{i-1,j-\frac{1}{2}}\sigma_{xy}}{\delta x} - \frac{u_{i-2,j-\frac{1}{2}}}{6} \frac{\Delta_{i-2,j-\frac{1}{2}}\sigma_{xy}}{\delta x} + \quad \text{when } u_{i-1,j-\frac{1}{2}} > 0 \\
 & \frac{u_{i-1,j-\frac{1}{2}}}{3} \frac{\Delta_{i-1,j-\frac{1}{2}}\sigma_{xy}}{\delta x} + \frac{5u_{i,j-\frac{1}{2}}}{6} \frac{\Delta_{i,j-\frac{1}{2}}\sigma_{xy}}{\delta x} - \frac{u_{i+1,j-\frac{1}{2}}}{6} \frac{\Delta_{i+1,j-\frac{1}{2}}\sigma_{xy}}{\delta x} + \quad \text{when } u_{i,j-\frac{1}{2}} < 0 \\
 & \frac{v_{i-\frac{1}{2},j}}{3} \frac{\Delta_{i-\frac{1}{2},j}\sigma_{xy}}{\delta y} + \frac{5v_{i-\frac{1}{2},j-1}}{6} \frac{\Delta_{i-\frac{1}{2},j-1}\sigma_{xy}}{\delta y} - \frac{v_{i-\frac{1}{2},j-2}}{6} \frac{\Delta_{i-\frac{1}{2},j-2}\sigma_{xy}}{\delta y} + \quad \text{when } v_{i-\frac{1}{2},j-1} > 0 \\
 & \frac{v_{i-\frac{1}{2},j-1}}{3} \frac{\Delta_{i-\frac{1}{2},j-1}\sigma_{xy}}{\delta y} + \frac{5v_{i-\frac{1}{2},j}}{6} \frac{\Delta_{i-\frac{1}{2},j}\sigma_{xy}}{\delta y} - \frac{v_{i-\frac{1}{2},j+1}}{6} \frac{\Delta_{i-\frac{1}{2},j+1}\sigma_{xy}}{\delta y} \quad \text{when } v_{i-\frac{1}{2},j} < 0
 \end{aligned} \tag{3.64}$$

Where $u_{i,j-\frac{1}{2}} = \frac{u_{i-\frac{1}{2},j} + u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j-1} + u_{i+\frac{1}{2},j-1}}{4}$ and $v_{i,j-\frac{1}{2}} = \frac{v_{i-1,j+\frac{1}{2}} + v_{i-1,j+\frac{1}{2}} + v_{i,j-\frac{1}{2}} + v_{i,j-\frac{1}{2}}}{4}$ are obtained by linear interpolation, $\Delta_{i,j-\frac{1}{2}}\sigma_{xy} = \sigma_{xy_{i+\frac{1}{2},j-\frac{1}{2}}} - \sigma_{xy_{i-\frac{1}{2},j-\frac{1}{2}}}$, $\Delta_{i-\frac{1}{2},j}\sigma_{xy} = \sigma_{xy_{i-\frac{1}{2},j+\frac{1}{2}}} - \sigma_{xy_{i-\frac{1}{2},j-\frac{1}{2}}}$.

For the last component, we also can approximate the equation at point (i, j) by:

$$\begin{aligned}
 & \frac{3\sigma_{yyi,j}^n}{2\delta t} + \frac{1}{We}(\sigma_{yyi,j}^n - 1) + \frac{\sigma_{yyi,j}^n}{K} - \kappa \left(\frac{\sigma_{yyi+1,j}^n - 2\sigma_{yyi,j}^n + \sigma_{yyi-1,j}^n}{\delta x^2} + \frac{\sigma_{yyi,j+1}^n - 2\sigma_{yyi,j}^n + \sigma_{yyi,j-1}^n}{\delta y^2} \right) = \frac{2\sigma_{yyi,j}^{n-1}}{\delta t} \\
 & - \frac{\sigma_{yyi,j}^{n-2}}{2\delta t} - 2tt^{n-1} + tt^{n-2} + 4(\sigma_{yyi,j}^{n-1} \left(\frac{v_{i,j+\frac{1}{2}}^{n-1} - v_{i,j-\frac{1}{2}}^{n-1}}{\delta y} \right) + \frac{\sigma_{xyi+\frac{1}{2},j+\frac{1}{2}}^{n-1} + \sigma_{xyi+\frac{1}{2},j-\frac{1}{2}}^{n-1} + \sigma_{xyi-\frac{1}{2},j-\frac{1}{2}}^{n-1} + \sigma_{xyi-\frac{1}{2},j+\frac{1}{2}}^{n-1}}{4} \\
 & \frac{(v_{i+1,j+\frac{1}{2}}^{n-1} + v_{i+1,j-\frac{1}{2}}^{n-1}) - (v_{i-1,j+\frac{1}{2}}^{n-1} + v_{i-1,j-\frac{1}{2}}^{n-1})}{4\delta x}) - 2(\sigma_{yyi,j}^{n-2} \left(\frac{v_{i,j+\frac{1}{2}}^{n-2} - v_{i,j-\frac{1}{2}}^{n-2}}{\delta y} \right) + \\
 & \frac{\sigma_{xyi+\frac{1}{2},j+\frac{1}{2}}^{n-2} + \sigma_{xyi+\frac{1}{2},j-\frac{1}{2}}^{n-2} + \sigma_{xyi-\frac{1}{2},j-\frac{1}{2}}^{n-2} + \sigma_{xyi-\frac{1}{2},j+\frac{1}{2}}^{n-2}}{4} \cdot \frac{(v_{i+1,j+\frac{1}{2}}^{n-2} + v_{i+1,j-\frac{1}{2}}^{n-2}) - (v_{i-1,j+\frac{1}{2}}^{n-2} + v_{i-1,j-\frac{1}{2}}^{n-2})}{4\delta x})
 \end{aligned} \tag{3.65}$$

Similarly, the transport terms of tensor $u \cdot \partial_x \sigma_{yy} + v \cdot \partial_y \sigma_{yy}$ can be approximated at point (i, j) by:

$$\begin{aligned}
 & \frac{u_{i+\frac{1}{2},j}}{3} \frac{\Delta_{i+\frac{1}{2},j} \sigma_{yy}}{\delta x} + \frac{5u_{i-\frac{1}{2},j}}{6} \frac{\Delta_{i-\frac{1}{2},j} \sigma_{yy}}{\delta x} - \frac{u_{i+\frac{3}{2},j}}{6} \frac{\Delta_{i+\frac{3}{2},j} \sigma_{yy}}{\delta x} + \quad \text{when } u_{i-\frac{1}{2},j} > 0 \\
 & \frac{u_{i-\frac{1}{2},j}}{3} \frac{\Delta_{i-\frac{1}{2},j} \sigma_{yy}}{\delta x} + \frac{5u_{i+\frac{1}{2},j}}{6} \frac{\Delta_{i+\frac{1}{2},j} \sigma_{yy}}{\delta x} - \frac{u_{i+\frac{3}{2},j}}{6} \frac{\Delta_{i+\frac{3}{2},j} \sigma_{yy}}{\delta x} + \quad \text{when } u_{i+\frac{1}{2},j} < 0 \\
 & \frac{v_{i,j+\frac{1}{2}}}{3} \frac{\Delta_{i,j+\frac{1}{2}} \sigma_{yy}}{\delta y} + \frac{5v_{i,j-\frac{1}{2}}}{6} \frac{\Delta_{i,j-\frac{1}{2}} \sigma_{yy}}{\delta y} - \frac{v_{i,j-\frac{3}{2}}}{6} \frac{\Delta_{i,j-\frac{3}{2}} \sigma_{yy}}{\delta y} + \quad \text{when } v_{i,j-\frac{1}{2}} > 0 \\
 & \frac{v_{i,j-\frac{1}{2}}}{3} \frac{\Delta_{i,j-\frac{1}{2}} \sigma_{yy}}{\delta y} + \frac{5v_{i,j+\frac{1}{2}}}{6} \frac{\Delta_{i,j+\frac{1}{2}} \sigma_{yy}}{\delta y} - \frac{v_{i,j+\frac{3}{2}}}{6} \frac{\Delta_{i,j+\frac{3}{2}} \sigma_{yy}}{\delta y} \quad \text{when } v_{i,j+\frac{1}{2}} < 0
 \end{aligned} \tag{3.66}$$

where $\Delta_{i+\frac{1}{2},j} \sigma_{yy} = (\sigma_{yyi+1,j} - \sigma_{yyi,j})$ and $\Delta_{i,j+\frac{1}{2}} \sigma_{yy} = (\sigma_{yyi,j+1} - \sigma_{yyi,j})$.

3.4 Multigrid solver

The discretization of the coupled velocity-pressure-tensor system of equations yields to solve a discrete linear system $L_h V_h^n = B_h^{n-1}$ where L_h represents the discrete operator, B_h^{n-1} is the discrete equivalent of the right hand side of (3.52) and $V_h^n = (\mathbf{u}_h^n, p_h^{n+1/2}, \boldsymbol{\sigma}_h^n)$ is the approximate solution to compute. We have used a time splitting to first solve the coupling pressure and velocity equations then the constitutive equations. We solve this discrete linear system by the FAS (Full Approximation Scheme) multigrid solver on a sequence of grids from the fine grid chosen for the approximation G_p to a coarsest grid G_1 as coarse as 16×4 cells. The multigrid algorithm uses a V-cycle procedure

illustrated below for the computation of the fine grid solution $V_h^n = V_p^n$ on grid G_p .

$$\left\{ \begin{array}{l} \text{For } k=1 \text{ to number_of_V-cycles do} \\ \tilde{V}_p^k = S_p^{(\nu_1)}(L_p, V_p^{k-1}, B_p^{n-1}) \\ \text{Correction on coarse grids} \\ \left\{ \begin{array}{l} \text{For } q=p-1, 1, -1 \text{ do} \\ \bar{V}_q^k = R_q^{q+1} \tilde{V}_{q+1}^k \\ B_q^k = R_q^{q+1}(B_{q+1}^k - L_{q+1} \tilde{V}_{q+1}^k) + L_q \bar{V}_q^k \\ \tilde{V}_q^k = S_q^{(\nu_1)}(L_q, \bar{V}_q^k, B_q^k) \end{array} \right. \\ \text{Updating of the fine grids} \\ \left\{ \begin{array}{l} \text{For } q=2, p \text{ do} \\ \hat{V}_q^k = \tilde{V}_q^k + P_{q-1}^q(V_{q-1}^k - \bar{V}_{q-1}^k) \\ V_q^k = S_q^{(\nu_2)}(L_q, \hat{V}_q^k, B_q^k) \end{array} \right. \\ \text{Convergence test} \\ \text{if } \|B_p^{n-1} - L_p V_p^k\| \leq \xi \text{ stop iterations} \end{array} \right.$$

In this algorithm $S_q^{(\nu)}$ denotes the smoother used on grid G_q to compute an approximate solution of the linear system doing ν iterations. The restriction R_q^{q+1} and prolongation P_{q-1}^q operators describe the linear interpolation operators from fine-to-coarse and from coarse-to-fine grids respectively.

For the Newtonian fluid, the constitutive equations of the tensor is absent. The smoothing operator S performs ν_1 or ν_2 iterations of a Gauss-Seidel cell-by-cell procedure that leads to solve a 5×5 linear system corresponding to the 5 unknowns of a cell.

$$\begin{pmatrix} \alpha & 0 & 0 & 0 & 1/\delta x \\ 0 & \alpha & 0 & 0 & -1/\delta x \\ 0 & 0 & \alpha & 0 & 1/\delta y \\ 0 & 0 & 0 & \alpha & -1/\delta y \\ -1/\delta x & 1/\delta x & -1/\delta y & 1/\delta y & 0 \end{pmatrix} \begin{pmatrix} u_{i-\frac{1}{2},j}^n \\ u_{i+\frac{1}{2},j}^n \\ v_{i,j-\frac{1}{2}}^n \\ v_{i,j+\frac{1}{2}}^n \\ p_{i,j}^n \end{pmatrix} = \begin{pmatrix} (C_h^{m-1})_{i-\frac{1}{2},j} \\ (C_h^{m-1})_{i+\frac{1}{2},j} \\ (C_h^{m-1})_{i,j-\frac{1}{2}} \\ (C_h^{m-1})_{i,j+\frac{1}{2}} \\ 0 \end{pmatrix}$$

In this system $\alpha = \frac{1}{\delta t} + \frac{2}{Re}(\frac{1}{\delta x} + \frac{1}{\delta y})$ and the other quantities of the linear operator are relaxed in the second member. So $(C_h^{m-1})_{i-\frac{1}{2},j}$ represents the sum of these relaxed terms and $(B_h^{n-1})_{i-\frac{1}{2},j}$. We solve this 5×5 coupled system directly by Cramer method. It is noted that the fifth equation ensures the free divergence constraint in each cell.

For the viscoelastic fluids, we still solve this 5×5 coupled system directly by Cramer method to get the velocity and the pressure firstly. Then we solve the conformation tensor components one by one directly by the linear equations of $L_h V_h^n = B_h^{n-1}$.

3.5 Numerical validation

3.5.1 Comparison for confined cylinder in a channel

The flow past a cylinder is becoming more and more frequent to study computational rheology and used as a numerical benchmark problem since the proposal by Brown and McKinley [19]. Numerous works on this problem have emerged [6, 33, 31, 45, 52, 69, 71, 70, 93, 101, 103, 111, 109, 138] using different viscoelastic models and computational methods. The main interest is the drag coefficient with the different Weissenberg number under creeping flow condition ($Re = 0$). In this benchmark flow, the viscosity ratio of the polymer ϵ is 0.41, the blockage ratio, i.e. the ratio of the cylinder diameter to the channel width, is $B = D/H = 0.5$.

Most of the authors adopt a very long channel, for instance L/D are set to 15 [71], 18 [31], 40 [109], 14 [45] respectively. Here, we use a shorter channel with $L/D = 8$. The lengths upstream and downstream of the cylinder are chosen to be $2D$ and $6D$, this is the trial purpose for our artificial boundary condition. For the steady creeping flow, the advective inertial forces are small compared to viscous forces. Hence Stokes equations, which ignore the inertial forces term of the Navier-Stokes equations, are widely adopted in the references [31, 71, 111]. However, the high Reynolds number (Re) flows are our interest. Therefore we use the complete momentum equation to simulate the benchmark question at $Re = 0.5$ even though a lot of cpu resources are wasted and the convective term may lead to some numerical instabilities. The Reynolds number and Weissenberg number for this flow are defined respectively by

$$Re = \frac{UD}{\eta}, \quad (3.67)$$

$$Wi = \frac{\tau U}{R}, \quad (3.68)$$

where U is the average velocity, D and R are the diameter and the radius of the cylinder, η and τ denote the kinematic shear viscosity and relaxation time of the polymer solution.

Author	Methods	hp	N_c	Kd
Huang et al. [70]	FVM	0.069		10.063
	FVM	0.035		10.318
	FVM	0.021		10.431
	FVM	0.00873		10.454
	FVM	0.00714		10.455
Dou et al. [45]	FVM	0.157	40	10.670
	FVM	0.105	60	10.570
	FVM	0.074	84	10.530
	FVM	0.040	156	10.490
Oliveira et al. [109]	FVM	0.0314	230	10.539
	FVM	0.0207	304	10.783
	FVM	0.0157	475	10.783
Sun et al. [138]	FEM	0.056		10.532
Mitsoulis [103]	FEM		62	10.538
	FEM		98	10.529
Fan et al. [52]	FEM			10.533
Liu et al. [93]	FEM	0.0136		10.532
Caola et al. [31]	FEM	0.002		10.528
Owens et al. [111]	SEM			10.532
Hulsen et al. [71]	FEM	0.0148		10.533
Present simulations	FDM	0.03125		10.213
	FDM	0.02		10.354
	FDM	0.01		10.429
	FDM	0.008		10.462
	FDM	0.0078125		10.529

FVM, finite volume method; FEM, finite element method; SEM, spectral element method; FDM, finite difference method; hp, minimum cell spacing (normalized by R); N_c , boundary nodes on the cylinder.

Table 3.2: Drag coefficient Kd for the Newtonian fluid with $B = 0.5$ at Stokes regime.

Because the inertial force is very small for the Stokes flow, the drag coefficient at this section is defined as

$$Kd = \frac{Fx}{2\pi\eta UD} \quad (3.69)$$

where Fx is the drag force acting on the cylinder. It is used as an index to judge whether we have obtained a steady state flow.

Wi	This work	Ref. [31]	Ref. [71]	Ref. [45]1	Ref. [45]2	Ref. [52]	Ref. [111]
0.1	0.98459		0.98493	0.98410	0.98416	0.98489	
0.2	0.95734		0.95670	0.95773	0.95900	0.95663	
0.3	0.93272		0.93076	0.93433	0.93706	0.93072	
0.4	0.91474		0.91114	0.91812	0.92224	0.91108	
0.5	0.90376	0.89711	0.89784	0.91057	0.91477	0.89778	0.89778
0.6	0.89967		0.88995	0.90952	0.91407	0.88985	0.88983
0.7	0.90209		0.88654	0.91499	0.91896	0.88637	0.88617
0.8	0.90322		0.88678	0.92631	0.92991	0.88667	0.88576
0.9	0.91130		0.88991	0.94230	0.94438	0.89000	0.88777
1.0	0.92001	0.88971	0.89531		0.96263	0.89521	0.89175

Ref. [45]1 and 2 represent the drag data obtained with the DEVSS- ω formulation and DAVSS- ω formulation respectively.

Table 3.3: Drag coefficient variation for the Oldroyd-B fluid with $B = 0.5$ at Stokes regime (normalized by the drag of the Newtonian fluid).

The drag coefficients of the cylinder for the Newtonian fluid are listed in Table 3.2 in order to compare with that of the literature. At a similar level of discretization, the FDM results seem to be below the FVM results but are consistent with the results of FEM. Let us point out that with the FDM the geometry of the cylinder is different from the other methods. Indeed, with a penalization method the cylinder is approximated by a cloud of points without a representation of its boundary. On the other way the cylinder is polygonal approximation for the other methods, which give a different cylinder geometry. Nevertheless, the results are in good agreement with the literature.

The computed drag coefficients for the Oldroyd-B fluid have been made dimensionless as for the Newtonian fluid and compared with published data in Table 3.3. All the data in the table give a similar trend: with the increase of the Weissenberg number, the drag coefficients first decrease and then rebound. The drag coefficients are similar for small Weissenberg number, but some discrepancies appear for higher ones. The maximum drag reduction for this kind of flow happens at $Wi = 0.6$ or $Wi = 0.7$ with

about 10% drag reduction. There is no available experimental data for comparison in two-dimensions, this will be an important topic of the next chapter.

3.5.2 Comparison with three-dimensional experimental results

Verhelst and Nieuwstadt [155] have performed experimental measurements of velocity and drag for the flow past a circular cylinder in a rectangular channel with a width of 20mm and a height of 160mm. They mounted a cylinder with a diameter of 10mm in the symmetry plane of this channel. This configuration gives an aspect ratio (AR), the ratio of the channel height to the width, $AR = 8$ and a blockage ratio $B = 0.5$. The section planes in the spanwise give the similar configuration of the benchmark problem. They have measured the velocity profiles of the stokes flow on the symmetry section plane in height. Concretely, five different locations in the streamwise X are $(X - X_c)/R = -21, -3, -1.5, 1.5$ and 3 , where X_c is the position of the cylinder center in the streamwise coordinate and R is the radius of the cylinder.

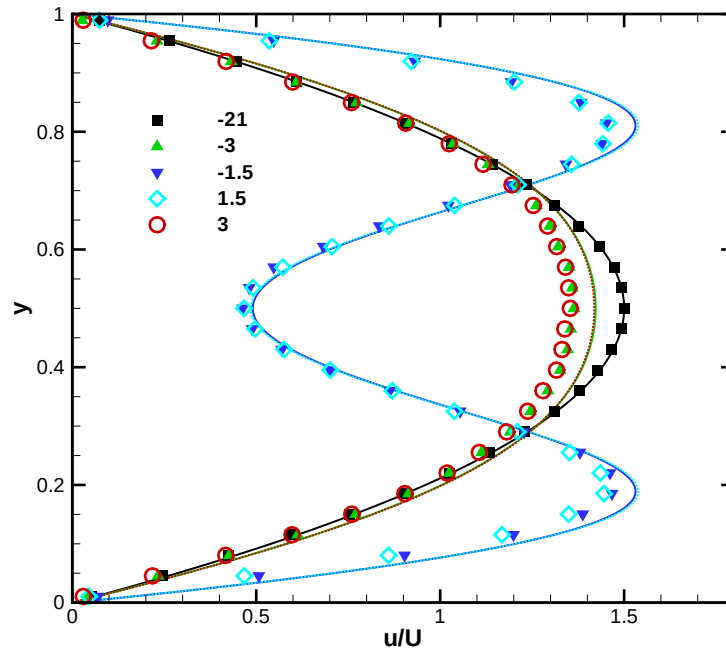


Figure 3.4: Comparison of predicted velocity profiles for Newtonian fluid flow with measurements by Verhelst and Nieuwstadt [155]; the numerical results have the curve superposition at ± 1.5 and ± 3 .

Figure 3.4 shows the comparison of the velocity profiles between the two dimensional simulation and the three dimensional experiment for the Newtonian glucose/water

solution at $(X - X_c)/R = -21, -3, -1.5$ (upstream of the cylinder) and $(X - X_c)/R = 1.5, 3$ (downstream of the cylinder). For the creeping flow, the fore-aft symmetry is well captured by both the experimental and numerical results (coincidence for $(X - X_c)/R = \pm 1.5, \pm 3$). The velocity profile at the inlet part $(X - X_c)/R = -21$ are completely consistent for the numerical and experimental results, but the profiles near the cylinder are different. The numerical results are always larger than the experimental results. It means that the three-dimensional effects make the fluids near the cylinder flow from the symmetric plane to both the lateral walls. Let's point out the secondary flow caused by the viscosity of the fluid and the wall boundary is strong and can not be ignored even though there isn't three-dimensional effect of turbulence for the creeping flow. The stagnation pressure at the symmetric plane has the largest value, larger than the value near the spanwise walls. The pressure difference make the fluids flow toward the spanwise walls. Because the three dimensional effects, the drag reports in the experiment is much different with the two dimensional data, so we will not compare them here.

Figure 3.5 shows the comparison of predicted velocity profiles for Oldroyd-B fluid at $Wi = 1.813$ with experimental measurements by Verhelst and Nieuwstadt [155] for a polyacrylamide solution at a flow rate of 0.052l/s. The secondary flow effects still make the numerical velocity larger than the experimental results especially for the upstream flow. The fore-aft symmetry present in the Newtonian flow is lost for the polymer solutions and Oldroyd-B fluid in our simulation. The downstream velocity profile is modified dramatically. There are a local velocity maximum on the centreline at $(X - X_c)/R = 1.5$ and a local velocity minimum on the centreline at $(X - X_c)/R = 3$ for the velocity profiles behind the cylinder. The qualitative modifications induced by elasticity are well represented by the predictions of the present two-dimensional simulations, even though we have not accounted for the slight shear thinning in the experiments.

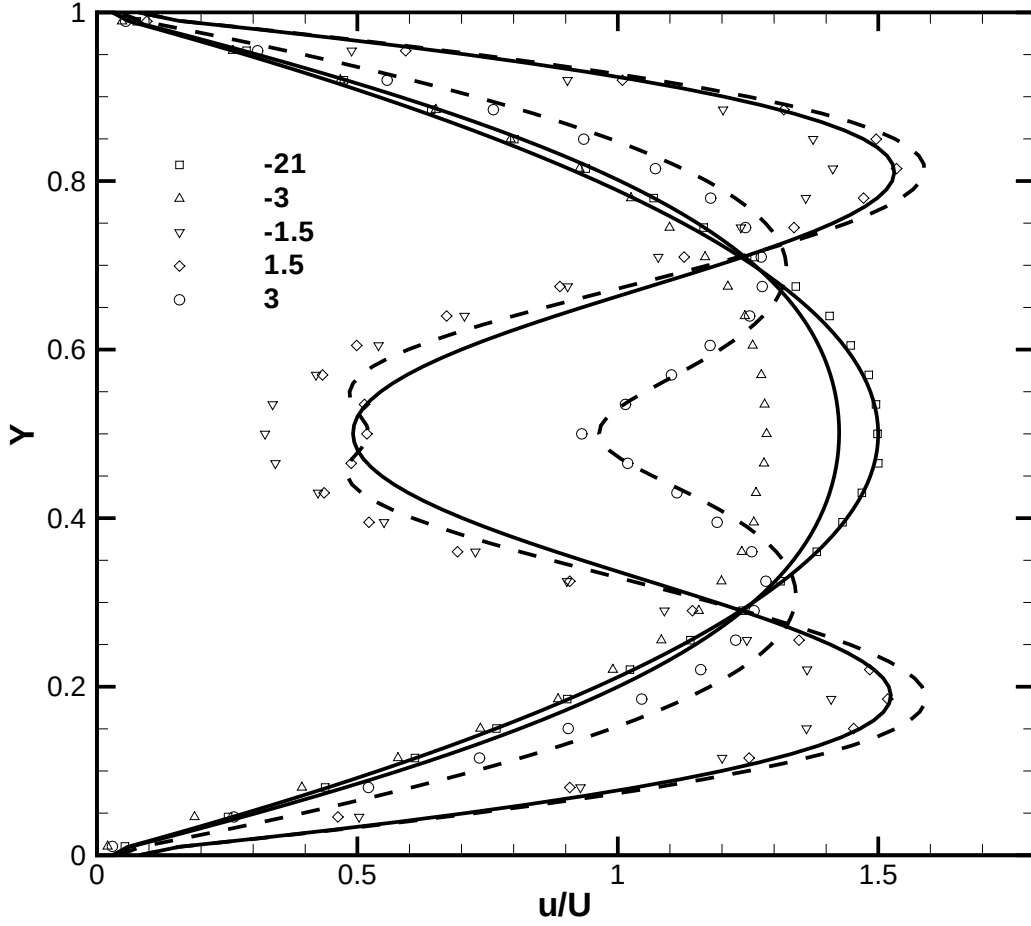


Figure 3.5: Comparison of predicted velocity profiles for two-dimensional numerical simulation of Oldroyd-B fluid at $Wi = 1.813$ with three-dimensional experimental measurements by Verhelst and Nieuwstadt [155] for a polyacrylamide solution at a flow rate of 0.052l/s.(the solid lines show the numerical results upstream of the cylinder, the dashed lines represent the downstream results)

Chapter 4

Flows past a circular cylinder

In this chapter we now study a standard problem to understand flows of complex fluids and notably those giving rise to drag reduction for the flow around a cylinder in two dimensions [101]. Let us recall that such a situation is widely studied, experimentally, for low Reynolds numbers where important drag enhancement has been observed and documented for different fluids [32, 55, 155]. Simulations have also addressed the issue and observations of such enhancement [93, 138]. A complication with some of the experiments is the presence of shear thinning effects which may influence the drag. Experiments with Boger fluids [73, 55] minimize such effects. For high Reynolds numbers, few simulations have been carried out and there is no report on the drag behavior affected by fluid elasticity neither experimentally nor numerically. An important question is how does the drag on a cylinder vary in the parameter space described by the two dimensionless numbers namely the Weissenberg number and the Reynolds number. We here study the simple case of the flow around a cylinder of a viscoelastic fluid in a two dimensional setting using direct numerical simulations to cover a wide range in both the Reynolds number Re and the Weissenberg number Wi which measures the ability of the fluid to change properties with respect to external solicitations. This is the most comprehensive study up to date to the author's knowledge. We measure the drag on the cylinder and show that this simple flow situation presents a nontrivial transition from drag enhancement to drag reduction. A full phase diagram in the parameter space $Re - Wi$ extending for 5 decades in Reynolds number is presented showing the extent of the different regimes. Attention is then brought to the stresses, the pressure, the elongation, and the shear distributions in the flow domain.

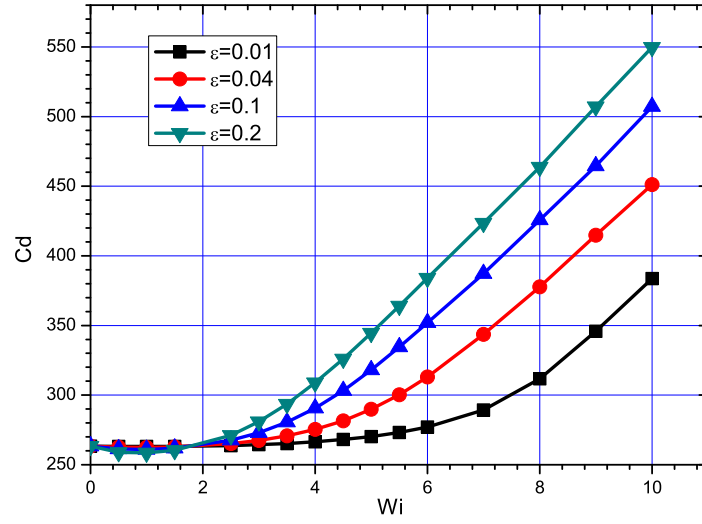
4.1 The effects of polymer viscosity

The character of the diluted polymer solution can be described by its concentration and the characteristic of the polymer molecules. In our governing equations, the polymer solutions are featured by the polymer viscosity ratio and the characteristic relax time. Even though they have not the same conception, the concentration of the polymer solution can effect both the viscosity and the relaxation time, the characteristics of polymer molecules affect both too; but the two parameter spaces give the main features of the polymer solutions. Hence the drag force map of the cylinder should be plotted in the three dimensional parameter spaces: the Reynolds number Re , the Weissenberg number Wi and the polymer viscosity ratio ϵ .

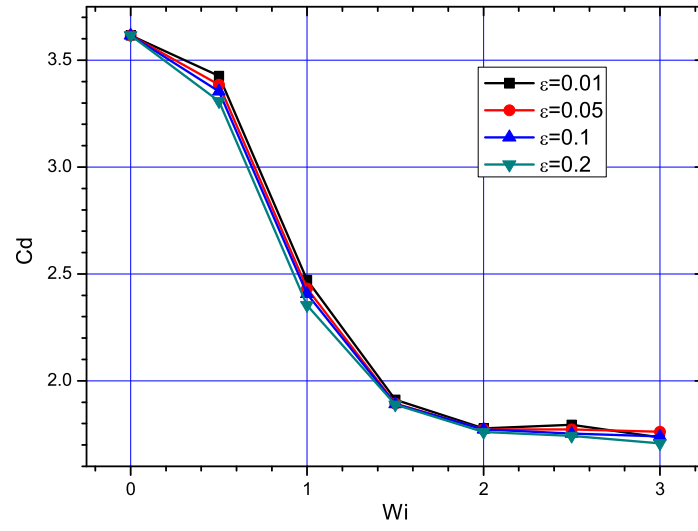
It is difficult to study drag force of viscoelastic fluids in this three dimensions parameter space by direct numerical simulations which need a lot of computational time. However we have not found any report about the effect of these parameters. In this section we study the effect of the polymer viscosity ratio on the drag force of the cylinder by the selected parameter positions. Taking four values of ϵ for diluted polymer solutions, we will discover their performances for two Reynolds numbers $Re = 0.1, 2000$ with different Weissenberg numbers and at a given Weissenberg number $Wi = 2.0$ with different Reynolds numbers. As before, the definition of Reynolds number and Weissenberg number are given by $Re = UD/\eta$ and $Wi = \tau U/R$ respectively, where U is the average velocity in the channel, D and R are the diameter and radius of the cylinder, η and τ are the viscosity and the relax time of the polymer solutions. For the drag coefficient Cd , we use an usual definition: $Cd = F_x/0.5\rho U^2 D$, where F_x is the drag force and ρ is the density of the polymer solution.

The only difference of the computational geometry to that of the previous chapter is the blockage ratio, we adopt $B = 0.1$ in this chapter. Huang and Feng [69] had reported the effects of the blockage ratio on the drag force. In their numerical investigations, there isn't drag reduction at a small blockage ratio $B < 0.1$, however the trend is completely reversed for flow with stronger blockages. Hence we hope to separate the effect on drag of the channel wall by using a smaller blockage ratio $B = 0.1$. The simulations are performed on a 2048×512 Cartesian mesh by a multigrid method in our finite difference code.

Figure 4.1 shows the drag coefficient of the cylinder for four different values of ϵ versus the Weissenberg number at $Re = 0.1$ and $Re = 2000$. For $Re = 0.1$, the variation of the drag force is very small for low Weissenberg numbers, and the drag force has a minimum value at $Wi = 1.0$ by a careful observation. The drag enhances above $Wi = 2.0$. The drag are divergent at the large Weissenberg numbers for different



(a) $Re=0.1$



(b) $Re=2000$

Figure 4.1: The effects of polymer viscosity ratio on drag coefficient of the cylinder

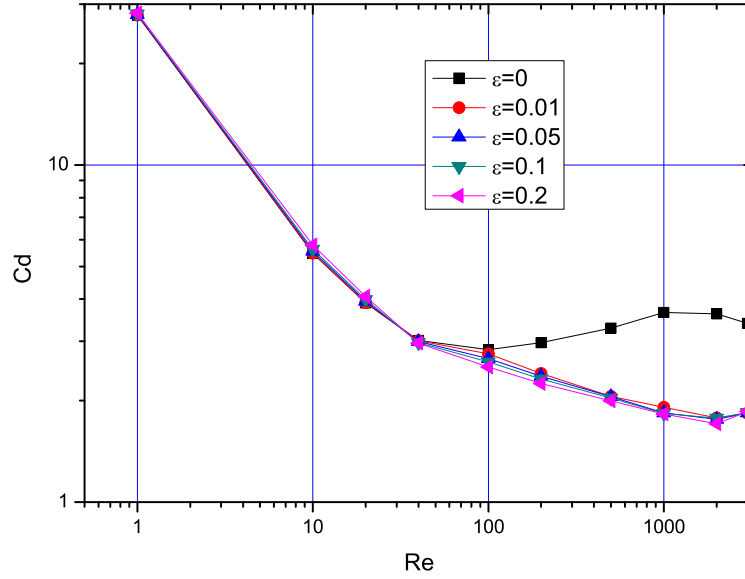


Figure 4.2: The effects of polymer viscosity ratios on the drag coefficient of the cylinder at $Wi = 2.0$

polymer viscosity ratios. The higher is the value of ϵ , the more the drag deviates from Newtonian fluid ($Wi = 0$ in the figure). For $Re = 2000$, the drag coefficient decreases as the Weissenberg number increases. The different values of ϵ don't have a strong impact on the drag coefficient. Taking one with another, the drag coefficient behavior versus the Weissenberg number for both Reynolds numbers is not modified by the polymer viscosity ratio.

Figure 4.2 displays a qualitative difference of the effect of the polymer viscosity ratio at $Wi = 2.0$ with different Reynolds numbers. The case $\epsilon = 0$ represents the Newtonian fluid. The drag force of these polymer viscosity ratios is consistent for the small Reynolds numbers, but a large difference occurs at the large Reynolds number between Newtonian fluid and non-Newtonian fluid. The important difference happens only between $\epsilon = 0$ and $\epsilon = 0.01$, the drag difference is negligible for other different polymer viscosity ratios.

In order to describe the effect of ϵ on the change of drag, let's define a correlative coefficient C_ϵ by

$$C_\epsilon = \frac{dCd(\epsilon_1) - dCd(\epsilon_2)}{dCd(\epsilon_2)} \frac{\epsilon_1}{\epsilon_2} \quad (4.1)$$

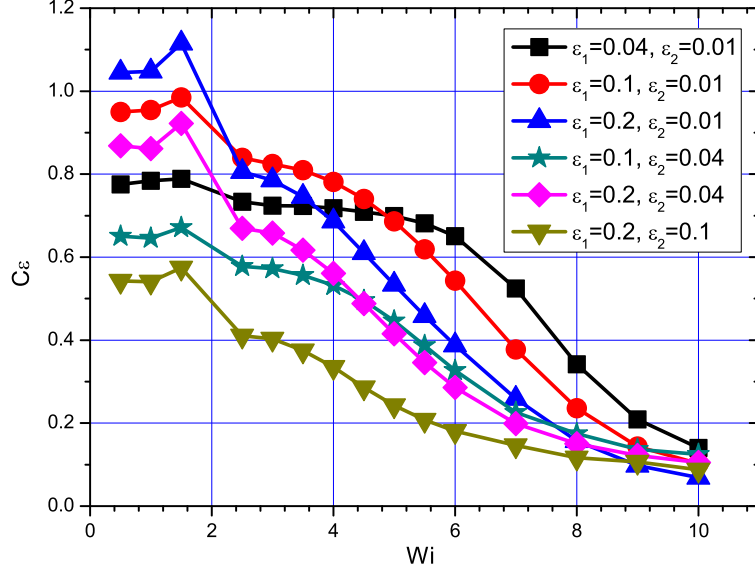


Figure 4.3: The drag correlative coefficient of the polymer viscosity at $Re = 0.1$

where the function $dCd(\epsilon)$ reads

$$dCd(\epsilon) = Cd(\epsilon) - Cd(\epsilon = 0) \quad (4.2)$$

and represents the variation of the drag coefficient compare to that of the Newtonian fluid. The value of C_ϵ reveals the effect of ϵ on the modification of the drag coefficient by the polymer. If the drag modification is linear with respect to ϵ , it gives $C_\epsilon = 1$, on the other hand $C_\epsilon = 0$ suggests that the modification of the drag coefficient is independent on the polymer viscosity ratio.

Figure 4.3 shows the evolution of C_ϵ at $Re = 0.1$ for different Weissenberg numbers. Contrarily to the drag coefficient (figure 4.1), C_ϵ is convergent to a small amount at high Weissenberg numbers. It means that the effects of ϵ on the drag force are less important at high Weissenberg number. Let us point out that we have deleted the data at $Wi = 2.0$, because there is a little drag increment at this position, which makes the denominator of C_ϵ close to zero and causes a singularity. For the small Weissenberg numbers, the values of C_ϵ are ranked by the ratio of polymer viscosity (ϵ_1/ϵ_2), C_ϵ increases as ϵ_1/ϵ_2 increases; while for high Weissenberg numbers, the values of C_ϵ are ranked by the product of polymer viscosity ($\epsilon_1 * \epsilon_2$), C_ϵ decreases as $\epsilon_1 * \epsilon_2$ increases. It suggests that the increase of the polymer viscosity has the ability to change the flow and the drag for small Weissenberg numbers, but its effects are weak for high Weissenberg numbers.

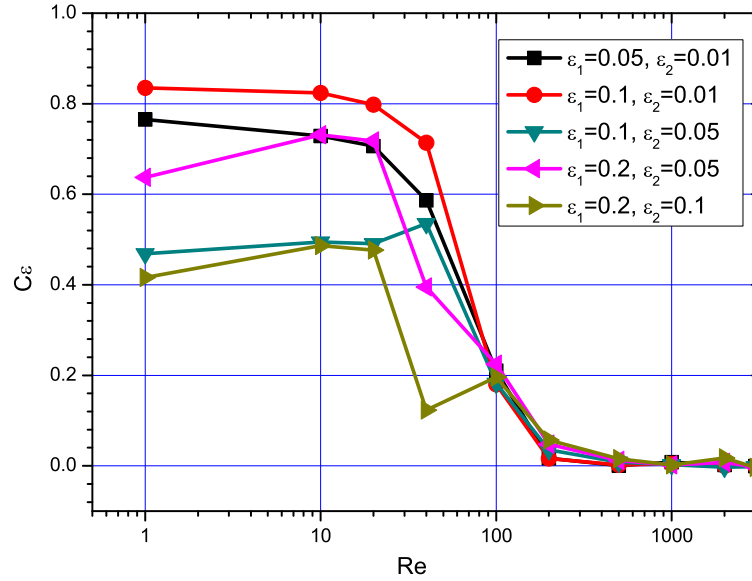


Figure 4.4: The drag correlative coefficient of the polymer viscosity at $Wi = 2.0$

In the same way, figure 4.4 shows the evolution of the correlative coefficient at $Wi = 2.0$ to different Reynolds numbers. We can see the effect of ϵ on the drag can be ignored at high Reynolds numbers. Let's point out the surprising value of C_ϵ is at $Re = 40$ for the high value of ϵ , this is because the shedding in the wake is inhibited. But for $\epsilon = 0.01$, the shedding is also inhibited for higher Weissenberg numbers as we will present in the next section. It suggests that the polymer viscosity ratio may play the similar role as Weissenberg number at low Reynolds number.

As a summary, the polymer viscosity ratio plays a significant role on the drag force at both low Reynolds numbers and low Weissenberg numbers, but its effect is negligible for high Reynolds number flows. In the Reynolds number parameter space, it only changes a small amount of the drag force at low Reynolds numbers, but does not change the general trend. At low Reynolds numbers, it plays a similar role as the Weissenberg number, but its effects is lower than the increase of the Weissenberg number.

4.2 The drag behavior with polymer

4.2.1 Drag enhancement and drag reduction

Our main result about drag is illustrated in figure 4.5 which shows a phase diagram of the different regimes observed in the parameter space $Re - Wi$. This graph shows

that three regions can be identified for the behavior of the drag versus the Reynolds number for varying values of the Weissenberg number. For $Re \leq 40$, two regimes are observed: either the fluid is acting as its Newtonian counterpart for values of Wi below a threshold Wi_2 or the fluid shows enhanced drag on the cylinder. For $Re > 40$, three different regions are observed: below a first threshold value which we call Wi_1 , the fluid behaves as a Newtonian fluid. Between Wi_1 and a second threshold value Wi_2 , drag reduction is obtained. This drag reduction goes through a maximum at the center of this region. Above Wi_2 , only drag enhancement is observed. This enhancement increases as the Weissenberg number increases. This diagram which spans roughly 5 decades in Reynolds number summarizes our findings and shows that both drag enhancement and drag reduction can be obtained but only for specific parts of the parameter space. To our knowledge, no such insight into both drag reduction and drag enhancement has been brought forth before and our results show that both phenomena can be related in a full parameter space representation. Both drag reduction and drag enhancement are present in our simulations but in different regions of the phase diagram. The phase diagram of this behavior is an important result which though has been shown before for particular values of the Reynolds number has not been put forth in such a systematic manner. In addition to these considerations, the maximum drag reduction turns out to be near 50% and is practically independent of the Reynolds number in the asymptotic large Reynolds number limit. It should be noted here that the transition from drag enhancement to both drag reduction and drag enhancement occurs at a value of the Reynolds number close to $Re = 40$, which coincides with the transition to the vortex shedding regime for the wake behind the cylinder pointing to a strong dependence on the structure of the flow and the presence of vortices. Other interesting features emerge such as the suppression of the vortex shedding for Weissenberg numbers in the drag enhancement regime.

Figure 4.8 displays the main flow features observed in this study: The vorticity contour plots are shown for a Reynolds number of 1000 and different Weissenberg numbers. Perhaps the most striking observation is the disappearance of vortex shedding behind the cylinder for high Weissenberg numbers. Another interesting feature is the increase in the length of the wake in the drag reduction regime pushing away the pressure well. These two aspects of the flow structure can be further appreciated through the power spectrum of transverse velocity variations downstream of the cylinder. We also noted that for the drag reduction regime, the amplitude of the higher harmonics seems to increase with respect to the Newtonian case indicating a large deformation of the wake. Experiments using soap films have discussed such an effect as the shedding length increases upon addition of polymer [38]. Here, the shedding length becomes

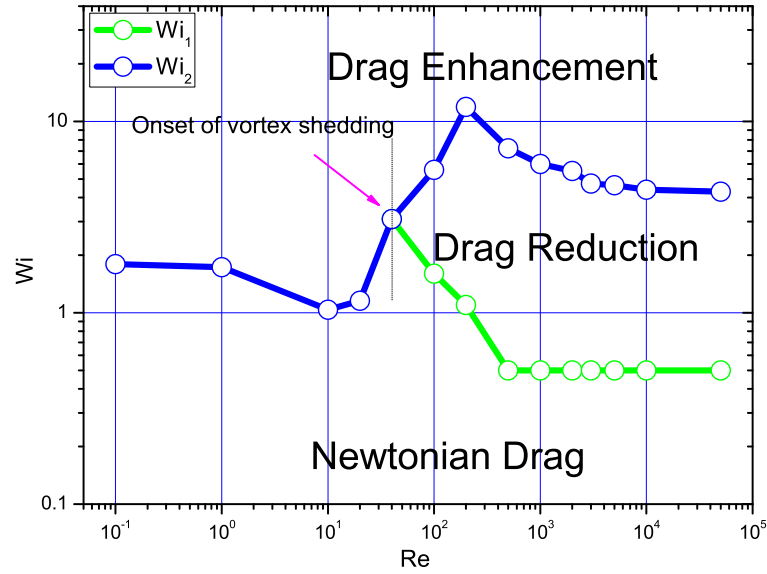


Figure 4.5: The drag phase diagram

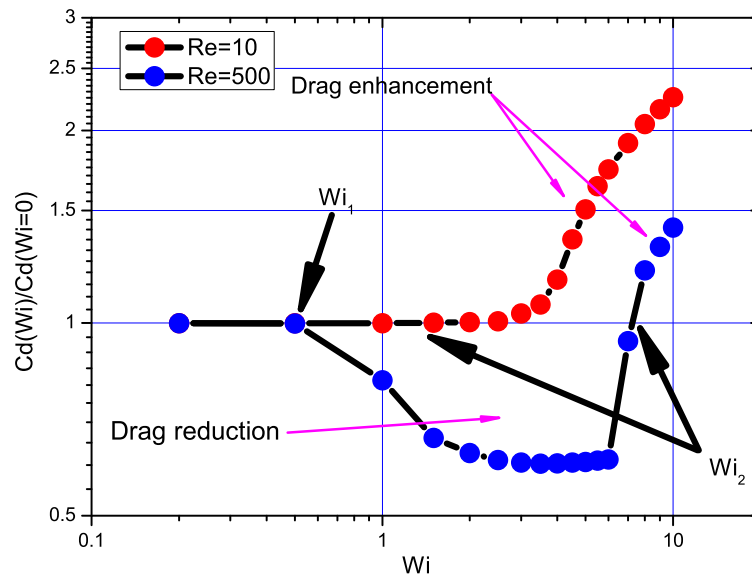
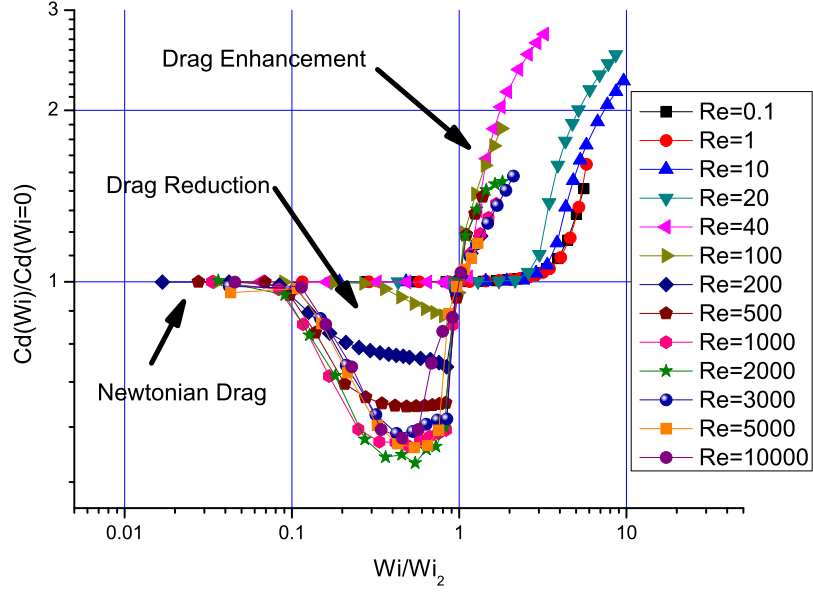


Figure 4.6: The definition of Wi_1 and Wi_2

Figure 4.7: The drag versus Wi/Wi_2

large enough that no shedding is observed in the channel length used. When this occurs, the drag starts to increase considerably.

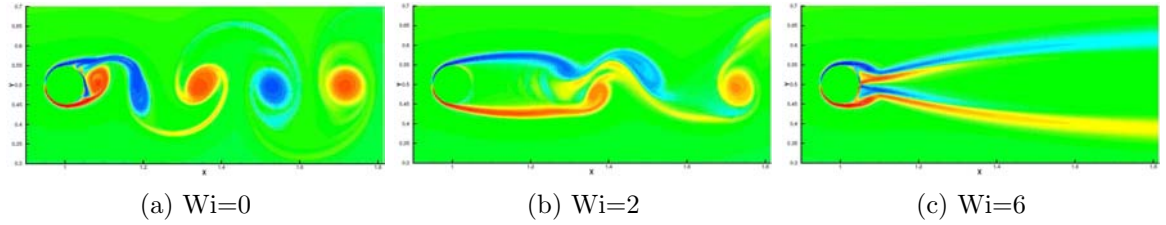
Figure 4.8: Vorticity snapshot at $Re = 1000$

Figure 4.6 and figure 4.7 display a quantitative measurement of the effect of the viscoelasticity. Here we plot the drag on the cylinder (normalized by the Newtonian drag) for different Reynolds numbers and varying Weissenberg numbers. Two different behaviors are observed as shown in figure 4.6. Below a Reynolds number of 40, the drag is similar to that of the Newtonian fluid for small Weissenberg numbers. However, above a threshold value which we denote Wi_2 , the drag increases versus the Weissenberg number. This regime therefore shows the well known drag enhancement observed in three dimensional experiments using spheres or cylinders in low Reynolds number flows [32, 55, 155]. This effect is therefore well captured by the simulations presented here for the model we have used. The drag is enhanced by a factor which can be as large

as 250%. The value of Wi_2 , however, is rather close to 1.5 than the traditional value of $1/2$ or 1 usually found in experiments. When the Reynolds number is above 40, a different behavior emerges: drag reduction (reaching 50%) is observed above a first threshold value of the Weissenberg number which we note as Wi_1 before the appearance of drag enhancement above a second threshold value which (as for the low Re) we denote by Wi_2 . This behavior is now shown in figure 4.7 for several Reynolds numbers versus the reduced Weissenberg number: Wi/Wi_2 . This figure clearly shows the difference between the low Reynolds numbers for which only drag enhancement is observed above a value of the reduced Weissenberg number of 1 and the high Reynolds numbers for which drag reduction is observed in an interval of reduced Weissenberg numbers below 1 and whose extent increases as the Reynolds number increases then saturates for even higher Re as shown in figure 4.5. The values of the increase in drag do not collapse perfectly for values of the reduced Wi/Wi_2 above 1. Below a value of 1, the low Reynolds numbers show a plateau at a relative drag of 1 showing Newtonian behavior while for high values of the Reynolds number ($Re > 40$) drag reduction is obtained. The reduction in drag increases as the Reynolds number increases and seems to reach a saturation value for Re near 2000. The drag reduction window (values of Wi for which drag reduction occurs) varies with Re as figure 4.5 shows. Figure 4.9 shows the maximum drag reduction versus the Reynolds number and as remarked above, this drag reduction sets in only near $Re = 40$ and increases for values of Re between 40 and 1000 from nearly 0 to 50 percent. This reduction then goes through a mild maximum at $Re = 2000$ before stabilizing on a plateau at about 48%.

To further examine the behavior of viscoelastic flow around the cylinder, we plot in figure 4.10, figure 4.11 and figure 4.12, the elongation rate, the longitudinal elastic stress, as well as the pressure variation along the center line of the channel. These quantities change qualitatively as either transition is crossed. We illustrate this behavior using $Re = 10$ and $Re = 100$. For $Re = 10$ and for Wi smaller than Wi_2 , the elongation rate is close to the Newtonian case. As Wi increases, the elongation rate becomes less negative (contraction) upstream while on the downstream side the location of maximum elongation (which is at about one cylinder diameter downstream) migrates towards the cylinder. As Wi increases above Wi_2 , the maximum elongation increases and its location migrates towards the cylinder wall. By contrast, for Wi below Wi_2 , the elongation rate near the cylinder wall is negative so as the transition from Newtonian drag to enhanced drag is crossed, the elongation near the cylinder wall changes sign. For $Re = 100$, for which three regimes are observed, as Wi_1 is crossed (to the drag reduction zone), the maximum elongation actually decreases and its location migrates away from the cylinder wall. It is only after crossing Wi_2 towards the drag

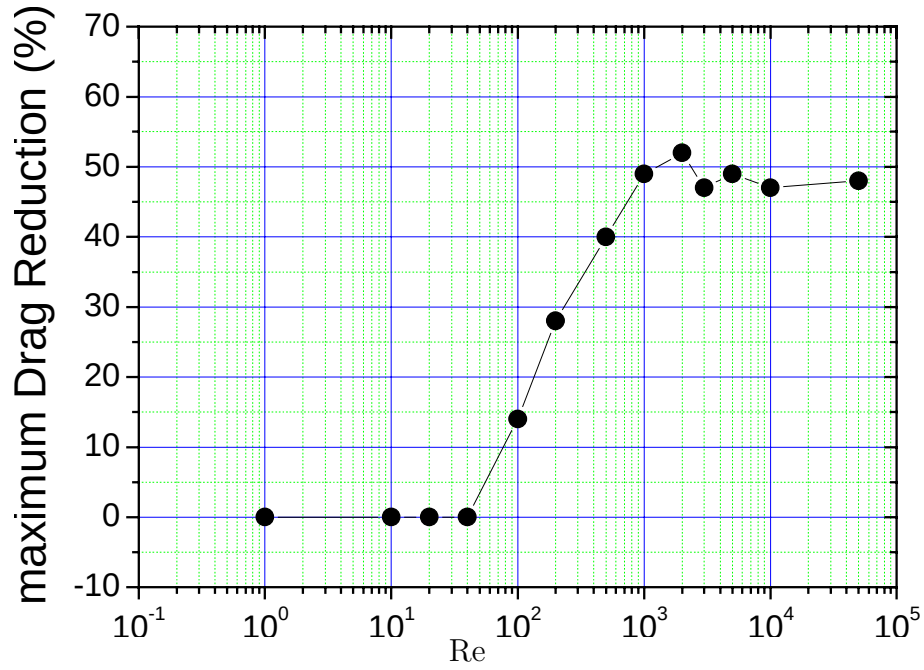


Figure 4.9: Maximum drag reduction of the cylinder by the polymer additive

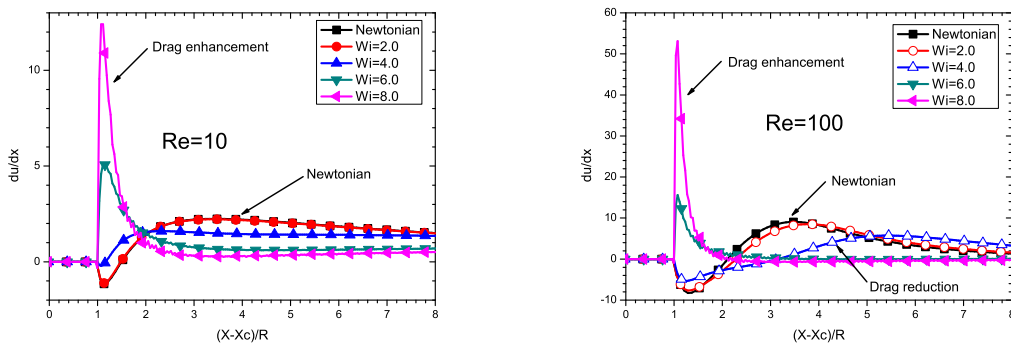


Figure 4.10: The elongation along the center line of the channel for two different Re and different Wi

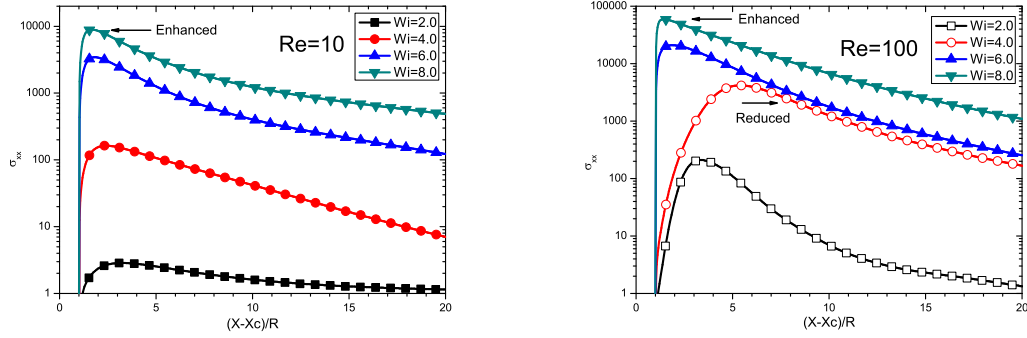


Figure 4.11: The longitudinal elastic stress along the center line of the channel for two different Re and different Wi

enhancement regime that the maximum elongation starts to increase and its location moves towards the cylinder wall. The signature of either transition is therefore visible in the elongation rate and its variation along the center line. On the other hand, the maximum of the stress, which occurs near the cylinder wall in the downstream region, increases as the Weissenberg number increases for $Re = 10$. At the crossing of the transition to drag enhancement, the maximum stress location migrates towards the cylinder wall. That the maximum stress location is at the cylinder wall is true also for $Re = 100$ in the drag enhancement regime. Both the elongation rate and the normal stress are maximal near the cylinder wall for the drag enhancement regime. Experiments in microchannels for small Reynolds numbers in the drag enhancement regime have observed large elongations of the polymers in the vicinity of the cylinder wall on the downstream side [55]. Our simulations seem to capture this essential feature of the experiments despite the fact that our simulations are two dimensional while the experiments are in three dimensions. In the drag reduction regime however, the location of maximum stress and maximum elongation is rather farther away from the cylinder (1 to 2 cylinder diameters). Another feature associated with the transition from drag enhancement to drag reduction versus the Weissenberg number can also be appreciated through the variation of the pressure along a cut in the flow direction and in the center line of the channel. The most prominent feature here is the pressure on the downstream side of the cylinder. Note that the pressure first increases with respect to the Newtonian case as the Weissenberg number increases, (this is the drag reduction regime) before decreasing to much lower values (with a sharp peak) for Weissenberg numbers greater than Wi_2 indicating a strong increase in the drag on the cylinder. The decrease in pressure is localized near the cylinder wall and its importance increases as

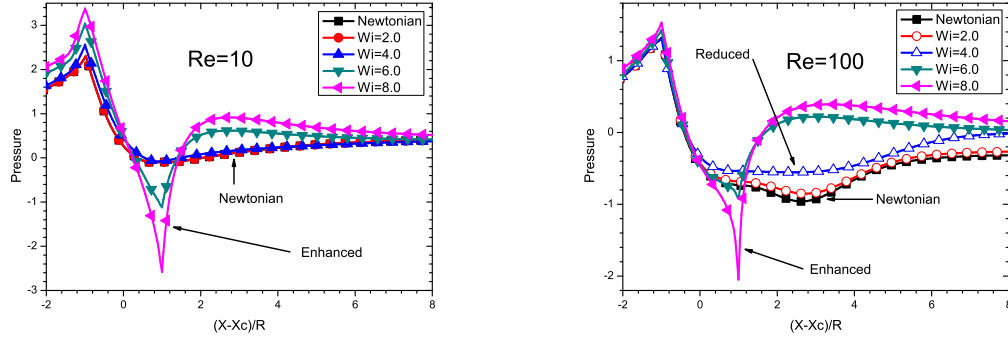


Figure 4.12: The pressure variation along the center line of the channel for two different Re and different Wi

the Weissenberg number increases.

4.2.2 The evolution of the drag and lift coefficient with polymer

In the previous section, we have presented the mean drag behavior for the polymer solution. When the wake is unsteady, the lift force is also important for the immersed body. The lift and drag forces are unsteady and vibrational. The vibration of the forces is important and interesting for many engineering problem.

Figure 4.13 shows the phase portrait of the drag and lift forces. For the two-dimensional flow, the vortex shedding is regular when the Reynolds number is less than 1000, the drag and lift force show a periodic vibration in the phase portrait. From these figures, the amplitudes of both drag and lift forces are increased as the Reynolds number increases for the Newtonian fluid ($Wi=0$). The amplitude of the lift force is larger than the amplitude of the drag force, a low drag force is corresponding to a low lift force. For the polymer solutions, the force phase portrait is similar to the one obtained with the Newtonian fluid when the Weissenberg number is smaller than Wi_1 , then both the absolute value of the drag and the amplitude of the drag or lift forces reduce quickly when the Weissenberg number is larger than Wi_1 . In fact, the phase portrait of the drag and lift forces degenerates to a point before the Weissenberg number exceeds Wi_2 .

Figure 4.14 shows the phase portrait of the drag and lift forces at $Re = 2000$. From this figure, we can observe that even though the force is still periodic in time, the lift force is not symmetric any more. A double bifurcation for the force also suggests the bifurcation for the wake flow, which is thought as an important step for the transition

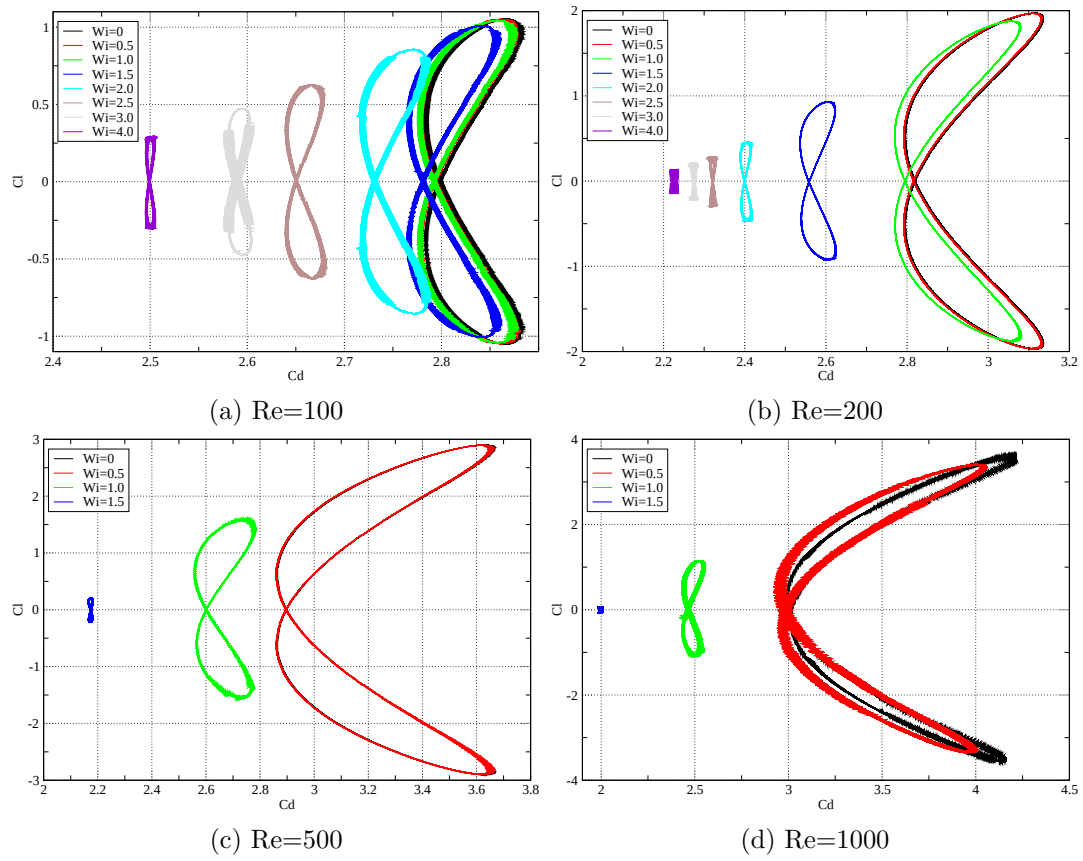
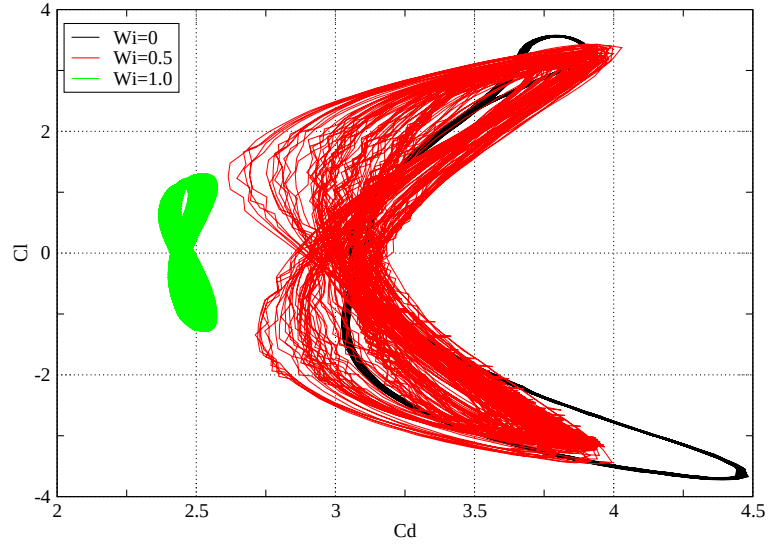


Figure 4.13: Phase portrait of the drag and lift forces at four different values of the Reynolds number

Figure 4.14: Phase portrait of the drag and lift force at $Re = 2000$

to a turbulent flow. For the polymer solutions, the forces show a better symmetry in the plane but the distribution is a mess. It suggests that the action of the polymer is not only make the chaotic flow to return to the laminar flow or a more regular flow but introduces a particular mode.

When the Reynolds number increases, the flow becomes turbulent, the drag and lift forces are too chaotic to be plotted. Figure 4.15 shows the evolution of the drag coefficient with time. The lift coefficient has a similar evolution. We can observe that the drag is very chaotic, the amplitude of the drag fluctuations is very large for the Newtonian fluid and for $Wi = 0.2$. As the Weissenberg number increases, the amplitude of the drag fluctuations decreases ($Wi = 0.5$), even if it doesn't change the mean drag force. Whereafter, both the mean value and the fluctuations decrease at $Wi = 1.0$, the drag keeps a constant value with time for higher Weissenberg numbers. This suggests several steps for the drag behavior for the viscoelastic turbulent flow. The flow is not affected by elasticity at very small Weissenberg numbers. After the Weissenberg number exceeds this threshold Wi_0 but not reaches Wi_1 , the drag fluctuations reduce and the mean drag doesn't change. Then as the Weissenberg number increases beyond Wi_1 , both the mean drag and the fluctuations reduce until the drag keeps a constant value with time.

Figure 4.16 shows the root mean square lift force reduction due to the polymer additive. When the Reynolds number is greater than 200, the root mean square lift force (Cl_{rms}) becomes negligible as soon as the Weissenberg number is greater than $0.25Wi_2$. Especially, for the Reynolds numbers 40 or 100, the Cl_{rms} first reduces and

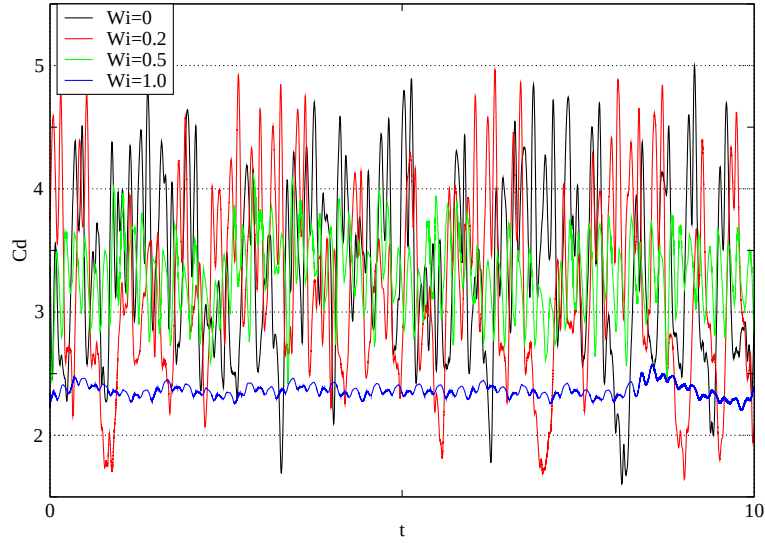


Figure 4.15: The evolution of the drag coefficient at $Re = 5000$

gets a minimum value nearby Wi_2 , then it increases with the Weissenberg number. We have performed a very long time simulation, but the lift force keeps a constant value with time at high Weissenberg numbers. It means the flow field is steady but loses up-down symmetry. We have no explanation for but this kind of unsymmetrical results was already observed in former experimental and numerical works [74, 33].

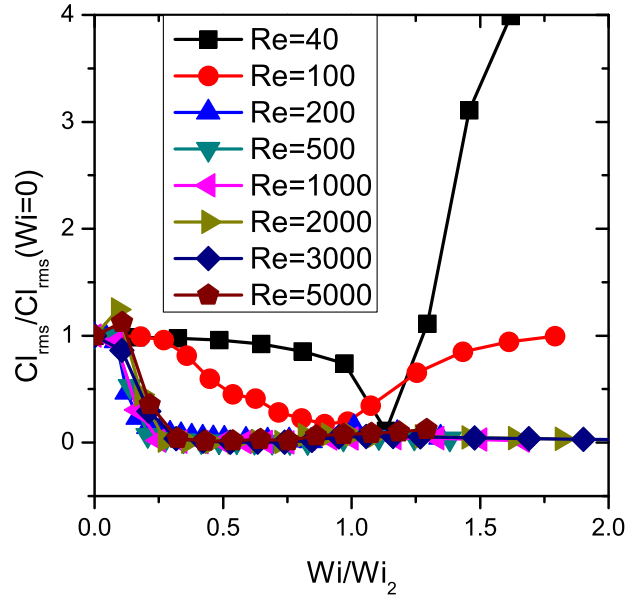


Figure 4.16: Root mean square lift force reduction by the polymer additive

4.3 The flow modification by polymer

As we have seen in previous section, the drag behavior shows the hand-in-hand interaction with the flow behavior. The nontrivial drag reduction happens only for the shedding wake flow, drag enhancement follows along with the inhibition of the shedding. In this section, we will present the flow behavior in $Re - Wi$ spaces. Let's review the flow behavior of cylinder wake in Re space for Newtonian fluid, then separate these flows into several zones. When the Reynolds number is less than 4, it is Stokes flow, the flow keeps fore-aft symmetry. From $Re = 4$ and to about $Re = 49$, there is a pair of recirculation regions behind the cylinder; the flow keeps up-down symmetry, the length of the symmetric recirculation regions increases as the Reynolds number increases. At Re above around 49, the onset of the wake instability makes the vortex shedding parallel to the centerline of the flow, the frequency of periodical shedding increases with the Reynolds number until $Re = 190$. Whereafter the flows show a chaotic behavior, then generate the turbulent wake at higher Reynolds numbers. More details can be found in [161]. Let us note that the flow onsets shedding since $Re = 40$ in our simulation, the primary reason is that we used Poiseuille flow at the inlet, and so the velocity at the centerline is much larger than the mean velocity of the channel.

4.3.1 Stokes flow

Figure 4.17 displays the vorticity fields at $Re = 1$. At Weissenberg numbers below 6, the vorticity field is very similar to that of Newtonian fluid, hence we haven't shown them. Because $Re = 1$ is close to the threshold of the Stokes flow, the fore-aft symmetry is about to completely break. The vorticity field for Newtonian fluid shows some symmetric behavior of the Stokes flow. However, as the Weissenberg number increases, we can see the small levels of vorticity behind the cylinder are stretched. The stretching is stronger for the higher Weissenberg numbers. From this figure, we know the polymer will affect the flow behind the cylinder, the fore-aft symmetry of Stokes flow can be broken by the viscoelasticity. Figure 4.18 shows the streamlines for Newtonian flow and a high Weissenberg flow. It is hard to discern the difference between them.

Figure 4.19 gives the streamwise velocity profiles at five different positions. The position is given by the abscissa of the no-slip wall boundary condition, that is $x = 0, 1, 2, 3$ and 4 . The streamwise velocity at $x = 0$ is the Poiseuille flow, which gives a parabola profile (inlet boundary condition). The maximum velocity locates at the centerline, which is 1.5 times the mean velocity. The center of the cylinder is located at $x = 1$. At this place, the streamwise velocity separate to two approximative parabolas, the maximum of the velocity is closer to the cylinder than to the wall of the channel; it is slightly greater than the maximum velocity at the inlet boundary. It is still very hard to discern the difference of the velocity profiles for the Newtonian and viscoelastic flow. Downstream of the cylinder, we observe the obvious streamwise velocity difference beyond $Wi = 6.0$, the velocity at high Weissenberg number is smaller than the velocity of the Newtonian fluid or with small Weissenberg numbers at the center part. Especially there is a local minimum on the centerline for the velocity profiles at $Wi = 10.0$. It is similar to the velocity profile for the validation at the previous chapter, but the position of the minimum is $20R$ behind the cylinder, that is much farther behind the cylinder than that shown in the previous chapter. As the position increases to $x = 3.0$, the streamwise velocity difference for Newtonian fluid and viscoelastic fluid reduces. At last part of the channel, the fluid with different viscoelasticity keep the consistent streamwise velocity profile which is recovered to the Poiseuille profile by the effect of the viscosity.

In order to observe the continuing trend near the cylinder, we have plotted the pressure and the streamwise velocity profiles on the centerline in figure 4.20. The abscissa is dimensionless by the radius of the cylinder, hence -1 in abscissa represents the stagnant point upstream the cylinder, 0 in abscissa represents the center of the cylinder and 1 the rear of the cylinder. The pressure increases at the upstream of the cylinder with the Weissenberg number, while the base pressure (at the rear of the cylinder) is entirely

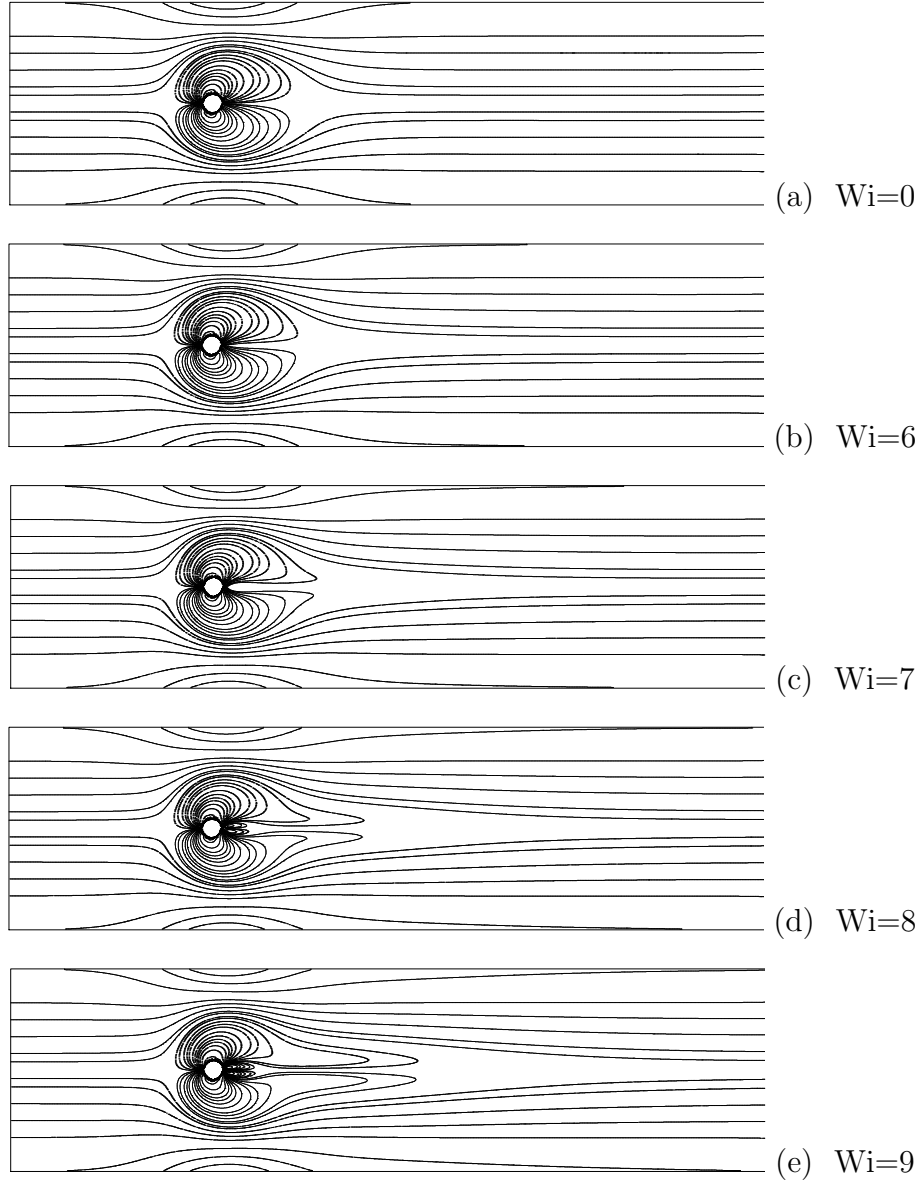


Figure 4.17: Vorticity fields at $Re = 1$

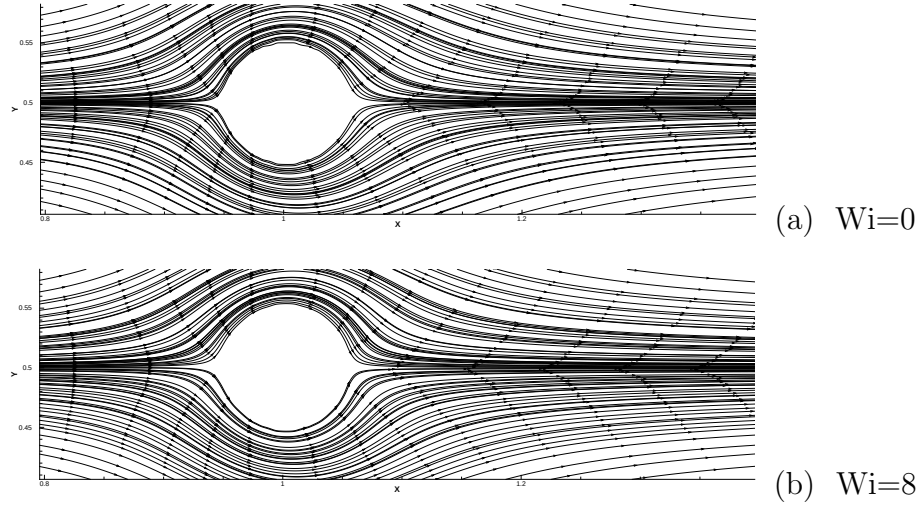


Figure 4.18: Streamlines at $Re = 1$

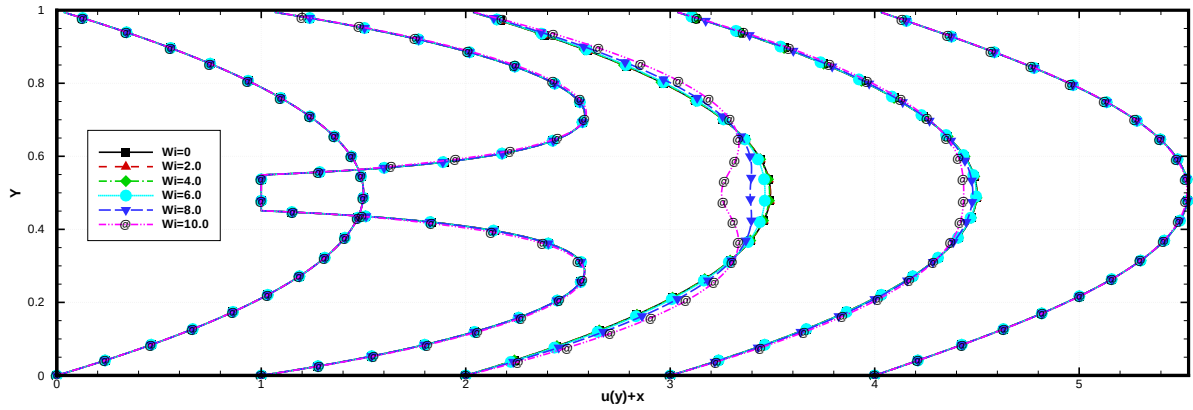


Figure 4.19: Streamwise velocity profiles at different positions. The Reynolds number is 1

opposite. For the high Weissenberg number flow, the enhanced stagnant pressure and the reduced base pressure enhance the drag of the cylinder. But for the low Weissenberg number, we haven't found the difference for the pressure and the velocity with the Newtonian fluid, of course there isn't non-trivial drag modification compared with the Newtonian fluid. Downstream of the cylinder, the base pressure recovers in a short distance, then reaches a local maximum; at the last part the pressure decreases slowly by the reason of pressure lost of the channel. At the outlet, the pressure will recover to zero which is set the reference pressure in our computational code. In the subfigure of streamwise velocity profile on the centerline, it is hard to find any difference upstream of the cylinder. But downstream of the cylinder, the velocity profiles are different beyond $Wi = 6.0$. Comparing with the Newtonian fluid, the fluid with higher elasticity accelerates quickly and results in a higher elongation rate near the rear of the cylinder. Whereafter their speedups are much smaller than the Newtonian fluid so that they need a longer distance to recover the bulk velocity.

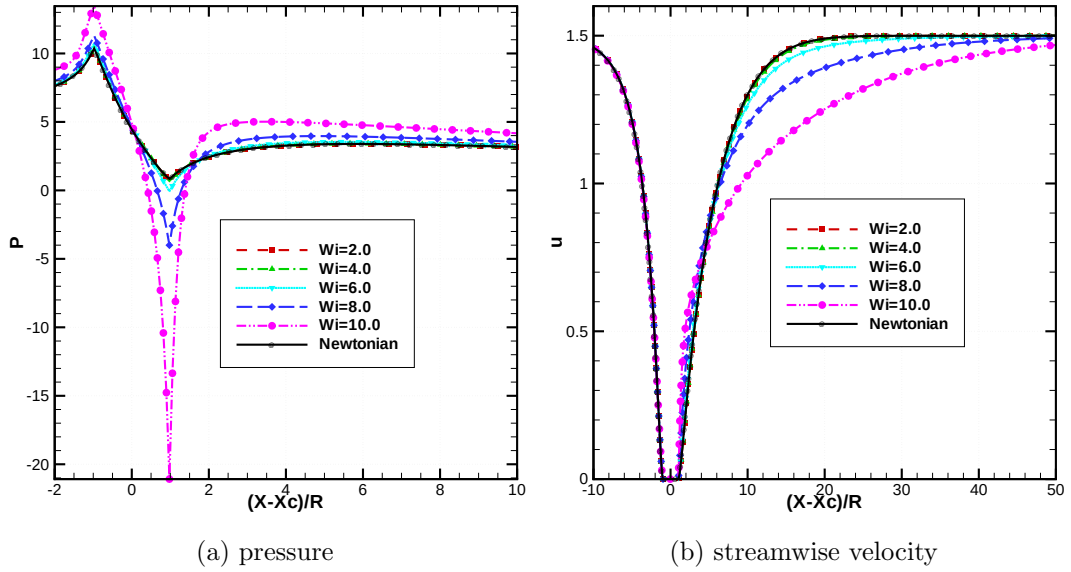
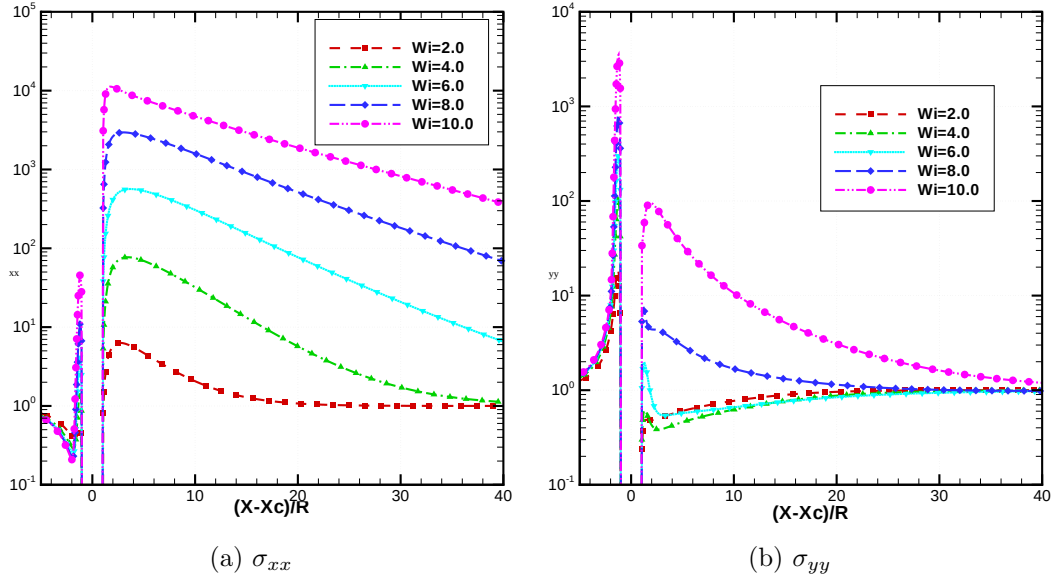


Figure 4.20: Pressure and streamwise velocity profiles on the centerline $y = 0.5$ at $Re = 1$

From figure 4.20, the streamwise velocity recover quickly than the Newtonian fluid at the rear of the cylinder. But the pressure also increases in this region, it means the pressure gradient doesn't help the speedup of the fluid but decelerates the fluid. From the profile of the velocity, it is impossible to have much higher viscous force for the high Weissenberg number flow than that of the Newtonian fluid flow. Therefore we plotted the figure 4.21 to find the driving force for the local fluid speedup. From the subfigure on σ_{xx} , we can find the driving force, the σ_{xx} get the maximum near the rear of the

Figure 4.21: The extra-stress profiles on the centerline $y = 0.5$ at $Re = 1$

cylinder, and this value is extraordinary high but the increase process is very short, these will produce a very strong driving force to increase the pressure and accelerate the local fluid. After the peak, σ_{xx} decreases while the distance to the cylinder increases, it will make a deceleration for the local fluid, which can explain the speedup of the streamwise velocity is much lower than for the Newtonian fluid, this part the main force comes from the viscous forces and is almost the same for both Newtonian fluids and viscoelastic fluids. As the polymer is diluted, the elastic force can not dominate the flow, but the extra-stress increases exponentially at the rear part of the cylinder as Weissenberg number increases. This shows that the elastic force can dominate locally the flow. Contrarily to σ_{xx} that has a maximum at the elongational flow at the rear of the cylinder, σ_{yy} has a maximum at the contractive flow in front of the cylinder. It will affect transverse velocity, as a secondary effect of the streamwise acceleration at high Weissenberg numbers; there is a local peak at the rear of the cylinder which is quite different from the profile at low Weissenberg numbers.

4.3.2 Laminar flow with steady wake

As the Reynolds number increases, the wake includes a steady recirculation region of two symmetrically placed vortices on each side of the wake as shown in figure 4.22. The length of the recirculation bubbles grows as the Reynolds number increases. However, it is difficult to define the variation of the bubble shape with Reynolds number as Re becomes large. From figure 4.22, the vortices at the end of the wake are stretched

comparing with the Stokes flow. For the viscoelastic fluid, the lengths of the vortices are shorter than for the Newtonian fluid. We have plotted the streamlines in figure 4.23. It is clear that two recirculation bubbles at the rear of the cylinder disappear at $Wi = 6.0$ for the viscoelastic fluid. In fact, they disappear beyond $Wi = 4.0$, and the scale and the shape of the bubbles are the same with the Newtonian fluid when the Weissenberg number is less than 3.0. The transition of the bubbles locates in the Weissenberg zone between 3.0 and 4.0.

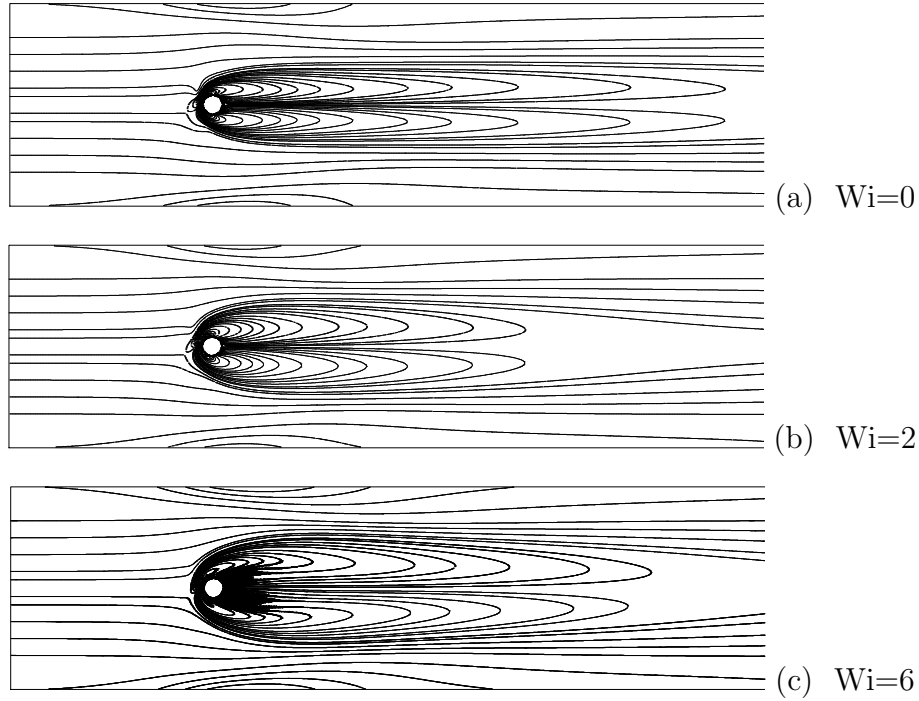


Figure 4.22: Vorticity fields at $Re = 10$

In figure 4.24, we plot the streamwise velocity profiles at the same five positions. The differences between the Newtonian fluid and the viscoelastic fluids are clear. At $x = 1.0$, the velocity profiles are very different from the Stokes flow, the maximum velocity is much near the cylinder. From the zoom of the velocity profile shown in figure 4.25, the profile for the Newtonian fluid is the same than those at $Wi = 2.0$. For higher Weissenberg numbers the velocity decreases as the Weissenberg number increases. Downstream of the cylinder at $x = 2$, the streamwise velocity has a local minimum near the centerline of the channel for every profile. As the Weissenberg number increases, the local minimum velocity decreases. The difference of velocity profiles also reduces during this process. Farther downstream of the cylinder at $x = 3.0$ and $x = 4.0$, the flow are on the way to recover the Poiseuille flow, and their difference also reduces during this process.

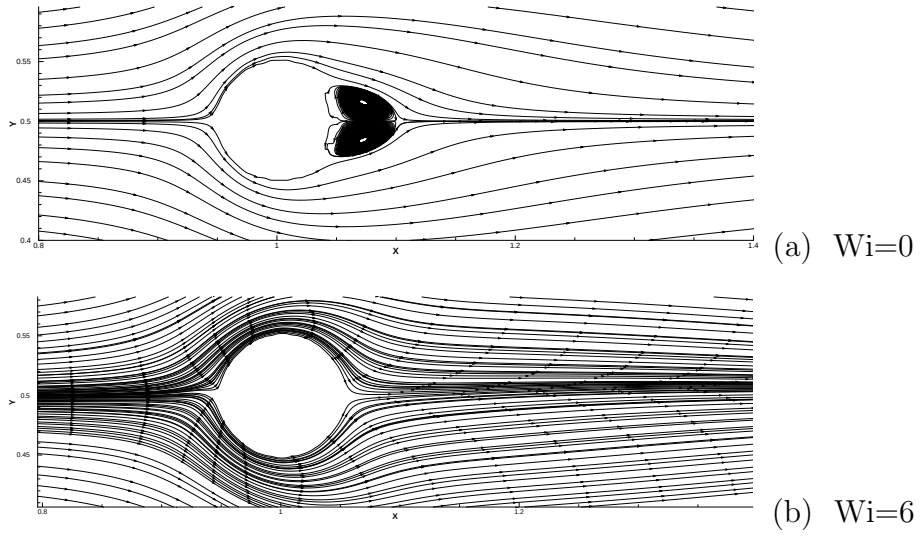


Figure 4.23: Streamlines at $Re = 10$

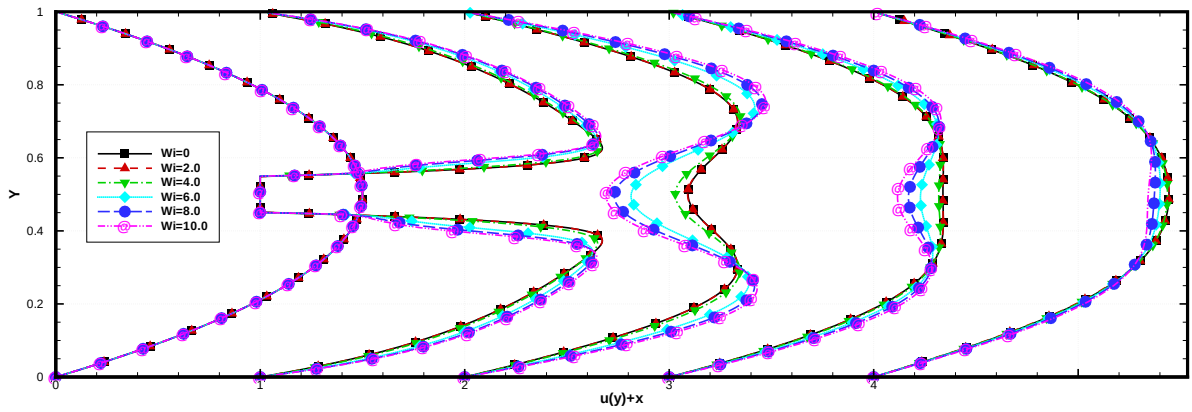


Figure 4.24: Streamwise velocity profiles at $Re = 10$ for different positions.

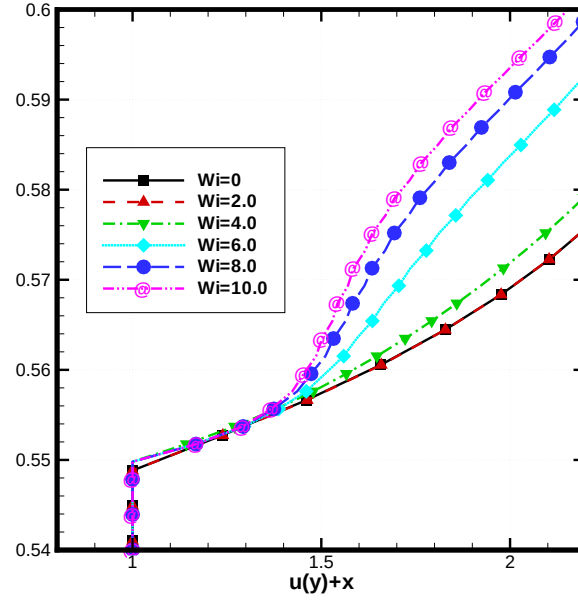


Figure 4.25: Zoom of streamwise velocity profile above the cylinder at $Re = 10$

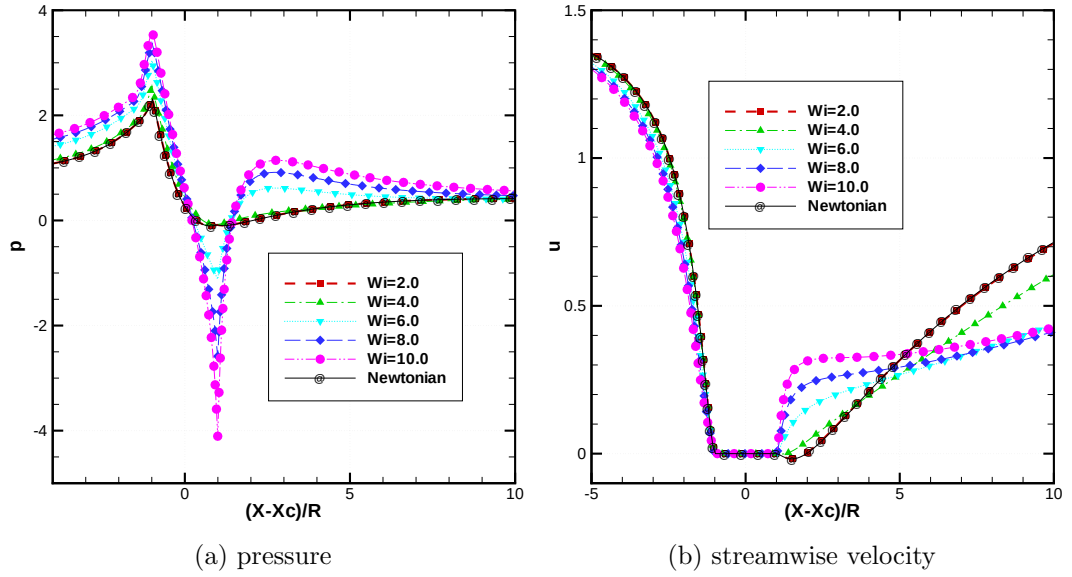


Figure 4.26: Pressure and streamwise velocity profiles on the centerline $y = 0.5$ at $Re = 10$

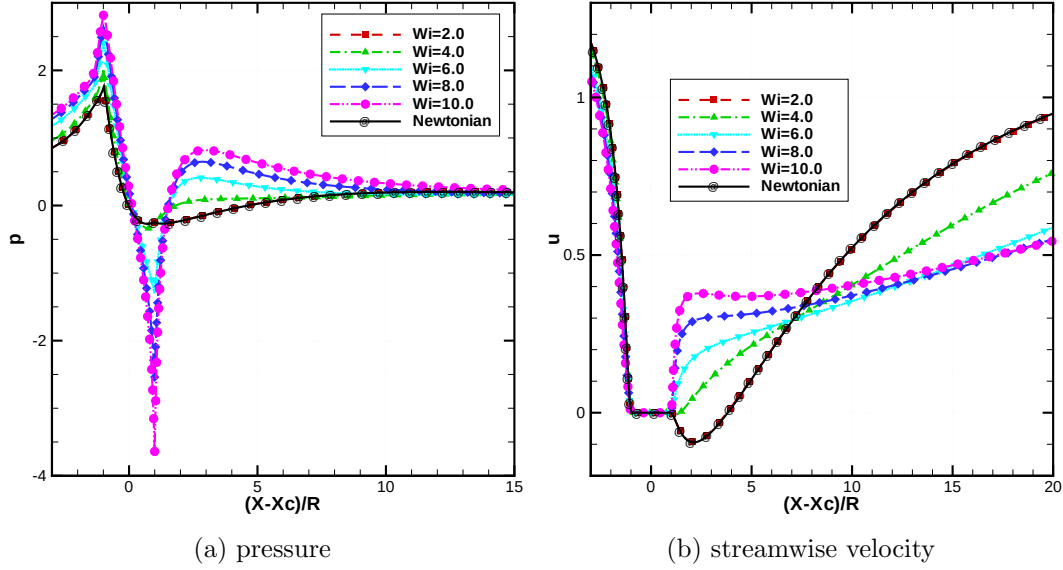


Figure 4.27: Pressure and streamwise velocity profiles on the centerline $y = 0.5$ at $Re = 20$

In figures 4.26 and 4.27 we plot the pressure and the streamwise velocity profiles along the channel centerline for $Re = 10$ and $Re = 20$ respectively. The pressure profiles are very close to those observed for Stokes flow. The superposition of the pressure and the velocity profiles suggests the flow with small Weissenberg number doesn't depart from Newtonian behavior. There is a negative part of the streamwise velocity for small Weissenberg numbers at the rear of the cylinder which is not observed with Stokes flow. It gives the length of the recirculation bubble; which increases as the Reynolds number increases. It disappears rapidly when the Weissenberg number increases. There is a local speedup at the rear of the cylinder for the flow with high Weissenberg numbers. It even may result in a local maximum velocity at the velocity recovery zone. The velocities for high Weissenberg numbers recover slower than in the Newtonian fluid flow. The velocity of the Newtonian fluid flow decreases more rapidly than the flow with high Weissenberg number.

Figure 4.28 shows the extra polymer stress profiles on the centerline at $Re = 10$, σ_{xx} also exhibits a strong local stress to increase the pressure upstream and downstream of the cylinder. Because the inertial force increases with the Reynolds number, the contractive rate increases upstream of the cylinder, the maximum value of σ_{yy} is larger than the maximum of σ_{xx} .

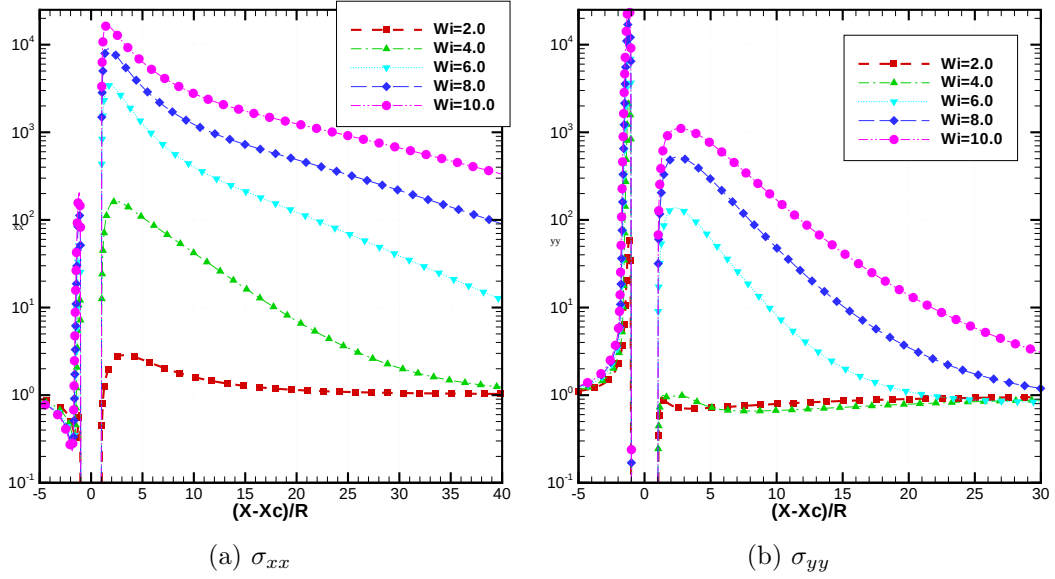


Figure 4.28: The extra polymer stress profiles on the centerline $y = 0.5$ at $Re = 10$

4.3.3 Laminar flow with unsteady wake

As the Reynolds number increases further, the recirculation region develops instabilities, the flow becomes unsteady as shown in figures 4.29 and 4.30, the vortices begin to oscillate, but the flow is still laminar. The onset of time-periodic flow indicates a supercritical Hopf bifurcation. As the Reynolds number increases further, vortices are shed alternately from the separated streamline above and below the cylinder, the well known Karman vortex street forms. As the Reynolds number is larger than about 190, the two-dimensional flow becomes more chaotic. In this section, we discuss the flow with Reynolds number less than 190, the wake oscillations are purely periodic in this regime.

From figure 4.29, we can observe the shedding of the vortices for the Newtonian fluid, as well as the inhibition of the vortex shedding for the high Weissenberg number flow. The vorticity field at high Weissenberg number looks like the vorticity field at small Reynolds number flow with high elasticity. The same conclusion can be found from the streamline snapshots shown in figure 4.31. The Strouhal number is an important dimensionless number to describe the oscillating mechanism of the unsteady time-periodic flow. The Strouhal number is given by $St = fD/U$, where f is the frequency of vortex shedding, D is the diameter of the cylinder and U is the mean velocity at the inlet boundary. We find $St = 0.2$ for the Newtonian flow at $Re = 40$. In the literature the Strouhal number for the flow past a cylinder is about 0.13 when there is an uniform inlet velocity. Let's note that the cylinder is at the center of the channel, where the

velocity is 1.5 times the mean velocity for the Poiseuille flow. If we use this velocity in the definition of the Strouhal number, we get $St = 0.133$ at $Re = 40$.



Figure 4.29: Vorticity snapshots at $Re = 40$

Figures 4.30 and 4.32 show the snapshots of the vorticity fields and of the streamlines at $Re = 100$. There is vortex shedding when the Weissenberg number is less than 5.0, but not any more at higher Weissenberg numbers. The snapshots of the vorticity fields show that the distance between vortices increases at $Wi = 5.0$ compared to the distance for the Newtonian fluid. This suggests a vortex spends longer sweeping time to a point for the elastic fluid than for the Newtonian fluid. Hence the Strouhal number decreases as the Weissenberg number increases. In the same way, one can point out the Strouhal number at $Re = 100$ is higher than that at $Re = 40$ for the Newtonian fluid. The Strouhal number is equal to 0.27 for the Newtonian fluid and 0.25 when $Wi = 5.0$. Furthermore, there is a slight difference of the streamline snapshots fields between $Wi = 5.0$ and $Wi = 0$.

Figure 4.33 shows the streamwise velocity profiles both at $Re = 40$ and at $Re = 100$. The streamwise velocity can be divided into three regimes according to the Weissenberg number. At low Weissenberg numbers when $Wi \leq Wi_1$, the velocity profiles are consistent with that of the Newtonian fluid; there is a similar streamwise velocity profile when $Wi \geq Wi_2$. It corresponds to the same three regimes observed for the drag. In the wake of the cylinder, the streamwise velocity profiles at $Re = 40$ is very similar to that at $Re = 10$, near the cylinder there is a local minimum on the centerline but Poiseuille profile is recovered far away from the cylinder. Because the vortex shedding is helpful for the momentum redistribution, the velocity recovery length is shorter than that at $Re = 10$. The velocity profile at $Re = 100$ for the Newtonian fluid flow or for small Weissenberg number flows is different to that at small Reynolds numbers, there

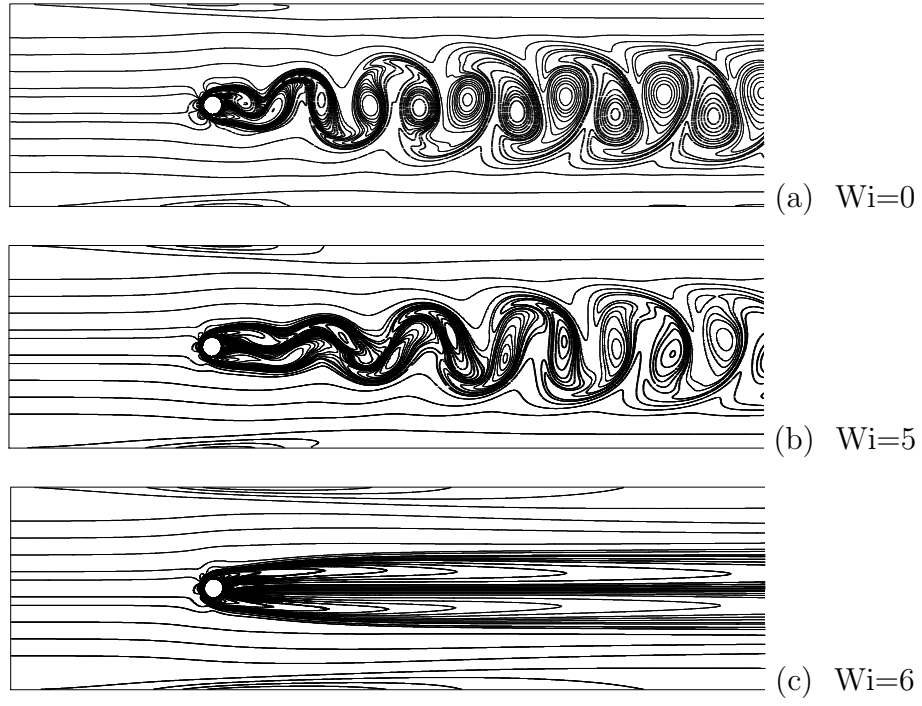


Figure 4.30: Vorticity snapshots at $Re = 100$

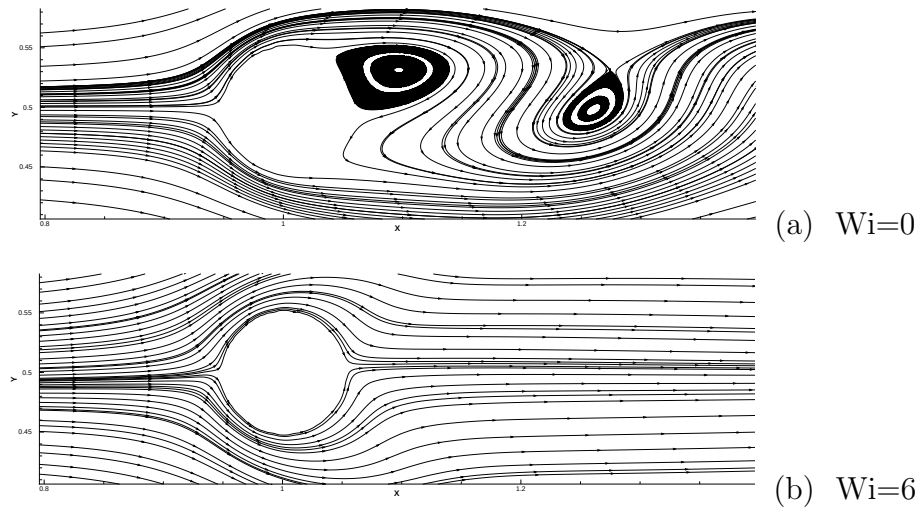


Figure 4.31: Streamline snapshots at $Re = 40$

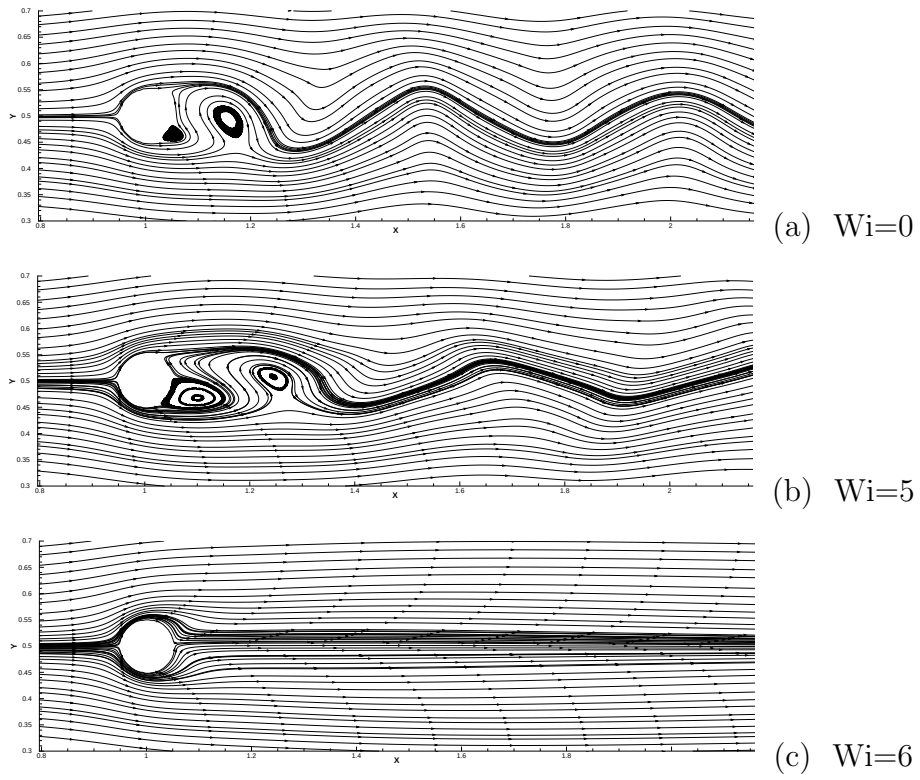


Figure 4.32: Streamline snapshots at $Re = 100$

is a local maximum velocity on the centerline due to vortex shedding. The velocity profiles at high Weissenberg number are alike both at $Re = 40$ and at $Re = 100$, the time-averaged streamwise velocity show a minimum value on the centerline. It seems that the fluid at high Weissenberg numbers has memory; it looks like the fluid can memorize the velocity profile upstream.

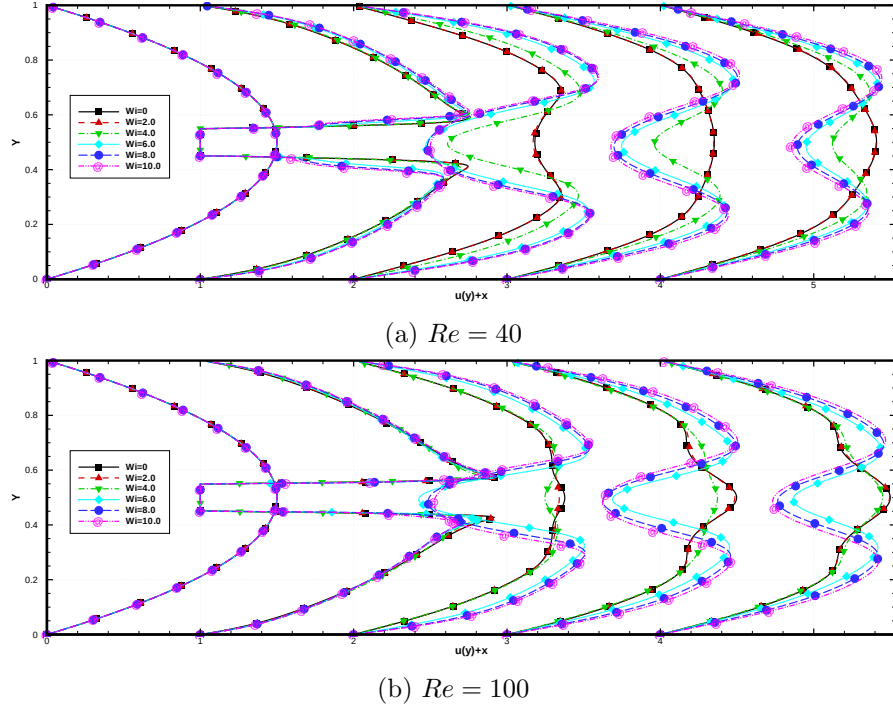


Figure 4.33: Mean streamwise velocity profiles for different positions.

Figure 4.34 shows the time-averaged pressure and streamwise velocity profiles on the centerline at both $Re = 40$ and $Re = 100$. When the Reynolds number is 40, the profiles look like that at small Reynolds number flows. Even though the wake becomes unsteady from $Re = 40$, there exists a recirculation bubble for the mean flow. The recirculation bubble has a maximum length near $Re = 40$ for the Newtonian fluid. At $Re = 100$ the flow at different Weissenberg numbers reduces upstream of the cylinder both the pressure and the velocity profiles. Downstream of the cylinder, the base pressure increases obviously at $Wi = 4.0$ compared to the Newtonian base pressure, and the length of the mean recirculation bubble increases. The profiles at high Weissenberg number show the elongation of the recirculation bubble for the mean flow. In the drag enhancement regime, the base pressure decreases and the mean recirculation bubble disappears.

We have plotted the polymer stress tensor profiles on the centerline in figure 4.35, the peak of σ_{xx} at the rear of the cylinder changes position by the effect of the mean length

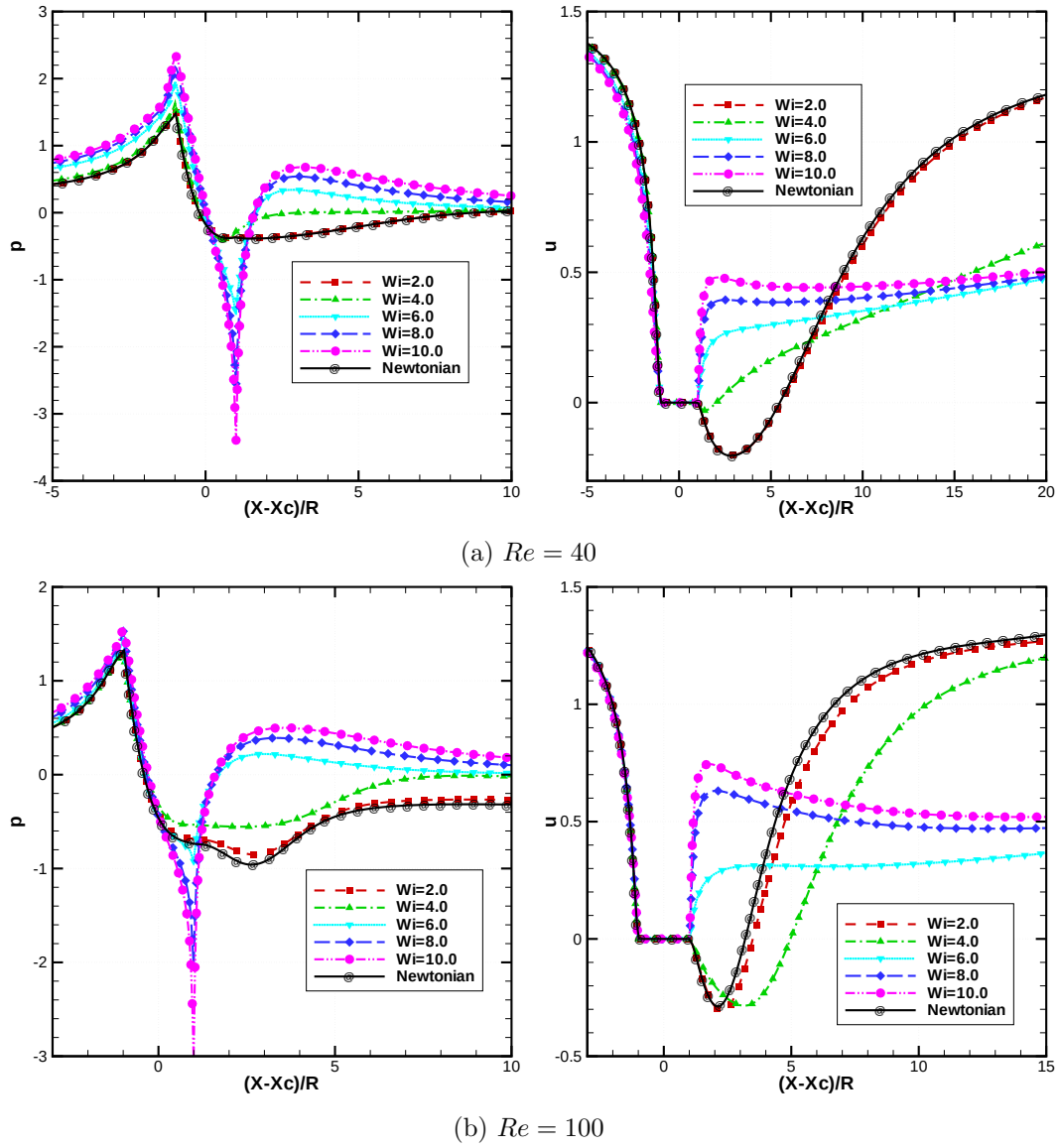


Figure 4.34: Mean pressure and streamwise velocity profiles on the centerline $y = 0.5$

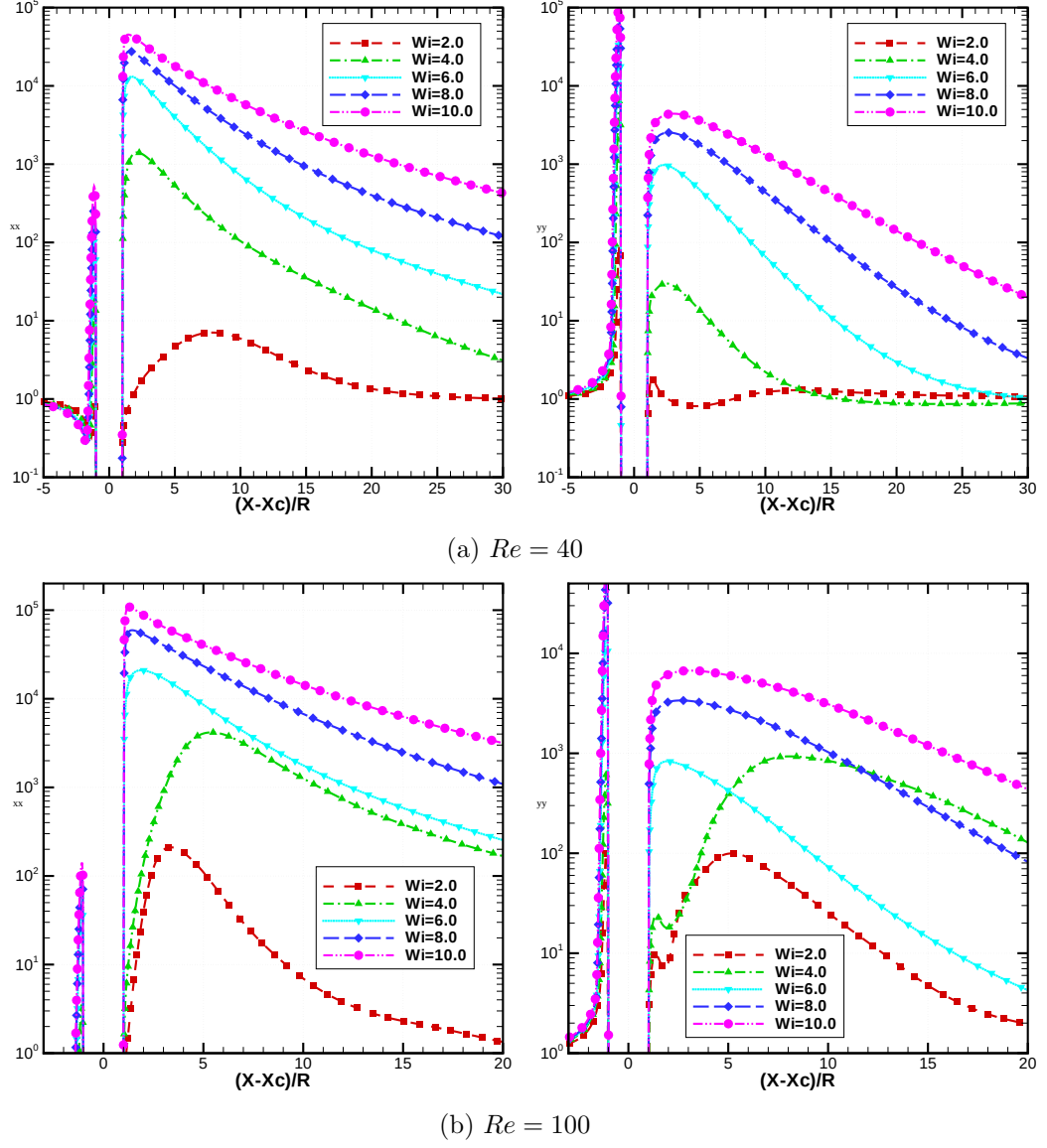


Figure 4.35: The mean extra-stress profiles on the centerline $y = 0.5$

of the recirculation bubble. The maximum of σ_{yy} in front of the cylinder is larger than the maximum of σ_{xx} at $Re = 40$, just like for $Re = 10$. But as the Reynolds number increases, the situation reverses, the enhancement of the vortex shedding at the wake affects the contraction upstream of the cylinder. Downstream of the cylinder, the peak in σ_{xx} shifts backwards for the drag reduction Weissenberg numbers.

4.3.4 Transitional flow

For Reynolds numbers beyond about $Re = 190$, the flow downstream of the cylinder transits from two-dimensional to three-dimensional by the vortex dislocations, whereafter the increase of the disorder in the fine scales make the wake flow turbulent. However, here we are interested in the two-dimensional flow which restricts the transition to the three dimensional flow. The flow in two dimensions also shows its particularity at $Re = 200$. As we reported, the drag reduction shows there is a maximum in Wi_2 at this Reynolds number. Figure 4.36 shows vorticity snapshots, they contain several different shedding modes at different Weissenberg numbers. Near the cylinder, the wake width decreases as the Weissenberg number increases. Further downstream the transverse distance of the vortex core increases, but the streamwise distance of vortex core does not distribute at the same distance around $Wi = 5$. The vortex shedding is inhibited at $Wi = 12$. In figure 4.37, we again plot vorticity snapshots at $Re = 1000$. The length of the vortex shedding increases as the Weissenberg number increases below Wi_2 . When the Weissenberg number increases, the shedding may lose symmetry which is difficult to explain. In figure 4.38, we can see the streamlines are modified by the elasticity of the fluid, the streamline patterns far away the cylinder are rigid zigzag comparing to the Newtonian fluid. Near the cylinder, the recirculation zone is more complex. When the Weissenberg number exceeds Wi_2 , there are a pair of dead vortices behind the cylinder, and the flow reattaches behind the cylinder. They show that the polymer inhibits the shedding.

Figures 4.39 and 4.40 show the mean streamwise velocity profiles at both $Re = 200$ and $Re = 1000$ and a zoom of the mean velocity profiles in order to observe the shear layer near the cylinder. As the Reynolds number increases, the ratio of the inertial forces in the flow increases, and the shear rate near the cylinder increases. The lower shear rate corresponds to the drag reduction elastic fluid and the upper shear rate corresponds to the fluid with Weissenberg number smaller than Wi_1 or larger than Wi_2 . We can see that as the Reynolds number increases, the difference between the Newtonian fluid and the viscoelastic fluid is magnified.

Figure 4.41 shows the mean pressure and the streamwise velocity profiles on the center-

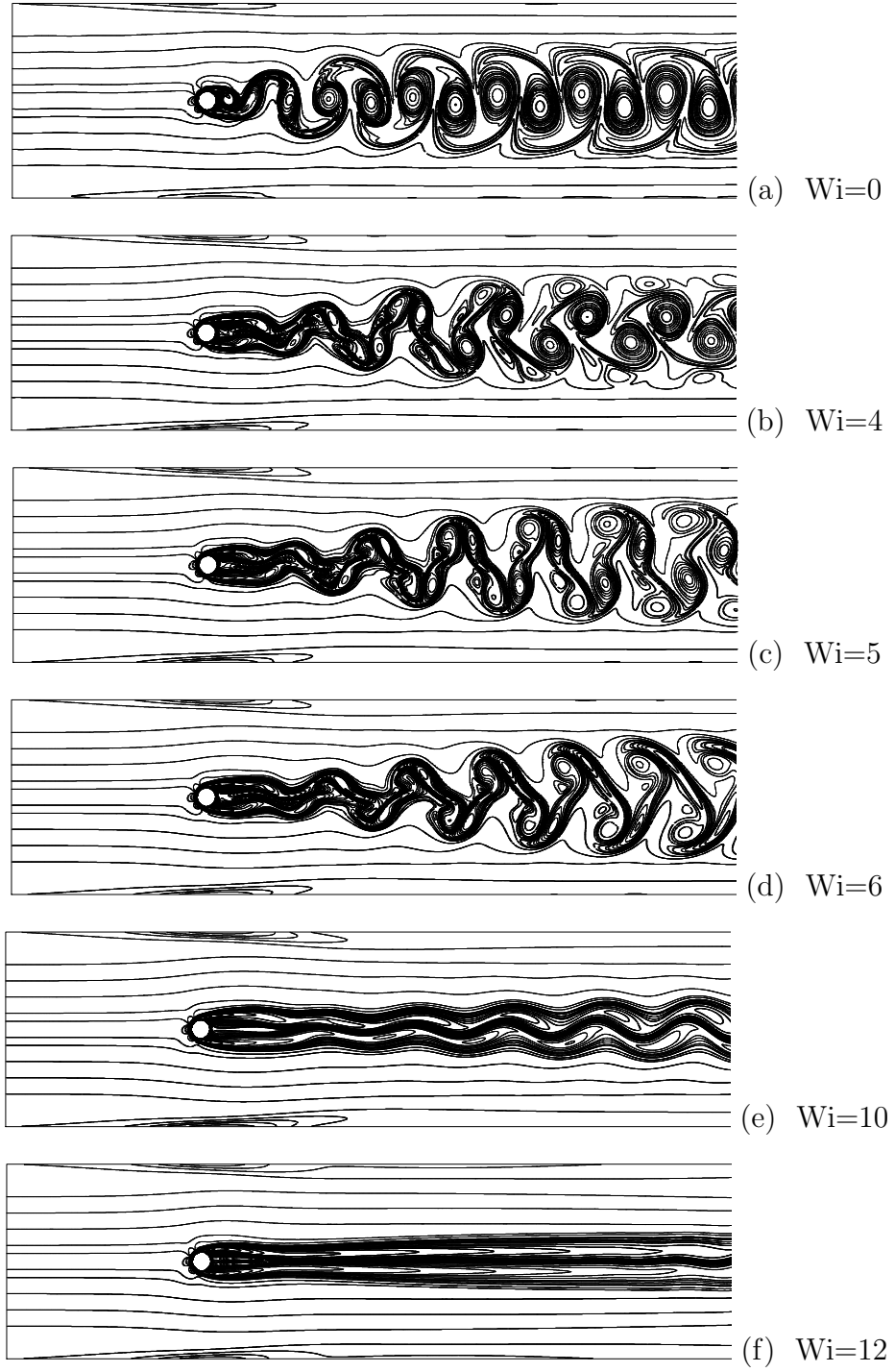


Figure 4.36: Vorticity snapshots at $Re = 200$

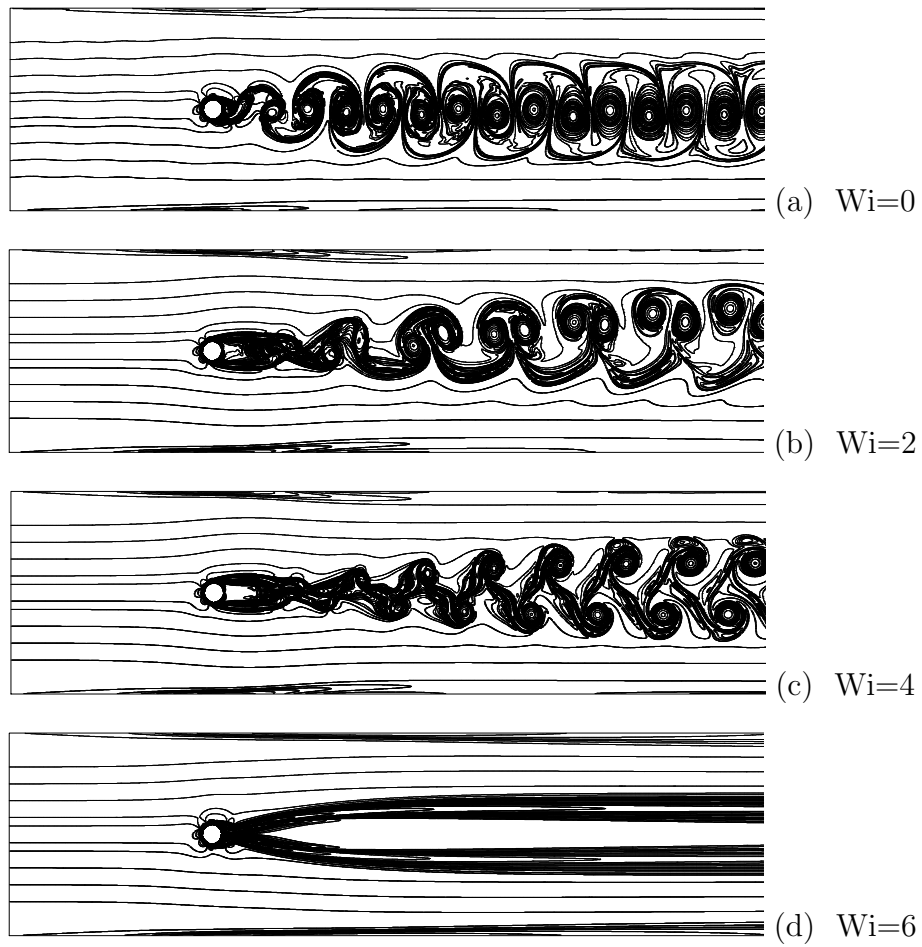


Figure 4.37: Vorticity snapshots at $Re = 1000$

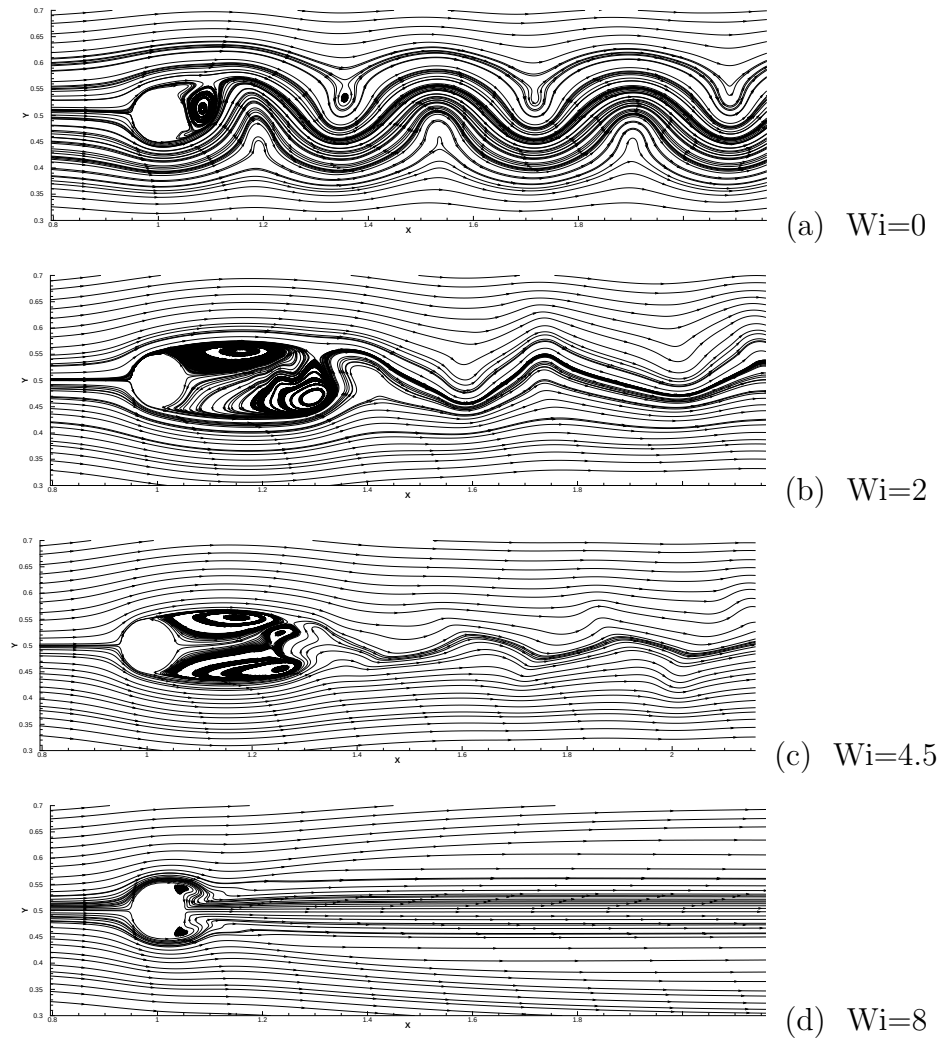


Figure 4.38: Streamline snapshots at $Re = 1000$

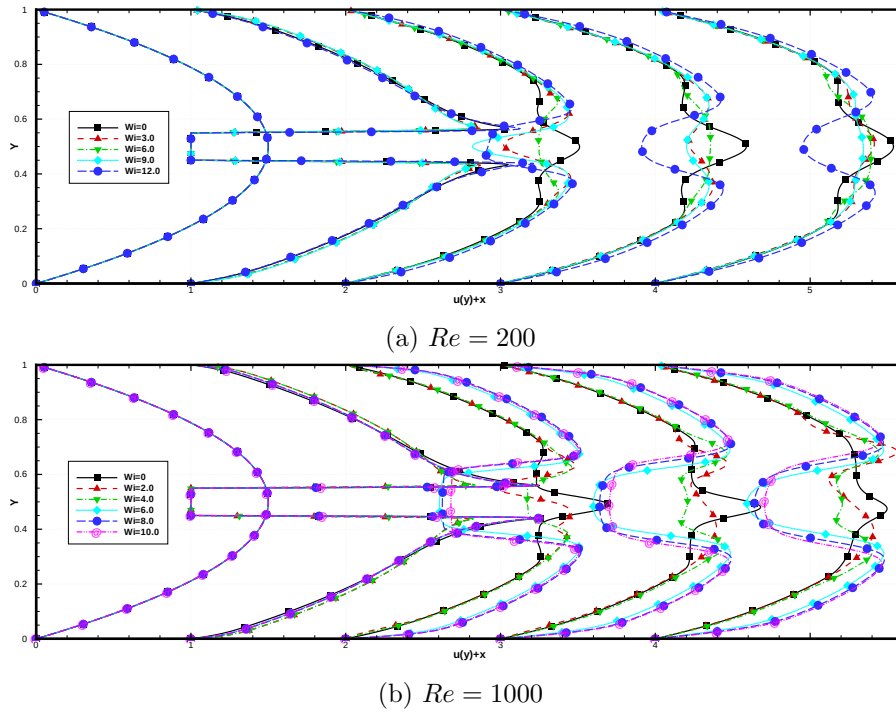


Figure 4.39: Mean streamwise velocity profiles for different positions.

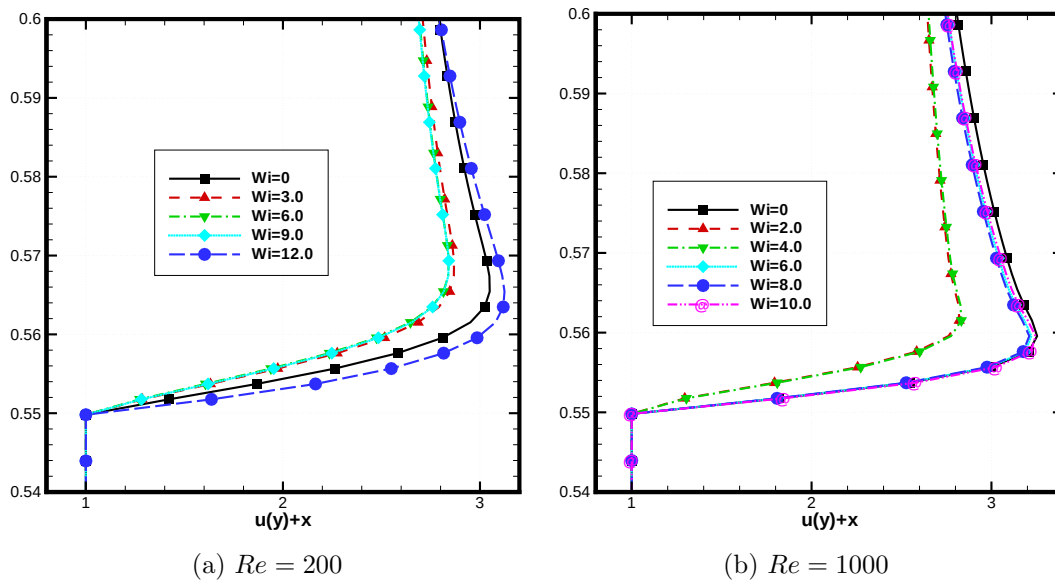


Figure 4.40: The zoom of mean streamwise velocity profile above the cylinder

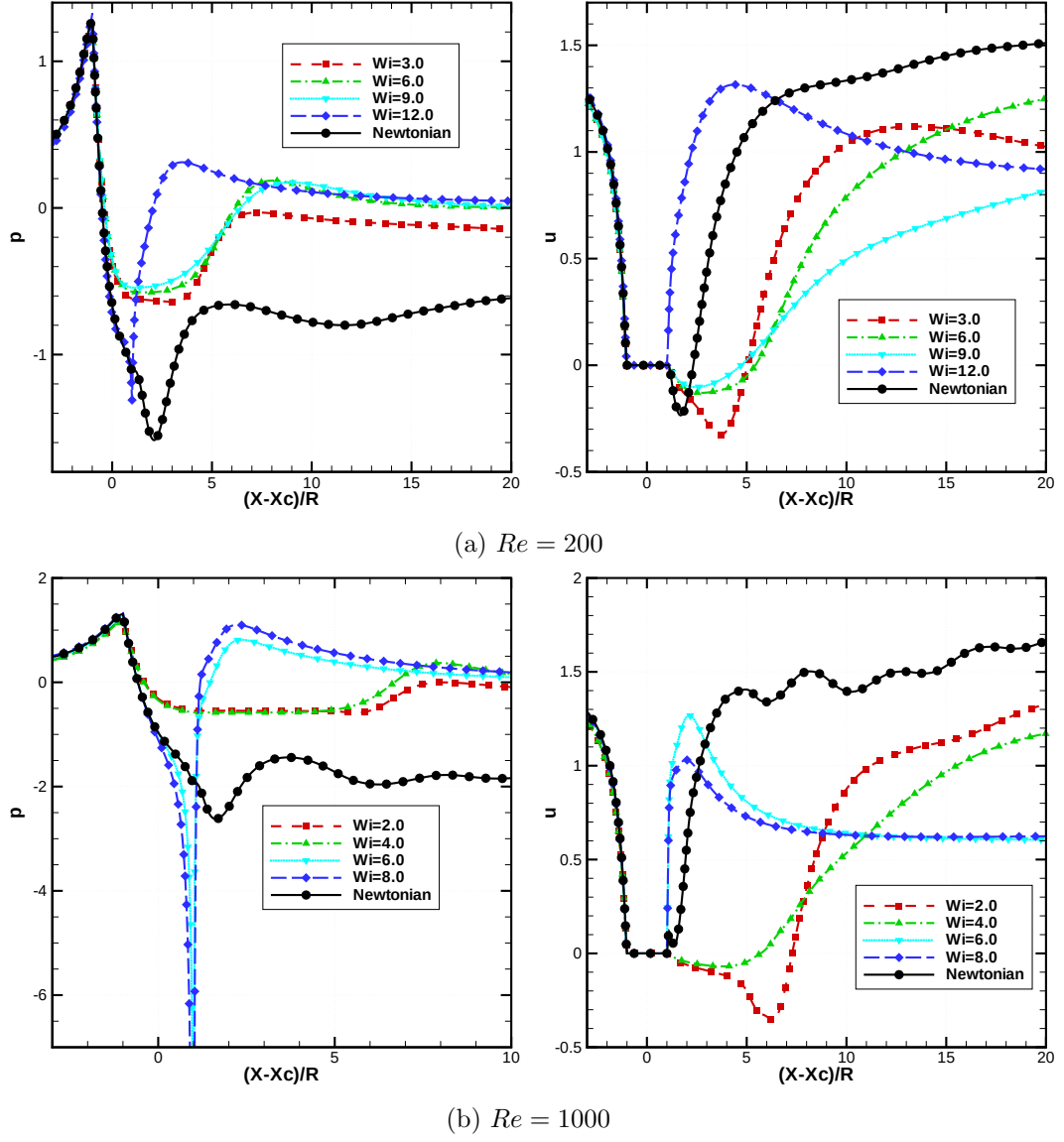


Figure 4.41: Mean pressure and streamwise velocity profiles on the centerline $y = 0.5$

line. As we discussed above, the base pressure of the cylinder increases for the elastic fluid with the Weissenberg number between Wi_1 and Wi_2 . When the Weissenberg number is greater than Wi_2 , the base pressure drops dramatically. The mean stream-wise velocity presents two similar behaviors. The drag reduction corresponds to the elongation of the recirculation while the drag enhancement corresponds to a contraction due to the elastic forces.

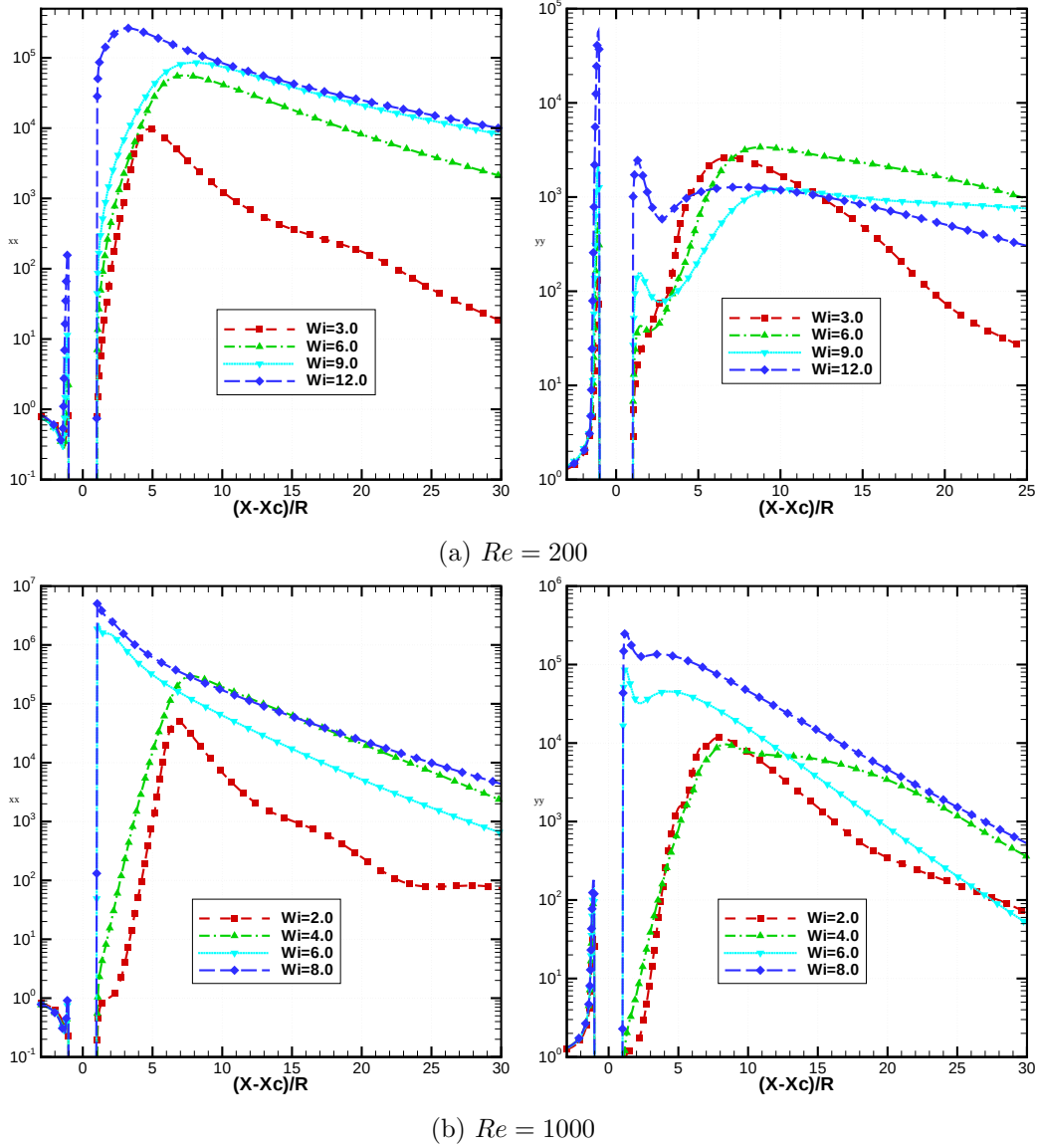


Figure 4.42: The mean conformation tensor profiles on the centerline $y = 0.5$

In figure 4.42, we see that the maximum of the mean conformation tensor profiles on the centerline increases as the Reynolds number grows. The peak of σ_{xx} shifts downstream when the drag reduces, but move towards the rear of the cylinder when

the drag enhances.

4.3.5 Turbulent flow

As the Reynolds number grows in Newtonian fluid, the inertial force increases, causing shear rate enhancements in the shear layer downstream of the wake. Then Kelvin-Helmholtz instability induces the vortices spread into the whole channel as shown in figure 4.43. When the Weissenberg number increases to $Wi = 2.0$, the wake becomes regular, the vortices don't spread everywhere even if the vortices have strong interactions. When the Weissenberg number increases again the vortex shedding is inhibited. At $Wi = 5$, the Kelvin-Helmholtz instability induces small vortices at the high shear rate regions.

In Figure 4.44 are plotted the streamline snapshots at different Weissenberg numbers. The streamlines near the centerline become rigid when the Weissenberg number is less than Wi_2 as we noticed before and the attached vortex elongates at the rear of the cylinder. Beyond Wi_2 , the streamlines are much alike the streamlines at $Re = 1000$, except there are some oscillations in the wake.

Figure 4.45 shows the mean streamwise velocity profiles at both $Re = 5000$ and $Re = 3000$. The shear rate is very high near the cylinder. The zoom in figure 4.46 shows that under Wi_2 , the shear rate reduces; and over Wi_2 , the shear rate increases. Once again when the Weissenberg number exceeds Wi_2 , there is a local low velocity zone in the middle of the channel. The shear rate is very strong between the central low velocity zone and the outer of the free shear layer.

We also have plotted the mean pressure, the streamwise velocity and the conformation tensor profiles in figures 4.47 and 4.48. The base pressure increases at the drag reduction Weissenberg numbers, it drops dramatically at the rear surface of the cylinder when the Weissenberg number exceeds Wi_2 , but it recovers and forms a peak in a very short distance by the effect of σ_{xx} . The mean streamwise velocity also recovers in very short distance and forms a peak at high Weissenberg numbers. At the drag reduction Weissenberg numbers, the mean recirculation zone elongates as the profile becomes negative. The maximum the σ_{yy} is also located at the rear of the cylinder, and has two peaks downstream of the cylinder at high Weissenberg numbers.

Figure 4.49 shows the probability density function of du/dx in the square window located from 2.5 to 3 in the streamwise position and from 0.25 to 0.75 in the transverse position. The range of the elongation rate shortens as the Weissenberg number increases.

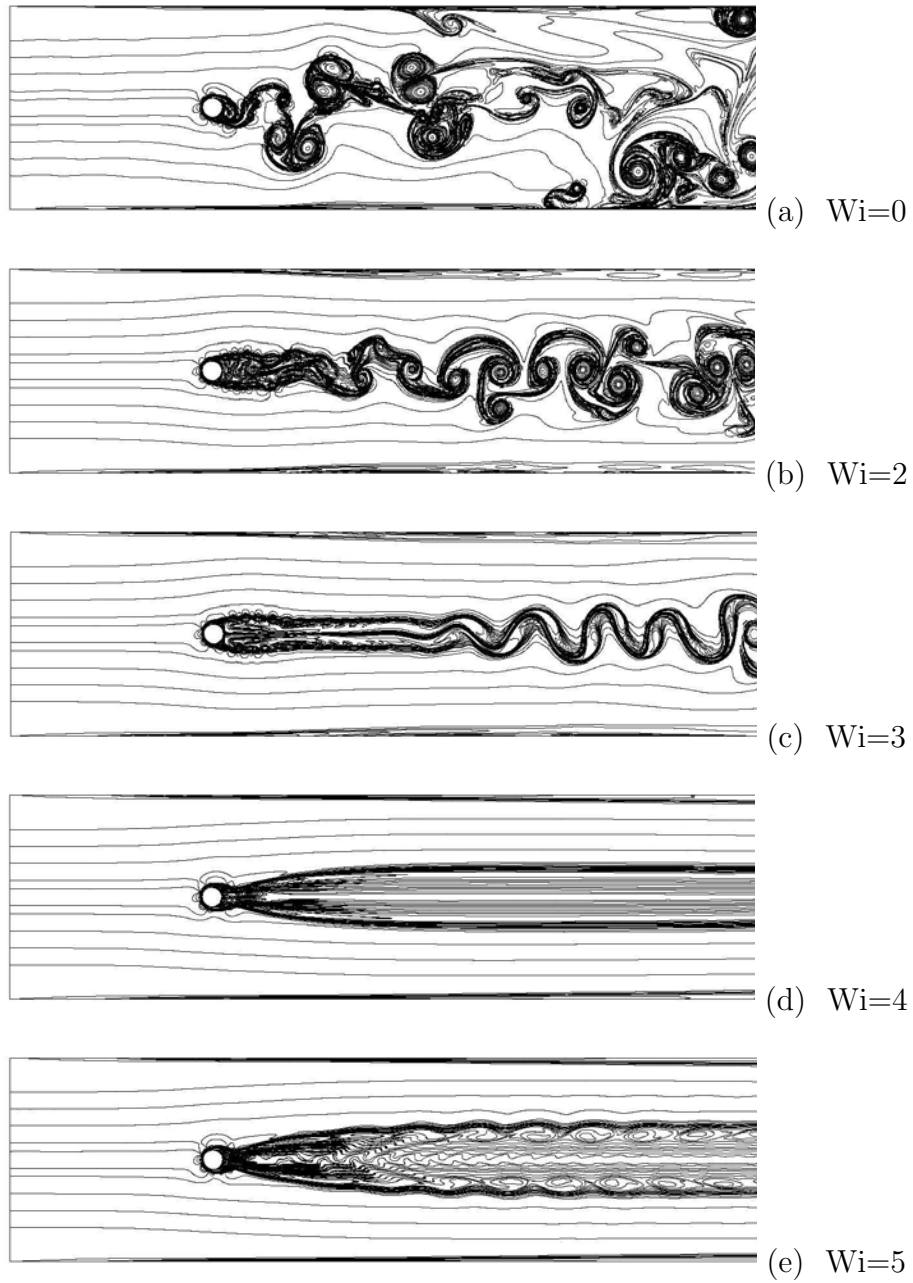


Figure 4.43: Vorticity snapshots at $Re = 5000$

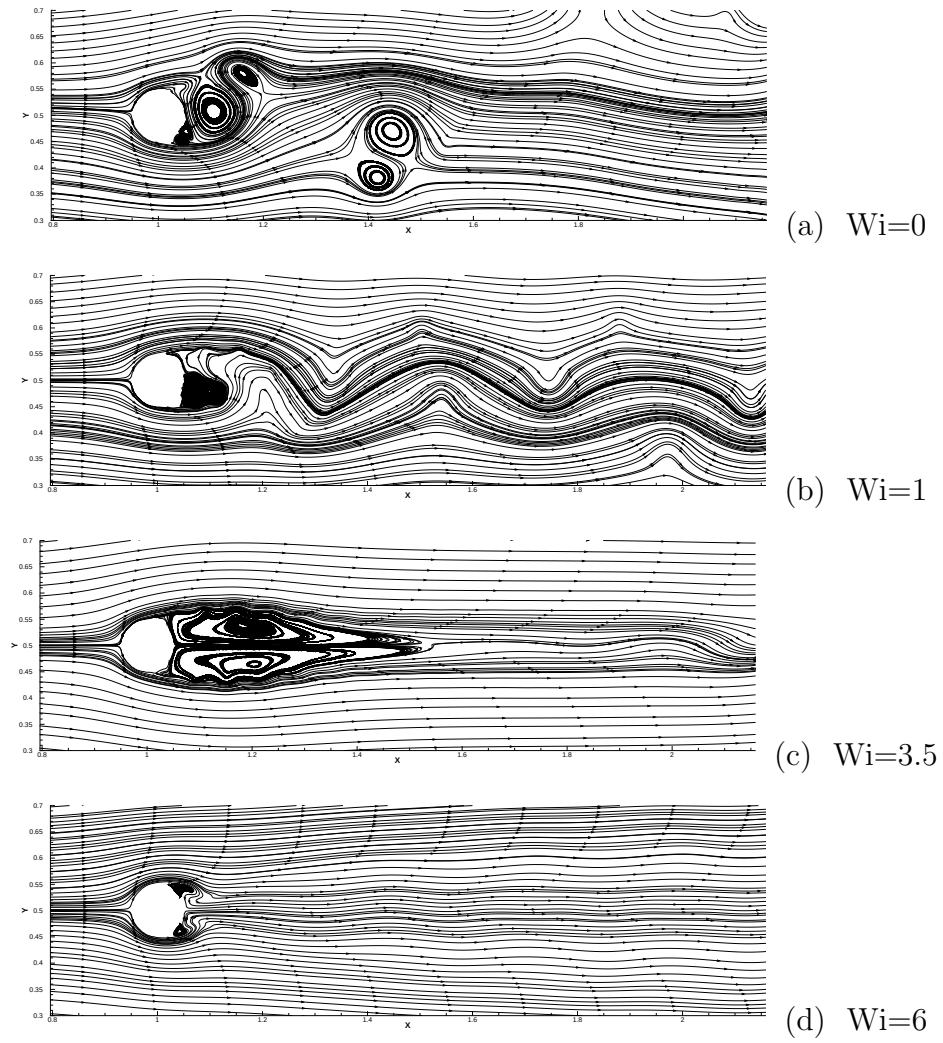


Figure 4.44: Streamline snapshots at $Re = 5000$

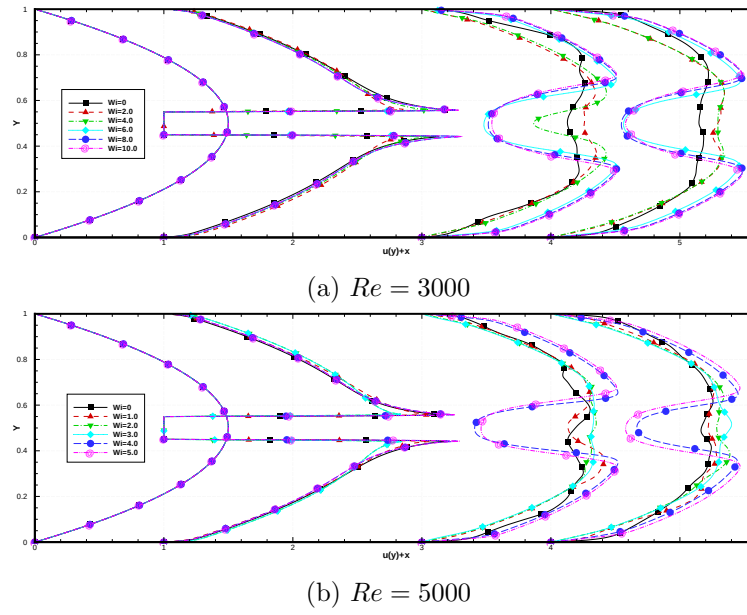


Figure 4.45: Mean streamwise velocity profiles for different positions.

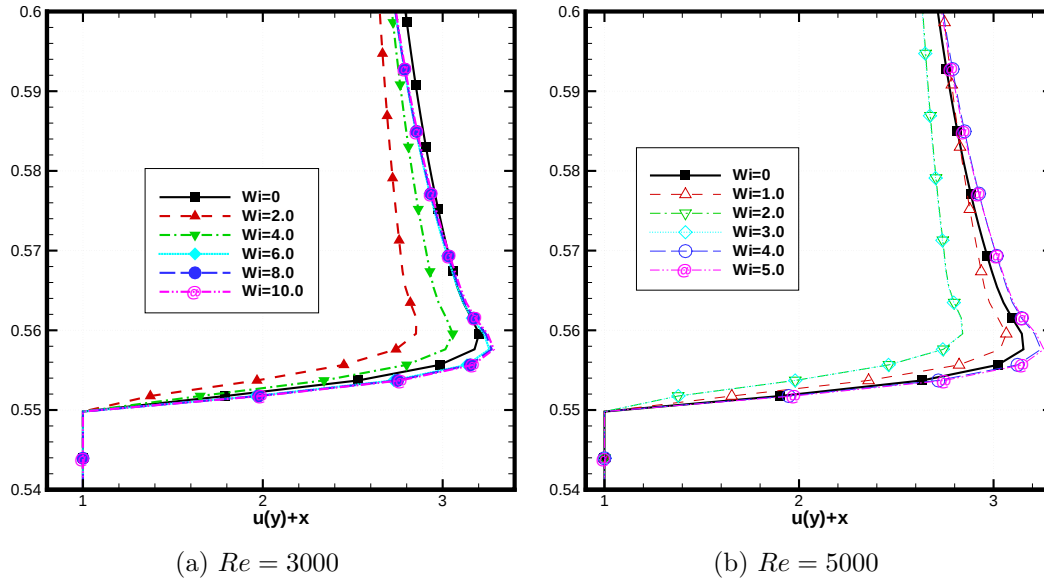


Figure 4.46: The zoom of mean streamwise velocity profile above the cylinder

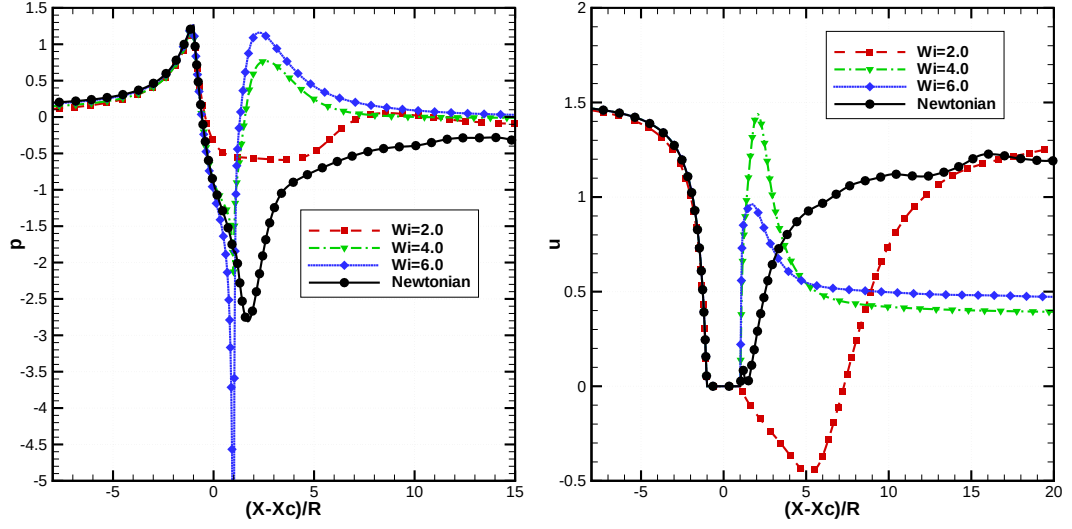


Figure 4.47: Pressure and the streamwise velocity profiles on the centerline $y = 0.5$ at $Re = 5000$

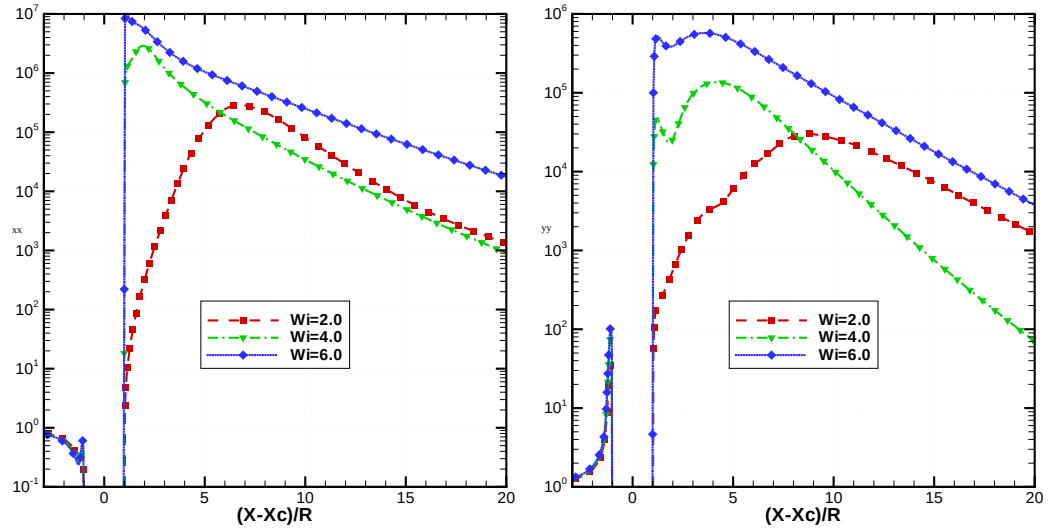


Figure 4.48: The conformation tensor profiles on the centerline $y = 0.5$ at $Re = 5000$

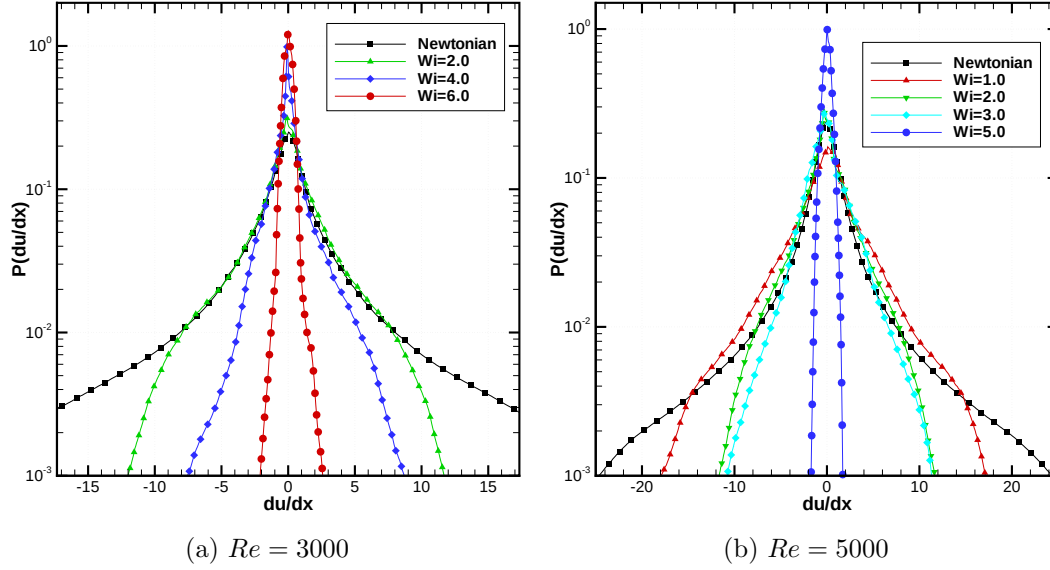


Figure 4.49: Probability density function of the elongation rate in the square window

4.4 High Weissenberg number instability

4.4.1 Stokes flow instability

As the Weissenberg number increases, the elasticity of the fluid increases. It may result in a pure elastic instability or even elastic turbulence at vanishingly small Reynolds number [62, 63]. The wake instability in viscoelastic flow past confined circular cylinders were reported by experimental observation [101] and finite volume simulation [109] at small Reynolds number. Oliveira and Miranda simulated the flow past a cylinder placed symmetrically in a channel, with a blockage ratio of 0.5, with FENE-CR constitutive model. They observed the flow transited from steady to unsteady at a critical Weissenberg number 1.3 when the solvent to the polymer solution viscosity ratio is 0.59. In our results, we found the similar unsteady drag from $Wi = 10.0$ in diluted polymer solution at $Re = 0.1$ with a blockage ratio 0.1 (see figure 4.50). The amplitude of the drag is very low at $Wi = 10$, but quite important at $Wi = 24$. As we see in the figure, the amplitude of the drag is about six percents of the mean drag at $Wi = 24$. Furthermore, the frequency of the drag oscillation also increases from figure 4.51.

Figure 4.51 shows the discrete Fourier transform of the drag and of the velocity at monitor point (1.3, 0.4). Both the drag and the streamwise velocity have the same frequency, it means the unsteady flow doesn't behave as for high Reynolds number, where the frequency of the drag is twice that of the velocity. As the Weissenberg number increases, the frequency increases quickly. Unfortunately, due to numerical

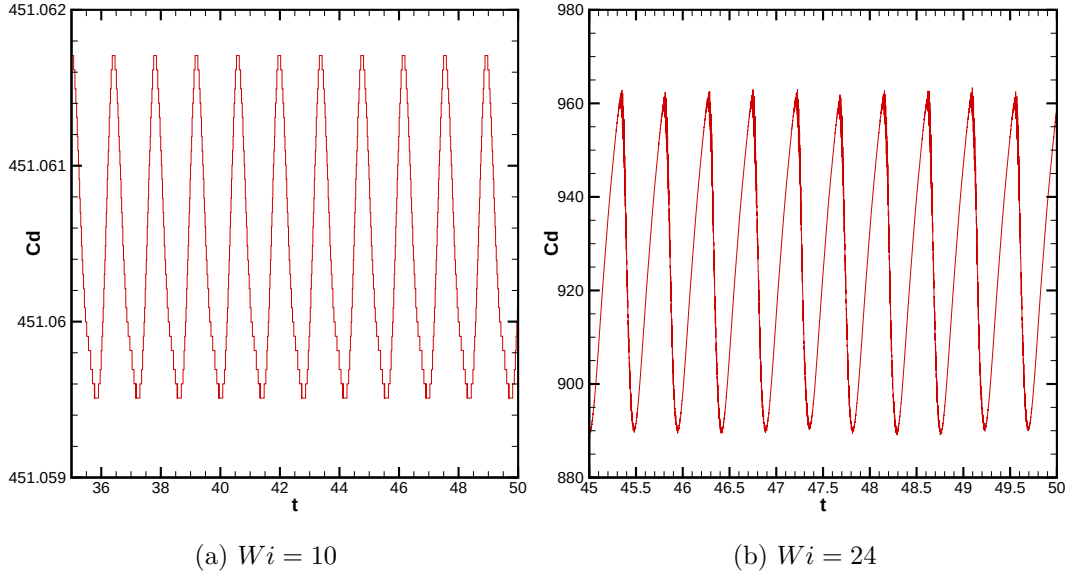


Figure 4.50: The evolution of the drag coefficient on the cylinder

instability, it is not possible to get result at higher Weissenberg numbers.

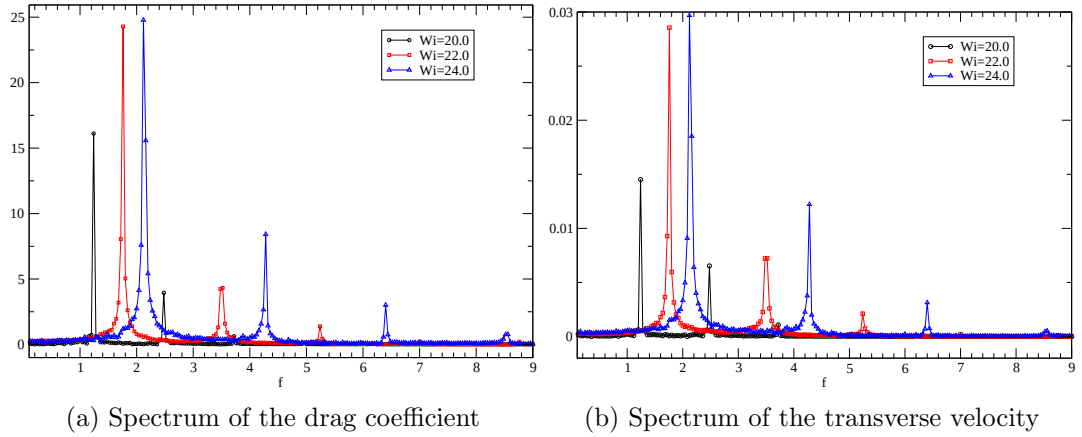


Figure 4.51: The discrete Fourier transform (DFT) of the drag and of the streamwise velocity at monitor point (1.3, 0.4)

Figure 4.52 shows the streamwise velocity and the streamline isovalues around the cylinder at $Wi = 24$. The coordinates are normalized by the radius of the cylinder. There is no symmetry any more, neither up-down nor fore-aft. At the rear of the cylinder, the regular Stokes flow is broken by the effect of the elasticity.

The flow is time-periodic in the wake. We have plotted the phase portrait at several monitoring points shown in figure 4.53. Three cylinder diameters downstream, both the

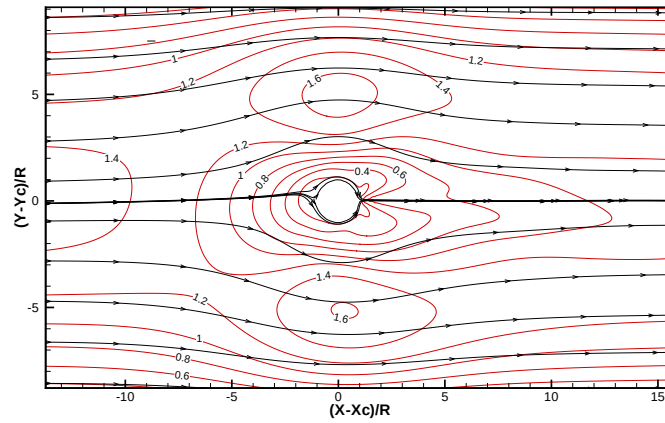


Figure 4.52: The streamwise velocity and the streamline isovals near the cylinder at $Wi = 24$; $Re = 0.1$

streamwise velocity and the transverse velocity oscillate with a significant amplitude that increases as the Weissenberg number grows. The flow shows unsteadiness far away downstream the cylinder.

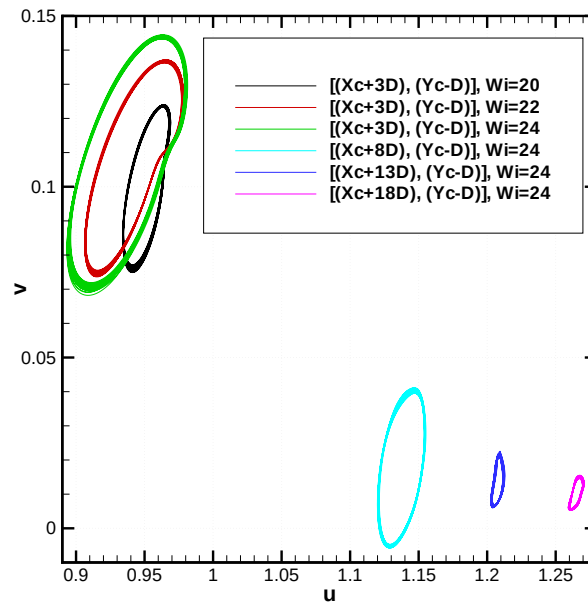


Figure 4.53: The phase portrait at the monitor points

4.4.2 The up-down symmetry broken by polymer

As we have shown in figures 4.16, 4.24 and 4.52 the up-down symmetry of the flow is broken by the polymer elasticity when the Weissenberg number is high despite the fact that the problem is geometrically up-down symmetric. This phenomenon never happens for the Newtonian fluid. So this behavior is not due to the computational process. A notable character of the fluid is to never lack small disturbance at any time and any place. The fluid molecules keep random motion and show a well-known Brownian motion at tiny scale. The fluid can keep the balance of forces after it loses the up-down symmetry. Even though it is impossible for the newtonian fluid, but the viscoelastic fluid includes a new elastic force which may change the balance pattern of the force act on the fluid.

Figure 4.54 shows the transverse velocity on the centerline. The mean transverse velocity is zero for the newtonian fluid or for the viscoelastic fluid at small Weissenberg number. Beyond $Wi = 2.0$, the transverse velocity increases near the cylinder as the Weissenberg number increases. Furthermore, the asymmetric transverse velocity grows as the Reynolds number increases for low Reynolds number flows. In the same way, the conformation tensor component σ_{xy} on the centerline is not symmetric when the Weissenberg number is high, as shown in figure 4.55. The asymmetric part mainly appears in front and at the rear of the cylinder. The variation of σ_{xy} along the centerline produces an elastic force, which makes the transverse velocity at the centerline. In fact the value of σ_{xy} is strongly coupled to the other components σ_{xx} and σ_{yy} , they have a maximum value on both side of the cylinder as we have reported, hence after a small disturbance, the shear may not zero, it is easy to produce a proper σ_{xy} value, which can effect the balance and symmetry. Let's point out the up-down symmetry is known to break at Reynolds number from 10 to 40 from early experimental works [74], even though it was not reported in this paper. We can observe the up-down asymmetric streamlines on their flow visualization as the polymer concentration grows. This also indicates the up-down symmetry could be broken by the high elasticity of the polymer.

4.4.3 The instability at high Reynolds numbers

At high Reynolds numbers, several investigations show that the polymer suppresses the vortices at different scales [100, 105, 82, 75], but very few give comment on the flow instability. As we have seen in figure 4.43, the Kelvin's 'cat's-eye' pattern of the vorticity appears in the free shear layer at $Re = 5000$ as the Weissenberg number increases. When the Weissenberg number is higher than Wi_2 , the shedding is inhibited and there isn't instability visible in the wake. The Kelvin-Helmholtz instability occurs

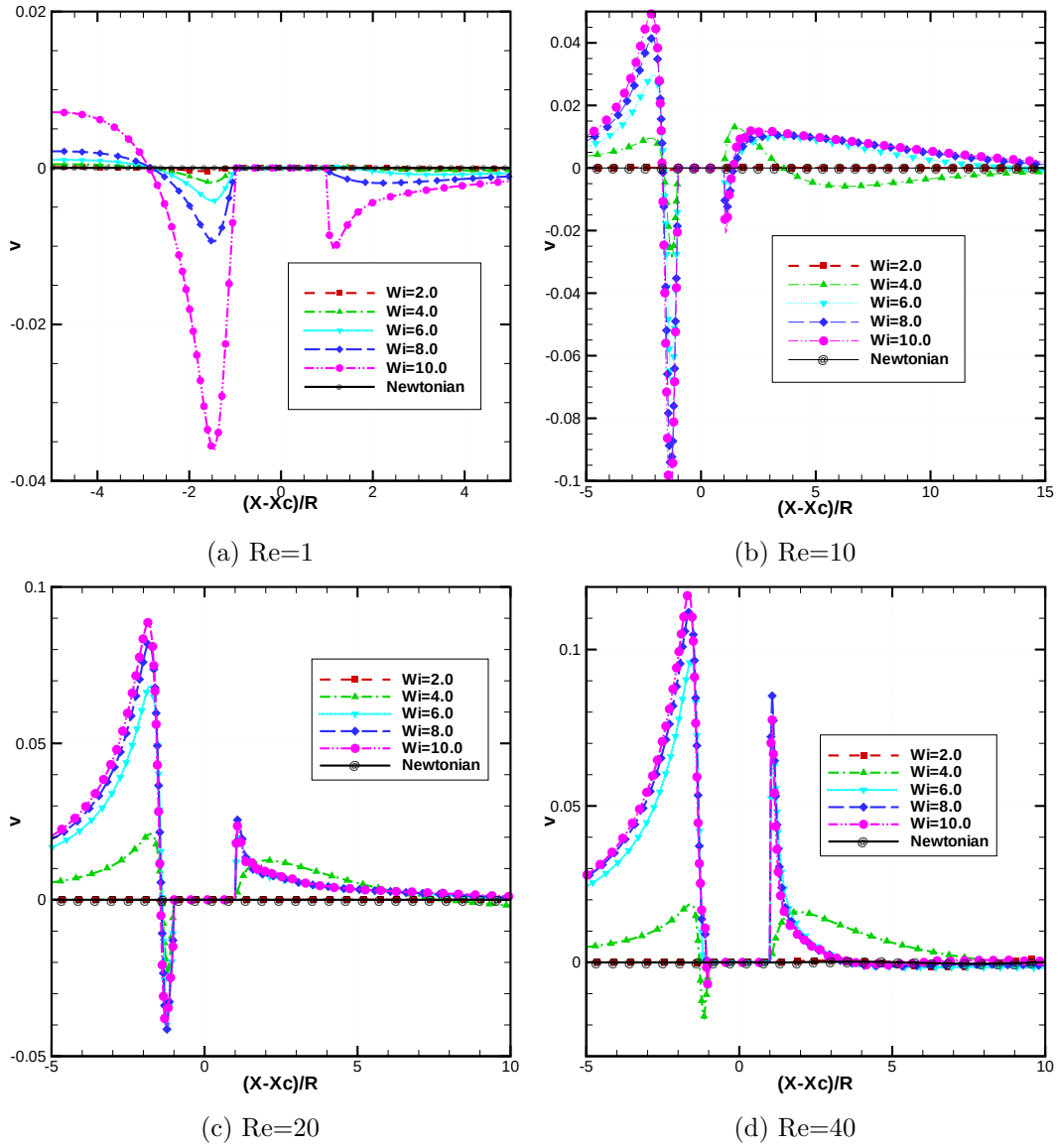


Figure 4.54: The transverse velocity on the centerline

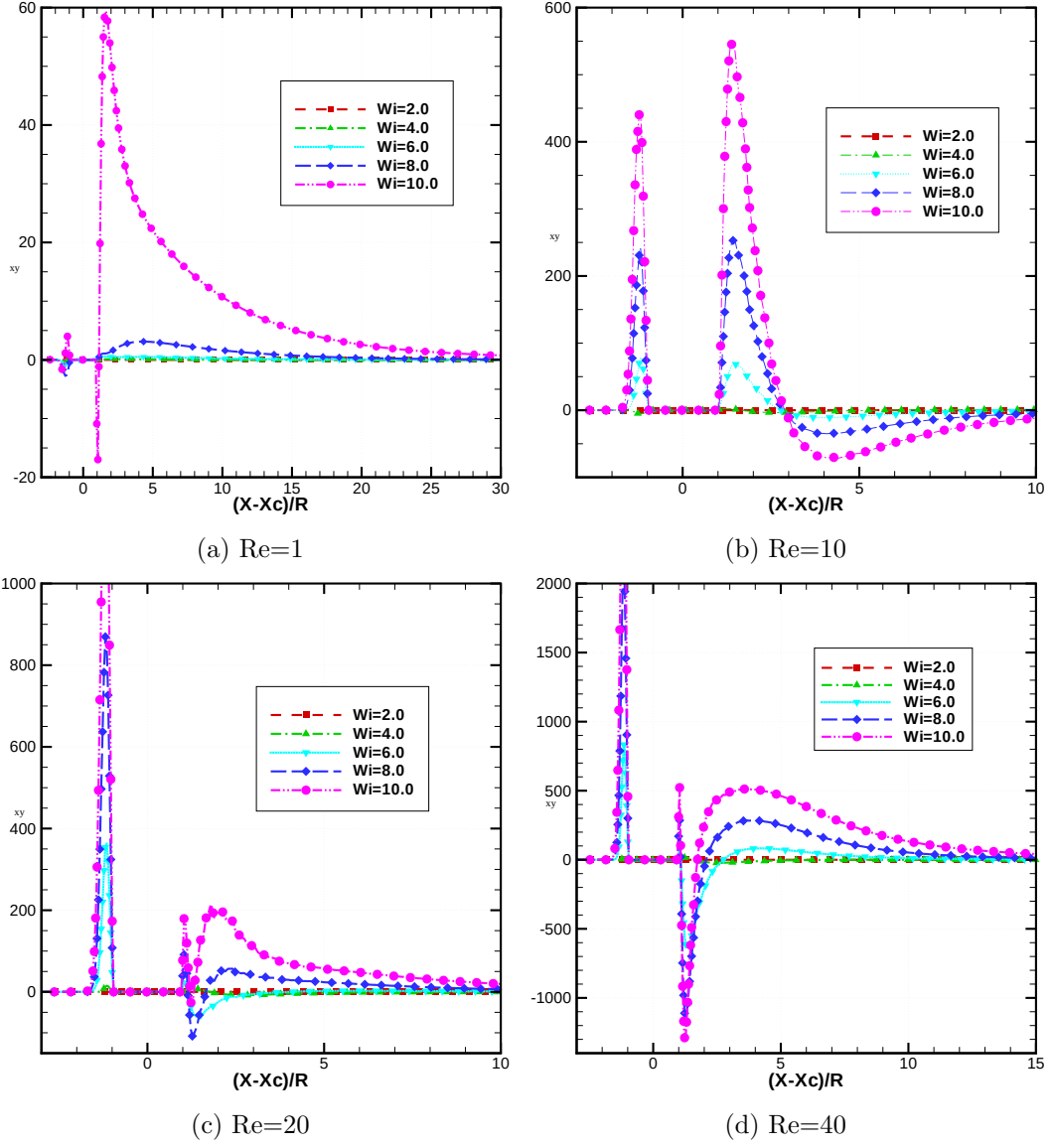


Figure 4.55: The conformation tensor component σ_{xy} on the centerline

at other Reynolds number when the Weissenberg number is high. As an example the flow at $Re = 3000$ and $Wi = 6.0$ is plotted in figure 4.56. Figure 4.57 shows the evolution of the velocity at the monitor point $(2.3, 0.4)$. Both streamwise velocity and transverse velocity oscillate at the same frequency about 3.8, the transverse velocity having larger oscillating amplitude than the streamwise velocity.

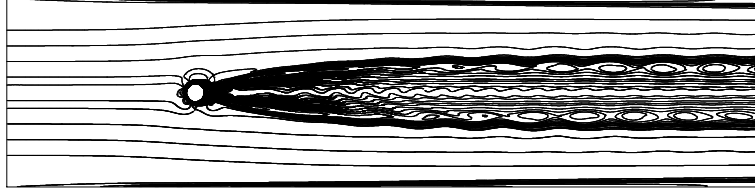


Figure 4.56: An instantaneous vorticity field snapshot at $Re = 3000$ and $Wi = 6.0$

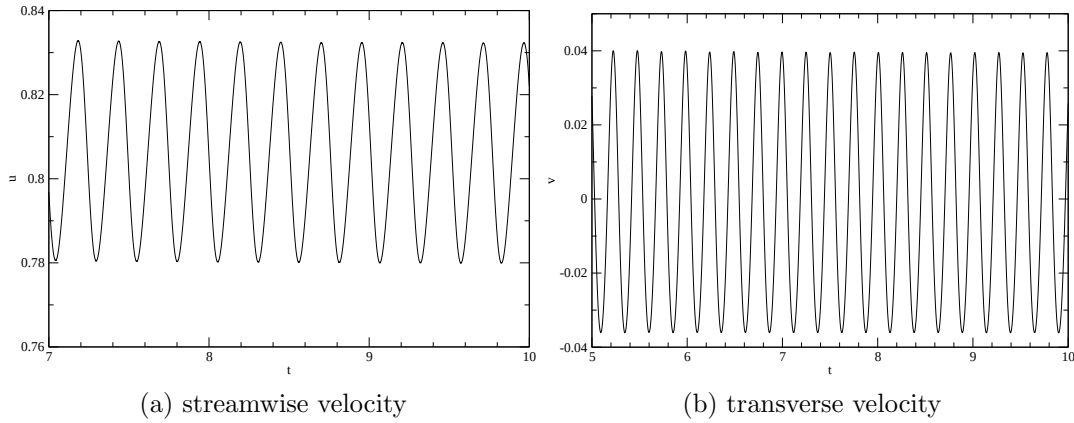


Figure 4.57: The evolution of the velocity at the monitor point $(2.3, 0.4)$ at $Re = 3000$ and $Wi = 6.0$

The Kelvin-Helmholtz instability can occur when a velocity shear is present within a unique fluid, or when there is sufficient velocity difference across the interface between two fluids. One example is wind blowing over a water surface, where the wind causes the relative motion between the stratified layers (i.e., water and air). The instability will manifest itself in the form of waves being generated on the water surface. In a homogeneous fluid, the instability can come from the high shear rate in the fluid. In the wake at high Weissenberg number, du/dy is the prominent contribution to the vorticity as shown in figure 4.58. The maximum shear rate in the wake at high Weissenberg number is 10 times bigger than in Newtonian fluid. However, the shear rate hasn't increased as the Weissenberg number grows beyond Wi_2 , as shown in figure 4.45. Hence, the shear rate is not the only reason to induce instability in the viscoelastic

wake flow. Indeed the shear rate at $Wi = 4$ is even larger than that at $Wi = 5$ when the Reynolds number is 5000, but we haven't observed the instability at $Wi = 4$, as shown in figure 4.43. The variation of the conformation tensor, especially σ_{xy} making the fluids to swirl, may induce the instability. As we see in figure 4.59, the mean σ_{xy} at $y = 0.6$ when $Re = 3000$, increases as the Weissenberg number grows beyond Wi_2 . If the shear rate is not the direct reason of the instability, one may expect a similar instability happens at lower Reynolds number and higher Weissenberg number. However, it is difficult to simulate viscoelastic flows at very high Weissenberg numbers. Thanks to the two-dimensional soap film experiments [80], we observe a similar instability at $Re = 700$ and $Wi = 30$, as shown in figure 4.60.

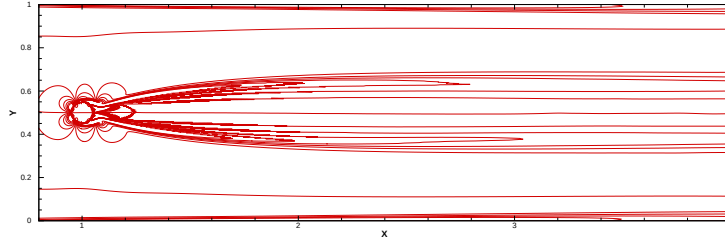


Figure 4.58: The mean du/dy field at $Re = 3000$ and $Wi = 6.0$

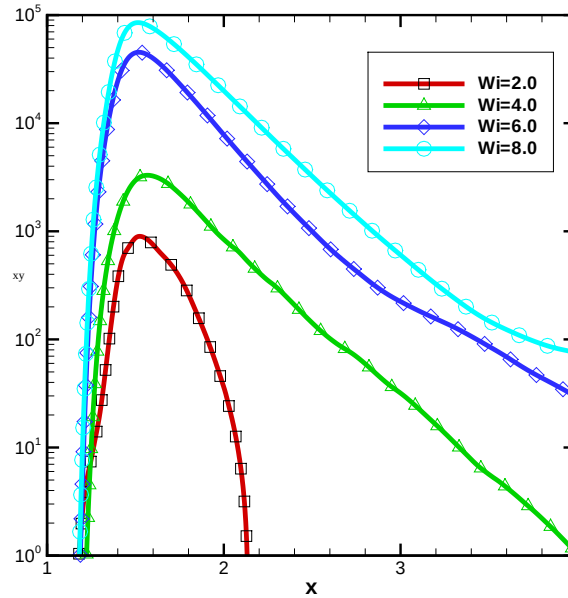


Figure 4.59: The mean conformation tensor component σ_{xy} at $y = 0.6$ and $Re = 3000$

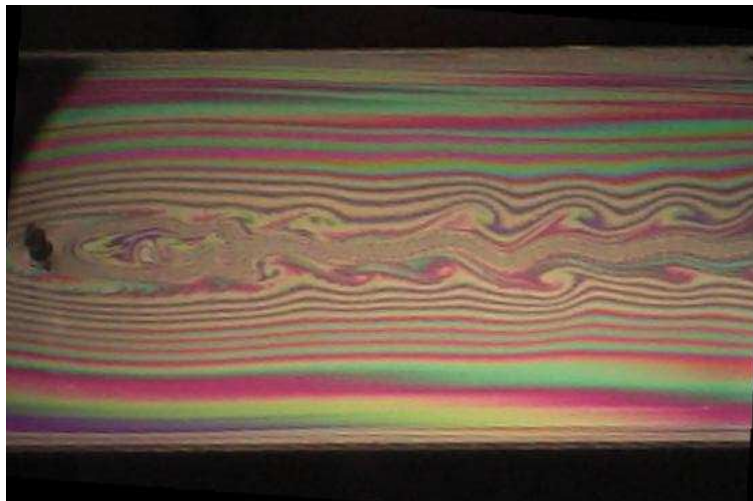


Figure 4.60: Flow visualization of a two dimensional soap film flow over a cylinder at $Re = 700$ and $Wi = 30$

Chapter 5

Two dimensional grid turbulence

As we have reported in previous chapters, a small amount of polymer added in solutions has an important effect on the flow behavior and in particular on the drag coefficient of the cylinder. For other flow situations like pipe flow or channel flow, the polymer also plays an important role on the turbulent flow structure and on the pressure losses. Understanding the polymer-turbulence interaction is thought to be the route to understand the drag reduction mechanism. Many research articles have appeared on the interaction between turbulence and additives [100, 75, 82, 105, 135, 136]. One of the most important topics is how the polymer affects both the near-wall turbulent boundary layer and the homogeneous turbulent flow far away from the wall region.

In this chapter, our goal is to examine the effect of the polymer on two-dimensional grid turbulence in a channel. Numerous experimental studies on three dimensional grid turbulence with polymer additives have been reported [100, 44]. McComb et al. [100] found that the polymer additive reduced both the turbulent intensity and the rate of decay behind the grid; the polymer attenuates the dissipation-range spectra after a threshold concentration. However the threshold concentration hasn't been found in the experimental works by Doorn et al. [44], they found that the turbulent structure in polymer solutions becomes more anisotropic than that in Newtonian fluids. Two dimensional soap film grid turbulence shows the important suppression of large scale fluctuations [2, 82]. Following on the soap film experiment, our numerical experiment setups are shown in figure 5.1.

Let Ω denote a rectangular bounded domain in two dimensions. The union of all the solid cylinders are represented by Ω_s and $\Omega_f = \Omega \setminus \Omega_s$ is the incompressible fluid domain. The boundary of Ω_f is defined by $\partial\Omega_f = \partial\Omega_s \cup \Gamma_i \cup \Gamma_w \cup \Gamma_o$, the boundaries are equally set with the flow past a cylinder in previous chapters. The two dimensional channel is represented by the computational domain $\Omega = (0, 4) \times (0, 1)$. To generate grid

turbulence an array of circular cylinders with the same diameter 0.05 is set at $x = 0.9$ equidistantly. The number of cylinders is 5, 8 and 10 in order to produce different blockage ratios, and are referred to as case A, case B, case C and case C' respectively. Two horizontal arrays of additional cylinders are added in order to increase the number of the vortices in case C'. The injection scale k_{inj} is given by the diameter of the cylinders and corresponds to $k_{inj} = 20$ in order to capture the inverse energy cascade. In all the simulations, the evolution in time of the velocity $\mathbf{u} = (u, v)$, of the vorticity w and of the pressure p is recorded at six monitoring points, which are located on the row $y = 0.4$ and between 1.3 and 3.8 equidistantly as shown by stars in the case C of figure 5.1.

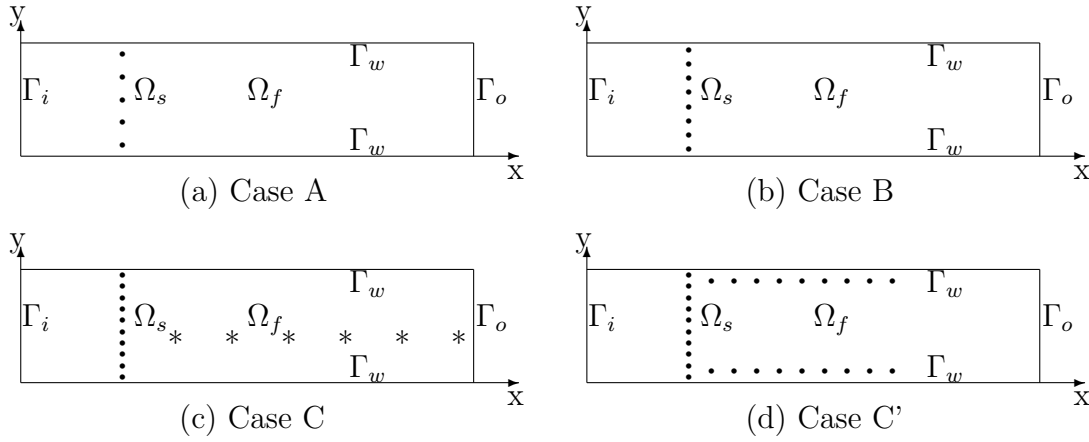


Figure 5.1: Numerical experiment setups

5.1 The flow fields and the modification by polymer

Vorticity snapshots in the wake of the cylinders for the case C of figure 5.1 at $Re = 5000$ are plotted in figure 5.2. When the Weissenberg number is zero, the fluid is Newtonian, a great number of vortices are created behind the cylinders, and these vortices interact with others in the decaying channel (the definition of the Weissenberg number is given in the previous chapters, the radius of the cylinders is used as the reference length). The size of the vortices increases in the channel, it indicates the merging of the vortices that is very different to the three-dimensional phenomenon. As we have pointed out, there are two ranges in the two dimensional turbulence, the inverse energy cascade and the direct enstrophy cascade. The vortical structures play an important role for the inverse energy cascade; the merging of vortices in the figure suggests the inverse energy transfer from the injection scale to larger scales. The filamentary structures can be

associated to the forward enstrophy transfer. As the Weissenberg number increases, we do not observe the growth of the vortical structures as obviously as in the Newtonian flow. It may suggest the suppression of the inverse energy cascade by polymer which was reported in two-dimensional soap film experiments [2, 82]. As the Weissenberg number grows, we can observe that the vortices are suppressed in the close wake. As the distance to the cylinders increases, the vortices begin to interact with others. If we look at the figures at high Weissenberg numbers carefully, the instability of the shear layer is caused by the jets emanating from the region between the cylinders but not by the cylinder wakes. This is the reason why we adopted the cases A and B for which the effect of the strong jets is reduced.

The turbulent boundary layer is crucial for understanding drag reduction. However the two-dimensional turbulent boundary layer has not received much attention until a recent and important finding [64, 145]. One of the most important reason, is that most turbulent flows in laboratory or in nature are three-dimensional. However, the two-dimensional boundary layer may shed some light on the drag reduction mechanism.

In three-dimensional turbulence, the boundary layer can be divided into the inner layer and the outer layer. In addition the inner layer can be divided into the linear layer, the transitional layer and the logarithmic layer. In the linear layer, the viscosity of the fluid plays an important role, the local frictional Reynolds number is less than 10. The frictional Reynolds number (y^+) is defined by:

$$y^+ = \frac{u_\tau y}{\nu} \quad (5.1)$$

where y represents the distance to the wall, ν denotes the kinematic viscosity of the fluid, u_τ is the frictional velocity which can be expressed by $\sqrt{\tau_w/\rho}$, where τ_w is the shear stress on the wall and ρ the density of the fluid. Hence the frictional Reynolds number reflects the effect of the viscosity with the variation of y . Prandtl thought that the mean velocity profile in the turbulent boundary inner layer can be expressed by

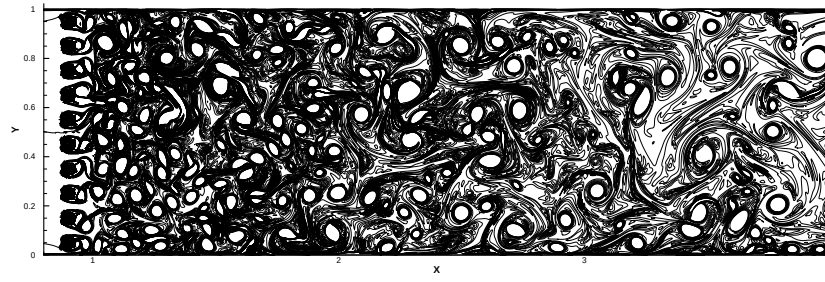
$$u^+ = \phi_1(y^+) \quad (5.2)$$

where the dimensionless velocity u^+ is linked to the local mean velocity by $u^+ = u/u_\tau$. Equation (5.2) yields

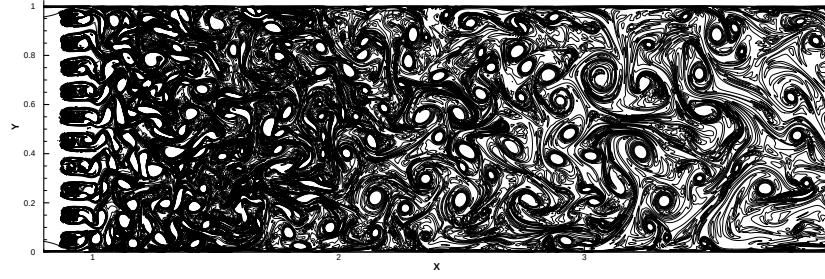
$$\frac{du}{dy} = \frac{u_\tau^2}{\nu} \frac{d\phi_1}{dy^+}. \quad (5.3)$$

When the frictional Reynolds number y^+ grows, the velocity gradient du/dy should be independent of ν . So the right-hand side of equation 5.3 should exclude ν . Thus we have

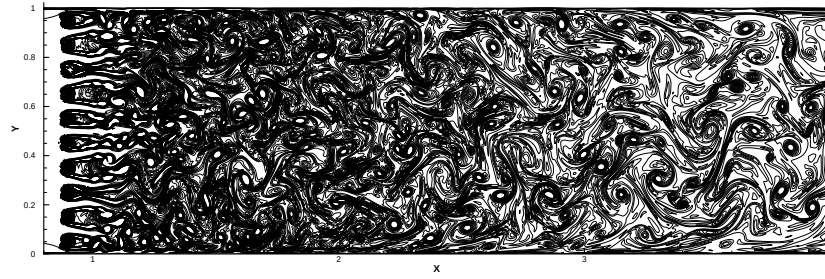
$$\frac{d\phi_1}{dy^+} = \frac{C_1}{dy^+}. \quad (5.4)$$



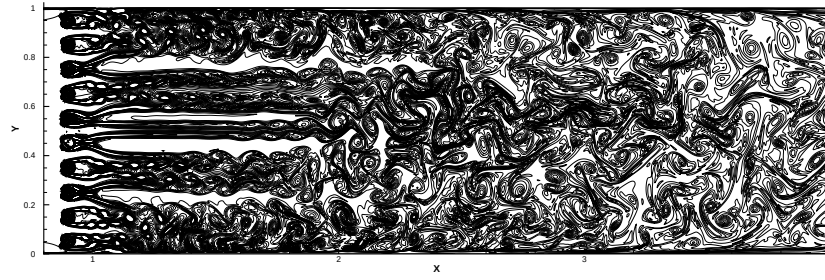
(a) $Wi=0$



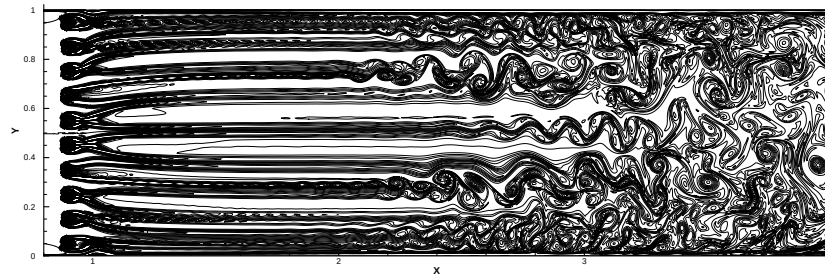
(b) $Wi=1$



(c) $Wi=2$



(d) $Wi=3$



(e) $Wi=4$

Figure 5.2: Vorticity snapshots for case C at $Re = 5000$

Consequently there is the following integral velocity:

$$\text{either} \quad u^+ = C_1 \ln y^+ + C_2 \quad \text{or} \quad u^+ = C'_1 \log_{10} y^+ + C_2 \quad (5.5)$$

The three dimensional experimental results have given $C_1 = 2.5$ or $C'_1 = 5.75$ and $C_2 = 5.5$. It is clear that experimental scatter should allow some variations in the values of C_1 and C_2 , for example Hinze prefer to take $C_1 = 2.44$ and $C_2 = 4.9$ that represent a good average of the various investigations.

To our knowledge, few people report the values of C_1 and C_2 in two dimensions, neither numerical simulations nor two-dimensional experiments except a recent theoretical prediction [156]. In figure 5.3, we plot the mean velocity distributions in two dimensional turbulent boundary layer for the Newtonian fluid at $Re = 5000$. Even though the upstream cylinders distributions are different for the three cases, the turbulent boundary layer keeps a similar character. Although different from the three dimensional turbulent boundary layer, there are probably two distinct logarithmic layers in two dimensions. From the figure, we can observe the velocity profiles are far from the three dimensional logarithmic layer which is also plotted in the figure. Both logarithmic layers in two dimensions have larger constants. The constant C_1 is about 3.65. However at $Re = 500$, we get $C_1 = 2.75$ as shown in figure 5.4. This suggests that the constant may not be universal, which is in agreement with the theory prediction [156]. Furthermore the size of the linear layer and of the logarithmic layers are shorter than that in three dimensions as the constants are larger. It is not easy to determine whether two logarithmic layers exist in two-dimensional turbulent boundary layer (there is a second short logarithmic layer in figure 5.3). In three dimensions, few relation about energy cascade in boundary layer was reported, however let's remind the logarithmic layer velocity expression also can be derived through the balance between turbulent energy production from the mean flow and turbulent energy dissipation. Recently, the difference of the friction drag between the two-dimensional flow in a soap film experiment and the three-dimensional turbulent flow was found [145]. They suggest that there may be a link between the macro friction drag and the turbulent energy spectrum. In two dimensions, the injection turbulent production may balance with the sum of the turbulent dissipation and the inverse energy transfer. Because two dimensional turbulent boundary is still a new problem in fluid mechanics, few literatures can be found, some consideration should wait for the further studies by experiment and simulation.

In three dimensions, the turbulent boundary layer is dramatically modified by adding a small amount of polymer. Virk [152] found the friction in polymer-solution pipe flow

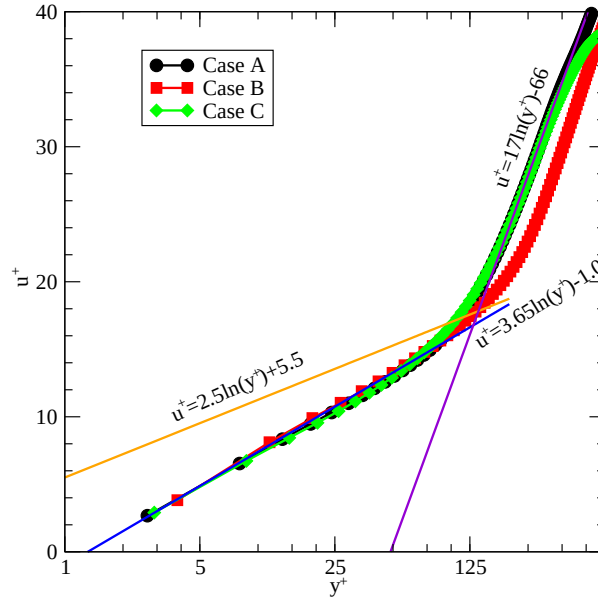


Figure 5.3: Mean velocity distributions in two dimensional Newtonian turbulent boundary layer at $Re = 5000$

can be rationalized by assuming the following expressions for the velocity profile:

$$\begin{aligned}
 u^+ &= y^+ & \text{if } 0 < y^+ < 11.6 \\
 u^+ &= 11.7 \ln y^+ - 17 & \text{if } 11.6 < y^+ < 11.6 + \delta_e^+ \\
 u^+ &= 2.5 \ln y^+ + 9.2 \ln (11.6 + \delta_e^+) - 17 & \text{if } 11.6 + \delta_e^+ < y^+ < Re_\tau/2
 \end{aligned} \tag{5.6}$$

where δ_e^+ denotes an elastic transitional layer and $y^+ = Re_\tau/2$ represents the centerline of the pipe flow. Thus the velocity profile at logarithmic layer adds an extra elastic layer from the Newtonian fluid, that induces drag reduction. The maximum drag reduction limit has the velocity profile $u^+ = 11.7 \ln y^+ - 17$. Figure 5.4 shows the mean velocity distribution for the viscoelastic fluids in the two dimensional turbulent boundary layer. As the Weissenberg number increases, the velocity increases at the same dimensionless position (y^+) until a critical Weissenberg number. This suggests that the friction drag reduces as the Weissenberg number increases and reach a maximum at a critical Weissenberg number. There is no so-called ‘elastic layer’ which is reported in three dimensional numerical simulations and experiments. The way the drag reduction occurs is also different in three dimensions. There is a parallel upward shift for the logarithmic velocity layer to increase the velocity in three dimensions, but here the length of the logarithmic layer is reduced and ‘second logarithmic layer’ shifts toward the left in two dimensions as shown in the figure.

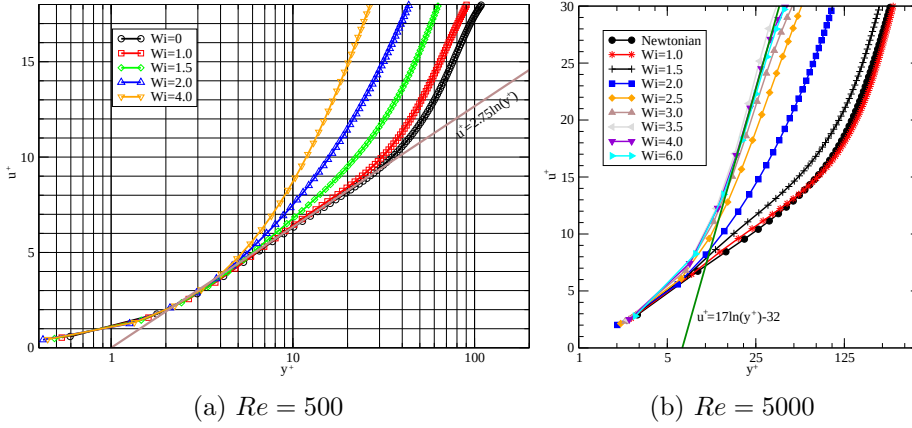


Figure 5.4: Mean boundary layer dimensionless velocity for viscoelastic fluids in two dimensions

5.2 The effects of the polymer to turbulent structures and the spectra

We plot the energy spectra at the last monitoring point for the Newtonian fluid in Figure 5.5. The spectra of the two velocity components and the energy are shown in the figure, which shows the flow far away from the cylinder is almost isotropic. The spectra were computed from the time series under the Taylor frozen turbulence hypothesis, which has been proven reasonable in the soap film experiments [79]. The time series signal are separated into many parts. Each part is multiplied by a Hanning function in order to remove the discontinuous effects on the boundary of the signal, then they are transferred to Fourier space. The spectra shown in the figure are averaged by many parts of the spectra (see [24]). In the figure, the spectra of both longitudinal velocity and transverse velocity are consistent at the small scales (or high wave numbers), larger than the injective wave number around 20, it suggests the small scales are isotropic at this position. But at low frequencies the spectrum of the longitudinal velocity is larger than that of the transverse velocity as already shown in the soap film experiments [2]. The reason is that the channel walls restrict the development of the large scales in the transverse direction. On both sides of the injection scale, the spectra have different slopes which correspond to the inverse energy cascade range at large scales and to the direct enstrophy cascade at small scales [9, 81, 84]. The theory also expects the inverse energy cascade range has a slope of $-5/3$ while the direct enstrophy cascade range has a slope of -3 . Our simulation have an approximative slope of $-5/3$ at the large scales which suggests the vortices merge to larger ones as observed in the vorticity fields

shown in figure 5.2. In the enstrophy cascade range, we have a slope of -2.7 instead of -3 predicted by the theory. On the one hand the decaying flow used in the simulation makes it different, and on the other hand wide deviations of -3 are reported in other numerical and experimental works [141].

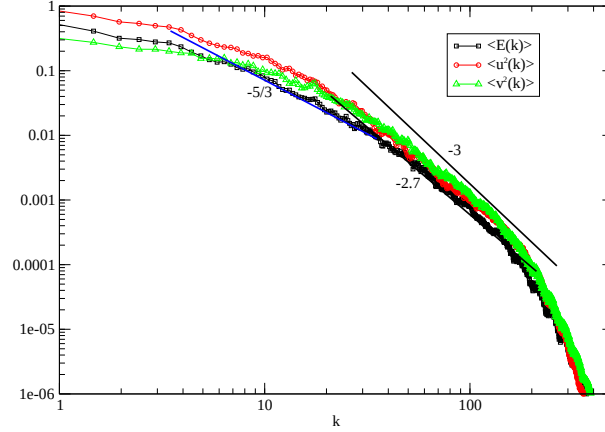


Figure 5.5: The energy spectra of the case C at $Re = 5000$ for the Newtonian fluid at the monitoring point (3.8,0.4)

In this part, we firstly focus on the grid turbulence with 5 cylinders, i.e. case A in our setup (figure 5.6). It has a small blockage ratio ($B=0.25$), which is the widely used setup in the soap film experiments.

Figure 5.7 shows the turbulent energy power spectra for case A. They are obtained in the same way from time series at the downstream monitoring point that we will use till the end of the chapter. As can be seen on the two subfigures, the energy spectrum is suppressed selectively as the Weissenberg number increases. When the Weissenberg number is small, the energy spectrum is almost the same at small scales but is significantly attenuated at large scales by the polymer. This is very similar to the soap film experimental observation [2, 82]. On the other hand, the whole energy power spectrum is suppressed at high Weissenberg number. The same conclusion can be made from the enstrophy spectra shown in figure 5.8. The almost horizontal enstrophy spectra in the first decade decrease significantly when the Weissenberg number grows. The decrease at the small scales is less pronounced.

The application of wavelets to fluid dynamics can decompose the vorticity field of the turbulent flow into coherent and incoherent parts by the orthogonal character of the wavelet. The vortices and the vorticity filaments can be separated as the coherent and incoherent parts by the strongest and the weakest wavelet coefficients [21]. We hence may use the wavelet packets to analyse the turbulent flow in the last part of

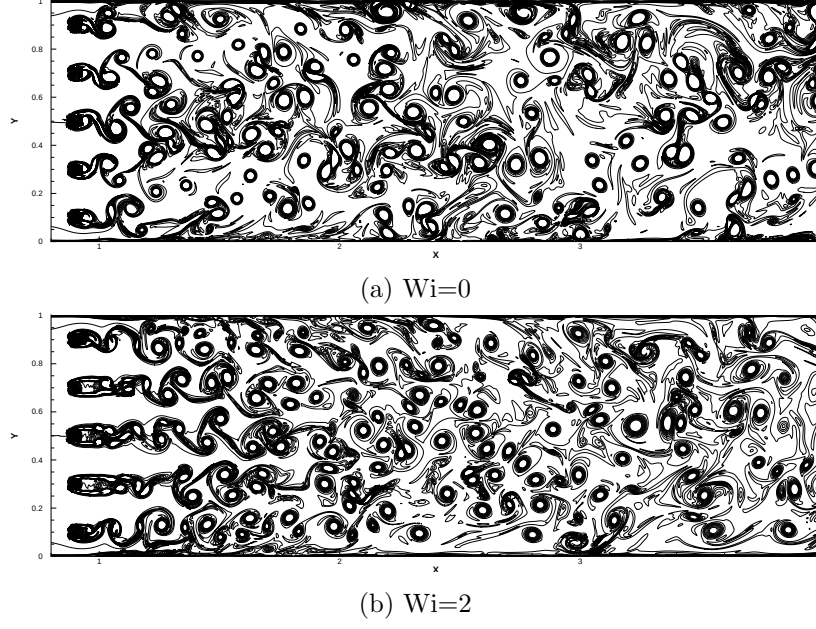


Figure 5.6: Vorticity snapshots for case A at $Re = 5000$

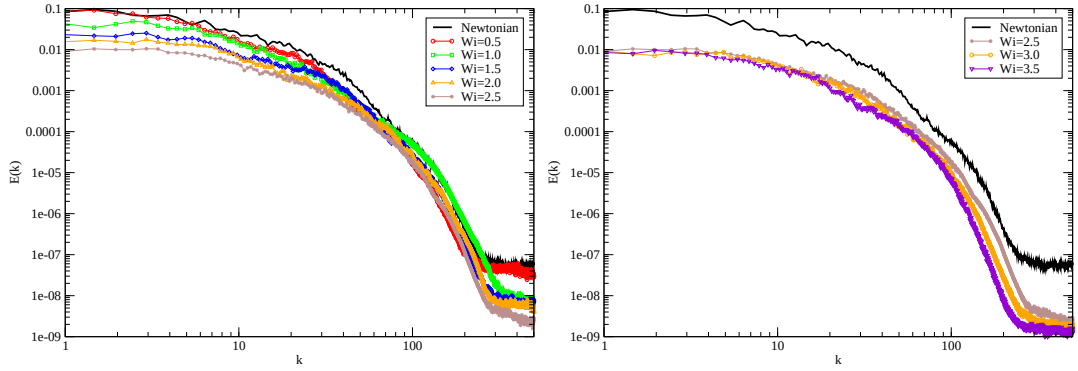
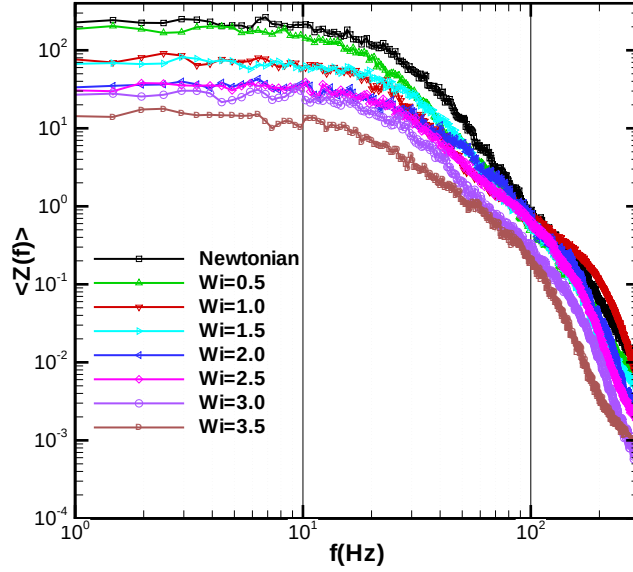


Figure 5.7: The turbulent energy power spectra for case A at $Re = 5000$

Figure 5.8: The enstrophy power spectra for case A at $Re = 5000$

the channel, i.e. the window with $x > 3.0$. The detailed method concerning the wavelet packets can be found in [24]. Let's look at the vorticity filament subfields thanks to the wavelet packets filtering in figure 5.9, these filaments were proved to play an important role in the creation of the direct enstrophy cascade. From these photos, one can see the vorticity filaments are suppressed as the Weissenberg number increases. These filaments may also appear inside of the vortices. Even though the turbulent energy is small for these filament, but they own a significant part of and so play a significant role for the enstrophy transfer [25]. The ratio $Z(k)/E(k)$ in the same window is plotted in figure 5.10. In the Fourier space this ratio should approximately be equal to $(2\pi k)^2$. This relation is well satisfied and is not affected by the polymer. From the energy spectra and the enstrophy spectra shown in figures 5.7 and 5.8, we might think that the polymer suppresses more enstrophy than energy at the large scales, but the suppression is pretty much the same at all scales.

Let us now focus on the moments of longitudinal velocity and vorticity differences across a scale r . They are defined by $\delta u^n(r) = \langle [u(r_0 + r) - u(r)]^n \rangle$ and $\delta w^n(r) = \langle [w(r_0 + r) - w(r)]^n \rangle$, with r the longitudinal increment and the angle brackets indicate an ensemble or time average. These differences are calculated from the time series by using the frozen turbulence assumption. For three dimensional turbulence, the velocity differences have the exponent $n/3$ increase with the increment r from the

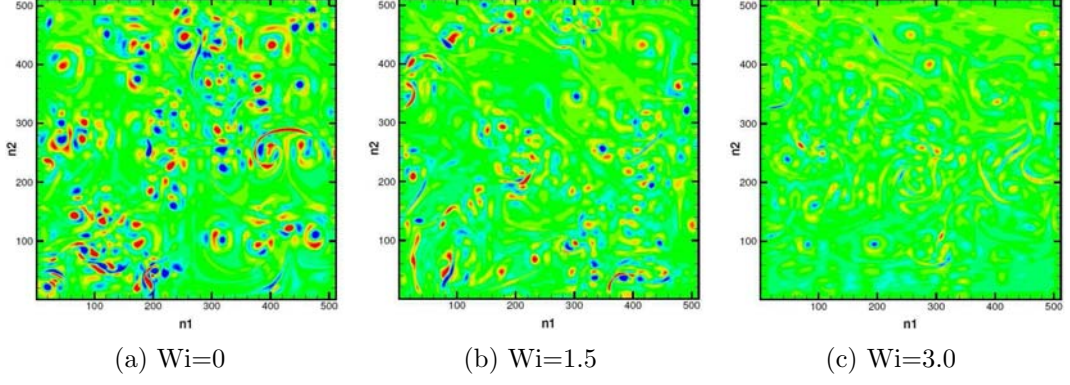


Figure 5.9: Wavelet packets filtering of a snapshot filaments subfield at the end of the channel for the case A at $Re = 5000$, ten levels from -20 to 20.

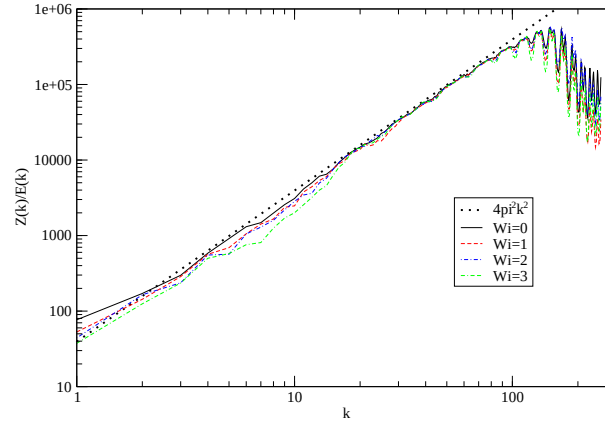


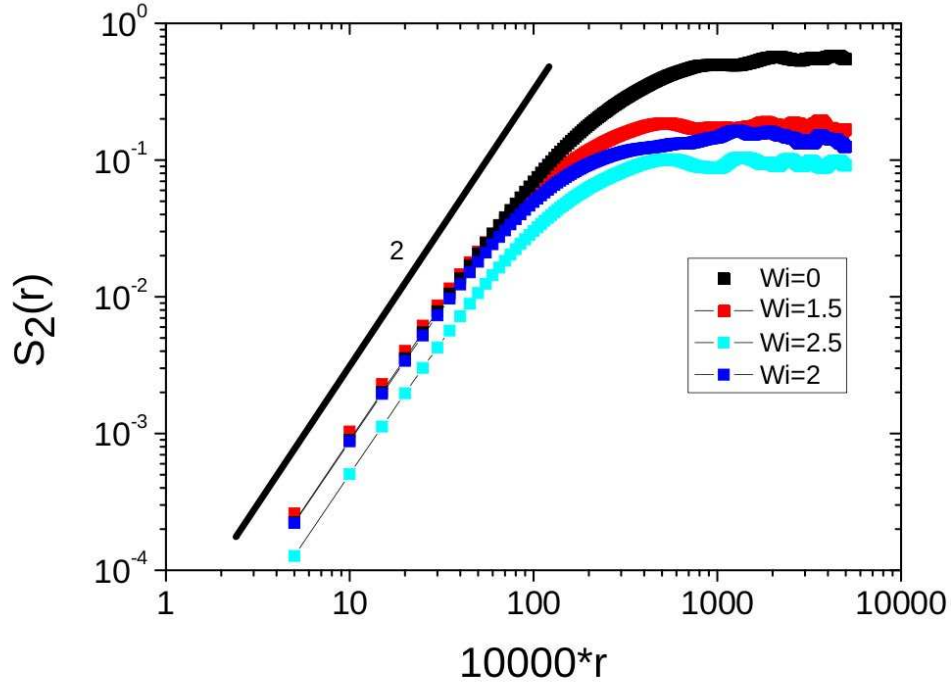
Figure 5.10: $Z(k)/E(k)$ in the given window

early turbulence theory, but in fact the exponent deviates obviously from $n/3$ for n larger than 4 due to intermittency. This was observed in experiments and simulations which are in agreement with the original doubt of Landau [56]. When the large scale vortices break up into smaller eddies, the small eddies have the shorter time period in three dimensions, hence the smaller eddies dissipate in the anisotropic small space. However the smaller eddies have the same time period than their mother vortices in the enstrophy cascade; they are supposed to dissipate in the full space and to be less intermittent. In two dimensional turbulence, the n -th velocity difference is expected to vary with the increment r as a power law with exponent equal to n at small scales and $n/3$ at large scales. We have plotted the second moment longitudinal velocity and vorticity differences for case A in figure 5.11. The second moment of longitudinal velocity differences ($S_2(r)$) increases versus r with an exponent 2 at small scales, and it has a plateau at large scales. As the Weissenberg number increases, only the large scales are suppressed. For high Weissenberg flow ($Wi=2.5$), the whole scale are suppressed, these features are in excellent agreement with several experimental observations [2, 82, 75]. The second moment vorticity difference (W_2) increases versus r with an exponent 2 at small scales, and there is no an exponential relation with r at larger scales. They are also suppressed as the Weissenberg number increases.

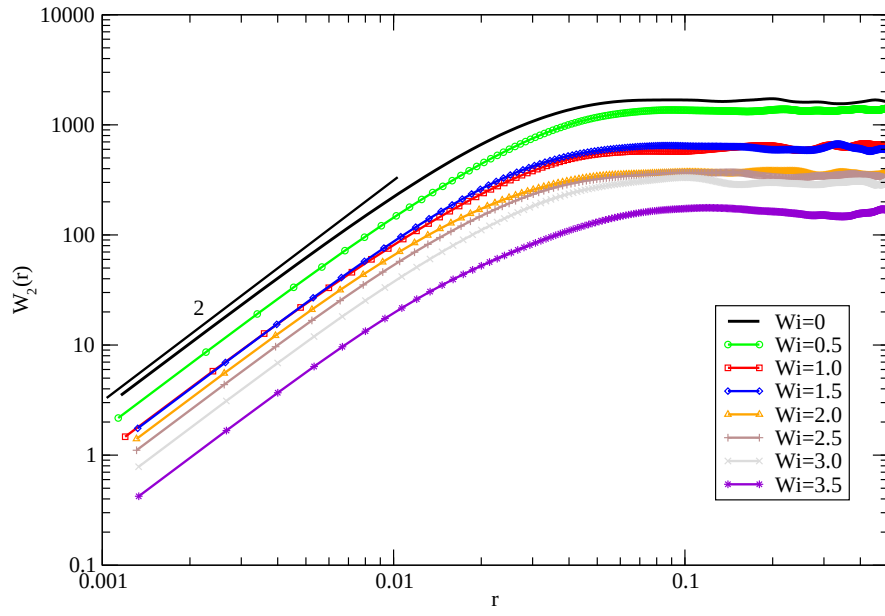
Figure 5.12 shows the probability density function of the elongation rate in the window defined by x between 3.0 to 3.5, and y between 0.25 and 0.75. The long time statistical results give a very symmetric curve. We also see the elongation rate fluctuations are restricted as the Weissenberg number increases.

The case A has five cylinders in the channel, the wakes behind the cylinders do not interact very strongly as we have seen in figure 5.6. More cylinders may produce more vortex filaments, the enstrophy injection increasing almost linearly with the number of cylinders, the energy injection may comparatively smaller. So it is also interesting to observe the polymer effect on the grid turbulence with more cylinders. In this part, we focus on the grid turbulence with 10 cylinders, i.e. case C in our setup.

The energy spectra for the viscoelastic fluids are plotted in figure 5.13. As the Weissenberg number grows, the spectra decrease. When the Weissenberg number is less than 2, the small scales decrease more than the large scales range. This is opposite to the observations in the soap film experiments. The difference of the spectra between the Weissenberg number 2 and 3 is small, this is due to the suppression of the large scale fluctuations. We also show the enstrophy spectra in figure 5.14. The spectrum flattens at low frequencies, above the injection wave number, the amplitude of the spectra decrease as the frequency grows. This decrease is steeper at high frequencies like for the energy spectrum. In the same way, the enstrophy spectra decrease as the



(a) $S_2(r)$



(b) $W_2(r)$

Figure 5.11: The second moment longitudinal velocity and vorticity differences for case A at $Re = 5000$

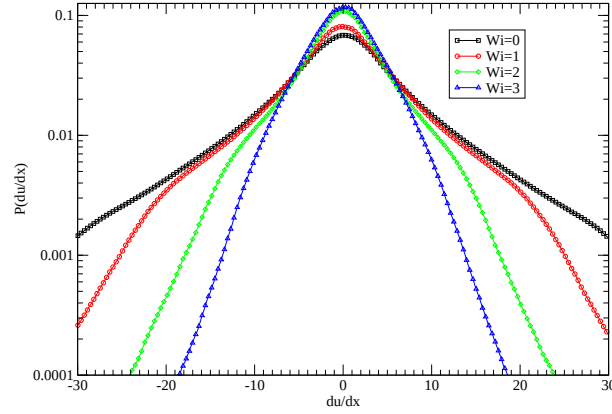


Figure 5.12: The probability density function of the elongation rate in case A at $Re = 5000$

Weissenberg number grows. The main decrease from $Wi = 3$ to $Wi = 4$ happens in the small scale range.

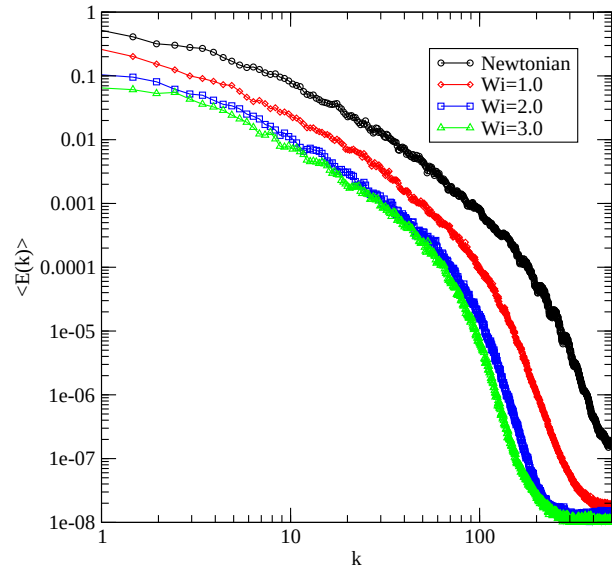


Figure 5.13: The energy spectra of the streamwise velocity of case C at $Re = 5000$ for Newtonian and viscoelastic flows

Figure 5.15 shows the filaments subfield for the case C at $Re = 5000$. They have the same level as the filament subfields for the case A. It seems the magnitude of the filaments is more than that of case A for the Newtonian fluid. However, as the Weissenberg number increases to 1.5, the filaments are more affected by the polymer compared to the case A. As we have mentioned, the filaments play an important role in the direct enstrophy cascade subrange, so the small scales should be more affected

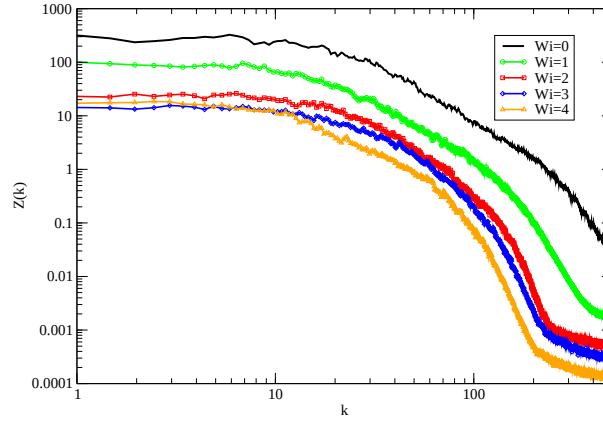


Figure 5.14: The enstrophy spectra of case C at $Re = 5000$ for the Newtonian and viscoelastic flows

by the polymer comparing to case A.

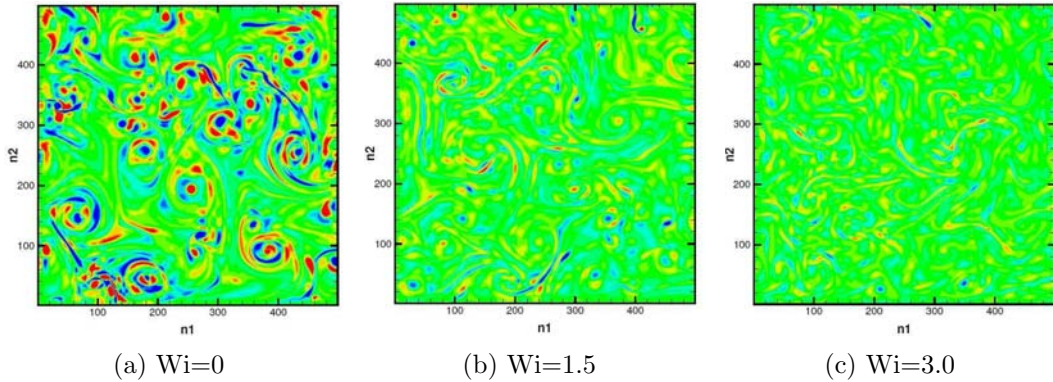


Figure 5.15: Vorticity filaments subfields at the end of the channel for the case C at $Re = 5000$ obtained by wavelet packets filtering

Figure 5.16 shows the energy spectra obtained in the given two-dimensional window by two-dimensional Fourier transform. They are different to the energy spectra obtained by time series and show a steeper enstrophy cascade range. As the Weissenberg number increases, the spectra decrease especially at large scales. The enstrophy cascade range is almost the same for the high Weissenberg numbers. We also plot the second moment of longitudinal velocity and vorticity differences with r in the streamwise direction in figure 5.17. As we know the second moment velocity differences has a direct relationship with the energy spectrum in isotropic turbulence. So this corresponds to the suppression by the polymer at large scales. From the figure, all the scales are suppressed at small Weissenberg numbers, which is different to case A. When the Weissenberg number is

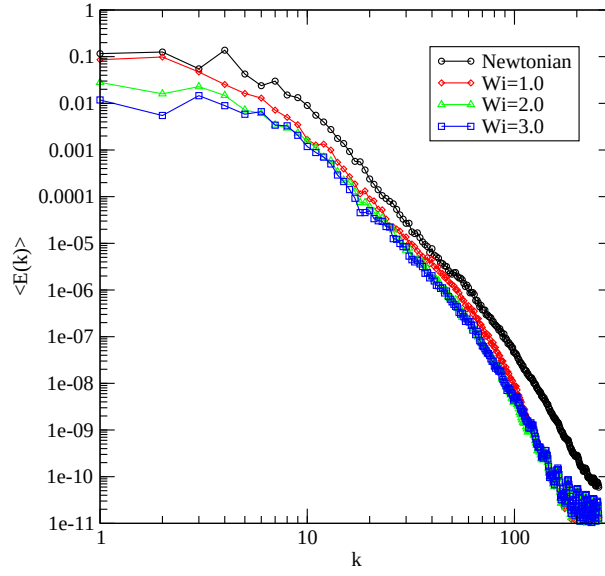


Figure 5.16: The energy spectra in the window by the wavelet packets

higher than 1.5, the structure function decreases slightly at the small scales and has a plateau at the large scales. This indicates that the large scales fluctuations only are suppressed instead of the whole scales, and the inverse energy transfer in two-dimensional turbulence is inhibited by the polymer. The second moment of vorticity differences (W_2) show a similar behavior by the polymer additive, it has an approximate exponent 2 at the small scale, and there is no exponent power law relation at large scales r . It is less affected by the increase of the Weissenberg number above $Wi = 1.5$.

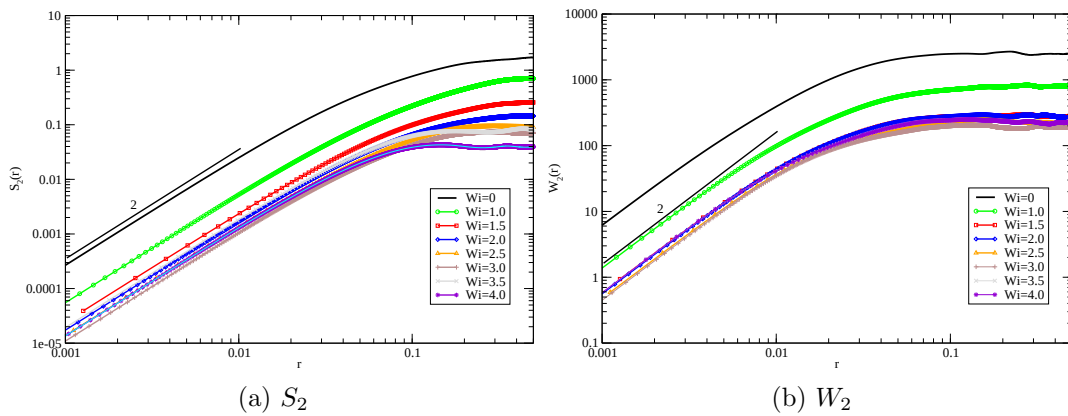

 Figure 5.17: The second moment of longitudinal velocity and vorticity differences with r in the streamwise direction for the case C at $Re = 5000$

Figure 5.18 shows the probability density functions of the elongation rate; it has a

narrower distribution as the Weissenberg number increases. The figure indicates the turbulent fluctuation is suppressed as the Weissenberg number increases.

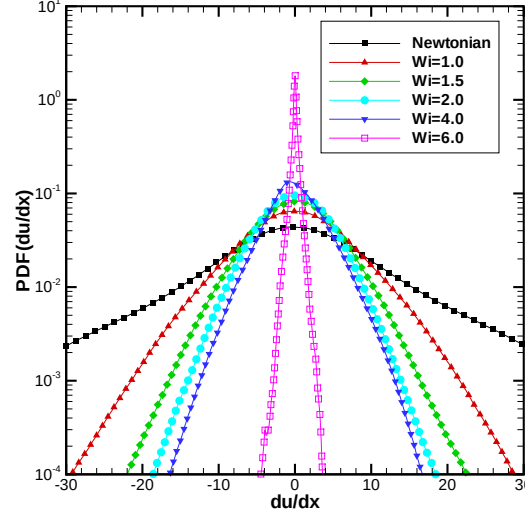


Figure 5.18: The probability density functions of the elongation rate for the case C at $Re = 5000$

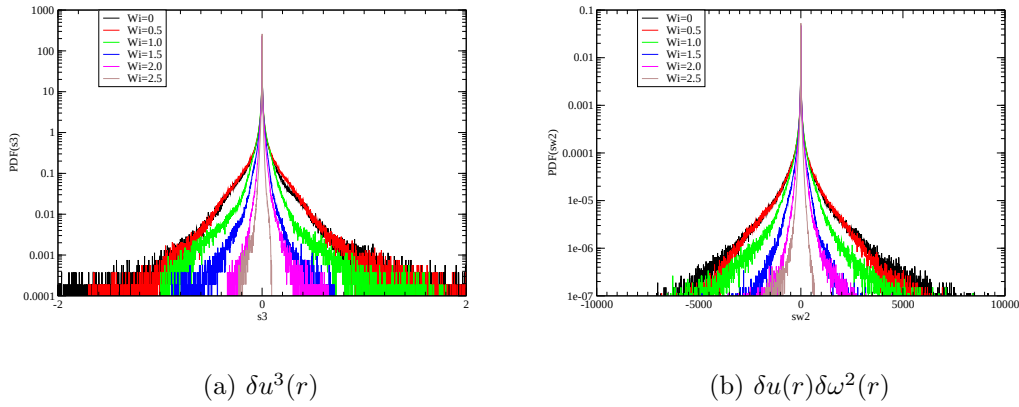


Figure 5.19: The probability density functions for case A

Besides these features, we have examined the pdfs of $\delta u^3(r)$ and $\delta u(r)\delta\omega^2(r)$. Since in 2D, the average of these two quantities is proportional to r and to the energy transfer rate and the enstrophy transfer rate respectively, an examination of the pdfs of these quantities amounts to an examination of the fluctuations of the transfer rates. It turns out that these fluctuations are much less important for the non Newtonian case. The addition of polymer reduces these fluctuations as has been found in experiments.

One can argue that a reduction of such fluctuations is a sign that elongation rates in turbulence are reduced upon the addition of polymers. In fact, and as figure 5.19 and 5.20 show, both pdfs show a large reduction of large fluctuations. The pdfs of du/dx (or elongation rate) confirms directly that large elongations are suppressed upon the addition of polymer as found experimentally.

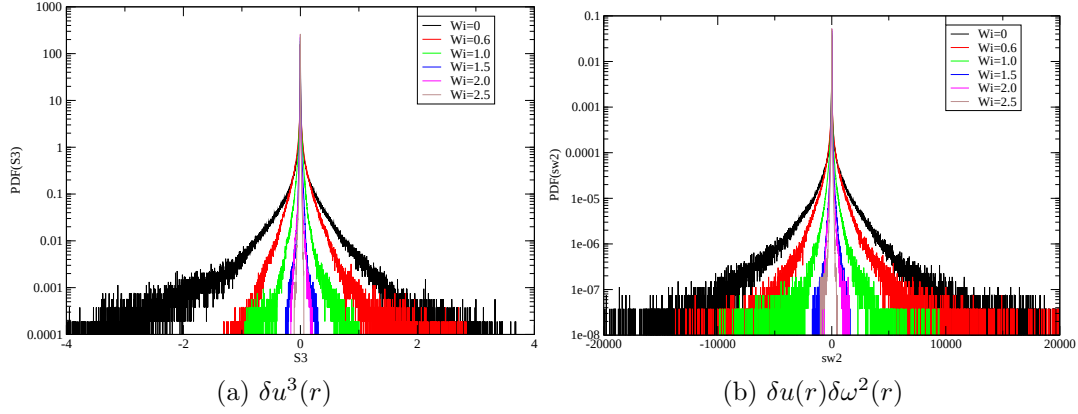


Figure 5.20: The probability density functions for case C

Note that the regions of positive elongation are strongly correlated to regions of large normal stresses. Elongation and high stresses therefore coexist at the same locations pointing out that the large positive elongation rates are at the heart of the interaction between the complex fluid and the turbulent flow. Since the large elongations are reduced one may conclude that the large increase in stresses is at the origin of this decrease. In fact, a close examination of the correlation between elongation and normal stresses is shown in figure 5.21. Here, from a time trace of both signals at a chosen location, it turns out that a large positive peak is obtained for the cross correlation function. It shows that the fluids elongate before the polymer stress increases as the maximum is located in the negative part. Therefore the polymer increases its potential energy by absorbing the turbulent kinematic energy and hence reduces the elongation of the fluid.

From the above observations on both cases A and C, the polymer inhibits both the turbulent energy flux and enstrophy flux in two-dimensional turbulence. The turbulent energy power spectra and the enstrophy spectra are attenuated to some extent for the different cases. The turbulent fluctuations decrease as the Weissenberg number increases, it induces the decrease of the turbulent intensity shown in figure 5.22, where $\mathbf{u}'$ is the velocity fluctuations. For small Weissenberg numbers, the case C has more cylinders which induces the high turbulent fluctuations. Beyond $Wi = 1.5$, the two cases are very similar.

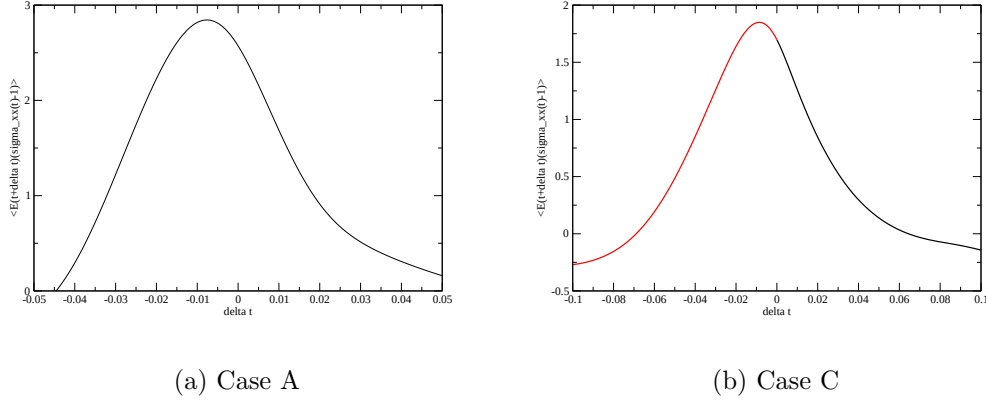


Figure 5.21: The cross correlation of the elongation rate and the polymer extra normal stress at $Wi = 1.5$

In the case A, the polymer suppresses the larger scale fluctuations in agreement with the two-dimensional experimental observation; while in case C, the polymer first reduces the turbulent intensity, then suppresses the large scale fluctuations more than the small scales. When the Weissenberg number is high enough, the whole scales are suppressed for all kinds of geometry configurations.

In figure 5.23, we plot the turbulent energy power spectra at four different Weissenberg numbers. For the Newtonian fluid, the turbulent energy power spectrum is very disperse at the small scales. Its exponent in the enstrophy cascade subrange varies between -2.7 and -4 for the different Reynolds numbers. More cylinders will inject more enstrophy, but not so much energy. This yields the different exponents in the enstrophy cascade subrange. But with polymer, these different flows have about the same spectra for different Weissenberg numbers. This process can explain why the behavior is different between cases A and C. That is: if the enstrophy is strong in the flow, the polymer suppresses fluctuations in the direct enstrophy cascade range, contrarily the polymer suppresses the larger scale fluctuations.

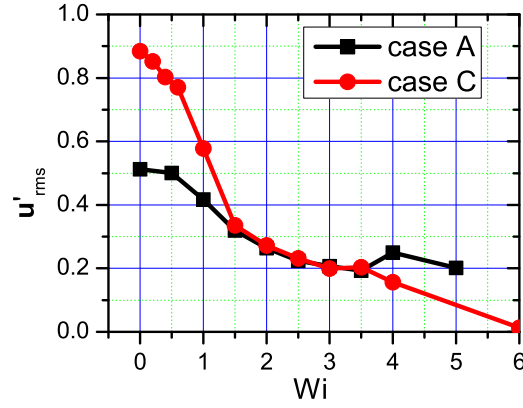


Figure 5.22: The root mean square of the fluctuating velocity

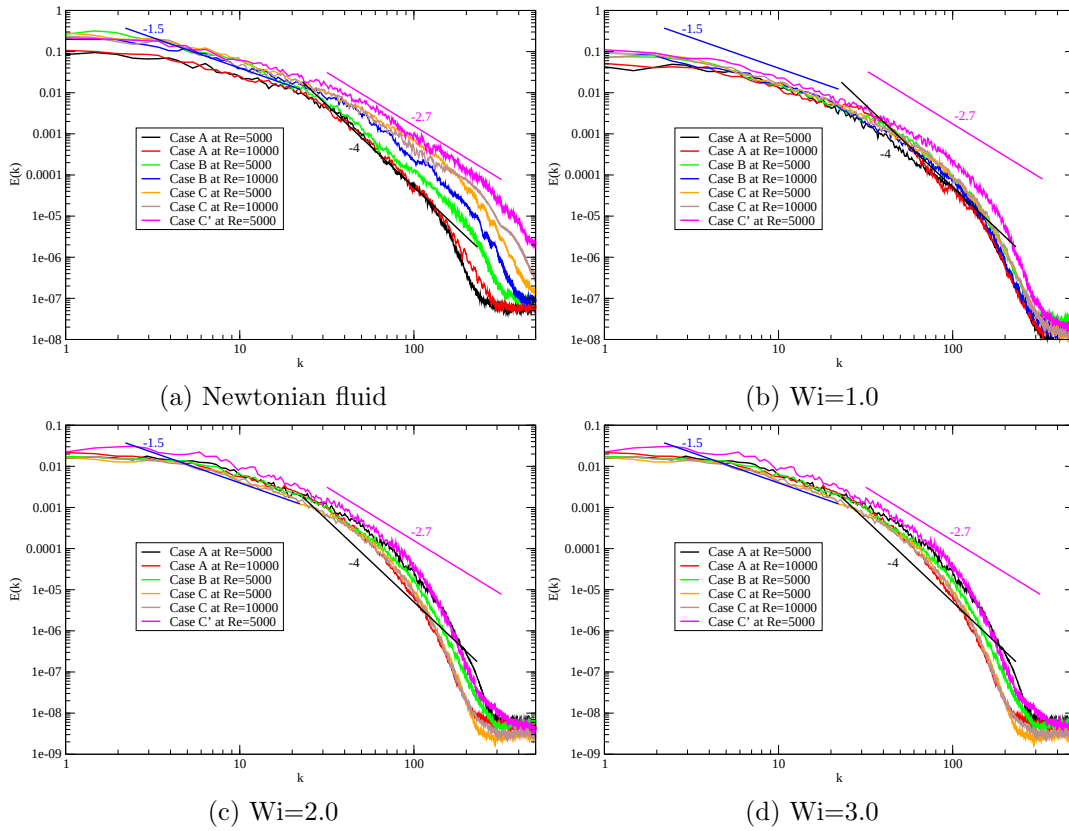


Figure 5.23: The turbulent energy power spectra at the same Weissenberg number for different flow setups and Reynolds numbers

Chapter 6

Conclusions and future works

6.1 Conclusions

In this dissertation, we use direct numerical simulations with the Oldroyd-B model to investigate the impact on the flow of the viscoelasticity of diluted polymers in solution. An artificial boundary condition for the viscoelastic fluid has been derived and embedded into our finite difference code. A penalization method is used to take into account the obstacles. The numerical validation shows our tool is accurate and can capture the important flow characteristics.

With the flow over a circular cylinder, we study the effect of polymer viscosity ratio on the drag reduction in the Reynolds number (Re) and Weissenberg number (Wi) parameter space. The results show that the polymer viscosity ratio has a linear effect on the drag behavior when the Weissenberg number is small at low Reynolds number flows, the effect being reduced as the Weissenberg number increases. Then, as the Reynolds number grows, its effect on the drag reduces, and is almost negligible when the Reynolds number exceeds 200.

Our numerical simulations in two dimensions show that the drag on a cylinder can be enhanced for low Reynolds numbers but can be reduced for higher ones. This behavior is captured in a phase diagram in Reynolds number-Weissenberg number space showing a transition from two to three regimes. The drag enhancement regime at large Reynolds numbers is accompanied by a total suppression of the vortex shedding instability as well as by large elongation rates and a large dip in pressure at the rear of the cylinder. Besides the important effect on the mean drag forces, the simulations show the polymer also suppresses the time oscillations in the drag and lift forces.

Our numerical simulations show the polymer modifies the wake flow of the cylinder in

different ways in the $Re - Wi$ space. In stokes regime, the flow becomes unsteady at very high Weissenberg numbers. When the Reynolds number is less than 100, the up-down symmetry is broken for the steady flow or the mean unsteady flow with shedding at high Weissenberg numbers. The cylinder drag is enhanced at these Weissenberg numbers. For flows with vortex shedding, the cylinder drag depends on the Weissenberg number. The mean wake length and the shedding period increase when the Weissenberg number is not too high and yields drag reduction. When the Weissenberg number is high enough, the shedding is inhibited and the drag is enhanced; the shear layer induces Kelvin Helmholtz instabilities at high Weissenberg numbers and high Reynolds numbers, this phenomenon is also observed in two dimensional experiments.

In the two-dimensional grid turbulent flow simulations, we found the turbulent boundary layer profile in two dimensions is different to that in three dimensions. The logarithmic law of the wall in the two-dimensional Newtonian turbulent boundary layer shows the ability to reduce the drag forces as the Weissenberg number increases. The two dimensional boundary layer drag reduction occurs in different ways in contrast to that in three dimensions.

Numerical results show the polymer affects two-dimensional turbulence to a large extent. By using different numerical set-ups using a various number of cylinders, we found the polymer suppresses the velocity fluctuations of both small and large scales at high Weissenberg number. At small Weissenberg number, numerical simulations show the polymer suppresses the energy power spectrum selectively. When the enstrophy injection is comparatively larger, the polymer mainly suppresses the small scales fluctuations, in contrast the large scales are mainly suppressed as the enstrophy injection is less.

6.2 Future works

In this work, we mainly focused on the difference between the Oldroyd-B constitutive fluid and Newtonian fluid for the two-dimensional flows. The Oldroyd-B model is a linear and physical sensitive model, which can capture the flow behaviors of the polymer solutions qualitatively. However, the real polymer solutions have many kinds of diversity. Some more complex non-linear models with more parameter spaces are designed to describe the polymer solutions, such as FENE models, PTT models and Giesekus model [13]. Hence it is worth to discover the effects of different viscoelastic model in order to fit in with real polymer solutions.

Besides the different viscoelastic models, the polymer solution may suffer from some

more complex physical phenomena in the flow or complex situations in real engineering questions, such as the degradation [124] or the scission [150] of the polymer molecules in high shear stress flows and inhomogeneous polymer solutions [42] around the body. These problems need to be investigated to establish a new model to observe their effects on drag and flow behavior.

Considering the three-dimensional character of the high Reynolds number flows in nature, it would be interesting to compare the drag behavior of the cylinder between three dimensions and two dimensions for high Reynolds number flows. Before performing three-dimensional simulations, it is necessary to develop a more efficient parallel computing code. Then we could also simulate viscoelastic two-dimensional grid turbulence at higher Reynolds number or higher resolution.

For industrial purposes, the flows often accompany other physical phenomena, such as heat transfer, cavitation, fluid acoustics, structure mechanics, etc. One should understand on the one side the flow behavior with polymers and on the other hand the other physical behavior with polymer, even the coupled problem. As an example, putting some polymers in the pipe flow of district heating systems may reduce the drag or the pump cost, but the polymer may change the heat transfer at the same time. Hence it may be necessary to embed other scalar equations to explore the multi-physical field effects of the polymer.

Bibliography

- [1] B. U. Achia, D.W. Thompson, *Structure of the turbulent boundary in drag-reducing pipe flow*, J. Fluid Mech. **81**, p. 439-464, 1977
- [2] Y. Amarouchene, H. Kellay, *Polymers in 2D turbulence: suppression of large scale fluctuation*, Phys. Rev. Lett. **89**, 104502, 2002
- [3] E. De Angelis, C. M. Casciola, V.S.L'vov, R. Piva, I. Procaccia *Drag reduction by polymers in turbulent channel flows: Energy redistribution between invariant empirical modes*, Phys. Rev. E **67**, 056312, 2003
- [4] E. De Angelis, C. M. Casciola, R. Benzi, R. Piva, *Homogeneous isotropic turbulence in dilute polymers*, J. Fluid Mech. **531**, p. 1-10, 2005
- [5] Ph. Angot, Ch.-H. Bruneau, P. Fabrie, *A penalization method to take into account obstacles in an incompressible flow*, Num. Math.**81**, p. 497-520, 1999
- [6] F. P. T. Baaijens, S.H.A. Selen, H.P.W. Baaijens, G.W.M. Peters, H.E.H. Meijer, *Viscoelastic flow past a confined cylinder of a low density polyethylene melt*, J. Non-Newtonian Fluid Mech. **68**, p. 173-203, 1997
- [7] E. Balkovsky, A. Fouxon, V. Lebedev, *Turbulent dynamics of polymer solutions*, Phys. Rev. Lett. **84**, p. 4765-4768, 2000
- [8] A. Baron, S. Sibilla *DNS of the turbulent channel flow of a dilute polymer solution*, Flow, Turbulence and Combustion **59**, p. 331-352, 1997
- [9] G. K. Batchelor, *Computation of the energy spectrum in homogeneous two-dimensional turbulence*, High-speed computing in fluid dynamics, the physics of fluids supplement , 1969
- [10] G. K. Batchelor, *An introduction to fluid dynamics*, Cambridge University Press, 1967
- [11] R. Benzi, E. S.C. Ching, N. Horesh, I. Procaccia, *Theory of Concentration Dependence in Drag Reduction by Polymers and of the Maximum Drag Reduction Asymptote*, Physical Review Letters **92**, 078302, 2004

- [12] C.Bergins, M.Nowak, M. Urban, *The flow of a dilute cationic surfactant solution past a circular cylinder*, Experiments in Fluids **30**, p. 410-417, 2001
- [13] R. B. Bird, C. F. Curtiss, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric liquids*, Wiley, New York, 1987, Vol.2.
- [14] G. Boffetta, A. Celani, S. Musacchio, *Two-dimensional turbulence of dilute polymer solutions*, Phys. Rev. Lett. **91**, 034501, 2003
- [15] G. Boffetta, A. Celani, M. Vergassola, *Two-dimensional turbulence: deviations from Gaussian behavior*, Phys. Rev. E **61**, R29, 2000
- [16] D. Bonn, Y. Amarouchene, C. Wagner, S. Douady, Olivier Cadot, *Turbulent drag reduction by polymers*, J. Phys.: Condens. Matter **17**, p. S1195-S1202, 2005
- [17] C. Brennen, *Some cavitation experiments with dilute polymer solutions*, J. Fluid Mech. **44**, p.51-63, 1970
- [18] J.M. Broadbent, B. Mena, *Slow flow of an elastico-viscous fluid past cylinders and spheres*, J. chem. Engng **8**, p. 11-19, 1974
- [19] R.A.Brown, G.H.McKinley, *Report on the VIIIth International Workshop on Numerical Methods in Non-Newtonian Flows*, J. Non-Newtonian Fluid Mech. **52**, p.407-413, 1994
- [20] C. H. Bruneau, O. Greffier, H. Kellay, *Numerical study of grid turbulence in two dimensions and comparison with experiments on turbulent soap films*, Phys. Rev. E **60**, R1165, 1999
- [21] Ch.-H. Bruneau, P. Fabrie, *New efficient boundary conditions for incompressible Navier-Stokes equations : a well-posedness result*, M2AN **30**, 1996
- [22] Ch.-H. Bruneau, P. Fabrie, P. Rasetarinera, *Accurate finite differences schemes to solve convection - dominated diffusion equations*, Int. J. Num. Methods Fluids **24** n° 2, 1997.
- [23] C. H. Bruneau, P. Fischer, *Influence of the filtering tools on the analysis of two-dimensional turbulent flows*, Computer and Fluids **35**, 2008
- [24] C. H. Bruneau, P. Fischer, *Spectra and filtering: a clarification*, Int. J. Wavelets, Multiresolution and Information Processing, **5**, p. 465 2007
- [25] C. H. Bruneau, P. Fischer, H. Kellay, *The structures responsible for the inverse energy and the forward enstrophy cascades in two-dimensional turbulence*, Europhys. Lett. **78**, 34002, 2007
- [26] Ch.-H. Bruneau, C. Jouron, *An efficient scheme for solving steady incompressible Navier-Stokes equations*, J. Comput. Phys. **89** n° 2, 1990.

-
- [27] C. H. Bruneau, M. Saad, *The 2D lid-driven cavity problem revisited*, Computers Fluids **35**,326-348, 2006
- [28] T. Burghlea, E. Segre, V, Steinberg, *Role of elastic stress in statistical and scaling properties of elastic turbulence*, Phys. Rev. Lett. **96**,214502, 2006
- [29] O. Cadot, S. Kumar, *Experimental characterization of viscoelastic effects on two- and three-dimensional shear instabilities*, J. Fluid Mech. **416**, p. 151-172, 2000
- [30] O. Cadot, M. Lebey, *Shear instability inhibition in a cylinder wake by local injection of a viscoelastic fluid*, Phys. Fluids **11**,p. 494-496, 1999
- [31] A.E. Caola, Y.L. Joo, R.C. Armstrong, R.A. Brown, *Highly parallel time integration of viscoelastic flows*, J. non-Newtonian Fluid Mech. **100**, p. 191-216, 2001
- [32] R. P. Chhabra, Kirti Ramia , P. H. T. Uhlherr *Drag on cylinders in shear thinning viscoelastic liquids*, Chem. Eng. Sci. **56** (2001),p. 2221-2227.
- [33] M.D.Chilcott, J.M.Rallison, *Creeping flow of dilute polymer solutions past cylinders and spheres*, J. non-Newtonian Fluid Mech. **29**, p. 381-432, 1988
- [34] A. J. Chorin, *Spectrum, dimension, and polymer analogies in fluid turbulence*, Phys. Rev. Lett. **60**, p. 1947-1949, 1988
- [35] P.M. Coelho, F.T.Pinho, *Vortex shedding in cylinder flow of shear-thinning fluids I . Identification and demarcation of flow regimes*, J. Non-Newtonian Fluid Mech. **110**, p. 143-176, 2003
- [36] P.M. Coelho, F.T.Pinho, *Vortex shedding in cylinder flow of shear-thinning fluids II . Flow characteristics*, J. Non-Newtonian Fluid Mech. **110**, p. 177-193, 2003
- [37] P.M. Coelho, F.T.Pinho, *Vortex shedding in cylinder flow of shear-thinning fluids III . Pressure measurements*, J. Non-Newtonian Fluid Mech. **121**, p. 55-68, 2004
- [38] J. R. Cressman, Q. Bailey, W.I. Goldburg, *Modification of a vortex street by a polymer additive*, Phys. Fluids **13**, p. 867-871, 2001
- [39] D.O.A.Cruz, F.T.Pinho, *Turbulent pipe flow predictions with a low Reynolds number $k - \epsilon$ model for drag reducing fluids*, J. Non-Newtonian Fluid Mech. **114**, p. 109-148, 2003
- [40] D.O.A.Cruz, F.T.Pinho, P.R.Resende, *Modelling the new stress for improved drag reduction predictions of viscoelastic pipe flow*, J. Non-Newtonian Fluid Mech. **121**, p. 127-141, 2004
- [41] M. M. Deen, *Fifty years of non-Newtonian fluid dynamics*, AIChE Journal. **50**, p. 2335-2345, 2004

- [42] C. D. Dimitropoulos, Y. Dubief, E. S. G. Shaqfeh, P. Moin, *Direct numerical simulation of polymer-induced drag reduction in turbulent boundary layer flow of inhomogeneous polymer solutions*, J. Fluid Mech. **566**, p. 153-162, 2006
- [43] C. D. Dimitropoulos, Y. Dubief, E. S. G. Shaqfeh, P. Moin, S. K. Lele, *Direct numerical simulation of polymer-induced drag reduction in turbulent boundary layer flow*, Phys. Fluids **17**, 200
- [44] E. Doorn, C. M. White, K. R. Sreenivasan *The decay of grid turbulence in polymer and surfactant solutions*, Phys. Fluids **11**, p. 2387-2393, 2000
- [45] H. S. Dou, N. Phan-Thien, *The flow of an oldroyd-B fluid past a cylinder in a channel: adaptive viscosity vorticity formulation*, J. Non-Newtonian Fluid Mech. **87**, p. 47-73, 1999
- [46] Y. Dubief, V. Terrapon, C. M. White, E. S. G. Shaqfeh, P. Moin, S. K. Lele, *On the coherent drag-reducing and turbulence-enhancing behavior of polymers in wall flows*, J. Fluid Mech. **514**, p. 271-280, 2004
- [47] Y. Dubief, C. M. White, V. Terrapon, E. S. G. Shaqfeh, P. Moin, S. K. Lele, *new answers on the interaction between polymers and vortices in turbulent flows*, Flow, Turbulence and Combustion, **74**, p. 311-329, 2005
- [48] C.D.Dimitropoulos, R. Sureshkumar, A. N. Beris, *Direct numerical simulation of viscoelastic turbulent channel flow exhibiting drag reduction: effect of the variation of rheological parameters*, J. Non-Newtonian Fluid Mech. **79**, p. 433-468, 1998
- [49] C.D.Dimitropoulos, R. Sureshkumar, A. N. Beris, R. A. Handler, *Budgets of Reynolds stress, kinetic energy and streamwise enstrophy in viscoelastic turbulent channel flow*, Phys. Fluids **13**, p. 1016-1027, 2001
- [50] G. L. Eyink, *Exact results on stationary turbulence in 2D: consequences of vorticity conservation*, Physica D **91**, p. 97-142, 1996
- [51] G. Falkovich, V. Lebedev, *Nonlocal vorticity cascade in two dimensions*, Phys. Rev. E **49**, p. 1800-1803, 1994
- [52] Y.R. Fan, R.I. Tanner, N. Phan-Thien , *Galerkin/least-squares finite-element methods for steady viscoelastic flows*, J. Non-Newtonian Fluid Mech. **84**, p. 233-256, 1999
- [53] P.J.Flory, *Statistical mechanics of chain molecules*, Hanser Publisher, New York, 1989
- [54] A.A.Fontaine, H.L.Petrie, T.A.Brungart, *Velocity profile statistics in a turbulent boundary layer with slot-injected polymer*, J. Fluid Mech. **238**, p. 435-466, 1991
- [55] N. Francois, D. Lasne, Y. Amarouchene, B. Lounis, H. Kellay, *Drag Enhancement with Polymers*, Phys. Rev. Lett. **100**, 018302, 2008

-
- [56] U. Frisch, *Turbulence*, Cambridge university press, Cambridge, U.K.,1995
- [57] U. Frisch, P.L.Sulem, *Numerical simulation of the inverse cascade in two-dimensional turbulence*, Phys. Fluid **27**, p. 1921-1923, 1984
- [58] P. Fischer, C. H. Bruneau, *Multiresolution analysis for 2D turbulence. Part 2: a physical interpretation*, Discrete and Continuous dynamical systems. **7**, p. 717-734, 2007
- [59] J.H. Gerrard, *The mechanism of the formation region of vortices behind bluff bodies*, J. Fluid Mech. **25**, p. 401-413, 1965
- [60] J. J. J. Gillissen, B. J. Boersma, P. H. Mortensen, H. I. Andersson, *Fibre-induced drag reduction*, J. Fluid Mech. **602**, p. 209-218, 2008
- [61] R. Govindarajan, V. S. L'vov, I. Procaccia, *Retardation of the onset of turbulence by minor viscosity contrasts*, Phys. Rev. Lett. **87**, 174501, 2001
- [62] A. Groisman, V. Steinberg, *Elastic turbulence in a polymer solution flow*, Nature **405**, p. 53-55, 2000
- [63] A. Groisman, V. Steinberg, *Elastic turbulence in curvilinear flows of polymer solutions*, New Journal of Phys. **6**, 29, 2004
- [64] N. Guttenberg, N. Goldenfeld *Friction factor of two-dimensional rough-boundary turbulent soap film flows*, Phys. Rev. E **79**, 065306(R) , 2009
- [65] M. Hammache, M. Gharib, *An experimental study of the parallel and oblique vortex shedding from circular cylinders*, J. Fluid Mech. **232**, p. 567-590, 1991
- [66] K. J. Harder, W. G. Tiederman, *Drag reduction and turbulent structure in two-dimensional channel flows*, Phil. Trans. R. Soc. Lond. A **336**, p. 19-34, 1991
- [67] J. Herweijer, W. van de Water, *Universal Shape of Scaling Functions in Turbulence*, phys. Rev. Lett. **74**, 4651, 1995
- [68] Y. Hou, V. S. R. Somandepalli, M. G. Mungal *A technique to determine total shear stress and polymer stress profiles in drag reduced boundary layer flows*, Exp. Fluid **40**, p. 589-600, 2006
- [69] P. Y. Huang, J. Feng, *Wall effects on the flow of viscoelastic fluids around a circular cylinder*, J. Non-Newtonian Fluid Mech. **60**, p. 179-198, 1995
- [70] X. Huang, N. Phan-Thien, R.I. Tanner, *Viscoelastic flow between eccentric rotating cylinders: unstructured control volume method* , J. Non-Newtonian Fluid Mech. **64**, p. 71-92, 1996

- [71] M. A. Hulsén, Raanan Fattal, Raz Kupferman, *Flow of viscoelastic fluids past a cylinder at high Weissenberg number: Stabilized simulations using matrix logarithms*, J. Non-Newtonian Fluid Mech. **127**, p. 27-39, 2005
- [72] A. K. M.F. Hussain, *Coherent structures - reality and myth*, Phys. Fluids **26**, p. 2816-2850, 1983
- [73] D. F. James, *Boger fluids*, Annual Rev. Fluid Mech. **41**, p. 129-142, 2009
- [74] D. F. James, A. J. Acosta, *The laminar flow of dilute polymer solutions around circular cylinders*, J. Fluid Mech. **42**, p. 269-288, 1970
- [75] Y. Jun, J. Zhang, X. L. Wu, *Polymer effects on small- and large-scale two-dimensional turbulence*, phys. Rev. Lett. **96**, 024502, 2006
- [76] G. Karniadakis, G. S. Triantafyllou, *Three-dimensional dynamics and transition to turbulence in the wake of bluff objects*, J. Fluid Mech. **238**, p. 1-30, 1992
- [77] H. Kato, Y. Mizuno, *An experimental investigation of viscoelastic flow past a circular cylinder*, Bulletin of the JSME **36**, p.529-536, 1983
- [78] H. Kellay, C. H. Bruneau, X. L. Wu, *Probability density functions of the enstrophy flux in two dimensional grid turbulence*, Phys. Rev. Lett. **84**, p. 1696-1699, 2000
- [79] H. Kellay, W. Goldburg, *Two-dimensional turbulence: a review of some recent experiments*, Rep. Prog. Phys. **65**, p. 845-894, 2002
- [80] H. Kellay, X.L. Wu, W. Goldburg, *Experiments with Turbulent Soap Films*, Phys. Rev. Lett. **74**, p. 3975-3978, 1995
- [81] R.H. Kraichnan, D. Montgomery, *Two-dimensional turbulence*, Rep. Prog. Phys. **43**, p. 547-619, 1980
- [82] H. Kellay, *Polymers suppress the inverse transfers of energy and the enstrophy flux fluctuations in two-dimensional turbulence*, Phys. Rev. E **70**, 036310, 2004
- [83] K. Kim, R.J.Adrian, S. Balachandar, R. Sureshkumar, *Dynamics of Hairpin vortices and polymer-induced turbulent drag reduction*, phys. Rev. Lett. **100**, 134504, 2008
- [84] R. H.Kraichnan, *Inertial ranges in two-dimensional turbulence*, The physics of fluids **10**, p. 1417-1423, 1967
- [85] R. G. Larson, E. S. G. Shaqfeh, S. J. Muller, *A purely elastic instability in Taylor-Couette flow*, J. Fluid Mech. **218**, p. 573-600., 1990
- [86] R. Leighton, D.T.Walker, T.Stephens, G.Garwood, *Reynolds stress modeling for drag reducing viscoelastic flows*, Proceedings of the ASME FEDSM 2003, Honolulu, Hawaii, USA

-
- [87] C.E. Leith, *Diffusion Approximation for Two-Dimensional Turbulence*, The physics of fluids **11**, p. 671-673, 1968
- [88] C.E. Leith, *Atmospheric Predictability and Two-Dimensional Turbulence*, Journal of the Atmospheric Sciences **28**, p. 145-161, 1971
- [89] M. Lesieur, *Turbulence in fluids*, Kluwer academic publishers, 1997
- [90] C.F.Li, R. Sureshkumar, B.Khomani, *Influence of rheological parameters on polymer induced turbulent drag reduction*, J. Non-Newtonian Fluid Mech. **140**, p. 23-40, 2006
- [91] A. Liberzon, M. Holzner, B. Luthi, M. Guala, W. Kinzelbach, *On turbulent entrainment and dissipation in dilute polymer solutions*, Physics of fluids **21**, 035107, 2009
- [92] A. Liberzon, M. Guala, B. Luthi, W. Kinzelbach, A. Tsinober, *Turbulence in dilute polymer solutions*, Physics of fluids **17**, 031707, 2005
- [93] A. W. Liu, D. E. Bornside, R. C. Armstrong, Robert A. Brown, *Viscoelastic flow of polymer solutions around a periodic, linear array of cylinders: comparisons of predictions for microstructure and flow fields*, J. Non-Newtonian Fluid Mech. **77**, p. 153-190, 1998
- [94] T.S.Luchik, W.G.Tiederman, *Turbulent structure in low-concentration drag reducing channel flows*, J. Fluid Mech. **190**, p. 241-263, 1988
- [95] V. S. L'vov, A. Pomyalov, I. Procaccia, V. Tiberkevich, *Drag reduction by polymers in wall bounded turbulence*, Phys. Rev. Lett. **92**, 244503, 2004
- [96] V. S. L'vov, A. Pomyalov, I. Procaccia, V. Tiberkevich, *Polymer stress tensor in turbulent shear flows*, Phys. Rev. E **71**, 106305, 2005
- [97] J. L. Lumley, *Drag reduction by additives*, Annu. Rev. Fluid Mech. **1**, p. 367, 1969
- [98] O. Manero, B. Mena, *On the slow flow of viscoelastic liquids past a circular cylinder*, J. Non-Newtonian Fluid Mech. **9**, p. 379-387, 1981
- [99] W. D. McComb, *The physics of fluid turbulence*, Clarendon press, Oxford, 1990
- [100] W. D. McComb, J. Allan, C. A. Greated *Effect of polymer additives on the small-scale structure of grid-generated turbulence*, The physics of fluid **20**, p. 873-879, 1977
- [101] J. H. McKinley, R. C. Armstrong, R.A. Brown, *The wake instability in viscoelastic flow past confined circular cylinders*, Philos. Trans. R. Soc. London **A344**, p. 265-304, 1993
- [102] B. Mena, B. Caswell, *Slow flow of a visco-elastic fluid past an immersed body*, J. chem. Engng **8**, p. 125-124, 1974

- [103] E. Mitsoulis, *Numerical simulation of confined flow of polyethylene melts around a cylinder in a planar channel*, J. Non-Newtonian Fluid Mech. **76**, p. 327-350, 1998
- [104] R. Nadolink, W. Haigh, *Bibliography on the skin friction reduction with polymers and other boundary-layer additives*, ASME Appl. Mech. Rev. **48**, 351, 1995
- [105] Theo Odijk, *Polymer-induced vortex modification in decaying two-dimensional turbulence*, Physica A **258**, p. 329-340, 1998
- [106] AM. Obukhov, *Spectral energy distribution in a turbulent flow*, Izv Akad Nauk SSR Ser Geogr Geofiz **32**, p. 22-24, 1941
- [107] S. Ogata, Y. OSano, K. Watanabe, *Effect of surfactant solutions on the drag and the flow pattern of a circular cylinder*, AIChE Journal. **52**,p. 49-57, 2006
- [108] J.G. Oldroyd, *On the formulation of rheological equations of state*, Proc. R. Soc. Lond. A **200**,p. 523-541, 1950
- [109] P. J. Oliveira, A. I.P. Miranda, *A numerical study of steady and unsteady viscoelastic flow past bounded cylinders*, J. Non-Newtonian Fluid Mech. **127**,p. 51-66, 2005
- [110] P. J. Oliveira, *Method for time-dependent simulations of viscoelastic flows: vortex shedding behind cylinder*, J. Non-Newtonian Fluid Mech. **101**, p. 113-137, 2001
- [111] Robert G. Owens, Cedric Chauviere, Timothy N. Philips *A locally-upwinded spectral technique (LUST) for viscoelastic flows*, J. Non-Newtonian Fluid Mech. **108**, p. 49-71, 2002
- [112] O. Paireau, D. Bonn, *Drag reduction in liquid-liquid friction*, Phys. Rev. Lett. **97**,264501, 1999
- [113] J. Paret, P. Tabeling, *Experimental observation of the two-dimensional inverse energy cascade*, Phys. Rev. Lett. **79**, p.4162-4165, 1997
- [114] J. S. Paschkewitz, C. D. Dimitropoulos, Y. X. Hou, V. S. R. Somandepalli, M. G. Mungal, E. S. G. Shaqfeh, P. Moin, *An experimental and numerical investigation of drag reduction in a turbulent boundary layer using a rigid rodlike polymer*, Phys. Fluids **17**, 085101, 2005
- [115] P. Perlekar, D. Mitra, R. Pandit, *Manifestation of drag reduction by polymer additives in decaying, homogeneous, isotropic turbulence*, Phys. Rev. Lett. **83**, p. 5591-5594, 2006
- [116] H.L.Petrie, S.Deutsch, T.A. Brungart, A.A.Fontaine *Polymer drag reduction with surface roughness in flat-plate turbulent boundary layer flow*, Exp. Fluids, **35**, p. 8-23, 2003

-
- [117] H.L.Petrie, A.A.Fontaine, *Comparison of turbulent boundary layer modification with slot-injected and homogeneous drag -reducing polymer solutions*, ASME Fluids Eng. Div. Conf. **237**, p. 205-210, 1996
- [118] C. J. Pipe, P. A. Monkewitz, *Vortex shedding in flows of dilute polymer solutions*, J. Non-Newtonian Fluid Mech. **139**, p. 54-67, 2006
- [119] A. Prasad, C. H. K. Williamsion, *The instability of the shear layer separating from a bluff body*, J. Fluid Mech. **333**, p. 375-402, 2006
- [120] P.K.Ptakinski, B.J.Boersma,F.T.M.Nieuwstadt,M.A.Hulsen, B.H.A.A.V.D. Brule, J.C.R. Hunt, *Turbulent channel flow maximum drag reduction: simulations,experiments and mechanisms*, J. Fluid Mech. **490**, p. 251-291, 2003
- [121] Y. Rabin, B. J. A. Zielinska, *Scale-dependent enhancement and damping of vorticity disturbances by polymers in elongational flow*, Phys. Rev. Lett. **63**, p.512-515, 1989
- [122] A. Roy, A. Morozov, W. van Saarloos, R. G. Larson, *Mechanism of polymer drag reduction using a low-dimensional model*, Phys. Rev. Lett. **97**, 234501, 2006
- [123] G. Ryskin, *Turbulent drag reduction by polymers: a quantitative theory*, Phys. Rev. Lett. **59**, p. 2059-2062, 1987
- [124] T. Sarpkaya, P. G. Rainey, R. E. Kell, *Flow of dilute polymer solutions about circular cylinders*, J. Fluid Mech. **57**, p. 177-208, 1973
- [125] S.Sibilla, A. Baron, *Polymer stress statistics in the near-wall turbulent flow of a drag-reducing solution*, Phys. Fluid **14**, p. 1123-1136, 2002
- [126] K. R. Sreenivasan, *On the scaling of the turbulence energy dissipation rate*, Phys. Fluid **27**, p. 1048-1051, 1984
- [127] L.M.Smith, V. Yakhot, *Finite size effects in forced two-dimensional turbulence* , J. Fluid Mech. **274**, p. 115-138, 1994
- [128] M.J. Solomon, S.J. Muller, *Flow past a sphere in polystyrene-based Boger fluids: the effect on the drag coefficient of finite extensibility, solvent quality and polymer molecular weight* , Journal of Non-Newtonian Fluid Mechanics, **62**, p. 81-94, 1996
- [129] R. Sureshkumar, Antony N. Beris, *Effect of artificial stress diffusivity on the stability of numerical calculations and the flow dynamics of time-dependent viscoelastic flows*, J. Non-Newtonian Fluid Mech. **60**, p. 53-80, 1995
- [130] M. Sahin, R. G. Owens, *On the effects of viscoelasticity on two-dimensional vortex dynamics in the cylinder wake*, J. Non-Newtonian Fluid Mech. **123**, p. 121-139, 2004

- [131] A. K. Sahu, R. P. Chhabra, V. Eswaran, *Two-dimensional unsteady laminar flow of a power law fluid across a square cylinder*, J. Non-Newtonian Fluid Mech. **160**, p. 157-167, 2009
- [132] A. H. Shiang, J. C. Lin, A. Owtekin, *Viscoelastic flow around a confined circular cylinder: measurements using high-image-density particle image velocimetry*, J. Non-Newtonian Fluid Mech. **73**, p. 29-49, 1997
- [133] A. H. Shiang, A. oztekin, D. Rockwell, *Hydroelastic instabilities in viscoelastic flow past a cylinder confined in a channel*, Exp. Fluids **28**, p. 128-142., 2000
- [134] V.S.R.Soomandepalli, C.M.White,M.G.Mungal, *Boundary layer studies on polymer drag reduction using PIV and PLIF*, ASME FEDSM2003-45659, Proceedings of the ASME FEDSM 2003
- [135] K. R. Sreenivasan, C. M. White, *The onset of drag reduction by dilute polymer additives, and the maximum drag reduction asymptote*, J. Fluid Mech. **409**, p. 149-164, 2000
- [136] P. A. Stone, F. Waleffe, M.D. Graham, *Toward a structural understanding of turbulent drag reduction: nonlinear coherent states in viscoelastic shear flows*, Phys. Rev. Lett. **89**, 208301, 2002
- [137] P. A. Stone, A.Roy, R.G.Larson, F. Waleffe, M.D. Graham, *Polymer drag reduction in exact coherent structures of plane shear flow*, Phys. Fluid **16**, p. 3470-3482, 2004
- [138] J. Sun, M. D. Smith, R. C. Armstrong, R. A. Brown, *Finite element method for viscoelastic flows based on the discrete adaptive viscoelastic stress splitting and the discontinuous Galerkin method: DAVSS-G/DG*, J. Non-Newtonian Fluid Mech. **86**, p. 281-307, 1999
- [139] M. Tabor, P. G. DE Gennes, *A cascade theory of drag reduction*, Europhys. Lett. **2**, p. 519-522, 1986
- [140] P. Tong, W. I. Goldburg, J. S. Huang, T. A. Witten, *Toward a structural understanding of turbulent drag reduction: nonlinear coherent states in viscoelastic shear flows*, Phys. Rev. Lett. **65**, p. 2780-2783, 1990
- [141] P. Tabeling, *Two-dimensional turbulence: a physicist approach*, Phys. Rep.**362**, p. 1-62, 2002
- [142] B.A. Toms, *Some observation on the flow of linear polymer solutions through straight tubes at large Reynolds number*, In:Proceedings of the First International Congress of Rheology, vol. 2, North Holland, Amsterdam, 1949, p.135-141.
- [143] J. M. J.Toonder, M. A. Hulsen, G. D. C. Kuiken, F. T. M. Nieuwstadt, *Drag reduction by polymer additives in a turbulent pipe flow: numerical and laboratory experiments*, J. Fluid Mech. **337**, p. 193-231, 1997

-
- [144] H. Tennekes, J. Lumley, *A First Course in Turbulence*. MIT Press, Cambridge, 1972
 - [145] T. Tran, P. Chakraborty, N. Guttenberg, A. Prescott, H. Kellay, W. Goldburg, N. Goldenfeld, G. Gioia *Macroscopic effects of the spectral structure in turbulent flows*, Nature phys. **6**, p. 438 - 441, 2010
 - [146] K. K. Tung, E. Gkioulekas, *Is the subdominant part of the energy spectrum due to downscale energy cascade hidden in quasi-geostrophic turbulence?*, Discrete Contin. Dyn. Syst. B. **7**, p. 293-314 2007
 - [147] J. S. Uhlmann, M. M. Denn, *Slow viscoelastic flow past submerged objects*, J. chem. Engng **2**, p. 81-89, 1971
 - [148] M. F. Unal, D. Rockwell, *On vortex formation from a cylinder. Part 1. The initial instability*, J. Fluid Mech. **190**, p. 491-512, 1988
 - [149] M. F. Unal, D. Rockwell, *On vortex formation from a cylinder. Part 2. Control by splitter-plate interference*, J. Fluid Mech. **190**, p. 513-529, 1987
 - [150] S. A. Vanapalli, S. L. Ceccio, M. J. Solomon, *Universal scaling for polymer chain scission in turbulence* Proc. National Academy of Science, **103**, p. 16660-16665, 2006
 - [151] P. S. Virk, E. W. Merrill, H. S. Mickley, K. A. Smith, *The Toms phenomenon: turbulent pipe flow of dilute polymer solutions*, J. Fluid Mech. **30**, p. 305-328, 1967
 - [152] P. S. Virk, *Drag reduction in rough pipes*, J. Fluid Mech. **45**, p. 225-246, 1970
 - [153] P. S. Virk, *Drag reduction fundamentals*, AIChE J. **21**, p. 625-656, 1975
 - [154] P. S. Virk, D. C. Sherman, S. L. Wagger, *Additive equivalence during turbulent drag reduction*, AIChE J. **43**, p. 3257-3259, 1997
 - [155] J. M. Verhelst, F. T. M. Nieuwstadt, *Visco-elastic flow past circular cylinders mounted in a channel: experimental measurements of velocity and drag*, J. Non-Newtonian Fluid Mech. **116**, p. 301-328, 2004
 - [156] V. S. L'vov, I. Procaccia, O. Rudenko, *Velocity and energy profiles in two- versus three-dimensional channels: Effects of an inverse-versus a direct-energy cascade*, Phys. Rev. E **79**, 045304, 2009
 - [157] Z. Warhaft, *Turbulence in Nature and in the laboratory*, Proc. Nat. Academy of Science, **99**, p. 2481-2486, 2002
 - [158] M. D. Warholic, H. Massah, T. J. Hanratty, *Influence of drag-reducing polymers on turbulence: effects of Reynolds number, concentration and mixing*, Exp. Mech. **27**, p. 461-472, 1999

- [159] M.D.Warholic, D.K.Heist, M. Katcher, T.J. Hanratty, *A study with particle image velocimetry of the influence of drag reducing polymers on the structure of turbulence*, Exp. Mech. **31**, p. 474-483, 2001
- [160] J. C. Mc Williams, *The vortices of two dimensional turbulence*, J. Fluid Mech. **219**, p. 361-385, 1990
- [161] C. H.K. Williamson, *Vortex dynamics in the cylinder wake*, Annu. Rev. Fluid Mech. **28**, p. 477-539, 1996
- [162] C. M. White, M. G. Mungal, *Mechanics and prediction of turbulent drag reduction with polymer additives*, Annu. Rev. Fluid Mech. **40**, p. 235-256, 2008
- [163] C. M. White, V. S. R. Somandepalli, M. G. Mungal, *The turbulence structure of drag reduced boundary layer flow*, Exp. Fluids **36**, p. 62-69, 2004
- [164] E. Zana, G. Tiefenbruck, L.G. Leal, *A note on the creeping motion of a viscoelastic fluid past a sphere*, Rheol. Acta **14**, p. 891-898, 1975
- [165] I. Zhu, C. S. Peskin, *Simulation of a flapping flexible filament in a flowing soap film by the immersed boundary method*, J. Comput. Phys. **179**, p. 452-468, 2002
- [166] B.H.Zimm, *Dynamics of polymer molecules in dilute solution: viscoelasticity, flow birefringence and dielectric loss*, J. Chem. Phys. **24**, p. 269-278, 1956