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INTERPOLATION AVEC CONTRAINTES SUR DES ENSEMBLES FINIS DU DISQUE
(Constrained Interpolation on finite subsets of the disc)

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2

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4

Contents

Introduction

(1) **General framework.** The problem considered in this Thesis is the following: given two Banach spaces X and Y of holomorphic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $X \supset Y$, and a finite set $\sigma \subset \mathbb{D}$, what is the best possible interpolation by functions of the space Y for the traces $f_{|\sigma}$ of functions of the space X, in the worst case? Precisely, the matter is to compare the restriction (or quotient) norms on the finite dimensional space $X_{|\sigma} = Y_{|\sigma}$, namely the norms

$$\|f\|_{X_{|\sigma}} = inf\left(\| g \|_X \colon g \in X, \, g_{|\sigma} = f_{|\sigma}\right),$$

$$\|f\|_{Y_{|\sigma}} = inf\left(\| g \|_Y \colon g \in Y, \, g_{|\sigma} = f_{|\sigma}\right).$$

The classical interpolation problems- those of Nevanlinna-Pick and Carathéodory-Schur (on the one hand) and Carleson's free interpolation (on the other hand)- are of this nature. Two first are "individual", in the sens that one looks simply to compute the norms $\|f\|_{H^\infty_{|\sigma}}$ or $\|f\|_{H^\infty/z^nH^\infty}$ for a given f, whereas the third one is to compare the norms $\|a\|_{l^\infty(\sigma)} = max_{\lambda\in\sigma}|a_\lambda|$ and

$$inf\left(\| g \|_\infty \colon g(\lambda) = a_\lambda, \, \lambda \in \sigma\right).$$

Here and everywhere below, H^∞ stands for the space (algebra) of bounded holomorphic functions in the unit disc $\mathbb{D}$ endowed with the norm $\|f\|_\infty = sup_{z\in\mathbb{D}}|f(z)|$. Looking at this comparison problem, say, in the form of computing/estimating the interpolation constant

$$c\,(\sigma, \, X, \, Y) = sup_{f\in X, \, \|f\|_X \leq 1} inf\left\{\|g\|_Y \, : \, g_{|\sigma} = f_{|\sigma}\right\},$$

which is nothing but the norm of the embedding operator $\left(X_{|\sigma}, \, \|.\|_{X_{|\sigma}}\right) \to \left(Y_{|\sigma}, \, \|.\|_{Y_{|\sigma}}\right)$, one can think, of course, on passing (after) to the limit- in the case of an infinite sequence $\{\lambda_j\}$ and its finite sections $\{\lambda_j\}_{j=1}^n$- in order to obtain a Carleson type interpolation theorem $X_{|\sigma} = Y_{|\sigma}$. But not necessarily. In particular, even the classical Pick-Nevanlinna theorem (giving a necessary and sufficient condition on a function a for the existence of $f \in H^\infty$ such that $\|f\|_\infty \leq 1$ and $f(\lambda) = a_\lambda, \, \lambda \in \sigma$), does not lead immediately to Carleson's criterion for $H^\infty_{|\sigma} = l^\infty(\sigma)$. (Finally, a direct deduction of Carleson's theorem from Pick's result was done by P. Koosis [K] in 1999 only). Similarly, the problem stated for $c\,(\sigma, \, X, \, Y)$ is of interest in its own. For this Thesis, the following question was especially stimulating (which is a part of a more complicated question arising in an applied situation in [BL1] and [BL2]): given a set $\sigma \subset \mathbb{D}$, how to estimate $c\,(\sigma, \, H^2, \, H^\infty)$ in terms of $n = card(\sigma)$ and $max_{\lambda\in\sigma}|\lambda| = r$ only? Here, H^2 is the standard Hardy space of the disc, see below for the formal definition. The last general remark before passing to a more detailed description of the Thesis: looking for a possible choice of an interpolating space Y, one can find many interesting candidates, which in addition are important for various applications (in matrix or numerical analysis), see a short list of such candidates below, in this Introduction. However, in this Thesis, we deal with H^∞ interpolation only. Therefore, conditions of the type $a_\lambda = f(\lambda)$, $f \in X$ play the role of constraints for interpolation expressed in form of a norm inequality $\|f\|_X \leq 1$. Now, we pass to technical details.

(2) **The spaces considered** in this Thesis. For our interpolation problem, we consider the following scales of Banach spaces:

- $X = H^p = H^p(\mathbb{D})$, $1 \leq p \leq \infty$, the standard Hardy spaces on the disc $\mathbb{D}$,

• $X = l^2_a\left(\frac{1}{(k+1)^{\alpha-1}}\right)$, $\alpha \geq 1$, the Hardy weighted spaces of all $f(z) = \sum_{k\geq 0} \hat{f}(k) z^k$ satisfying

$$\sum_{k\geq 0} \left|\hat{f}(k)\right|^2 \frac{1}{(k+1)^{2(\alpha-1)}} < \infty,$$

• $X = L^2_a\left(\left(1-|z|^2\right)^\beta dA\right)$, $\beta > -1$, where dA stands for the area measure, the Bergman weighted spaces of all holomorphic functions f such that

$$\int_{\mathbb{D}} |f(z)|^2 \left(1-|z|^2\right)^\beta dA < \infty.$$

For the case $\beta = 0$, we shorten the notation to $X = L^2_a$. Notice that two latter series of spaces coincide:

$$l^2_a\left(\frac{1}{(k+1)^{\alpha-1}}\right) = L^2_a\left(\left(1-|z|^2\right)^{2\alpha-3} dA\right), \ \alpha > 1.$$

For all three series we show that

$$c_1\varphi_X\left(1-\frac{1-r}{n}\right) \leq sup\left\{c\left(\sigma,\, X,\, H^\infty\right) :\, \#\sigma \leq n,\, |\lambda| \leq r,\, \lambda \in \sigma\right\} \leq c_2\varphi_X\left(1-\frac{1-r}{n}\right),$$

where $\varphi_X(t)$, $0 \leq t < 1$ stands for the norm of the evaluation functional $f \mapsto f(t)$ on the space X. Other spaces considered are the following:

• $X = l^p_a\left(\frac{1}{(k+1)^{\alpha-1}}\right)$, $\alpha \geq 1$, $1 \leq p \leq \infty$.

• $X = L^p_a\left(\left(1-|z|^2\right)^\beta dA\right)$, $\beta > -1$, $1 \leq p \leq 2$.

For these spaces we also found upper and lower bounds for $c\left(\sigma,\, X,\, H^\infty\right)$ (sometimes for special sets σ) but with some gaps between these bounds. (See details below, in this Introduction).

(3) **Principal results.** Let $\sigma = \{\lambda_1, ..., \lambda_1, \lambda_2, ..., \lambda_2, ..., \lambda_t, ..., \lambda_t\}$ be a finite sequence in the unit disc, where every λ_s is repeated according its multiplicity m_s , $\sum_{s=1}^t m_s = n$ and $r = max_{i=1..t}|\lambda_i|$. Let X, Y be Banach spaces of holomorphic functions continuously embedded into the space $Hol(\mathbb{D})$ of holomorphic functions in the unit disc $\mathbb{D}$. In what follows, we systematically use the following conditions for the spaces X and Y,

$$(P_1) \qquad Hol((1+\epsilon)\mathbb{D}) \; is\; continuously\; embedded\; into\, Y\; for\; every\; \epsilon > 0,$$

$$(P_2) \qquad Pol_+ \subset X \; and \; Pol_+ \; is\; dense\; in\; X,$$

where Pol_+ stands for the set of all complex polynomials p, $p(z) = \sum_{k=0}^N a_k z^k$,

$$(P_3) \qquad [f \in X] \Rightarrow \left[z^n f \in X\,,\, \forall n \geq 0 \; and \; \overline{lim}\, \|z^n f\|^{\frac{1}{n}} \leq 1\right],$$

$$(P_4) \qquad [f \in X,\, \lambda \in \mathbb{D},\, and\, f(\lambda) = 0] \Rightarrow \left[\frac{f}{z-\lambda} \in X\right].$$

We are interested in estimating the quantity

$$c\,(\sigma,\, X,\, Y) = sup_{\|f\|_X \le 1} inf\left\{\|g\|_Y\, :\, g \in Y,\, g^{(j)}\,(\lambda_i) = f^{(j)}\,(\lambda_i)\ \forall i,\, j,\, 1 \le i \le t,\, 0 \le j < m_i\right\}.$$

In order to simplify the notation, the condition

$$g^{(j)}\,(\lambda_i) = f^{(j)}\,(\lambda_i)\ \forall i,\, j,\, 1 \le i \le t,\, 0 \le j < m_i$$

will also be written as

$$g_{|\sigma} = f_{|\sigma}.$$

Supposing X verifies property (P_4) and $Y \subset X$, the quantity $c\,(\sigma,\, X,\, Y)$ can be written as follows,

$$c\,(\sigma,\, X,\, Y) = sup_{\|f\|_X \le 1} inf\left\{\|g\|_Y\, :\, g \in Y,\, g - f \in B_\sigma X\right\},$$

where B_σ is the Blaschke product

$$B_\sigma = \Pi_{i=1..n} b_{\lambda_i},$$

corresponding to σ, $b_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z}$ being an elementary Blaschke factor for $\lambda \in \mathbb{D}$.

The interesting case is obviously when X is larger than Y, and the sens of the issue lies in comparing $\|\,.\,\|_X$ and $\|\,.\,\|_Y$ when Y interpolates X on the set σ. For example, we can wonder what happens when $X = H^p$, the classical Hardy spaces of the disc or $X = L^p_a$, the Bergman spaces, etc..., and when $Y = H^\infty$, but also $Y = W$ the Wiener algebra (of absolutely converging Fourier series) or $Y = B^0_{\infty,1}$, a Besov algebra (an interesting case for the functional calculus of finite rank operators, in particular, those satisfying the so-called Ritt condition). Here, H^p stands for the classical Hardy space of the disc (see below).

It is also important to understand what kind of interpolation we are going to study when bounding the constant $c(\sigma,\, X,\, Y)$. Namely, comparing with the Carleson free interpolation, we can say that the latter one deals with the interpolation constant defined as

$$c\,(\sigma,\, l^\infty(\sigma),\, H^\infty) = sup\left\{inf\left(\|\, g\,\|_\infty\!:\, g \in H^\infty,\, g_{|\sigma} = a\right)\, :\, a \in l^\infty(\sigma),\, \|\, a\,\|_{l^\infty} \le 1\right\}.$$

We also can add some more motivations to our problem:

(a) One of the most interesting cases is $Y = H^\infty$. In this case, the quantity $c\,(\sigma,\, X,\, H^\infty)$ has a meaning of an intermediate interpolation between the Carleson one (when $\|f\|_{X_{|\sigma}} \asymp sup_{1 \le i \le n} |f\,(\lambda_i)|$) and the individual Nevanlinna-Pick interpolation (no conditions on f).

(b) There is a straight link between the constant $c\,(\sigma,\, X,\, Y)$ and numerical analysis. For example, in matrix analysis, it is of interest to bound the norm of an H^∞-calculus $\|f(A)\| \le c\,\|f\|_\infty$, $f \in H^\infty$, for an arbitrary Banach space n-dimensional contraction A with a given spectrum $\sigma(A) \subset \sigma$. The best possible constant is $c = c\,(\sigma,\, H^\infty,\, W)$, so that

$$c\,(\sigma,\, H^\infty,\, W) = sup_{\|f\|_\infty \le 1} sup\left\{\|f(A)\| \,:\, A : (\mathbb{C}^n,\, |.|) \to (\mathbb{C}^n,\, |.|)\,,\, \|A\| \le 1,\, \sigma(A) \subset \sigma\right\},$$

where $W = \left\{f = \sum_{k\ge 0} \hat{f}(k) z^k \,:\, \sum_{k\ge 0} \left|\hat{f}(k)\right| < \infty\right\}$ stands for the Wiener algebra, and the interior sup is taken over all contractions on $n-$dimensional Banach spaces. An interesting case occurs for $f \in H^\infty$ such that $f_{|\sigma} = \frac{1}{z}_{|\sigma}$ (estimation of condition numbers and the norms inverses of $n \times n$ matrices) or $f_{|\sigma} = \frac{1}{\lambda - z}_{|\sigma}$ (for estimation of the norm of the resolvent of an $n \times n$ matrix).

The Thesis is organised as follows.

Chapter 1 is devoted to upper bounds for generalized Nevanlinna-Pick interpolation (excepting for **Section 1.11**, which deals with generalized Carathéodory-Schur interpolation).

In Chapter 2, we discuss the sharpness of some of the upper bounds obtained in **Chapter 1**.

Finally, **Chapter 3** deals with an application of H^∞ interpolation to an estimate of condition numbers of Toeplitz matrices, this is the content of [Z].

Chapter 1 starts studying general Banach spaces X and Y and gives some sufficients conditions under which $C_{n,\,r}(X,Y) < \infty$, where

$$C_{n,\,r}(X,Y) = sup\left\{c(\sigma, X, Y) \;:\, \#\sigma \le n\,,\, \forall j = 1..n,\, |\lambda_j| \le r\right\}.$$

In particular, we prove the following basic fact.

Theorem. 1.1.1 *Let X, Y be Banach spaces verifying properties (P_i), $i = 1...4$, then*

$$C_{n,r}(X,Y) < \infty,$$

for every $n \ge 1$ and r, $0 \le r < 1$.

We also give in **Section 1.1** a general lower bound for the quantity $C_{n,r}(X, H^\infty)$ using the evaluation functionals φ_λ for $\lambda \in \mathbb{D}$,

$$\varphi_\lambda(f) = f(\lambda),\;\; f \in X,$$

and more generally, the evaluation functionals for the derivatives $\varphi_{\lambda,s}$ $(s = 0, 1, ...)$,

$$\varphi_{\lambda,s}(f) = f^{(s)}(\lambda),\;\; f \in X.$$

Theorem. 1.1.3 *(1) For every sequence $\sigma \subset \mathbb{D}$, we have*

$$c(\sigma,\, X,\, H^{\infty}) \geq max_{\lambda\in\sigma} \|\varphi_{\lambda}\|\,.$$

(2) Moreover, writing $\sigma = \{\lambda_1, ..., \lambda_1, \lambda_2, ..., \lambda_2, ..., \lambda_t, ..., \lambda_t\}$, *where each* λ_s *is repeated according its multiplicity* m_s, *we have*

$$c(\sigma,\, X,\, H^{\infty}) \geq max_{1\leq s\leq t,\, 0\leq j<m_s} \left(\|\varphi_{\lambda_s,\, j}\| \left(1-|\lambda_j|^2\right)^j \frac{1}{j!} \right).$$

Next, we add the condition that X is a Hilbert space, and give in this case a general upper bound for the quantity $C_{n,\,r}(X,\, Y)$.

Theorem. 1.2.1 *Let Y be a Banach space verifying property (P_1) and $X = (H,\, (.)_H)$ a Hilbert space satisfying properties (P_i) for $i = 2,\, 3,\, 4$. We moreover suppose that for every $0 < r < 1$ there exists $\epsilon > 0$ such that $k_{\lambda} \in Hol\,((1+\epsilon)\mathbb{D})$ for all $|\lambda| < r$, where k_{λ} stands for the reproducing kernel of X at point λ, and $\overline{\lambda} \mapsto k_{\lambda}$ is holomorphic on $|\lambda| < r$ as a $Hol((1+\epsilon)\mathbb{D})$-valued function. Let $\sigma = \{\lambda_1, ..., \lambda_1, \lambda_2, ..., \lambda_2, ..., \lambda_t, ..., \lambda_t\}$ be a sequence in $\mathbb{D}$, where λ_s are repeated according their multiplicity m_s, $\sum_{s=1}^{t} m_s = n$. Then we have,*
i)

$$c\,(\sigma,\, X,\, Y) \leq \left(\sum_{k=1}^{n} \|e_k\|_Y^2\right)^{\frac{1}{2}},$$

where $(e_k)_{k=1}^n$ stands for the Gram-Schmidt orthogonalization (in the space H) of the sequence

$$k_{\lambda_1,0},\ k_{\lambda_1,1},\ k_{\lambda_1,2}...,k_{\lambda_1,m_1-1},\ k_{\lambda_2,0},\ k_{\lambda_2,1},\ k_{\lambda_2,2}...,k_{\lambda_2,m_2-1},\ ...,\ k_{\lambda_t,0},\ k_{\lambda_t,1},\ k_{\lambda_t,2}...,k_{\lambda_t,m_t-1},$$

$k_{\lambda,\,i} = \left(\frac{d}{d\overline{\lambda}}\right)^i k_{\lambda}$ *and k_{λ} is the reproducing kernel of X at point λ for every $\lambda \in \mathbb{D}$.*

ii) For the case $Y = H^{\infty}$, we have

$$c(\sigma,\, H,\, H^{\infty}) \leq sup_{z\in\mathbb{D}} \|P_{B_{\sigma}} k_z\|_H\,,$$

where $P_{B_{\sigma}} = \sum_{k=1}^{n} (.,e_k)_H\, e_k$ stands for the orthogonal projection of H onto $K_{B_{\sigma}}$,

$$K_{B_{\sigma}} = span\left(k_{\lambda_j,\,i} :\ 1 \leq i < m_j,\ j = 1, ..., t\right).$$

After that, we specialize the upper bound obtained in Theorem 1.2.1 (ii) to the case $X = H^2$ and prove the following (see Corollary 1.3.0 and Proposition 1.3.1).

For every sequence $\sigma = \{\lambda_1,\, ...,\, \lambda_n\}$ of $\mathbb{D}$, we have

$$c\left(\sigma,\, H^2,\, H^{\infty}\right) \leq sup_{z\in\mathbb{D}} \left(\frac{1-|B_{\sigma}(z)|^2}{1-|z|^2}\right)^{\frac{1}{2}},$$

$$c\left(\sigma,\, H^2,\, H^\infty\right) \le \sqrt{2} sup_{|\zeta|=1} \left|B'(\zeta)\right|^{\frac{1}{2}} = \sqrt{2} sup_{|\zeta|=1} \left|\sum_{i=1}^{n} \frac{1-|\lambda_i|^2}{\left(1-\bar{\lambda}_i\zeta\right)^2} \frac{B_\sigma(\zeta)}{b_{\lambda_i}(\zeta)}\right|^{\frac{1}{2}}.$$

Corollary. 1.3.2 *Let* $\sigma = \{\lambda_1,\, ...,\, \lambda_n\}$ *and* $r = max_{1\le i\le n}|\lambda_i|$*. Then*

$$c\left(\sigma,\, H^2,\, H^\infty\right) \le 2\frac{\sqrt{n}}{\sqrt{1-r}}.$$

Next, we present a slightly different approach to the interpolation constant $c\left(\sigma,\, H^2,\, H^\infty\right)$ proving an estimate in the following form.

Theorem. 1.3.3 *For every sequence* $\sigma = \{\lambda_1,\, ...,\, \lambda_n\}$ *of* $\mathbb{D}$ *,*

$$c\left(\sigma,\, H^2,\, H^\infty\right) \le sup_{z\in\mathbb{T}} \left(\sum_{k=1}^{n} \frac{(1-|\lambda_k|^2)}{|z-\lambda_k|^2}\right)^{\frac{1}{2}},$$

and hence

$$c(\sigma,\, H^2,\, H^\infty) \le \left(\sum_{j=1}^{n} \frac{1+|\lambda_j|}{1-|\lambda_j|}\right)^{\frac{1}{2}}.$$

In particular, we get once more the same estimate for $c(\sigma,\, H^2,\, H^\infty)$, and hence for $C_{n,r}(H^2, H^\infty)$ as in Corollary 1.3.2.

Later on, (see **Chapter 2**), we show that the estimate of Corollary 1.3.2 is sharp proving the following theorem.

Theorem. 2.1.2 *We have*

$$\frac{1}{4\sqrt{2}}\frac{\sqrt{n}}{\sqrt{1-r}} \le C_{n,r}\left(H^2,\, H^\infty\right) \le \sqrt{2}\frac{\sqrt{n}}{\sqrt{1-r}},$$

for all $n \ge 1$*,* $0 \le r < 1$*.*

Then, we extend these results to the H^p spaces.

Theorem. 1.5.0 *Let* $1 \le p \le \infty$*. Then*

$$C_{n,r}\left(H^p,\, H^\infty\right) \le A_p \left(\frac{n}{1-r}\right)^{\frac{1}{p}},$$

for all $n \ge 1$*,* $0 \le r < 1$*, where* A_p *is a constant depending only on* p*.*

In particular, this gives yet another proof of the fact that $C_{n,r}(H^2,\, H^\infty) \le 2^{\frac{1}{2}}\left(\frac{n}{1-r}\right)^{\frac{1}{2}}$. Later on, (in **Chapter 2**), we show that the latter estimate is sharp for even p.

Theorem. 2.2.0 *Let* $p \in 2\mathbb{Z}_+$ *and* $\sigma_{\lambda, n} = \{\lambda, ..., \lambda\}$ *(n times), then*

$$c\left(\sigma_{\lambda,n}, H^p, H^\infty\right) \geq \frac{1}{32^{\frac{1}{p}}}\left(\frac{n}{1-|\lambda|}\right)^{\frac{1}{p}},$$

for all $\lambda \in \mathbb{D}$, $n \geq 1$, *where* $\sigma_{\lambda, n} = \{\lambda, \lambda, ..., \lambda\}$. *Hence, for every* $n \geq 1$, $0 \leq r < 1$,

$$\frac{1}{32^{\frac{1}{p}}}\left(\frac{n}{1-r}\right)^{\frac{1}{p}} \leq C_{n,r}\left(H^p, H^\infty\right) \leq \left(\frac{2n}{1-r}\right)^{\frac{1}{p}}.$$

We also consider the case where X is a weighted space $X = l^p_a(w_k)$,

$$l^p_a(w_k) = \left\{f = \sum_{k\geq 0} \hat{f}(k)z^k : \|f\|^p = \sum_{k\geq 0} |\hat{f}(k)|^p w_k^p < \infty\right\},$$

with a weight w satisfying $w_k > 0$ for every $k \geq 0$ and $\overline{lim}_k(1/w_k^{1/k}) = 1$. The latter condition implies that $l^p_a(w_k)$ is continuously embedded into the space of holomorphic functions $Hol(\mathbb{D})$ on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (and not on a larger disc, i.e. $l^p_a(w_k)$ does not contained in $Hol(r\mathbb{D})$ for every $r > 1$).
Our principal case is $p = 2$, where $l^2_a(w_k)$ is a reproducing kernel Hilbert space on the disc $\mathbb{D}$.

Theorem. 1.6.0 *Let* σ *be a sequence in* $\mathbb{D}$. *Then*

$$c\left(\sigma,\, l^2_a\left(\frac{1}{(k+1)^{\alpha-1}}\right),\, H^\infty\right) \leq A\left(\frac{n}{1-r}\right)^{\frac{2\alpha-1}{2}}.$$

Otherwise,

$$C_{n,r}\left(l^2_a\left(\frac{1}{(k+1)^{\alpha-1}}\right),\, H^\infty\right) \leq A\left(\frac{n}{1-r}\right)^{\frac{2\alpha-1}{2}},$$

$$C_{n,r}\left(L^2_a\left(\left(1-|z|^2\right)^\beta dA\right),\, H^\infty\right) \leq A^{'}\left(\frac{n}{1-r}\right)^{\frac{\beta+2}{2}},$$

for all $n \geq 1$, $0 \leq r < 1$, $\alpha \geq 1$, $\beta > -1$, *where* $A = A(\alpha - 1)$ *is a constant depending only on* α *and* $A^{'} = A^{'}(\beta)$ *is a constant depending only on* β.

Later on, (see **Chapter 2**), we show that for $\alpha = \frac{3}{2}$, (which is equivalent to $\beta = 0$), the latter estimate is sharp. Precisely, in the case of the Bergman space L^2_a, we have the following theorem.

Theorem. 2.1.3 *We have*

$$\frac{1}{32}\frac{n}{1-r} \leq C_{n,r}\left(L^2_a,\, H^\infty\right) \leq \sqrt{14}\frac{n}{1-r},$$

for all $n \geq 1$, $0 \leq r < 1$.

In a more general case of $X = l_A^2\left(\frac{1}{(k+1)^{\frac{N-1}{2}}}\right)$, where $N \geq 1$ is an integer, we prove a similar result but with quite a worse constant.

Theorem. 2.1.0 *Let $N \geq 1$ be an integer and $\sigma_{\lambda,n} = \{\lambda, ..., \lambda\}$ (n times). Then,*

$$c\left(\sigma_{\lambda,n},\, l_A^2\left(\frac{1}{(k+1)^{\frac{N-1}{2}}}\right),\, H^\infty\right) \geq a_N\left(\frac{n}{1-|\lambda|}\right)^{\frac{N}{2}}$$

for a positive constant a_N depending on N only. In particular,

$$a_N\left(\frac{n}{1-r}\right)^{\frac{N}{2}} \leq C_{n,r}\left(l_a^2\left(\frac{1}{(k+1)^{\frac{N-1}{2}}}\right),\, H^\infty\right) \leq A\left(\frac{n}{1-r}\right)^{\frac{N}{2}},$$

for all $n \geq 1$, $0 \leq r < 1$, where $A = A\left(\frac{N-1}{2}\right)$ is a constant defined in Theorem 1.6.0.

In **Sections 1.7, 1.8 and 1.9**, we deal with an upper estimate for $C_{n,r}(X,\, H^\infty)$ in the scale of spaces $X = l_A^p\left(\frac{1}{(k+1)^{\alpha-1}}\right)$, $\alpha \geq 1$, $1 \leq p \leq +\infty$. (The case $p = 2$ is solved in **Section 1.6** and **in Chapter 2** (for sharpness)). We start giving a result for $1 \leq p \leq 2$.

Theorem. 1.8.0 *Let $1 \leq p \leq 2$, $\alpha \geq 1$. Then*

$$B\left(\frac{1}{1-r}\right)^{\alpha-\frac{1}{p}} \leq C_{n,r}\left(l_a^p\left(\frac{1}{(k+1)^{\alpha-1}}\right),\, H^\infty\right) \leq A\left(\frac{n}{1-r}\right)^{\alpha-\frac{1}{2}},$$

for all $r \in [0,\, 1[$, $n \geq 1$, where $A = A(\alpha-1,\, p)$ is a constant depending only on α and p and $B = B(p)$ is a constant depending only on p.

It is very likely that the bounds of Theorem 1.8.0 are not sharp. The sharp one should be probably $\left(\frac{n}{1-r}\right)^{\alpha-\frac{1}{p}}$. In the same way, for $2 \leq p \leq \infty$, we give the following theorem, in which we feel again that the upper bound $\left(\frac{n}{1-r}\right)^{\alpha+\frac{1}{2}-\frac{2}{p}}$ is not sharp. The sharp one probably should be the lower bound $\left(\frac{n}{1-r}\right)^{\alpha-\frac{1}{p}}$.

Theorem. 1.10.0 *Let $2 \leq p \leq \infty$, $\alpha \geq 1$. Then*

$$B\left(\frac{1}{1-r}\right)^{\alpha-\frac{1}{p}} \leq C_{n,r}\left(l_a^p\left(\frac{1}{(k+1)^{\alpha-1}}\right),\, H^\infty\right) \leq A\left(\frac{n}{1-r}\right)^{\alpha+\frac{1}{2}-\frac{2}{p}},$$

for all $r \in [0,\, 1[$, $n \geq 1$, where $A = A(\alpha-1,\, p)$ is a constant depending only on α and p and $B = B(p)$ is a constant depending only on p.

In **Section 1.11**, we suppose that $X = L_a^p\left(\left(1-|z|^2\right)^\beta dA\right)$, $\beta > -1$ and $1 \leq p \leq 2$. Our goal in this section is to give an estimate for the constant for a generalized Carathéodory-Schur interpolation, (a partial case of the Nevanlinna-Pick interpolation),

$$c(\sigma_{\lambda,n}, X, H^\infty) = \ sup\left\{\|f\|_{H^\infty/b_\lambda^n H^\infty} : \ f \in X, \ \|f\|_X \leq 1\right\},$$

where $\|f\|_{H^\infty/b_\lambda^n H^\infty} = \ inf\left\{\|f + b_\lambda^n g\|_\infty : \ g \in X\right\}$, and $\sigma_{\lambda,n} = \ \{\lambda, \lambda, ..., \lambda\}$, $\lambda \in \mathbb{D}$. The corresponding interpolation problem is: given $f \in X$, to minimize $\|h\|_\infty$ such that $h^{(j)}(\lambda) = f^{(j)}(\lambda)$, $0 \leq j < n$.

For this partial case, we have the following generalization of the estimate from Theorem 1.6.0.

Theorem. 1.11.0 *Let $\lambda \in \mathbb{D}$, $\beta > -1$ and $1 \leq p \leq 2$. Then,*

$$c\left(\sigma_{\lambda,\,n},\ L_a^p\left(\left(1 - |z|^2\right)^\beta dA\right),\ H^\infty\right) \leq A'\left(\frac{n}{1 - |\lambda|}\right)^{\frac{\beta+2}{p}},$$

where $A' = A'(\beta,\, p)$ is a constant depending only on β and p.

We finish **Chapter 1** with **Section 1.12** in which we compare the method used in **Sections 1.2, 1.3, 1.4 and 1.6**, with those resulting from the Carleson-free interpolation. Especially, we are interested in the cases of circular and radial sequences σ (see below). Recall that given a (finite) set $\sigma = \{\lambda_1,\ ...,\ \lambda_n\} \subset \mathbb{D}$, the Carleson interpolation constant $C_I(\sigma)$ is defined by

$$C_I(\sigma) = sup_{\|a\|_{l^\infty} \leq 1} inf\left(\|\ g\ \|_\infty\!:\ g \in H^\infty,\ g_{|\sigma} = a\right).$$

Theorem. 1.12.0 *Let X be a Banach space, $X \subset Hol(\mathbb{D})$. Then, for all sequences $\sigma = \{\lambda_1,\ ...,\ \lambda_n\}$ of distinct points in the unit disc $\mathbb{D}$,*

$$max_{1\leq i\leq n}\|\varphi_{\lambda_i}\| \leq c(\sigma,\ X,\ H^\infty) \leq C_I(\sigma).max_{1\leq i\leq n}\|\varphi_{\lambda_i}\|,$$

where $C_I(\sigma)$ stands for the Carleson interpolation constant.

Theorem 1.12.0 tells us that, for σ with a "reasonable" interpolation constant $C_I(\sigma)$, the quantity $c(\sigma,\ X,\ H^\infty)$ behaves as $max_i\|\varphi_{\lambda_i}\|$. However, for "tight" sequences σ, the constant $C_I(\sigma)$ is so large that the estimate in question contains almost no information. On the other hand, an advantage of the estimate of Theorem 1.12.0 is that it does not contain $\#\sigma = n$ explicitly. Therefore, for well-separated sequences σ, Theorem 1.12.0 should give a better estimate than those of Corollary 1.3.2, and of Theorem 1.6.0.

Now, how does the interpolation constant $C_I(\sigma)$ behave in terms of the caracteristics r and n of σ? We answer this question for some particular sequences σ.

Example. 1.12.2 *Two points sets. Let $\sigma = \{\lambda_1,\ \lambda_2\}$, $\lambda_i \in \mathbb{D}$, $\lambda_1 \neq \lambda_2$. Then,*

$$\frac{1}{|b_{\lambda_1}(\lambda_2)|} \leq C_I(\sigma) \leq \frac{2}{|b_{\lambda_1}(\lambda_2)|},$$

and Theorem 1.12.0 implies

$$c(\sigma,\ X,\ H^\infty) \leq \frac{2}{|b_{\lambda_1}(\lambda_2)|} max_{i=1,2}\|\varphi_{\lambda_i}\|,$$

whereas a straightforward estimate gives (see Section 1.12)

$$c(\sigma,\ X,\ H^\infty) \leq \|\varphi_{\lambda_1}\| + max_{|\lambda|\leq r}\|\varphi_{\lambda,1}\|\,(1 + |\lambda_1|),$$

where $r = max\,(|\lambda_1|\,,\,|\lambda_2|)$ and the functional $\varphi_{\lambda,1}$ is defined above just after Theorem 1.1.1. The difference is that the first upper bound blows up when $\lambda_1 \to \lambda_2$, whereas the second one is still well-bounded.

Example. 1.12.3 *Circular sequences. Let $0 < r < 1$ and $\sigma = \{\lambda_1, \lambda_2, ..., \lambda_n\}$, $\lambda_i \neq \lambda_j$, $|\lambda_i| = r$ for every i, and let $\alpha = \frac{min_{i\neq j}|\lambda_i - \lambda_j|}{1-r}$. Then, $\frac{1}{\alpha} \leq C_I(\sigma) \leq 8e^{K'\left(1+\frac{K}{\alpha^3}\right)}$, where K, $K' > 0$ are absolute constants. Therefore,*

$$c(\sigma,\, X,\, H^\infty) \leq 8e^{K'\left(1+\frac{K}{\alpha^3}\right)} max_{|\lambda|=r}\, \|\varphi_\lambda\|$$

for every $r-circular$ set σ (an estimate does not depending on n explicitly). In particular, there exists an increasing function $\varphi\,:\,\mathbb{R}_+ \to \mathbb{R}_+$ such that, for any n uniformly distributed points $\lambda_1, ..., \lambda_n$, $|\lambda_i| = r$, $|\lambda_i - \lambda_{i+1}| = 2rsin\left(\frac{\pi}{2n}\right)$, we have

(1) $c(\sigma,\, H^2,\, H^\infty) \leq \varphi\left(\frac{n(1-r)}{r}\right)\frac{1}{(1-r)^{\frac{1}{2}}}$, for every n and r, $0 < r < 1$ and in particular, for $n \leq [r(1-r)^{-1}]$ we obtain

$$c(\sigma,\, H^2,\, H^\infty) \leq c\frac{1}{(1-r)^{\frac{1}{2}}},$$

whereas our specific Corollary 1.3.2, (which is sharp over all n elements sequences σ), gives

$$c(\sigma,\, H^2,\, H^\infty) \leq c\frac{1}{(1-r)}$$

only.

(2) $c(\sigma,\, L^2_a,\, H^\infty) \leq \varphi\left(\frac{n(1-r)}{r}\right)\frac{1}{(1-r)}$, for every n and r, $0 < r < 1$ and in particular, for $n \leq [r(1-r)^{-1}]$ we obtain

$$c(\sigma,\, L^2_a,\, H^\infty) \leq c\frac{1}{(1-r)},$$

whereas our specific Theorem 1.6.0, (which, again, is sharp over all n elements sequences σ), gives

$$c(\sigma,\, L^2_a,\, H^\infty) \leq c\frac{1}{(1-r)^2}$$

only.

We finally deal with a special case of radial sequences, in which we study *sparse sequences, condensed sequences,* and *long sequences,* and prove the following claim.

Example. 1.12.4 *Radial sequences.*

Claim. Let $\sigma = \left\{1 - \rho^{p+k}\right\}_{k=1}^n$, $0 < \rho < 1$, $p > 0$. The estimate of $c(\sigma,\, H^2,\, H^\infty)$ via the Carleson constant $C_I(\sigma)$ (using Theorem 1.12.0) is comparable with or better than the estimates from Corollary 1.3.2 (for $X = H^2$) and Theorem 1.6.0 (for $X = L^2_a$ and $X = L^2_a\left((1-|z|^2)^\beta\right)$) for sufficently small values of ρ (as $\rho \to 0$) and/or for a fixed ρ and $n \to \infty$. In all other cases, as for $p \to \infty$ (which means $\lambda_1 \to 1$), or $\rho \to 1$, or $n \to \infty$ and $\rho \to 1$, it is worse.

In **Chapter 3**, we study the condition numbers $CN(T) = \|T\|\,.\,\|T^{-1}\|$ of Toeplitz and analytic Toeplitz $n \times n$ matrices T. It is shown that the supremum of $CN(T)$ over all such matrices with $\|T\| \leq 1$ and the given minimum of eigenvalues $r = min_{i=1..n}|\lambda_i| > 0$ behaves as the corresponding supremum over all $n \times n$ matrices (i.e., as $\frac{1}{r^n}$ (Kronecker)), and this equivalence is uniform in n and

r (see [Z]). The proof is based on a use of the Sarason-Sz.Nagy-Foias commutant lifting theorem and the Carathéodory-Schur interpolation.

Let H be a Hilbert space of finite dimension n and T an invertible operator acting on H such that $\| T \| \leq 1$. We are interested in estimating the norm of the inverse of T :

$$\| T^{-1} \| .$$

More precisely, given a family $\mathcal{F}$ of $n-$dimensional operators and a $T \in \mathcal{F}$, we set

$$r_{min}(T) = min_{i=1..n}|\lambda_i| > 0,$$

where $\{\lambda_1, ..., \lambda_n\} = \sigma(T)$ is the spectrum of T. We are looking for "the best possible" majorant $\Phi_n(r)$ such that

$$\left\|T^{-1}\right\| \leq \Phi_n(r)$$

for every $T \in \mathcal{F}$, $\|T\| \leq 1$. This leads to define the following bound $c_n(\mathcal{F}, r)$, where $0 < r < 1$,

$$c_n(\mathcal{F}, r) = sup\left\{\left\|T^{-1}\right\| : \ T \in \mathcal{F}, \ \|T\| \leq 1, \ r_{min}(T) \geq r\right\}.$$

The following classical result is attributed to Kronecker (XIX c.)

Theorem. 3.0 (Kronecker):

Let $\mathcal{F}$ be the set of all n-dimensional operators defined on an euclidean space. Then

$$c_n(r) := c_n(\mathcal{F}, r) = \frac{1}{r^n}$$

Since obviously the upper bound in $c_n(r)$ is attained (by a compactness argument), a natural question arises: how to describe the extremal matrices T such that $\|T\| \leq 1$, $r_{min}(T) \geq r$ and $\|T^{-1}\| = \frac{1}{r^n}$. The answer is contained in N. Nikolski [N3], where it is shown that such $T's$ are of a very special form (the so-called model operators) and are never Toeplitz. Recall that T is a Toeplitz matrix if and only if there exists a sequence $(a_k)_{k=-n+1}^{k=n-1}$ such that

$$T = T_a = \begin{pmatrix} a_0 & a_{-1} & . & . & a_{-n+1} \\ a_1 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & a_{-1} \\ a_{n-1} & . & . & a_1 & a_0 \end{pmatrix},$$

and that T is an analytic Toeplitz matrix if and only if there exists a sequence $(a_k)_{k=0}^{k=n-1}$ such that

$$T = T_a = \begin{pmatrix} a_0 & 0 & . & . & 0 \\ a_1 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & 0 \\ a_{n-1} & . & . & a_1 & a_0 \end{pmatrix}.$$

We denote by $\mathcal{T}_n$ the set of Toeplitz matrices of size n, and $\mathcal{T}_n^a$ will be the set of analytic Toeplitz matrices of size n. This leads to the following questions.

How behave the constants $c_n(\mathcal{T}_n, r)$ and $c_n(\mathcal{T}_n^a, r)$ when $n \to \infty$ and/or $r \to 0$? Are they uniformly comparable with the Kronecker bound $c_n(r)$? The answers seem not to be obvious, at least the obvious candidates like $T = \frac{\lambda + J_n}{\|\lambda + J_n\|}$, where J_n is the $n-$dimensional Jordan matrix, do not lead to the needed uniform (in n and r) equivalence. For short, we denote

$$t_n(r) = c_n(\mathcal{T}_n, r)$$

and

$$t_n^a(r) = c_n(\mathcal{T}_n^a, r).$$

Obviously we have,

$$t_n^a(r) \leq t_n(r) \leq c_n(r) = \frac{1}{r^n}.$$

The following theorem (see [Z]) answers the above questions.

Theorem. 3.1 *1) For all* $r \in]0, 1[$ *and* $n \geq 1$,

$$\frac{1}{2} \leq r^n t_n^a(r) \leq r^n c_n(r) = 1$$

2) For every $n \geq 1$

$$lim_{r \to 0} r^n t_n^a(r) = lim_{r \to 1} r^n t_n^a(r) = 1$$

and for every $0 < r \leq 1$

$$lim_{n \to \infty} r^n t_n^a(r) = 1.$$

The proof of the theorem is given in Section 3.2. It depends on a Carathéodory-Schur interpolation treated via the corona equation $fg + z^n h = 1$.

1. Upper bounds for $c(\sigma, X, Y)$, as a kind of the Nevanlinna-Pick problem

1.1. General Banach spaces X and Y satisfying properties (P_i) , $i = 1...4$

The following theorem shows that if X and Y satisfy properties (P_i) for $i = 1...4$, then our interpolation constant $c(\sigma, X, Y)$ is bounded by a quantity $M_{n,r}$ which depends only on $n = \#\sigma$ and $r = max_{1\leq i\leq n}|\lambda_i|$ (and of course on X and Y). In this generality, we cannot discuss the question of sharpness of the bounds obtained. First, we prove the following lemma.

Lemma. 1.1.0 *Under* (P_2), (P_3) and (P_4), $B_\sigma X$ *is a closed subspace of* X *and moreover,*

$$B_\sigma X = \{f \in X : f(\lambda) = 0, \, \forall\lambda \in \sigma \, (including \, multiplicities)\}.$$

Proof. Since $X \subset Hol(\mathbb{D})$ continuously, and evaluation functionals $f \mapsto f(\lambda)$ and

$$f \mapsto f^{(k)}(\lambda), \, k = 1, 2, ...,$$

are continous on $Hol(\mathbb{D})$, the subspace

$$M = \{f \in X : f(\lambda) = 0, \, \forall\lambda \in \sigma \, (including \, multiplicities)\},$$

is closed in X.

On the other hand, $B_\sigma X \subset X$, and hence $B_\sigma X \subset M$. Indeed, properties (P_2) and (P_3) imply that $h.X \subset X$, for all $h \in Hol((1+\epsilon)\mathbb{D})$ with $\epsilon > 0$; we can write $h(z) = \sum_{k\geq 0}\widehat{h}(k)z^k$ with $\left|\widehat{h}(k)\right| \leq Cq^n$, $C > 0$ and $q < 1$. Then $\sum_{n\geq 0}\left\|\widehat{h}(k)z^k f\right\|_X < \infty$ for every $f \in X$. Since X is a Banach space we can conclude that $hf = \sum_{n\geq 0}\widehat{h}(k)z^k f \in X$.

In order to see that $M \subset B_\sigma X$, it suffices to justify that

$$[f \in X \, and \, f(\lambda) = 0] \Longrightarrow \left[\frac{f}{b_\lambda} = (1-\overline{\lambda}z).\frac{f}{\lambda - z} \in X\right].$$

But this is obvious from (P_4) and the previous arguments. □

Theorem. 1.1.1 *Let* X, Y *be Banach spaces verifying properties* (P_i), $i = 1...4$ *, then*

$$C_{n,r}(X, Y) < \infty,$$

for every $n \geq 1$ *and* r, $0 \leq r < 1$.

Proof. For $k = 1..n$, we set

$$f_k(z) = \frac{1}{1-\overline{\lambda_k}z},$$

and define the family $(e_k)_{k=1}^n$, (which is known as Malmquist basis, see [N1] p.117), by

$$e_1 = \left(1-|\lambda_1|^2\right)^{\frac{1}{2}} f_1,$$

and

$$e_k = \left(1-|\lambda_k|^2\right)^{\frac{1}{2}} \left(\Pi_{j=1..k-1}b_{\lambda_j}\right) f_k = \frac{f_k}{\parallel f_k \parallel_2}\Pi_{j=1}^{k-1}b_{\lambda_j}$$

for $k = 2...n$. Now, taking $f \in X$, we set

$$g = \sum_{k=1}^{n} \left(\sum_{j \geq 0} \hat{f}(j) \overline{\widehat{e_k}(j)} \right) e_k,$$

where the series

$$\sum_{j \geq 0} \hat{f}(j) \overline{\widehat{e_k}(j)}$$

are absolutely convergent. Indeed,

$$\widehat{e_k}(j) = \frac{1}{2\pi i} \int_{R\mathbb{T}} \frac{e_k(w)}{w^{j+1}} dw,$$

for all $j \geq 0$ and for all $1 < R < \frac{1}{r}$. For a subset A of $\mathbb{C}$ and for a bounded function h on A, we define

$$\|h\|_A := sup_{z \in A} |h(z)|.$$

As a result,

$$|\widehat{e_k}(j)| \leq \frac{1}{2\pi} \frac{1}{R^{j+1}} \parallel e_k \parallel_{R\mathbb{T}}.$$

So

$$\sum_{j \geq 0} \left| \hat{f}(j) \overline{\widehat{e_k}(j)} \right| \leq \frac{\|e_k\|_{RT}}{2\pi R} \sum_{j \geq 0} \left| \hat{f}(j) \right| \left(\frac{1}{R} \right)^j < \infty,$$

since $R > 1$ and f is holomorphic in $\mathbb{D}$.
Next, we observe that the map

$$\Phi : \, Hol(\mathbb{D}) \rightarrow Y \subset Hol(\mathbb{D})$$

$$\Phi : \, f \mapsto \sum_{k=1}^{n} \left(\sum_{j \geq 0} \hat{f}(j) \overline{\widehat{e_k}(j)} \right) e_k,$$

is well defined and has the following properties.
(a) $\Phi_{|H^2} = P_{B_\sigma}$ where P_{B_σ} is the orthogonal projection on the n-dimensional subspace of H^2, K_{B_σ} defined by

$$K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \Theta B_\sigma H^2,$$

the last equality being a consequence of Lemma 1.2.0 of Section 1.2. Here, H^2 stands for the classical Hardy space $H^2(\mathbb{D})$ of the disc,

$$H^2(\mathbb{D}) = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \sum_{k \geq 0} \left| \hat{f}(k) \right|^2 < \infty \right\},$$

or equivalently,

$$H^2(\mathbb{D}) = \left\{ f \in Hol(\mathbb{D}) : sup_{0\leq r<1} \int_{\mathbb{T}} |f(rz)|^2 \, dm(z) < \infty \right\},$$

m being the normalized Lebesgue measure on $\mathbb{T}$. See [N2] p.31-p.57 for more details on the Hardy spaces H^p, $1 \leq p \leq \infty$.

(b) Φ is continuous on $Hol(\mathbb{D})$ for the uniform convergence on compact sets of $\mathbb{D}$.
Indeed, the point (a) is obvious since $(e_k)_{k=1}^n$ is an orthonormal basis of K_{B_σ} and

$$\sum_{j\geq 0} \widehat{f}(j)\overline{\widehat{e_k}(j)} = \langle f, e_k \rangle ,$$

where $\langle ., . \rangle$ means the Cauchy sesquilinear form $\langle h, g \rangle = \sum_{k\geq 0} \hat{h}(k)\overline{\hat{g}(k)}$. In order to check point (b), let $(f_l)_{l\in\mathbb{N}}$ be a sequence of $Hol(\mathbb{D})$ converging to 0 uniformly on compact sets of $\mathbb{D}$. We need to see that $(\Phi(f_l))_{l\in\mathbb{N}}$ converges to 0, for which it is sufficient to show that $lim_l \left| \sum_{j\geq 0} \widehat{f_l}(j)\overline{\widehat{e_k}(j)} \right| = 0$, for every $k = 1, 2, ..., n$. Let $\rho \in]0, 1[$, then

$$\widehat{f_l}(j) = \frac{1}{2\pi} \int_{\rho\mathbb{T}} \frac{f_l(w)}{w^{j+1}} dw,$$

for all $j, l \geq 0$. As a result,

$$\left| \widehat{f_l}(j) \right| \leq \frac{1}{2\pi} \frac{1}{\rho^{j+1}} \|f_l\|_{\rho\mathbb{T}} .$$

So

$$\left| \sum_{j\geq 0} \widehat{f_l}(j)\overline{\widehat{e_k}(j)} \right| \leq \sum_{j\geq 0} \left| \widehat{f_l}(j)\overline{\widehat{e_k}(j)} \right| \leq \frac{\|f_l\|_{\rho\mathbb{T}}}{2\pi\rho} \sum_{j\geq 0} |\widehat{e_k}(j)| \frac{1}{\rho^j}.$$

Now if ρ is close enough to 1, it satisfies the inequality $1 \leq \frac{1}{\rho} < \frac{1}{r}$, which entails

$$\sum_{j\geq 0} |\widehat{e_k}(j)| \frac{1}{\rho^j} < +\infty$$

for each $k = 1..n$. The result follows.

Let

$$\Psi = Id_{|X} - \Phi_{|X}.$$

Using point (a), since $Pol_+ \subset H^2$ (Pol_+ standing for the set of all complex polynomials p, $p(z) = \sum_{k=0}^N a_k z^k$), we get that $Im\left(\Psi_{|Pol_+}\right) \subset B_\sigma H^2$. Now, since $Pol_+ \subset Y$ and $Im(\Phi) \subset Y$, we deduce that

$$Im\left(\Psi_{|Pol_+}\right) \subset B_\sigma H^2 \cap Y \subset B_\sigma H^2 \cap X,$$

since $Y \subset X$. Now $\Psi(p) \in X$ and satisfies $(\Psi(p))_{|\sigma} = 0$ (that is to say $(\Psi(p))(\lambda) = 0$, $\forall \lambda \in \sigma$ *(including multiplicities)*) for all $p \in Pol_+$. Using Lemma 1.1.0, we get that $Im\left(\Psi_{|Pol_+}\right) \subset B_\sigma X$.

Now, Pol_+ being dense in X (property (P_2)), and Ψ being continuous on X, we can conclude that $Im\,(\Psi) \subset B_\sigma X$.

Now, we return to the proof of Theorem 1.1.1. Let $f \in X$ such that $\parallel f \parallel_X \leq 1$ and $g = \Phi(f)$. Since $Hol\left(\frac{1}{r}\mathbb{D}\right) \subset Y$, we have

$$g = \Phi(f) \in Y$$

and

$$f - g = \Psi(f) \in B_\sigma X.$$

Moreover,

$$\|g\|_Y \leq \sum_{k=1..n} |\langle f, e_k\rangle| \, \|e_k\|_Y .$$

In order to bound the right hand side, recall that for all $j \geq 0$ and for $R = \frac{2}{r+1} \in]1, \frac{1}{r}[$,

$$\sum_{j\geq 0} \left|\widehat{f}(j)\overline{\widehat{e_k}(j)}\right| \leq \frac{\|e_k\|_{\frac{2}{r+1}\mathbb{T}}}{2\pi} \sum_{j\geq 0} \left|\widehat{f}(j)\right| \left(\frac{r+1}{2}\right)^j .$$

Since the norm $f \mapsto \sum_{j\geq 0} \left|\widehat{f}(j)\right| \left(\frac{r+1}{2}\right)^j$ is continuous on $Hol(\mathbb{D})$, and the inclusion $X \subset Hol(\mathbb{D})$ is also continuous, there exists $C_r > 0$ such that

$$\sum_{j\geq 0} \left|\widehat{f}(j)\right| \left(\frac{r+1}{2}\right)^j \leq C_r \parallel f \parallel_X ,$$

for every $f \in X$. On the other hand,

$$Hol\left(\frac{2}{r+1}\mathbb{D}\right) \subset Y,$$

(continuous inclusion again), and hence there exists $K_r > 0$ such that

$$\|e_k\|_Y \leq K_r sup_{|z| < \frac{2}{r+1}} |e_k(z)| = K_r \, \|e_k\|_{\frac{2}{r+1}\mathbb{T}} .$$

It is more or less clear that the right hand side of the last inequality can be bounded in terms of r and n only. Let us give a proof to this fact. It is clear that it suffices to estimate

$$sup_{1 < |z| < \frac{2}{r+1}} |e_k(z)| .$$

In order to bound this quantity, notice that

$$|b_\lambda(z)|^2 \leq \left|\frac{\lambda - z}{1 - \bar{\lambda}z}\right|^2 = 1 + \frac{(|z|^2 - 1)(1 - |\lambda|^2)}{|1 - \bar{\lambda}z|^2}, \tag{1.1.0}$$

for all $\lambda \in \mathbb{D}$ and all $z \in \frac{1}{|\lambda|}\mathbb{D}$. Using the identity (1.1.0) for $\lambda = \lambda_j$, $1 \leq j \leq n$, and $z = \rho e^{it}$, $\rho = \frac{2}{r+1}$, we get

$$\left|e_k(\rho e^{it})\right|^2 = (1-|\lambda_k|^2)\left(\Pi_{j=1}^{k-1}\left|b_{\lambda_j}(\rho e^{it})\right|^2\right)\left|\frac{1}{1-\bar{\lambda}_k\rho e^{it}}\right|^2,$$

$$\left|e_k(\rho e^{it})\right|^2 \leq \left(\Pi_{j=1}^{k-1}\left|b_{\lambda_j}(\rho e^{it})\right|^2\right)\left(\frac{1}{1-|\lambda_k|\rho}\right)^2,$$

for all $k = 2..n$,

$$\left|e_k\left(\rho e^{it}\right)\right|^2 \leq 2\left(\Pi_{j=1}^{k-1}\left(1+\frac{(\rho^2-1)(1-|\lambda_j|^2)}{1-|\lambda_j|^2\rho^2}\right)\right)\left(\frac{1}{1-|\lambda_k|\rho}\right)^2.$$

Hence,

$$\left|e_k\left(\rho e^{it}\right)\right|^2 \leq 2\left(\Pi_{j=1}^{k-1}\left(1+\frac{2(\frac{1}{r^2}-1)}{1-r^2\frac{4}{(r+1)^2}}\right)\right)\left(\frac{1}{1-\frac{2r}{r+1}}\right)^2.$$

Finally,

$$\|e_k\|_{\frac{2}{r+1}\mathbb{T}} \leq$$

$$\leq \frac{1}{1-\frac{2r}{r+1}}\sqrt{2\left(\Pi_{j=1..n-1}\left(1+\frac{2(\frac{1}{r^2}-1)}{1-r^2\frac{4}{(r+1)^2}}\right)\right)} =: C_1(r,n).$$

and

$$\sum_{j\geq 0}\left|\hat{f}(j)\overline{\hat{e_k}(j)}\right| \leq \frac{C_r\|e_k\|_{\frac{2}{r+1}\mathbb{T}}}{2\pi}\parallel f\parallel_X \leq$$

$$\leq \frac{C_rC_1(r,n)}{2\pi}\parallel f\parallel_X.$$

On the other hand,

$$\|e_k\|_Y \leq K_r\|e_k\|_{\frac{2}{r+1}\mathbb{T}} \leq K_rC_1(r,n).$$

So

$$\|g\|_Y \leq \sum_{k=1}^{n}|\langle f, e_k\rangle|\,\|e_k\|_Y \leq$$

$$\leq \sum_{k=1}^{n}\frac{C_rC_1(r,n)}{2\pi}\|f\|_X\,K_rC_1(r,n) = \frac{nC_rK_r}{2\pi}\left(C_1(r,n)\right)^2\|f\|_X,$$

which proves that

$$c(\sigma,X,Y) \leq \frac{nC_rK_r}{2\pi}\left(C_1(r,n)\right)^2$$

and completes the proof of Theorem 1.1.1. $\square$

We now give some lower bounds for the quantity $C_{n,r}(X,\, H^\infty)$. A partial case of our general interpolation problem- the so-called generalized Caratheodory-Schur interpolation- will be useful (see Lemma 1.1.2 below). The latter correspons to the case $\lambda_1 = \lambda_2 = ... = \lambda_n = \lambda$, and hence consists in the following: given a function $f \in X$, to find $inf \left\{\| g \|_\infty \colon \, g^{(s)}(\lambda) = f^{(s)}(\lambda),\, 0 \leq s < n\right\}$. This means to compute or estimate the quantity

$$c(\sigma_{\lambda,n},\, X,\, H^\infty) = sup_{\|f\|_X \leq 1} \| f \|_{H^\infty / b_\lambda^n H^\infty}$$

where,

$$\sigma_{\lambda,n} := \{\lambda,\, \lambda, ...,\, \lambda\}\, with\, multiplicity\, n,$$

and

$$\| f \|_{H^\infty / b_\lambda^n H^\infty} = inf\{\| g \|_\infty \colon \, g - f \in b_\lambda^n X\} =$$

$$= inf \left\{\| g \|_\infty \colon \, g^{(s)}(\lambda) = f^{(s)}(\lambda),\, 0 \leq s < n\right\}.$$

The following simple lemma will be useful for studying $c(\sigma,\, X,\, H^\infty)$.

Lemma. 1.1.2 *Let $\lambda \in \mathbb{D}$, $n \geq 1$, and $f \in Hol(\mathbb{D})$. Then*

$$\left(\sum_{j=0}^{n-1} \left|\frac{f^{(j)}(\lambda)}{j!}\right|^2 (1-|\lambda|)^{2j}\right)^{\frac{1}{2}} \leq \|f\|_{H^\infty / b_\lambda^n H^\infty} \leq$$

$$\leq \sum_{j=0}^{n-1} \frac{\left|f^{(j)}(\lambda)\right|}{j!} (1+|\lambda|)^j .$$

Proof. Let $g \in (f + z^n Hol(\mathbb{D})) \bigcap H^\infty$ and $\rho = 1 - |\lambda|$. By Parseval,

$$\|g\|_\infty^2 \geq \frac{1}{2\pi} \int_0^{2\pi} \left|g(\lambda + \rho e^{it})\right|^2 dt = \sum_{j \geq 0} \left|\frac{g^{(j)}(\lambda)}{j!}\right|^2 \rho^{2j} \geq$$

$$\geq \sum_{j=0}^{n-1} \left|\frac{f^{(j)}(\lambda)}{j!}\right|^2 \rho^{2j},$$

which shows the left hand side inequality. For the right hand side one, we simply bound the polynomial $g = \sum_{j=0}^{n-1} \frac{f^{(j)}(\lambda)}{j!} (z-\lambda)^j$ replacing $z - \lambda$ by $1 + |\lambda|$. $\square$

To give the first application of this lemma, we introduce the evaluation functionals φ_λ for $\lambda \in \mathbb{D}$,

$$\varphi_\lambda(f) = f(\lambda), \;\; f \in X,$$

as well as the evaluation of the derivatives $\varphi_{\lambda,s}$ $(s = 0, 1, ...)$,

$$\varphi_{\lambda,s}(f) = f^{(s)}(\lambda), \;\; f \in X.$$

Theorem. 1.1.3 *(1) For every sequence $\sigma \subset \mathbb{D}$, we have*

$$c(\sigma,\, X,\, H^{\infty}) \geq max_{\lambda\in\sigma} \left\|\varphi_{\lambda}\right\|.$$

(2) Let $\sigma = \{\lambda_1, ..., \lambda_1, \lambda_2, ..., \lambda_2, ..., \lambda_t, ..., \lambda_t\}$, where each λ_s is repeated according its multiplicity m_s. Then

$$c(\sigma,\, X,\, H^{\infty}) \geq max_{1\leq s\leq t,\, 0\leq j<m_s} \left(\left\|\varphi_{\lambda_s,\, j}\right\| \left(1-|\lambda_j|^2\right)^j \frac{1}{j!}\right).$$

Indeed, by Lemma 1.1.2,

$$\left|\frac{f^{(j)}(\lambda_s)}{j!}\right| \left(1-|\lambda_s|^2\right)^j \leq \parallel f \parallel_{H^{\infty}/b_{\lambda}^{m_s}H^{\infty}} \leq \parallel f \parallel_{H^{\infty}/B_{\sigma}H^{\infty}},$$

for all $\lambda_s \in \sigma$ and $0 \leq j < m_s$ and for every $f \in X$. Taking a sup over f, $\parallel f \parallel_X \leq 1$, we get the result.

□

1.2. The case where X is a Hilbert space

In the following theorem, we suppose that X is a Hilbert space and both X, Y satisfy properties (P_i) for $i = 1...4$. In this case, we obtain a better estimate for $c\,(\sigma,\, X,\, Y)$ than in Theorem 1.1.1 (see point (i) of Theorem 1.2.1). For the case $Y = H^{\infty}$, (point (ii) of Theorem 1.2.1), we can considerably improve this estimate.

Lemma. 1.2.0 *Let $\sigma = \{\lambda_1, ..., \lambda_1, \lambda_2, ..., \lambda_2, ..., \lambda_t, ..., \lambda_t\}$ be a finite sequence of $\mathbb{D}$ where every λ_s is repeated according to its multiplicity m_s, $\sum_{s=1}^{t} m_s = n$. Let $(H,\, (.)_H)$ be a Hilbert space continuously emebedded into $Hol(\mathbb{D})$ and satisfying properties (P_i) for $i = 2,\, 3,\, 4$. Then*

$$K_{B_{\sigma}} =: (B_{\sigma}H)^{\perp} = span\left(k_{\lambda_j,\, i} :\, 1 \leq j \leq t,\, 0 \leq i \leq m_j - 1\right),$$

where $k_{\lambda,\, i} = \left(\frac{d}{d\overline{\lambda}}\right)^i k_{\lambda}$ and k_{λ} is the reproducing kernel of X at point λ for every $\lambda \in \mathbb{D}$.

Proof. First, we explain the notation. Namely, since $H \subset Hol(\mathbb{D})$ (with continuous inclusion), the function $\lambda \mapsto f(\lambda)$ is holomorphic and since $f(\lambda) = (f,\, k_{\lambda})_H$ for every f, the function $\overline{\lambda} \mapsto k_{\lambda}$ is (weakly, and hence strongly) holomorphic. We have $f^{'}(\lambda) = \left(f,\, \frac{d}{d\overline{\lambda}}k_{\lambda}\right)_H$, and by induction, $f^{(i)}(\lambda) = \left(f,\, \left(\frac{d}{d\overline{\lambda}}\right)^i k_{\lambda}\right)_H$ for every i, $i = 0,\, 1,\, ...$. Denote

$$\left(\frac{d}{d\overline{\lambda}}\right)^i k_{\lambda} = k_{\lambda,\, i},$$

we know, (see Lemma 1.1.0), that

$$B_{\sigma}H = \{f \in H :\, f^{(i)}\left(\lambda_j\right) = 0,\, \forall i,\, j,\, 1 \leq i < m_j,\, j = 1, ..., t\} =$$

$$= \{f \in H:\ \left(f,\, k_{\lambda_j, i}\right)_H = 0,\ \forall i,\, j,\, 1 \le i < m_j,\, j = 1, ..., t\}.$$

This means that

$$(B_\sigma H)^\perp = span\left(k_{\lambda_j, i}:\ 1 \le i < m_j,\, j = 1, ..., t\right).$$

□

Theorem. 1.2.1 *Let Y be a Banach space verifying property (P_1) and $X = (H, (.)_H)$ a Hilbert space satisfying properties (P_i) for $i = 2, 3, 4$. We moreover suppose that for every $0 < r < 1$ there exists $\epsilon > 0$ such that $k_\lambda \in Hol\left((1+\epsilon)\mathbb{D}\right)$ for all $|\lambda| < r$, where k_λ stands for the reproducing kernel of X at point λ, and $\overline{\lambda} \mapsto k_\lambda$ is holomorphic on $|\lambda| < r$ as a $Hol((1+\epsilon)\mathbb{D})$-valued function. Let $\sigma = \{\lambda_1, ..., \lambda_1, \lambda_2, ..., \lambda_2, ..., \lambda_t, ..., \lambda_t\}$ be a sequence in $\mathbb{D}$, where λ_s are repeated according their multiplicity m_s, $\sum_{s=1}^{t} m_s = n$. Then we have,*

i)

$$c\,(\sigma,\, X,\, Y) \le \left(\sum_{k=1}^{n} \|e_k\|_Y^2\right)^{\frac{1}{2}},$$

where $(e_k)_{k=1}^n$ stands for the Gram-Schmidt orthogonalization (in the space H) of the sequence

$$k_{\lambda_1,0},\, k_{\lambda_1,1},\, k_{\lambda_1,2}...,\, k_{\lambda_1,m_1-1},\, k_{\lambda_2,0},\, k_{\lambda_2,1},\, k_{\lambda_2,2}...,\, k_{\lambda_2,m_2-1},\, ...,\, k_{\lambda_t,0},\, k_{\lambda_t,1},\, k_{\lambda_t,2}...,\, k_{\lambda_t,m_t-1},$$

notation $k_{\lambda,i}$ is introduced in Lemma 1.2.0.

ii) For the case $Y = H^\infty$, we have

$$c(\sigma,\, H,\, H^\infty) \le sup_{z\in\mathbb{D}} \|P_{B_\sigma} k_z\|_H,$$

where $P_{B_\sigma} = \sum_{k=1}^{n} (., e_k)_H\, e_k$ stands for the orthogonal projection of H onto K_{B_σ},

$$K_{B_\sigma} = span\left(k_{\lambda_j, i}:\ 1 \le i < m_j,\, j = 1, ..., t\right).$$

Proof. i). Let $f \in X$, $\|f\|_X \le 1$. Lemma 1.2.0 shows that

$$g = P_{B_\sigma} f = \sum_{k=1}^{n} (f, e_k)_H\, e_k$$

is the orthogonal projection of f onto subspace K_{B_σ}. Function g belongs to Y because all $k_{\lambda_j,i}$ are in $Hol((1+\epsilon)\mathbb{D})$ for a convenient $\epsilon > 0$, and Y satisfies (P_1).
On the other hand,

$$g - f \in B_\sigma H,$$

again by Lemma 1.2.0.

Moreover,

$$\|g\|_Y \le \sum_{k=1}^{n} |(f, e_k)_H|\, \|e_k\|_Y,$$

and by Cauchy-Schwarz inequality,

$$\|g\|_Y \leq \left(\sum_{k=1}^{n} |(f, e_k)_H|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} \|e_k\|_Y^2\right)^{\frac{1}{2}} \leq$$

$$\leq \|f\|_H \left(\sum_{k=1}^{n} \|e_k\|_Y^2\right)^{\frac{1}{2}},$$

which proves i).
ii). If $Y = H^\infty$, then

$$|g(z)| = |(P_{B_\sigma} f, k_z)_H| = |(f, P_{B_\sigma} k_z)_H| \leq \|f\|_H \, \|P_{B_\sigma} k_z\|_H,$$

for all $z \in \mathbb{D}$, which proves ii). □

1.3. Upper bounds for $c\,(\sigma,\, H^2,\, H^\infty)$

In this section, we specialize the estimate obtained in point (ii) of Theorem 1.2.1 for the case $X = H^2$, the Hardy space of the disc. Later on, we will see that this estimate is sharp at least for some special sequences σ (see **Chapter 3**). We also develop a slightly different approach to the interpolation constant $c\,(\sigma,\, H^2,\, H^\infty)$ giving more estimates for individual sequences $\sigma = \{\lambda_1,\, ...,\, \lambda_n\}$ of $\mathbb{D}$.

Corollary. 1.3.0 *Let $\sigma = \{\lambda_1,\, ...,\, \lambda_n\}$ be a sequence in $\mathbb{D}$. Then,*

$$c\left(\sigma,\, H^2,\, H^\infty\right) \leq sup_{z\in\mathbb{D}} \left(\frac{1 - |B_\sigma(z)|^2}{1 - |z|^2}\right)^{\frac{1}{2}}.$$

Indeed, applying point (ii) of Theorem 1.2.1 for $X = H^2$ and $Y = H^\infty$, and using

$$k_z(\zeta) = \frac{1}{1 - \bar{z}\zeta}$$

and

$$\left(P_{B_\sigma} k_z\right)(\zeta) = \frac{1 - \overline{B_\sigma(z)} B_\sigma(\zeta)}{1 - \overline{z}\zeta},$$

(see [N1] p.199), we obtain

$$\|P_{B_\sigma} k_z\|_{H^2} = \left(\frac{1 - |B_\sigma(z)|^2}{1 - |z|^2}\right)^{\frac{1}{2}},$$

which gives the result.

□

Proposition. 1.3.1 *For every sequence $\sigma = \{\lambda_1, ..., \lambda_n\}$ of $\mathbb{D}$ we have*

$$c\left(\sigma,\, H^2,\, H^\infty\right) \leq \sqrt{2} sup_{|\zeta|=1} \left|B'(\zeta)\right|^{\frac{1}{2}} = \sqrt{2} sup_{|\zeta|=1} \left|\sum_{i=1}^{n} \frac{1-|\lambda_i|^2}{\left(1-\bar{\lambda}_i\zeta\right)^2} \frac{B_\sigma(\zeta)}{b_{\lambda_i}(\zeta)}\right|^{\frac{1}{2}}.$$

Proof. We use Corollary 1.3.0. The map $\zeta \mapsto \|P_B(k_\zeta)\| = sup\{|f(\zeta)| : f \in K_B, \|f\| \leq 1\}$, and hence the map

$$\zeta \mapsto \left(\frac{1-|B(\zeta)|^2}{1-|\zeta|^2}\right)^{\frac{1}{2}},$$

is a subharmonic function so

$$sup_{|\zeta|<1} \left(\frac{1-|B(\zeta)|^2}{1-|\zeta|^2}\right)^{\frac{1}{2}} \leq sup_{|w|=1} lim_{r\to 1} \left(\frac{1-|B(rw)|^2}{1-|rw|^2}\right)^{\frac{1}{2}}.$$

Now apply Taylor's Formula of order 1 for points $w \in \mathbb{T}$ and $u = rw$, $0 < r < 1$. (It is applicable because B is holomorphic at every point of $\mathbb{T}$). We get

$$\frac{B(u)-B(w)}{u-w} = B'(w) + o(1),$$

and since

$$|u-w| = 1-|u|,$$

$$\left|\frac{B(u)-B(w)}{u-w}\right| = \frac{|B(u)-B(w)|}{1-|u|} = |B'(w)+o(1)|.$$

Now,

$$|B(u)-B(w)| \geq |B(w)| - |B(u)| = 1-|B(u)|,$$

$$\frac{1-|B(u)|}{1-|u|} \leq \frac{|B(u)-B(w)|}{1-|u|} = |B'(w)+o(1)|,$$

and

$$lim_{r\to 1} \left(\frac{1-|B(rw)|}{1-|rw|}\right)^{\frac{1}{2}} \leq \sqrt{|B'(w)|}.$$

Since we have

$$B'(w) = -\sum_{i=1}^{n} \frac{1-|\lambda_i|^2}{(1-\bar{\lambda}_i w)^2} \Pi_{j=1,\, j\neq i}^{n} b_{\lambda_j}(w),$$

for all $w \in \mathbb{T}$. This completes the proof since

$$\frac{1-|B(rw)|^2}{1-|rw|^2} = \frac{(1-|B(rw)|)(1+|B(rw)|)}{(1-|rw|)(1+|rw|)} \leq 2\frac{1-|B(rw)|}{1-|rw|}. \square$$

Corollary. 1.3.2 *Let* $\sigma = \{\lambda_1, ..., \lambda_n\}$ *and* $r = max_{1\leq i\leq n}|\lambda_i|$. *Then*

$$c\left(\sigma,\, H^2,\, H^\infty\right) \leq 2\left(\frac{n}{1-r}\right)^{\frac{1}{2}},$$

and hence,

$$C_{n,r}(H^2, H^\infty) \leq 2\left(\frac{n}{1-r}\right)^{\frac{1}{2}}.$$

Indeed, we apply Proposition 1.3.1 and observe that

$$|B'(w)| \leq \left|\sum_{i=1..n}\frac{1-|\lambda_i|^2}{(1-|\lambda_i|)^2}\right| \leq n\frac{1+r}{1-r} \leq \frac{2n}{1-r}.$$

□

Now, we develop a slightly different approach to the interpolation constant $c\,(\sigma,\, H^2,\, H^\infty)$.

Theorem. 1.3.3 *For every sequence* $\sigma = \{\lambda_1, ..., \lambda_n\}$ *of* $\mathbb{D}$ *,*

$$c\left(\sigma,\, H^2,\, H^\infty\right) \leq sup_{z\in\mathbb{T}}\left(\sum_{k=1}^{n}\frac{(1-|\lambda_k|^2)}{|z-\lambda_k|^2}\right)^{\frac{1}{2}}$$

Proof. We give two proofs to this estimate. The first proof is shorter than the second one, but it contains an extra $\sqrt{2}$ factor.

First proof. Using Proposition 1.3.1, we obtain

$$c\left(\sigma,\, H^2,\, H^\infty\right) \leq \sqrt{2}sup_{|\zeta|=1}\left|\sum_{j=1}^{n}\frac{1-|\lambda_j|^2}{\left(1-\overline{\lambda_j}\zeta\right)^2}\frac{B_\sigma}{b_{\lambda_j}}\right|^{\frac{1}{2}} \leq$$

$$\leq \sqrt{2}sup_{|\zeta|=1}\left(\sum_{i=1}^{n}\frac{1-|\lambda_i|^2}{|1-\overline{\lambda_i}\zeta|^2}\right)^{\frac{1}{2}} = \sqrt{2}sup_{|\zeta|=1}\left(\sum_{i=1}^{n}\frac{1-|\lambda_i|^2}{|\bar{\zeta}-\overline{\lambda_i}|^2}\right)^{\frac{1}{2}}.$$

Second proof. In order to simplify the notation, we set $B = B_\sigma$. Consider K_B, the n-dimensional subspace of H^2 defined by

$$K_B = \left(BH^2\right)^\perp = H^2\Theta BH^2.$$

Then the family $(e_k)_{k=1}^{n}$ introduced in the proof of Theorem 1.1.1, (known as Malmquist's basis), is an orthonormal basis of K_B, (see [N1], Malmquist-Walsh Lemma, p.116). Recall that

$$e_1 = \frac{f_1}{\|f_1\|_2},$$

and

$$e_k = \frac{f_k}{\|f_k\|_2}\Pi_{j=1}^{k-1}b_{\lambda_j},$$

for all $k = 2..n$, where

$$f_k = \frac{1}{1 - \overline{\lambda_k} z},$$

is the reproducing kernel of H^2 associated to λ_k. Now, let $f \in H^2$ and

$$g = P_B f = \sum_{k=1}^{n} (f, e_k)_{H^2} \, e_k.$$

Function g belongs to H^∞ because it is a finite sum of H^∞ functions. Moreover,

$$g(\lambda_i) = f(\lambda_i)$$

for all $i = 1...n$, counting with multiplicities. (Indeed, we can write $f = P_B f + g_1$ with $g_1 \in K_B^\perp = BH^2$). We have

$$|g(\zeta)| \leq \sum_{k=1}^{n} |(f, e_k)_{H^2}| \, |e_k(\zeta)|,$$

for all $\zeta \in \mathbb{D}$. And by Cauchy-Schwarz inequality,

$$|g(\zeta)| \leq \left(\sum_{k=1}^{n} |(f, e_k)_{H^2}|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} \frac{(1 - |\lambda_k|^2)}{|1 - \lambda_k \zeta|^2} \right)^{\frac{1}{2}},$$

$$\|g\|_\infty \leq \|f\|_2 \, sup_{\zeta \in \mathbb{T}} \left(\sum_{k=1}^{n} \frac{(1 - |\lambda_k|^2)}{|1 - \lambda_k \zeta|^2} \right)^{\frac{1}{2}}.$$

Since f is an arbitrary H^2 function, we obtain

$$c(\sigma, \, H^2, \, H^\infty) \leq sup_{\zeta \in \mathbb{T}} \left(\sum_{k=1}^{n} \frac{(1 - |\lambda_k|^2)}{|\zeta - \lambda_k|^2} \right)^{\frac{1}{2}},$$

which completes the proof.

□

Corollary. 1.3.4 *For any sequence $\sigma = \{\lambda_1, \, ..., \, \lambda_n\}$ in $\mathbb{D}$,*

$$c(\sigma, \, H^2, \, H^\infty) \leq \left(\sum_{j=1}^{n} \frac{1 + |\lambda_j|}{1 - |\lambda_j|} \right)^{\frac{1}{2}}.$$

Indeed,

$$\sum_{k=1}^{n} \frac{(1 - |\lambda_k|^2)}{|\zeta - \lambda_k|^2} \leq \left(\sum_{k=1}^{n} \frac{(1 - |\lambda_k|^2)}{(1 - |\lambda_k|)^2} \right)^{\frac{1}{2}}$$

and the result follows from Theorem 1.3.3. □

Remark. 1.3.5 As a result, we get once more the same estimate for $C_{n,r}(H^2,\, H^\infty)$ as in Corollary 1.3.2, with the constant $\sqrt{2}$ instead of 2: since $1+|\lambda_j|\leq 2$ and $1-|\lambda_j|\geq 1-r$, applying Corollary 1.3.4, we get

$$C_{n,r}(H^2,H^\infty)\leq \sqrt{2}\frac{\sqrt{n}}{\sqrt{1-r}}.$$

It is natural to wonder if it is possible to improve the bound $\sqrt{2}\frac{\sqrt{n}}{\sqrt{1-r}}$. We return to this question in **Chapter 2** below. To finish this section it can be interesting to mention the following remark. We can apply explicitly Theorem 1.1.3 here in order to look for a lower bound for $C_{n,r}(H^2,H^\infty)$.

Remark 1.3.6 For $X=H^2$, $\|\varphi_\lambda\|=\frac{1}{\left(1-|\lambda|^2\right)^{\frac{1}{2}}}$ and since $\varphi_{\lambda,\,j}(f)=f^{(j)}(\lambda)=\frac{d^j}{d\lambda^j}\,(f,\,k_\lambda)$, (where k_λ is the reproducing kernel for H^2, $k_\lambda(z)=\frac{1}{1-\bar{\lambda}z}$) we get $\varphi_{\lambda,\,j}(f)=\left(f,\,j!\frac{z^j}{\left(1-\bar{\lambda}z\right)^{j+1}}\right)$ and $\|\varphi_{\lambda,\,j}\|^2=\frac{1}{\left(1-|\lambda|^2\right)^{2j+1}}\sum_{l=0}^{j}\left(\begin{array}{c} j\\ l\end{array}\right)^2|\lambda|^{2l}$, (see [N1] p.228). This gives, if $\sigma=\{\lambda_1,\ldots,\lambda_1,\lambda_2,\ldots,\lambda_2,\ldots,\lambda_t,\ldots,\lambda_t\}$, where each λ_s is repeated according its multiplicity m_s,

$$c(\sigma,\,X,\,H^\infty)\geq max_{1\leq s\leq t,\,0\leq j<m_s}\left(\frac{1}{\left(1-|\lambda_s|^2\right)^{j+\frac{1}{2}}}\left(\sum_{l=0}^{j}\left(\begin{array}{c} j\\ l\end{array}\right)^2|\lambda_s|^{2l}\right)^{\frac{1}{2}}\left(1-|\lambda_s|^2\right)^j\frac{1}{j!}\right)\geq$$

$$\geq\frac{1}{\sqrt{2}\sqrt{1-r}}max_{1\leq s\leq t,\,0\leq j<m_s}\left(\frac{1}{j!}\left(\sum_{l=0}^{j}\left(\begin{array}{c} j\\ l\end{array}\right)^2|\lambda_s|^{2l}\right)^{\frac{1}{2}}\right).$$

1.4. Estimation of $c\,(\sigma,\,H^2,\,H^\infty)$ for circular sequences σ

In this section, we consider with more details the case of circular sequences, that is $(\lambda_j)_{j=1}^n$ such that $|\lambda_j|=r\ \forall i=1..n,\, 0<r<1$.

Definition. 1.4.0 *We say that a sequence $\sigma=(\lambda_j)_{j=1}^n$ is r-circular ($0<r<1$) if and only if*

$$|\lambda_i|=r,\,\forall i=1..n.$$

Before studying this kind of sequences σ, we give a general lemma which is going to be useful in this section.

Lemma. 1.4.1 *Let $\sigma=\{\lambda_1,\,\ldots,\,\lambda_n\}$ and $r=max_i|\lambda_i|$, then*

$$c(\sigma,\,H^2,\,H^\infty)\geq\frac{1}{\sqrt{2}}\frac{1}{\sqrt{1-r}}.$$

Proof. We apply Theorem 1.1.3 (1), which implies

$$c(\sigma,\,H^2,\,H^\infty)\geq max_{1\leq j\leq n}\left\|\varphi_{\lambda_j}\right\|_2=max_j\frac{1}{\left(1-|\lambda_j|^2\right)^{\frac{1}{2}}}=\frac{1}{\left(1-r^2\right)^{\frac{1}{2}}}.$$

□

Lemma. 1.4.2 *Let $\sigma = \{\lambda_1, ..., \lambda_n\}$ be an r-circular sequence with $\#\sigma = n$. Then,*

$$c(\sigma, H^2, H^\infty) \le c(\sigma),$$

where,

$$c(\sigma) = sup_{z\in\mathbb{T}} \left(\sum_{k=1}^{n} \frac{(1-r^2)}{|z-\lambda_k|^2} \right)^{\frac{1}{2}}.$$

Proof. This is a straightforward consequence of Theorem 1.3.3. □

Remark. The quantity $c(\sigma)$, regarded for an *arbitrary r-circular* sequence σ, is comparable with $C_{n,r} := C_{n,r}(H^2, H^\infty)$ because, taking a "sup" over all $r-circular$ sequences, we get

$$sup\, c_{n,r}(\sigma) = sup\,(1-r^2)^{\frac{1}{2}} sup_{z\in\mathbb{T}} \left(\sum_{k=1}^{n} \frac{1}{|z-\lambda_k|^2} \right)^{\frac{1}{2}} =$$

$$= (1-r^2)^{\frac{1}{2}} sup_{z\in\mathbb{T}} sup \left(\sum_{k=1}^{n} \frac{1}{|z-\lambda_k|^2} \right)^{\frac{1}{2}} =$$

$$= (1-r^2)^{\frac{1}{2}} \left(\frac{n}{(1-r)^2} \right)^{\frac{1}{2}} = \sqrt{n} \left(\frac{1+r}{1-r} \right)^{\frac{1}{2}} \ge \frac{1}{\sqrt{2}} \left(\frac{2n}{1-r} \right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}} C_{n,r}.$$

To the contrary, for some special circular σ's , the constant $c(\sigma)$ is $\sqrt{n}$ times less than the general estimate for $C_{n,r} = C_{n,r}(H^2, H^\infty)$, $C_{n,r} \le \left(\frac{2n}{1-r}\right)^{\frac{1}{2}}$, from Corollary 1.3.2. This follows from the following proposition.

Proposition. 1.4.3 *Let $\sigma_\star = \{z_1, ..., z_n\}$ be a sequence of distincts unimodular numbers ($z_i \ne z_j$, $\forall i \ne j$, $|z_i| = 1$, $\forall i = 1...n$) and $\sigma_r = r\sigma_\star$, where $0 < r < 1$. Then,*

$$\sqrt{n} \le limsup_{r\to 1} \frac{\left(\frac{2n}{1-r}\right)^{\frac{1}{2}}}{c(\sigma_r)} \le 2\sqrt{n}.$$

Proof. Clearly

$$c(\sigma_r)^2 = max_{|z|=1} \left(\sum_{k=1}^{n} \frac{(1-r^2)}{|z-rz_k|^2} \right) = \sum_{k=1}^{n} \frac{(1-r^2)}{|z_0-rz_k|^2} \ge$$

$$\ge \frac{1+r}{1-r} \text{ (take } z = z_k),$$

which shows the right hand side inequality of Proposition 1.4.3.

Moreover, it implies that for a maximal term $\frac{(1-r^2)}{|z_0-rz_j|^2} = max_k \frac{(1-r^2)}{|z_0-rz_k|^2}$ we have $\frac{(1-r^2)}{|z_0-rz_j|^2} \ge \frac{1+r}{n(1-r)}$. This gives $|z_0 - rz_j| \le (1-r)\sqrt{n}$. Let now $a = min_{i\ne j}|z_i - z_j|$ and $0 < 1-r < \frac{ra}{2\sqrt{n}}$. Then $|z_0 - rz_j| < \frac{ra}{2}$, and

$$c(\sigma_r)^2 = \sum_{k=1}^{n} \frac{(1-r^2)}{|z_0-rz_k|^2} = \frac{(1-r^2)}{|z_0-rz_j|^2} + \sum_{k\ne j} \frac{(1-r^2)}{|z_0-rz_k|^2} \le$$

$$\leq \frac{1+r}{1-r} + \sum_{k\neq j} \frac{(1-r^2)}{(r|z_j - z_k| - |rz_j - z_0|)^2} \leq$$

$$\leq \frac{1+r}{1-r} + \sum_{k\neq j} \frac{(1-r^2)}{\left(\frac{ra}{2}\right)^2} \leq \frac{1+r}{1-r} + (n-1)\frac{(1-r^2)}{\left(\frac{ra}{2}\right)^2}.$$

Therefore, $(1-r)c\left(\sigma_r\right)^2 \leq 1 + r + (1-r)(n-1)\frac{(1-r^2)}{\left(\frac{ra}{2}\right)^2}$, and the last expression obviously tends to 2 as $r \to 1$. This shows the left hand side inequality of Proposition 1.4.3. □

Remark 1.4.4. In fact, there is a more general result than Proposition 1.4.3. Indeed, we can give a special estimate for circular sequences using a new parameter α, which is a kind of relative separation constant for σ (relative to the distance to $\mathbb{T}$):

$$c\left(\sigma_r\right) \leq \frac{1}{\sqrt{1-r}} \left(\frac{8bn}{\alpha^2} + 2\right)^{\frac{1}{2}},$$

where $b = (\frac{\pi}{2})^2$ and α is defined as

$$\alpha = \frac{min_{i\neq j}|\lambda_i - \lambda_j|}{1-r} = \frac{ra}{1-r},$$

with $a = min_{i\neq j}\left|z_i - z_j\right|$.

It is natural to wonder wether we can obtain a better bound using techniques of Carleson interpolation. We compare these approaches in Section 1.12 below.

1.5. The case $X = H^p$

The aim of this section is to extend Corollary 1.3.2 to all Hardy spaces H^p. This is the subject of the following theorem.

Theorem. 1.5.0 *Let $1 \leq p \leq \infty$. Then*

$$C_{n,r}\left(H^p,\, H^\infty\right) \leq A_p \left(\frac{n}{1-r}\right)^{\frac{1}{p}},$$

for all $n \geq 1$, $0 \leq r < 1$, where A_p is a constant depending only on p.

We first prove the following lemma.

Lemma. 1.5.1 *Let $\sigma = \{\lambda_1,\, ...,\, \lambda_n\}$ and $r = max_{1\leq i\leq n}|\lambda_i|$, then*

$$c(\sigma,\, H^1,\, H^\infty) \leq \frac{2n}{1-r}$$

and hence,

$$C_{n,r}(H^1, H^\infty) \leq \frac{2n}{1-r}.$$

Proof. Let $f \in H^1$ such that $\|f\|_1 \leq 1$ and let,

$$g = P_B f = \sum_{k=1..n} \langle f, e_k \rangle \, e_k,$$

where, as always, $(e_k)_{k=1}^n$ is the Malmquist basis corresponding to σ, and where $\langle ., . \rangle$ means the Cauchy sesquilinear form $\langle f, \, g \rangle = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)}$. That is to say that,

$$g(\zeta) = \sum_{k=1..n} \langle f, e_k \rangle \, e_k(\zeta) = \left\langle f, \sum_{k=1..n} e_k \overline{e_k(\zeta)} \right\rangle,$$

for all $\zeta \in \mathbb{D}$, which gives,

$$|g(\zeta)| \leq \|f\|_{H^1} \left\| \sum_{k=1..n} e_k \overline{e_k(\zeta)} \right\|_{H^\infty} \leq \left\| \sum_{k=1..n} e_k \overline{e_k(\zeta)} \right\|_{H^\infty}.$$

Now, we recall that

$$e_k = \frac{\left(1 - |\lambda_k|^2\right)^{\frac{1}{2}}}{\left(1 - \overline{\lambda_k} z\right)} \left(\Pi_{j=1}^{k-1} b_{\lambda_j}\right),$$

and, as we saw it in Theorem 1.3.3 (second proof),

$$\|e_k\|_{H^\infty} \leq \frac{(1 + |\lambda_k|)^{\frac{1}{2}}}{(1 - |\lambda_k|)^{\frac{1}{2}}}.$$

As a consequence,

$$|g(\zeta)| \leq \sum_{k=1}^{n} \|e_k\|_{H^\infty} \left| \overline{e_k(\zeta)} \right| = \sum_{k=1}^{n} \|e_k\|_{H^\infty}^2 \leq \sum_{k=1}^{n} \frac{(1 + |\lambda_k|)}{(1 - |\lambda_k|)} \leq \frac{2n}{1-r},$$

for all $\zeta \in \mathbb{D}$, which completes the proof. □

Proof of Theorem 1.5.0. Let $\sigma = \{\lambda_1, \, ..., \, \lambda_n\}$ be a sequence in the unit disc $\mathbb{D}$, $B_\sigma = \Pi_{i=1}^n b_{\lambda_i}$, and $T \, : \, H^p \longrightarrow H^\infty / B_\sigma H^\infty$ be the restriction map defined by

$$Tf = \{g \in H^\infty : \, f - g \in B_\sigma H^p\},$$

for every f. Then,

$$\| \, T \, \|_{H^p \to H^\infty / B_\sigma H^\infty} = c\left(\sigma, \, H^p, H^\infty\right).$$

There exists $0 \leq \theta \leq 1$ such that $\frac{1}{p} = 1 - \theta$, and since (we use the notation of the interpolation theory between Banach spaces see [Tr] or [Be]) $[H^1, H^\infty]_\theta = H^p$ (a topological identity: the spaces are the same and the norms are equivalent (up to constants depending on p only), see [S] Section 5.5), by a known interpolation Theorem (see [Tr] , Theorem 1.9.3, p.59),

$$\| \, T \, \|_{[H^1, H^\infty]_\theta \to H^\infty / B_\sigma H^\infty} \leq \left(A_1 c\left(\sigma, \, H^1, H^\infty\right)\right)^{1-\theta} \left(A_\infty c\left(\sigma, \, H^\infty, H^\infty\right)\right)^\theta,$$

where $A_1,\ A_\infty$ are numerical constants, and using both Lemma 1.5.1 and the fact that $c\,(\sigma,\ H^\infty, H^\infty) \leq 1$, we find

$$\parallel T \parallel_{[H^1,H^\infty]_\theta \to H^\infty/B_\sigma H^\infty} \leq \left(A_1 \frac{2n}{1-r}\right)^{1-\theta} A_\infty^\theta = (2A_1)^{1-\theta} A_\infty^\theta \left(\frac{n}{1-r}\right)^{\frac{1}{p}},$$

which completes the proof.

□

1.6. The case $X = l_a^2\,(w_k)$

In this section, we generalize Corollary 1.3.2 to the case of spaces X which contain H^2: $X = l_a^2\left(\frac{1}{(k+1)^{\alpha-1}}\right)$, $\alpha \geq 1$, the Hardy weighted spaces of all $f(z) = \sum_{k\geq 0} \hat{f}(k) z^k$ satisfying

$$\sum_{k\geq 0} \left|\hat{f}(k)\right|^2 \frac{1}{(k+1)^{2(\alpha-1)}} < \infty.$$

It is also important to recall that

$$l_a^2\left(\frac{1}{(k+1)^{\alpha-1}}\right) = L_a^2\left(\left(1-|z|^2\right)^{2\alpha-3} dA\right),\ \alpha > 1,$$

where $L_a^2\left(\left(1-|z|^2\right)^\beta dA\right)$, $\beta > -1$, stand for the Bergman weighted spaces of all holomorphic functions f such that

$$\int_{\mathbb{D}} |f(z)|^2 \left(1-|z|^2\right)^\beta dA < \infty.$$

Notice also that $H^2 = l_a^2(1)$ and $L_a^2(\mathbb{D}) = l_a^2\left(\frac{1}{(k+1)^{\frac{1}{2}}}\right)$, where $L_a^2(\mathbb{D})$ stands for the Bergman space of the unit disc $\mathbb{D}$.

Theorem. 1.6.0 *Let σ be a sequence in $\mathbb{D}$. Then*

$$c\left(\sigma,\ l_a^2\left(\frac{1}{(k+1)^{\alpha-1}}\right),\ H^\infty\right) \leq A\left(\frac{n}{1-r}\right)^{\frac{2\alpha-1}{2}}.$$

Otherwise,

$$C_{n,r}\left(l_a^2\left(\frac{1}{(k+1)^{\alpha-1}}\right),\ H^\infty\right) \leq A\left(\frac{n}{1-r}\right)^{\frac{2\alpha-1}{2}},$$

$$C_{n,r}\left(L_a^2\left(\left(1-|z|^2\right)^\beta dA\right),\ H^\infty\right) \leq A'\left(\frac{n}{1-r}\right)^{\frac{\beta+2}{2}},$$

for all $n \geq 1$, $0 \leq r < 1$, $\alpha \geq 1$, $\beta > -1$, where $A = A(\alpha-1)$ is a constant depending only on α and $A' = A'(\beta)$ is a constant depending only on α.

In particular, for $\alpha = \frac{3}{2}$ (or equivalently $\beta = 0$) we get

$$C_{n,r}\left(L_a^2,\ H^\infty\right) \leq \sqrt{14}\frac{n}{1-r},$$

for all $n \geq 1$, $0 \leq r < 1$.

First, we prove a following lemma. In fact, Lemma 1.6.1 below is a partial case ($p = 2$) of the following K. Dyakonov's result [D] (which is, in turn, a generalization of M. Levin's inequality [L] corresponding to the case $p = \infty$): for every p, $1 < p \leq \infty$ there exists a constant $c_p > 0$ such that

$$\left\|f^{'}\right\|_{H^p} \leq c_p \left\|B^{'}\right\|_{\infty} \|f\|_{H^p}$$

for all $f \in K_B$, where B is a finite Blaschke product (of order n) and $\|.\|_{\infty}$ means the norm in $L^{\infty}(\mathbb{T})$. For our partial case, our proof is different and the constant is slightly better.

Lemma. 1.6.1 *Let $B = \Pi_{j=1}^{n} b_{\lambda_j}$, be a finite Blaschke product (of order n), $r = max_j |\lambda_j|$, and $f \in K_B =: H^2 \Theta B H^2$. Then,*

$$\left\|f^{'}\right\|_{H^2} \leq \frac{5}{2} \frac{n}{1-r} \|f\|_{H^2}.$$

Proof. Since $f \in K_B$, $f = P_B f = \sum_{k=1}^{n} (f, e_k)_{H^2} e_k$. Noticing that,

$$e_k^{'} = \sum_{i=1}^{k-1} \frac{b_{\lambda_i}^{'}}{b_{\lambda_i}} e_k + \overline{\lambda_k} \frac{1}{(1-\overline{\lambda_k} z)} e_k,$$

for $k = 2..n$, we get

$$f^{'} = (P_B f)^{'} = (f, e_1)_{H^2} e_1^{'} + \sum_{k=2}^{n} (f, e_k)_{H^2} e_k^{'} =$$

$$= (f, e_1)_{H^2} \frac{\bar{\lambda}_1}{(1-\overline{\lambda_1} z)} e_1 + \sum_{k=2}^{n} (f, e_k)_{H^2} \sum_{i=1}^{k-1} \frac{b_{\lambda_i}^{'}}{b_{\lambda_i}} e_k + \sum_{k=2}^{n} (f, e_k)_{H^2} \overline{\lambda_k} \frac{1}{(1-\overline{\lambda_k} z)} e_k,$$

which gives

$$f^{'} = (f, e_1)_{H^2} \frac{\bar{\lambda}_1}{(1-\overline{\lambda_1} z)} e_1 + \sum_{k=2}^{n} \sum_{i=1}^{n-1} (f, e_k)_{H^2} \frac{b_{\lambda_i}^{'}}{b_{\lambda_i}} e_k \chi_{[1,\, k-1]}(i) + \sum_{k=2}^{n} (f, e_k)_{H^2} \overline{\lambda_k} \frac{1}{(1-\overline{\lambda_k} z)} e_k =$$

$$= (f, e_1)_{H^2} \frac{\bar{\lambda}_1}{(1-\overline{\lambda_1} z)} e_1 + \sum_{i=1}^{n} \frac{b_{\lambda_i}^{'}}{b_{\lambda_i}} \sum_{k=i+1}^{n-1} (f, e_k)_{H^2} e_k + \sum_{k=2}^{n} (f, e_k)_{H^2} \overline{\lambda_k} \frac{1}{(1-\overline{\lambda_k} z)} e_k,$$

where $\chi_{[1,\, k-1]}$ is the characteristic function of $[1,\, k-1]$. Now,

$$\left\|(f, e_1)_{H^2} \frac{\bar{\lambda}_1}{(1-\overline{\lambda_1} z)} e_1\right\|_{H^2} \leq |(f, e_1)_{H^2}| \left\|\frac{\bar{\lambda}_1}{(1-\overline{\lambda_1} z)}\right\|_{\infty} \|e_1\|_{H^2} \leq$$

$$\leq \|f\|_{H^2} \|e_1\|_{H^2} \frac{1}{1-r} \|e_1\|_{H^2} \leq \|f\|_{H^2} \frac{1}{1-r},$$

using both Cauchy-Schwarz inequality and the fact that e_1 is a vector of norm 1 in H^2. By the same reason, we have

$$\left\|\sum_{k=2}^{n} \overline{\lambda_k} (f, e_k)_{H^2} \frac{1}{(1-\overline{\lambda_k} z)} e_k\right\|_{H^2} \leq \sum_{k=2}^{n} |(f, e_k)_{H^2}| \left\|\overline{\lambda_k} \frac{1}{(1-\overline{\lambda_k} z)}\right\|_{\infty} \|e_k\|_{H^2} \leq$$

$$\leq \frac{1}{1-r}\sum_{k=2}^{n}|(f,\, e_k)_{H^2}| \leq \frac{1}{1-r}\left(\sum_{k=2}^{n}|(f,\, e_k)_{H^2}|^2\right)^{\frac{1}{2}}\sqrt{n-2} \leq \frac{1}{1-r}\left\|f\right\|_{H^2}\sqrt{n-2}.$$

Finally,

$$\left\|\sum_{i=1}^{n-1}\frac{b'_{\lambda_i}}{b_{\lambda_i}}\sum_{k=i+1}^{n}e_k\,(f,\, e_k)_{H^2}\right\|_{H^2} \leq \sum_{i=1}^{n-1}\left\|\frac{b'_{\lambda_i}}{b_{\lambda_i}}\right\|_{\infty}\left\|\sum_{k=i+1}^{n}(f,\, e_k)_{H^2}\,e_k\right\|_{H^2} =$$

$$= \left(max_{1\leq i\leq n-1}\left\|\frac{b'_{\lambda_i}}{b_{\lambda_i}}\right\|_{\infty}\right)\sum_{i=1}^{n-1}\left(\sum_{k=i+1}^{n}|(f,\, e_k)_{H^2}|^2\right)^{\frac{1}{2}} \leq max_i\left\|\frac{b'_{\lambda_i}}{b_{\lambda_i}}\right\|_{\infty}\sum_{i=1}^{n-1}\left\|f\right\|_{H^2}.$$

Moreover, since

$$\left\|\frac{b'_{\lambda_i}}{b_{\lambda_i}}\right\|_{\infty} = \left\|\frac{|\lambda_i|^2-1}{\left(1-\overline{\lambda_i}z\right)(\lambda_i-z)}\right\|_{\infty} \leq \frac{2}{1-|\lambda_i|} \leq \frac{2}{1-r},$$

we get,

$$\left\|\sum_{i=1}^{n-1}\frac{b'_{\lambda_i}}{b_{\lambda_i}}\sum_{k=i+1}^{n}(f,\, e_k)_{H^2}\,e_k\right\|_{H^2} \leq \frac{2(n-1)}{1-r}\left\|f\right\|_{H^2}.$$

Finally,

$$\left\|f'\right\|_{H^2} \leq \frac{1}{1-r}\left\|f\right\|_{H^2} + \frac{2(n-1)}{1-r}\left\|f\right\|_{H^2} + \frac{1}{1-r}\sqrt{n-2}\left\|f\right\|_{H^2} \leq \frac{\left(2n-1+\sqrt{n-2}\right)}{1-r}\left\|f\right\|_{H^2} \leq$$

$$\leq \frac{5}{2}\frac{n}{1-r}\left\|f\right\|_{H^2},$$

for all $n \geq 2$ and for every $f \in K_B$. (The case $n=1$ is obvious since $\left\|f'\right\|_{H^2} \leq \frac{1}{1-r}\left\|f\right\|_{H^2}$). $\square$

Corollary. 1.6.2 *Let $B = \Pi_{j=1}^{n}b_{\lambda_j}$, be a finite Blaschke product (of order n), $r = max_j\,|\lambda_j|$, and $f \in K_B =: H^2\Theta BH^2$. Then,*

$$\left\|f^{(k)}\right\|_{H^2} \leq k!\left(\frac{5}{2}\right)^k\left(\frac{n}{1-r}\right)^k\left\|f\right\|_{H^2},$$

for every $k = 0,\, 1,\, ...$

Indeed, since $f^{(k-1)} \in K_{B^k}$, we obtain applying Lemma 1.6.1 for B^k instead of B,

$$\left\|f^{(k)}\right\|_{H^2} \leq \frac{5}{2}\frac{kn}{1-r}\left\|f^{(k-1)}\right\|_{H^2},$$

and by induction,

$$\left\|f^{(k)}\right\|_{H^2} \leq k!\left(\frac{5}{2}\frac{n}{1-r}\right)^k\left\|f\right\|_{H^2}. \ \square$$

Corollary. 1.6.3 *Let $N \geq 0$ be an integer and σ a sequence in $\mathbb{D}$. Then,*

$$c\left(\sigma,\, l_a^2\left(\frac{1}{(k+1)^N}\right),\, H^\infty\right) \leq A\left(\frac{n}{1-r}\right)^{\frac{2N+1}{2}},$$

where $A = A(N)$ is a constant depending on N (of order N^{2N} from the proof below).
In particular,

$$c\left(\sigma,\, l_A^2\left(\frac{1}{(k+1)}\right),\, H^\infty\right) \leq 7\sqrt{2}\left(\frac{n}{1-r}\right)^{\frac{3}{2}}.$$

Indeed, if $f \in l_a^2\left(\frac{1}{(k+1)^N}\right) = H$ then $|P_B f(\zeta)| = |\langle P_B f,\, k_\zeta\rangle| = |\langle f,\, P_B k_\zeta\rangle|$, where $\langle .,\, .\rangle$ means the Cauchy pairing and $k_\zeta = \left(1-\overline{\zeta}z\right)^{-1}$. Denoting $H^\star$ the dual of H with respect to this pairing, $H^\star = l_a^2\left((k+1)^N\right)$, we get

$$|P_B f(\zeta)| \leq \|f\|_H \, \|P_B k_\zeta\|_{H^\star} \leq \|f\|_H \, K_N\left(\|P_B k_\zeta\|_{H^2} + \left\|(P_B k_\zeta)^{(N)}\right\|_{H^2}\right),$$

where

$$K_N = max\left\{N^N,\, sup_{k\geq N}\frac{(k+1)^N}{k(k-1)...(k-N+1)}\right\} =$$

$$= max\left\{N^N,\, \frac{(N+1)^N}{N!}\right\} = \left\{\begin{array}{c} N^N,\ if\ N\geq 3 \\ \frac{(N+1)^N}{N!},\ if\ N = 1,\, 2 \end{array}\right..$$

(Indeed, the sequence $\left(\frac{(k+1)^N}{k(k-1)...(k-N+1)}\right)_{k\geq N}$ is decreasing since $(1+x)^{-N} \geq 1 - Nx$ for all $x \in [0,\, 1]$, and $\left[N^N > \frac{(N+1)^N}{N!}\right] \Longleftrightarrow N \geq 3$). Since $P_B k_\zeta \in K_B$, Corollary 1.6.2 implies

$$|P_B f(\zeta)| \leq \|f\|_H \, \|P_B k_\zeta\|_{H^\star} \leq \|f\|_H \, K_N\left(\|P_B k_\zeta\|_{H^2} + N!\left(\frac{5}{2}\frac{n}{1-r}\right)^N \|P_B k_\zeta\|_{H^2}\right) \leq$$

$$\leq A(N)\left(\frac{n}{1-r}\right)^{N+\frac{1}{2}} \|f\|_H,$$

where $A(N) = \sqrt{2}K_N\left(1 + N!\left(\frac{5}{2}\right)^N\right)$, since $\|P_B k_\zeta\|_2 \leq \frac{\sqrt{2n}}{\sqrt{1-r}}$. Since $K_1 = 2$, $A(1) = 7\sqrt{2}$. $\square$

Proof of Theorem 1.6.0. Let $B_\sigma = \Pi_{i=1}^n b_{\lambda_i}$ and $T\ :\ l_A^2\left(\frac{1}{(k+1)^{\alpha-1}}\right) \longrightarrow H^\infty/B_\sigma H^\infty$ be the restriction map defined by

$$Tf = \left\{g \in H^\infty :\ f - g \in B_\sigma l_A^2\left(\frac{1}{(k+1)^{\alpha-1}}\right)\right\},$$

for every f. Then,

$$\| \, T \, \|_{l_A^2\left(\frac{1}{(k+1)^{\alpha-1}}\right)\to H^\infty/B_\sigma H^\infty} = c\left(\sigma,\, l_A^2\left(\frac{1}{(k+1)^{\alpha-1}}\right), H^\infty\right).$$

Moreover, there exists an integer N such that $N \leq \alpha \leq N+1$. In particular, there exists $0 \leq \theta \leq 1$ such that $\alpha - 1 = (1-\theta)(N-1) + \theta N$. And since (as in Theorem 1.5.0, we use the notation of the interpolation theory between Banach spaces see [Tr] or [Be])

$$\left[l^2_a\left(\frac{1}{(k+1)^{N-1}}\right), l^2_a\left(\frac{1}{(k+1)^{N}}\right)\right]_{\theta,2} = l^2_a\left(\left(\left(\frac{1}{(k+1)^{N-1}}\right)^{2\frac{1-\theta}{2}}\left(\frac{1}{(k+1)^{N}}\right)^{2\frac{\theta}{2}}\right)\right) =$$

$$= l^2_a\left(\frac{1}{(k+1)^{(1-\theta)(N-1)+\theta N}}\right) = l^2_A\left(\frac{1}{(k+1)^{\alpha-1}}\right),$$

this gives, using Corollary 1.6.3 and (again) [Tr] Theorem 1.9.3 p.59,

$$\| T \|_{l^2_a\left(\frac{1}{(k+1)^{\alpha-1}}\right) \to H^\infty / B_\sigma H^\infty} \leq$$

$$\leq \left(c\left(\sigma_{\lambda,n},\, l^2_a\left(\frac{1}{(k+1)^{N-1}}\right), H^\infty\right)\right)^{1-\theta}\left(c\left(\sigma_{\lambda,n},\, l^2_a\left(\frac{1}{(k+1)^{N}}\right), H^\infty\right)\right)^{\theta} \leq$$

$$\leq \left(A(N-1)\left(\frac{n}{1-r}\right)^{\frac{2N-1}{2}}\right)^{1-\theta}\left(A(N)\left(\frac{n}{1-r}\right)^{\frac{2N+1}{2}}\right)^{\theta} =$$

$$= A(N-1)^{1-\theta}A(N)^{\theta}\left(\frac{n}{1-r}\right)^{\frac{(2N-1)(1-\theta)}{2}+\frac{(2N+1)\theta}{2}}.$$

It remains to use $\theta = \alpha - N$ and set $A(\alpha-1) = A(N-1)^{1-\theta}A(N)^{\theta}$. In particular,

$$A\left(\frac{3}{2}\right) = A(0)^{(1-\frac{1}{2})}A(1)^{\frac{1}{2}} = \sqrt{2}^{\frac{1}{2}}(7\sqrt{2})^{\frac{1}{2}} = \sqrt{14}.$$

□

1.7. The case $X = l^1_a(w_k)$

The aim of this section is to prove the following theorem, in which the upper bound $\left(\frac{n}{1-r}\right)^{\alpha-\frac{1}{2}}$ is not as sharp as in Section 1.6. We suspect $\left(\frac{n}{1-r}\right)^{\alpha-1}$ is the sharp bound for the quantity $C_{n,r}\left(l^1_a\left(\frac{1}{(k+1)^{\alpha-1}}\right),\, H^\infty\right)$.

Theorem. 1.7.0 *Let $\alpha \geq 1$. Then,*

$$C_{n,r}\left(l^1_a\left(\frac{1}{(k+1)^{\alpha-1}}\right),\, H^\infty\right) \leq A_1\left(\frac{n}{1-r}\right)^{\alpha-\frac{1}{2}},$$

for all $r \in [0,\, 1[$, $n \geq 1$, where $A_1 = A_1(\alpha-1)$ is a constant depending only on α.

First, we prove the following lemma.

Lemma. 1.7.1 *Let $B = \Pi_{j=1}^n b_{\lambda_j}$, be a finite Blaschke product (of order n), $r = max_j |\lambda_j|$, and $f \in K_B$. Then,*

$$\left\|f^{(k)}\right\|_{H^1} \leq k!\left(\frac{2n}{1-r}\right)^k \|f\|_{H^1}$$

for every $k = 0,\, 1,\, ...$

Proof. By A. Baranov (see [B] Theorem 5.1 p.50),

$$\left\|f'\right\|_{H^1} \leq \left\|B'\right\|_{\infty} \|f\|_{H^1}$$

for every $f \in K_B$. (A private communication with A. Baranov shows that Theorem 5.1 of [B] is also true for the Hardy spaces of the unit disc $\mathbb{D}$). Since $f^{(k-1)} \in K_{B^k}$, we obtain, applying Baranov's inequality for B^k instead of B,

$$\|f^{(k)}\|_{H^1} \leq \left\|kB'B^{k-1}\right\|_{\infty} \|f^{(k-1)}\|_{H^1},$$

and by induction,

$$\|f^{(k)}\|_{H^1} \leq k! \left\|B'\right\|_{\infty}^{k} \|f\|_{H^1}.$$

On the other hand, $\left|B'\right| = \left|-\sum_j \frac{1-|\lambda_j|^2}{\left(1-\overline{\lambda_j}z\right)^2} \cdot \frac{B}{b_{\lambda_j}}\right| \leq \sum_j \frac{1+|\lambda_j|}{1-|\lambda_j|} \leq \frac{2n}{1-r}$, which completes the proof. □

Corollary. 1.7.2 *Let $N \geq 0$ be an integer. Then,*

$$C_{n,r}\left(l_a^1\left(\frac{1}{(k+1)^N}\right), H^{\infty}\right) \leq A_1\left(\frac{n}{1-r}\right)^{N+\frac{1}{2}},$$

for all $r \in [0, 1[$, $n \geq 1$, where $A_1 = A_1(N)$ is a constant depending only on N (of order N^{2N} from the proof below).

Indeed, if $f \in l_a^1\left(\frac{1}{(k+1)^N}\right) = H$ then $|P_Bf(\zeta)| = |\langle P_Bf, k_\zeta\rangle| = |\langle f, P_Bk_\zeta\rangle|$, where $\langle ., .\rangle$ means the Cauchy pairing and $k_\zeta = \left(1-\overline{\zeta}z\right)^{-1}$. Denoting $H^\star$ the dual of H with respect to this pairing, $H^\star = l_a^\infty\left((k+1)^N\right)$, we get,

$$|P_Bf(\zeta)| \leq \|f\|_H \|P_Bk_\zeta\|_{H^\star} \leq$$

$$\leq \|f\|_H K_N max\left\{sup_{0\leq k\leq N-1}\left|\widehat{P_Bk_\zeta}(k)\right|, sup_{k\geq N}\left|\widehat{(P_Bk_\zeta)^{(N)}}(k-N)\right|\right\} \leq$$

$$\leq \|f\|_H K_N max\left\{\|P_Bk_\zeta\|_{H^1}, \left\|(P_Bk_\zeta)^{(N)}\right\|_{H^1}\right\},$$

where

$$K_N = max\left\{N^N, sup_{k\geq N}\frac{(k+1)^N}{k(k-1)...(k-N+1)}\right\} =$$

$$= \begin{cases} N^N, \; if\; N \geq 3 \\ \frac{(N+1)^N}{N!}, \; if\; N = 1, 2 \end{cases},$$

(see the proof of Corollary 1.6.3 for the last equality). Since $P_Bk_\zeta \in K_B$, Lemma 1.7.1 implies

$$|P_Bf(\zeta)| \leq \|f\|_H \|P_Bk_\zeta\|_{H^\star} \leq \|f\|_H K_N\left(\|P_Bk_\zeta\|_{H^1} + N!2^N\left(\frac{n}{1-r}\right)^N \|P_Bk_\zeta\|_{H^1}\right) \leq$$

$$\leq K_N \|f\|_H \|P_Bk_\zeta\|_2\left(1 + N!2^N\left(\frac{n}{1-|\lambda|}\right)^N\right) \leq K_N \|f\|_H\left(\frac{2n}{1-r}\right)^{\frac{1}{2}}\left(1 + N!2^N\left(\frac{n}{1-r}\right)^N\right),$$

which completes the proof setting $A_1(N) = \sqrt{2}\left(1+N!2^N\right)K_N$.

□

Proof of Theorem 1.7.0. This is the same reasoning as in Theorem 1.6.0. Let $B_\sigma = \Pi_{i=1}^n b_{\lambda_i}$ and $T\,:\, l_a^1\left(\frac{1}{(k+1)^{\alpha-1}}\right) \longrightarrow H^\infty/B_\sigma H^\infty$ be the restriction map defined by

$$Tf = \left\{g \in H^\infty \,:\, f-g \in B_\sigma l_a^1\left(\frac{1}{(k+1)^{\alpha-1}}\right)\right\},$$

for every f. Then,

$$\parallel T \parallel_{l_a^1\left(\frac{1}{(k+1)^{\alpha-1}}\right)\to H^\infty/B_\sigma H^\infty} = c\left(\sigma,\, l_a^1\left(\frac{1}{(k+1)^{\alpha-1}}\right), H^\infty\right).$$

Moreover, there exists an integer N such that $N \le \alpha \le N+1$. In particular, there exists $0 \le \theta \le 1$ such that $\alpha - 1 = (1-\theta)(N-1) + \theta N$. Therefore, we can use the same kind of interpolation between Banach spaces as in the proof of Theorem 1.6.0. We get

$$\left[l_a^1\left(\frac{1}{(k+1)^{N-1}}\right), l_a^1\left(\frac{1}{(k+1)^{N}}\right)\right]_{\theta,1} = l_a^1\left(\left(\frac{1}{(k+1)^{N-1}}\right)^{1\frac{1-\theta}{1}}\left(\frac{1}{(k+1)^{N}}\right)^{1\frac{\theta}{1}}\right) =$$

$$= l_a^1\left(\frac{1}{(k+1)^{(1-\theta)(N-1)+\theta N}}\right) = l_a^1\left(\frac{1}{(k+1)^{\alpha-1}}\right).$$

This gives, using Corollary 1.7.2 and (again) [Tr] Theorem 1.9.3 p.59,

$$\parallel T \parallel_{l_a^1\left(\frac{1}{(k+1)^{\alpha-1}}\right)\to H^\infty/B_\sigma H^\infty} \le$$

$$\le \left(c\left(\sigma,\, l_a^1\left(\frac{1}{(k+1)^{N-1}}\right), H^\infty\right)\right)^{1-\theta}\left(c\left(\sigma,\, l_a^1\left(\frac{1}{(k+1)^{N}}\right), H^\infty\right)\right)^{\theta} \le$$

$$\le \left(A_1(N-1)\left(\frac{n}{1-|\lambda|}\right)^{N-\frac{1}{2}}\right)^{1-\theta}\left(A_1(N)\left(\frac{n}{1-|\lambda|}\right)^{N+\frac{1}{2}}\right)^{\theta} =$$

$$= A_1(N-1)^{(1-\theta)}A_1(N)^\theta\left(\frac{n}{1-r}\right)^{\left(N-\frac{1}{2}\right)(1-\theta)+\left(N+\frac{1}{2}\right)\theta} = A_1(N-1)^{(1-\theta)}A_1(N)^\theta\left(\frac{n}{1-r}\right)^{N+\theta-\frac{1}{2}}.$$

It remains to use $\theta = \alpha - N$ and set $A_1(\alpha-1) = A_1(N-1)^{(1-\theta)}A_1(N)^\theta$.

□

1.8. The case $X = l_a^p(w_k)$, $1 \le p \le 2$

The aim of this section is to prove the following theorem, in which the upper bound $\left(\frac{n}{1-r}\right)^{\alpha-\frac{1}{2}}$ is not sharp as sharp as in Section 1.6. We suppose that the sharp upper (and lower) bound here should be of the order of $\left(\frac{n}{1-r}\right)^{\alpha-\frac{1}{p}}$.

Theorem. 1.8.0 *Let $1 \le p \le 2$, $\alpha \ge 1$. Then*

$$B\left(\frac{1}{1-r}\right)^{\alpha-\frac{1}{p}} \le C_{n,r}\left(l^p_a\left(\frac{1}{(k+1)^{\alpha-1}}\right),\, H^\infty\right) \le A\left(\frac{n}{1-r}\right)^{\alpha-\frac{1}{2}},$$

for all $r \in [0,\, 1[$, $n \ge 1$, where $A = A(\alpha-1,\, p)$ is constant depending only on α and p and $B = B(p)$ is a constant depending only on p.

Proof. We first prove the right hand side inequality. The scheme of the proof is completely the same as in 1.6.0 and 1.7.0, but we simply use interpolation between l^1 and l^2 (the classical Riesz-Thorin theorem). Let $B_\sigma = \Pi_{i=1}^n b_{\lambda_i}$ and $T\,:\, l^p_a\left(\frac{1}{(k+1)^{\alpha-1}}\right) \longrightarrow H^\infty/B_\sigma H^\infty$ be the restriction map defined by

$$Tf = \left\{g \in H^\infty :\, f-g \in B_\sigma l^p_a\left(\frac{1}{(k+1)^{\alpha-1}}\right)\right\},$$

for every f. Then,

$$\| T \|_{l^p_a\left(\frac{1}{(k+1)^{\alpha-1}}\right)\to H^\infty/B_\sigma H^\infty} = c\left(\sigma,\, l^p_a\left(\frac{1}{(k+1)^{\alpha-1}}\right),H^\infty\right).$$

There exists $0 \le \theta \le 1$ such that

$$\frac{1}{p} = (1-\theta)\frac{1}{1} + \theta\frac{1}{2} = 1 - \frac{\theta}{2},$$

and then,

$$\left[l^1_a\left(\frac{1}{(k+1)^{\alpha-1}}\right),l^2_a\left(\frac{1}{(k+1)^{\alpha-1}}\right)\right]_\theta = l^p_a\left(\frac{1}{(k+1)^{\alpha-1}}\right).$$

This gives, using both Theorem 1.6.0&1.7.0 and (again) [Tr] Theorem 1.9.3 p.59,

$$\| T \|_{l^p_a\left(\frac{1}{(k+1)^{\alpha-1}}\right)\to H^\infty/B_\sigma H^\infty} \le$$

$$\le \left(c\left(\sigma,\, l^1_a\left(\frac{1}{(k+1)^{\alpha-1}}\right),H^\infty\right)\right)^{1-\theta}\left(c\left(\sigma,\, l^2_a\left(\frac{1}{(k+1)^{\alpha-1}}\right),H^\infty\right)\right)^{\theta} \le$$

$$\le \left(A_1(\alpha-1)\left(\frac{n}{1-r}\right)^{\alpha-\frac{1}{2}}\right)^{1-\theta}\left(A(\alpha-1)\left(\frac{n}{1-r}\right)^{\frac{2\alpha-1}{2}}\right)^{\theta} =$$

$$= A_1(\alpha-1)^{(1-\theta)}A(\alpha-1)^{\theta}\left(\frac{n}{1-|\lambda|}\right)^{(\alpha-\frac{1}{2})(1-\theta)+\theta(\alpha-\frac{1}{2})}.$$

In order to prove the right hand side inequality, it remains to use $\theta = 2(1-\frac{1}{p})$ and set $A(\alpha-1,\, p) = (A_1(\alpha-1)^{(1-\theta)}A(\alpha-1)^{\theta}$. Now, we prove the left hand side one. We use Theorem 1.1.3 (1). Indeed,

$$C_{n,r}\left(l^p_a\left(\frac{1}{(k+1)^{\alpha-1}}\right),\, H^\infty\right) \ge \|\varphi_r\|_{l^{p'}_a((k+1)^{\alpha-1})} =$$

$$= \left(\sum_{k\ge 0}(k+1)^{(\alpha-1)p'}\left(r^{p'}\right)^k\right)^{\frac{1}{p'}},$$

where φ_r is the evaluation functional

$$\varphi_r(f) = f(r),\ \ f \in X,$$

and $p^{'}$ is the conjugate of p: $\frac{1}{p} + \frac{1}{p^{'}} = 1$. Now, since

$$\sum_{k\geq 1} k^s x^k \sim \int_1^{\infty} t^s x^t dt \sim \Gamma(s+1)(1-x)^{-s-1}, \text{ as } x \to 1,$$

for all $s > -1$, we get

$$\sum_{k\geq 0} (k+1)^{(\alpha-1)p^{'}} \left(r^{p^{'}}\right)^k \sim \int_1^{\infty} t^{(\alpha-1)p^{'}} r^{p^{'}t} dt, \text{ as } r \to 1.$$

But

$$\int_1^{\infty} t^{(\alpha-1)p^{'}} r^{p^{'}t} dt = \left(\frac{1}{p^{'}}\right)^{1+(\alpha-1)p^{'}} \int_{p^{'}}^{\infty} t^{(\alpha-1)p^{'}} r^t dt \sim$$

$$\sim \left(\frac{1}{p^{'}}\right)^{1+(\alpha-1)p^{'}} \int_1^{\infty} t^{(\alpha-1)p^{'}} r^t dt \sim \left(\frac{1}{p^{'}}\right)^{1+(\alpha-1)p^{'}} \Gamma\left((\alpha-1)p^{'}+1\right)(1-r)^{-(\alpha-1)p^{'}-1}, \text{ as } r \to 1.$$

This gives

$$\left(\sum_{k\geq 0} (k+1)^{(\alpha-1)p^{'}} \left(r^{p^{'}}\right)^k\right)^{\frac{1}{p^{'}}} \sim \left(\frac{1}{p^{'}}\right)^{\frac{1}{p^{'}}+(\alpha-1)} \left(\Gamma\left((\alpha-1)p^{'}+1\right)\right)^{\frac{1}{p^{'}}} (1-r)^{-(\alpha-1)-\frac{1}{p^{'}}}, \text{ as } r \to 1.$$

This completes the proof since $\frac{1}{p^{'}} = 1 - \frac{1}{p}$. □

1.9. The case $X = l_a^{\infty}(w_k)$

The aim of this section is the following theorem, in which -again- the upper bound $\left(\frac{n}{1-r}\right)^{\alpha+\frac{1}{2}}$ is not as sharp as in Section 1.6. We can suppose here that the constant $\left(\frac{n}{1-r}\right)^{\alpha}$ is the sharp bound for the quantity $C_{n,r}\left(l_a^{\infty}\left(\frac{1}{(k+1)^{\alpha-1}}\right),\, H^{\infty}\right)$.

Theorem. 1.9.0 *Let $\alpha \geq 1$. Then*

$$C_{n,r}\left(l_a^{\infty}\left(\frac{1}{(k+1)^{\alpha-1}}\right),\, H^{\infty}\right) \leq A_{\infty}\left(\frac{n}{1-r}\right)^{\alpha+\frac{1}{2}},$$

for all $r \in [0,\, 1[$, $n \geq 1$, where $A_{\infty} = A_{\infty}(\alpha-1)$ is a constant depending only on α.

First, we prove the following partial case of Theorem 1.9.0.

Lemma. 1.9.1 *Let $N \geq 0$ be an integer. Then,*

$$C_{n,r}\left(l_a^{\infty}\left(\frac{1}{(k+1)^{N}}\right),\, H^{\infty}\right) \leq A_{\infty}\left(\frac{n}{1-r}\right)^{N+\frac{3}{2}},$$

for all $r \in [0,\, 1[$, $n \geq 1$, where $A_{\infty} = A_{\infty}(N)$ is a constant depending on N (of order N^{2N} from the proof below).

Proof. We use literally the same method as in Sections 1.6, 1.7 and 1.8. Indeed, if $f \in l_a^\infty\left(\frac{1}{(k+1)^N}\right) = H$ then $|P_B f(\zeta)| = |\langle P_B f,\, k_\zeta\rangle| = |\langle f,\, P_B k_\zeta\rangle|$, where $\langle .,\, .\rangle$ means the Cauchy pairing and $k_\zeta = \left(1-\overline{\zeta}z\right)^{-1}$. Denoting $H^\star$ the dual of H with respect to this pairing, $H^\star = l_a^1\left((k+1)^N\right)$, we get

$$|P_B f(\zeta)| \leq \|f\|_H \|P_B k_\zeta\|_{H^\star} \leq \|f\|_H K_N \left(\|P_B k_\zeta\|_W + \left\|(P_B k_\zeta)^{(N)}\right\|_W\right),$$

where $W = \left\{f = \sum_{k\geq 0}\hat{f}(k)z^k :\, \|f\|_W := \sum_{k\geq 0}\left|\hat{f}(k)\right| < \infty\right\}$ stands for the Wiener algebra, and

$$K_N = max\left\{N^N,\, sup_{k\geq N}\frac{(k+1)^N}{k(k-1)...(k-N+1)}\right\} =$$

$$= \begin{cases} N^N,\ if\ N \geq 3 \\ \frac{(N+1)^N}{N!},\ if\ N = 1,\, 2 \end{cases},$$

(see the proof of Corollary 1.6.3 for the last equality). Now, applying Hardy's inequality (see [N2] p.370, 8.7.4 (c)),

$$|P_B f(\zeta)| \leq \|f\|_H K_N\left(\pi\left\|(P_B k_\zeta)'\right\|_{H^1} + |(P_B k_\zeta)(0)| + \pi\left\|(P_B k_\zeta)^{(N+1)}\right\|_{H^1} + \left|(P_B k_\zeta)^{(N)}(0)\right|\right),$$

which gives using Lemma 1.7.1,

$$|P_B f(\zeta)| \leq$$

$$\leq \|f\|_H K_N\pi\left(\left(\frac{2n}{1-r}\right)\|P_B k_\zeta\|_{H^1} + |(P_B k_\zeta)(0)| + \right.$$

$$\left. + (N+1)!\left(\frac{2n}{1-r}\right)^{N+1}\|P_B k_\zeta\|_{H^1} + \left|(P_B k_\zeta)^{(N)}(0)\right|\right) \leq$$

$$\leq \|f\|_H K_N\pi\left(\left(\frac{2n}{1-r}\right)\|P_B k_\zeta\|_{H^2} + \|P_B k_\zeta\|_{H^2} + \right.$$

$$\left. +(N+1)!\left(\frac{2n}{1-r}\right)^{N+1}\|P_B k_\zeta\|_{H^2} + N!\,\|P_B k_\zeta\|_{H^2}\right).$$

This completes the proof since $\|P_B k_\zeta\|_{H^2} \leq \left(\frac{2n}{1-r}\right)^{\frac{1}{2}}$. □

Proof of Theorem 1.9.0. This is the same application of interpolation between Banach spaces, as before. Let $B_\sigma = \Pi_{i=1}^n b_{\lambda_i}$ and $T\ :\ l_A^\infty\left(\frac{1}{(k+1)^{\alpha-1}}\right) \longrightarrow H^\infty/B_\sigma H^\infty$ be the restriction map defined by

$$Tf = \left\{g \in H^\infty :\ f - g \in B_\sigma l_A^\infty\left(\frac{1}{(k+1)^{\alpha-1}}\right)\right\},$$

for every f. Then,

$$\parallel T \parallel_{l_A^\infty\left(\frac{1}{(k+1)^{\alpha-1}}\right)\to H^\infty/B_\sigma H^\infty} = c\left(\sigma,\, l_A^\infty\left(\frac{1}{(k+1)^{\alpha-1}}\right),\, H^\infty\right).$$

Moreover, there exists an integer N such that $N \leq \alpha \leq N+1$. In particular, there exists $0 \leq \theta \leq 1$ such that $\alpha - 1 = (1-\theta)(N-1) + \theta N$. We get

$$\left[l_a^\infty \left(\frac{1}{(k+1)^{N-1}} \right), l_a^\infty \left(\frac{1}{(k+1)^N} \right) \right]_{\theta,1} = l_a^\infty \left(\left(\frac{1}{(k+1)^{N-1}} \right)^{1^{\frac{1-\theta}{1}}} \left(\frac{1}{(k+1)^N} \right)^{1^{\frac{\theta}{1}}} \right) =$$

$$= l_a^\infty \left(\frac{1}{(k+1)^{(1-\theta)(N-1)+\theta N}} \right) = l_A^\infty \left(\frac{1}{(k+1)^{\alpha-1}} \right).$$

This gives, using Lemma 1.9.1 and (again) [Tr] Theorem 1.9.3 p.59,

$$\| T \|_{l_a^\infty \left(\frac{1}{(k+1)^{\alpha-1}} \right) \to H^\infty / B_\sigma H^\infty} \leq$$

$$\leq \left(c\left(\sigma,\, l_a^\infty \left(\frac{1}{(k+1)^{N-1}} \right), H^\infty \right) \right)^{1-\theta} \left(c\left(\sigma,\, l_a^\infty \left(\frac{1}{(k+1)^{N}} \right), H^\infty \right) \right)^{\theta} \leq$$

$$\leq \left(A_\infty(N-1) \left(\frac{n}{1-r} \right)^{N+\frac{1}{2}} \right)^{1-\theta} \left(A_\infty(N) \left(\frac{n}{1-r} \right)^{N+\frac{3}{2}} \right)^{\theta} =$$

$$= A_\infty(N-1)^{(1-\theta)} A_\infty(N)^\theta \left(\frac{n}{1-r} \right)^{\left(N+\frac{1}{2}\right)(1-\theta)+\left(N+\frac{3}{2}\right)\theta} = A_\infty(N-1)^{(1-\theta)} A_\infty(N)^\theta \left(\frac{n}{1-r} \right)^{N+\theta+\frac{1}{2}}.$$

It remains to use $\theta = \alpha - N$ and set $A_\infty(\alpha-1) = A_\infty(N-1)^{(1-\theta)} A_\infty(N)^\theta$. $\square$

1.10. The case $X = l_a^p(w_k)$, $2 \leq p \leq \infty$

The aim of this section is to prove the following theorem.

Theorem. 1.10.0 *Let* $2 \leq p \leq \infty$ *,* $\alpha \geq 1$*. Then*

$$B \left(\frac{1}{1-r} \right)^{\alpha - \frac{1}{p}} \leq C_{n,r} \left(l_a^p \left(\frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right) \leq A \left(\frac{n}{1-r} \right)^{\alpha+\frac{1}{2}-\frac{2}{p}},$$

for all $r \in [0, 1[$*,* $n \geq 1$*, where* $A = A(\alpha-1, p)$ *is a constant depending only on* α *and* p *and* $B = B(p)$ *is a constant depending only on* p*.*

Remark. As before, the upper bound $\left(\frac{n}{1-r}\right)^{\alpha+\frac{1}{2}-\frac{2}{p}}$ is not as sharp as in Section 1.6. We can suppose here the constant $\left(\frac{n}{1-r}\right)^{\alpha-\frac{1}{p}}$should be a sharp upper (and lower) bound for the quantity $C_{n,r}\left(l_a^p \left(\frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right)$, $2 \leq p \leq +\infty$.

Proof. We first prove the right hand side inequality. The proof repeates the scheme from previous theorems. Let $B_\sigma = \Pi_{i=1}^n b_{\lambda_i}$ and $T : l_a^p \left(\frac{1}{(k+1)^{\alpha-1}} \right) \longrightarrow H^\infty / B_\sigma H^\infty$ be the restriction map defined by

$$Tf = \left\{ g \in H^\infty : \ f - g \in B_\sigma l_a^p \left(\frac{1}{(k+1)^{\alpha-1}} \right) \right\},$$

for every f. Then,

$$\| T \|_{l_a^p\left(\frac{1}{(k+1)^{\alpha-1}}\right) \to H^\infty / B_\sigma H^\infty} = c\left(\sigma,\, l_a^p\left(\frac{1}{(k+1)^{\alpha-1}}\right),\, H^\infty\right).$$

There exists $0 \le \theta \le 1$ such that

$$\frac{1}{p} = (1-\theta)\frac{1}{2},$$

and then,

$$\left[l_a^2\left(\frac{1}{(k+1)^{\alpha-1}}\right),\, l_a^\infty\left(\frac{1}{(k+1)^{\alpha-1}}\right)\right]_\theta = l_a^p\left(\frac{1}{(k+1)^{\alpha-1}}\right),$$

this gives, using both Theorems 1.6.0&1.9.0, and using again [Tr] Theorem 1.9.3 p.59,

$$\| T \|_{l_a^p\left(\frac{1}{(k+1)^{\alpha-1}}\right) \to H^\infty / B_\sigma H^\infty} \le$$

$$\le \left(c\left(\sigma,\, l_a^2\left(\frac{1}{(k+1)^{\alpha-1}}\right),\, H^\infty\right)\right)^{1-\theta} \left(c\left(\sigma,\, l_a^\infty\left(\frac{1}{(k+1)^{\alpha-1}}\right),\, H^\infty\right)\right)^{\theta} \le$$

$$\le \left(A(\alpha-1)\left(\frac{n}{1-r}\right)^{\frac{2\alpha-1}{2}}\right)^{1-\theta} \left(A_\infty(\alpha-1)\left(\frac{n}{1-r}\right)^{\alpha+\frac{1}{2}}\right)^{\theta} =$$

$$= A(\alpha-1)^{(1-\theta)} A_\infty(\alpha-1)^{\theta} \left(\frac{n}{1-r}\right)^{(\alpha-\frac{1}{2})(1-\theta)+(\alpha+\frac{1}{2})\theta} = A(\alpha-1)^{(1-\theta)} A_\infty(\alpha-1)^{\theta} \left(\frac{n}{1-r}\right)^{\alpha+\theta-\frac{1}{2}}.$$

In order to prove the right hand side inequality, it remains to use $\theta = 1 - \frac{2}{p}$, and set $A(\alpha-1,\, p) = A(\alpha-1)^{(1-\theta)} A_\infty(\alpha-1)^{\theta}$. The proof of the left hand side inequality is exactly the same as in Theorem 1.8.0. □

1.11. Carathéodory-Schur Interpolation in weighted Bergman spaces

We suppose that $X = L_a^p\left(\left(1-|z|^2\right)^\beta dA\right)$, $\beta > -1$ and $1 \le p \le 2$. Our aim in this section is to give an estimate for the constant for a generalized Carathéodory-Schur interpolation, (a partial case of the Nevanlinna-Pick interpolation),

$$c(\sigma_{\lambda,n}, X, H^\infty) = \; sup\left\{\|f\|_{H^\infty / b_\lambda^n H^\infty} :\, f \in X,\, \|f\|_X \le 1\right\},$$

where $\|f\|_{H^\infty / b_\lambda^n H^\infty} = inf\left\{\|f + b_\lambda^n g\|_\infty :\, g \in X\right\}$, and $\sigma_{\lambda,n} = \{\lambda, \lambda, ..., \lambda\}$, $\lambda \in \mathbb{D}$. The corresponding interpolation problem is: given $f \in X$, to minimize $\|h\|_\infty$ such that $h^{(j)}(\lambda) = f^{(j)}(\lambda)\,,\, 0 \le j < n$.

For this partial case, we have the following generalization of the estimate from Theorem 1.6.0.

Theorem. 1.11.0 *Let $\lambda \in \mathbb{D}$, $\beta > -1$ and $1 \le p \le 2$. Then,*

$$c\left(\sigma_{\lambda,\,n},\, L_a^p\left(\left(1-|z|^2\right)^\beta dA\right),\, H^\infty\right) \le A'\left(\frac{n}{1-|\lambda|}\right)^{\frac{\beta+2}{p}},$$

where $A' = A'(\beta,\, p)$ is a constant depending only on β and p.

We first need a simple equivalent to $I_k(\beta) = \int_0^1 r^{2k+1}(1-r^2)^\beta dr$, $\beta > -1$.

Lemma. 1.11.1 *Let $k \geq 0$, $\beta > -1$ and $I_k(\beta) = \int_0^1 r^{2k+1}(1-r^2)^\beta dr$. Then,*

$$I_k(\beta) \sim \frac{1}{2}\frac{\Gamma(\beta+1)}{k^{\beta+1}},$$

for $k \to \infty$, where Γ stands for the usual Gamma function, $\Gamma(z) = \int_0^{+\infty} e^{-s}s^{z-1}ds$.

Proof. Let $a = \frac{1}{\sqrt{k+1}}$, $b = max(1, a^\beta)$. Since $1 - e^{-u} \sim u$ as $u \longrightarrow 0$, we have

$$\begin{aligned}
I_k(\beta) = \int_0^1 r^{2k+1}(1-r^2)^\beta dr &= \int_0^\infty e^{-(2k+1)t}(1-e^{-2t})^\beta e^{-t}dt = \\
&= \int_0^a e^{-2(k+1)t}(1-e^{-2t})^\beta dt + \int_a^\infty e^{-2(k+1)t}(1-e^{-2t})^\beta dt = \\
&= \int_0^a e^{-2(k+1)t}(1-e^{-2t})^\beta dt + O\left(\frac{b}{k+1}e^{-2a(k+1)}\right) = \\
&= (1+o(1)) \int_0^a e^{-2(k+1)t}(2t)^\beta dt + O\left(\frac{b}{k+1}e^{-2a(k+1)}\right) = \\
&= (1+o(1)) \int_0^{2(k+1)a} e^{-s}\left(\frac{s}{k+1}\right)^\beta \frac{ds}{2(k+1)} + O\left(\frac{b}{k+1}e^{-2a(k+1)}\right) = \\
&= \frac{1}{2}\frac{1}{(k+1)^{\beta+1}}(1+o(1)) \int_0^{2(k+1)a} e^{-s}s^\beta ds + O\left(\frac{b}{k+1}e^{-2a(k+1)}\right) = \\
&= \frac{1}{2}\frac{\Gamma(\beta+1)}{(k+1)^{\beta+1}}(1+o(1)) + O\left(\frac{b}{k+1}e^{-2a(k+1)}\right) = \\
&= \frac{1}{2}\frac{\Gamma(\beta+1)}{(k+1)^{\beta+1}}(1+o(1)) \sim \frac{1}{2}\frac{\Gamma(\beta+1)}{k^{\beta+1}},
\end{aligned}$$

which completes the proof. □

Proof of Theorem 1.11.0. **Step 1.** We start to prove the Theorem for $p = 1$.
Let $f \in X = L^1_a\left(\left(1-|z|^2\right)^\beta dA\right)$ such that $\|f\|_X \leq 1$. Since $X \circ b_\lambda = X$, we have $f \circ b_\lambda = \sum_{k \geq 0} a_k z^k \in X$. Let $p_n = \sum_{k=0}^{n-1} a_k z^k$ and $g = p_n \circ b_\lambda$. Then, $f \circ b_\lambda - p_n \in z^n X$ and $f - p_n \circ b_\lambda \in (z^n X) \circ b_\lambda = b_\lambda^n X$. Now, $p_n \circ b_\lambda = \sum_{k=0}^{n-1} a_k b_\lambda^k$ and

$$\|p_n \circ b_\lambda\|_\infty = \|p_n\|_\infty \leq A_n \|f \circ b_\lambda\|_X,$$

where $A_n = \left\|\sum_{k \geq 0} a_k z^k \mapsto \sum_{k=0}^{n-1} a_k z^k\right\|_{X \to H^\infty}$. Now,

$$\|f \circ b_\lambda\|_X \leq \int_{\mathbb{D}} |f(b_\lambda(z))| \left(1-|z|^2\right)^\beta dA = \int_{\mathbb{D}} |f(w)| \left(1-|b_\lambda(w)|^2\right)^\beta \left|b_\lambda'(w)\right|^2 dA =$$

$$\leq 2^{\beta}\int_{\mathbb{D}}|f(w)|\left(\frac{\left(1-|\lambda|^2\right)\left(1-|w|^2\right)}{\left|1-\overline{\lambda}w\right|^2}\right)^{\beta}\left(\frac{\left(1-|\lambda|^2\right)}{\left|1-\overline{\lambda}w\right|^2}\right)^2 dA =$$

$$= \int_{\mathbb{D}}|f(w)|\left(1-|w|^2\right)^{\beta}\left(\frac{\left(1-|\lambda|^2\right)}{\left|1-\overline{\lambda}w\right|^2}\right)^{2+\beta} dA \leq$$

$$\leq sup_{w\in\mathbb{D}}\left(\frac{\left(1-|\lambda|^2\right)}{\left|1-\overline{\lambda}w\right|^2}\right)^{2+\beta}\int_{\mathbb{D}}|f(w)|\left(1-|w|^2\right)^{\beta} dA \leq \left(\frac{\left(1-|\lambda|^2\right)}{\left(1-|\lambda|\right)^2}\right)^{2+\beta}\|f\|_X\,,$$

which gives,

$$\|f\circ b_\lambda\|_X \leq \left(\frac{1+|\lambda|}{1-|\lambda|}\right)^{2+\beta}\|f\|_X\,.$$

We now give an estimation for A_n. Let $g(z)=\sum_{k\geq 0}\hat{g}(k)z^k\in X$, then

$$\left\|\sum_{k=0}^{n-1}\hat{g}(k)z^k\right\|_{\infty}\leq\sum_{k=0}^{n-1}|\hat{g}(k)|\,.$$

Now, noticing that

$$\int_{\mathbb{D}}g(w)\,\overline{w}^k\left(1-|w|^2\right)^{\beta}dA=\int_0^1\int_0^{2\pi}f(re^{it})r^ke^{-ikt}\left(1-r^2\right)^{\beta}rdtdr=$$

$$=\int_0^1\left(1-r^2\right)^{\beta}r^{k+1}\int_0^{2\pi}f(re^{it})e^{-ikt}dtdr=\int_0^1\widehat{g_r}(k)r^{k+1}(1-r^2)^{\beta}dr,$$

where $g_r(z)=g(rz)$, $\widehat{g_r}(k)=r^k\widehat{g}(k)$. Setting $I_k(\beta)=\int_0^1 r^{2k+1}(1-r^2)^{\beta}dr$, we get

$$\widehat{g}(k)=\frac{1}{I_k(\beta)}\int_{\mathbb{D}}g(w)\,\overline{w}^k\left(1-|w|^2\right)^{\beta}dA.$$

Then,

$$|\widehat{g}(k)|=\frac{1}{I_k(\beta)}\left|\int_{\mathbb{D}}g(w)\,\overline{w}^k\left(1-|w|^2\right)^{\beta}dA\right|\leq\frac{1}{I_k(\beta)}\|g\|_X\,,$$

which gives

$$\left\|\sum_{k=0}^{n-1}\hat{g}(k)z^k\right\|_{\infty}\leq\left(\sum_{k=0}^{n-1}\frac{1}{I_k(\beta)}\right)\|g\|_X\,.$$

Now using Lemma 1.11.1,

$$\sum_{k=0}^{n-1}\frac{1}{I_k(\beta)}\sim_{n\to\infty}\frac{2}{\Gamma(\beta+1)}\sum_{k=0}^{n-1}k^{\beta+1}\sim\frac{2c_\beta}{\Gamma(\beta+1)}n^{\beta+2},$$

where c_β is a constant depending on β only. This gives

$$\left\|\sum_{k=0}^{n-1}\hat{g}(k)z^k\right\|_{\infty}\leq C_\beta n^{\alpha+2}\|g\|_X\,,$$

where C_β is also a constant depending on β only. Finally, we conclude that $A_n \le C_\beta n^{\beta+2}$, and as a result,

$$\|p_n \circ b_\lambda\|_\infty \le C_\beta n^{\beta+2} \left(\frac{1+|\lambda|}{1-|\lambda|}\right)^{2+\beta} \|f\|_X ,$$

which proves the Theorem for $p=1$.

Step 2. This step of the proof repetes the scheme from Theorems 1.8.0&1.10.0. Let $T : L^p_a\left(\left(1-|z|^2\right)^\beta dA\right) \longrightarrow H^\infty / b^n_\lambda H^\infty$ be the restriction map defined by

$$Tf = \left\{g \in H^\infty : \, f - g \in b^n_\lambda L^p_a\left(\left(1-|z|^2\right)^\beta dA\right)\right\},$$

for every f. Then,

$$\| \, T \, \|_{L^p_a\left(\left(1-|z|^2\right)^\beta dA\right) \to H^\infty / b^n_\lambda H^\infty} = c\left(\sigma, \, L^p_a\left(\left(1-|z|^2\right)^\beta dA\right), H^\infty\right).$$

Now, let $\gamma > \beta$ and $P_\gamma : L^p\left(\left(1-|z|^2\right)^\beta dA\right) \longrightarrow L^p_a\left(\left(1-|z|^2\right)^\beta dA\right)$ be the Bergman projection, (see [H], p.6), defined by

$$P_\gamma f = (\gamma+1)\int_{\mathbb{D}} \frac{\left(1-|w|^2\right)^\gamma}{(1-z\overline{w})^{2+\gamma}} f(w) dA(w),$$

for every f. P_γ is a bounded projection from $L^p\left(\left(1-|z|^2\right)^\beta dA\right)$ onto $L^p_a\left(\left(1-|z|^2\right)^\beta dA\right)$ (see [H], Theorem 1.10 p.12), (since $1 \le p \le 2$). Moreover, since $L^p_a\left(\left(1-|z|^2\right)^\beta dA\right) \subset L^p_a\left(\left(1-|z|^2\right)^\gamma dA\right)$, we have $P_\gamma f = f$ for all $f \in L^p_a\left(\left(1-|z|^2\right)^\beta dA\right)$, (see [H], Corollary 1.5 p.6). As a result,

$$\| \, T \, \|_{L^p_a\left(\left(1-|z|^2\right)^\beta dA\right) \to H^\infty / b^n_\lambda H^\infty} \le \| \, T P_\gamma \, \|_{L^p\left(\left(1-|z|^2\right)^\beta dA\right) \to H^\infty / b^n_\lambda H^\infty},$$

for all $1 \le p \le 2$. We set

$$c_i(\beta) = \|P_\gamma\|_{L^i\left(\left(1-|z|^2\right)^\beta dA\right) \to L^i_a\left(\left(1-|z|^2\right)^\beta dA\right)},$$

for $i = 1, \, 2$. Then,

$$\|TP_\gamma\|_{L^1\left(\left(1-|z|^2\right)^\beta dA\right) \to H^\infty / b^n_\lambda H^\infty} \le$$

$$\le \|T\|_{L^1_a\left(\left(1-|z|^2\right)^\beta dA\right) \to H^\infty / b^n_\lambda H^\infty} \|P_\gamma\|_{L^1\left(\left(1-|z|^2\right)^\beta dA\right) \to L^1_a\left(\left(1-|z|^2\right)^\beta dA\right)} =$$

$$= c\left(\sigma, \, L^1_a\left(\left(1-|z|^2\right)^\beta dA\right), H^\infty\right) c_1(\beta) \le$$

$$\leq A^{'}(\beta,\,1)\left(\frac{n}{1-|\lambda|}\right)^{\beta+2}c_1(\beta),$$

using Step 1. In the same way,

$$\|TP_\gamma\|_{L^2\left(\left(1-|z|^2\right)^\beta dA\right)\to H^\infty/b_\lambda^n H^\infty}\leq \|T\|_{L_a^2\left(\left(1-|z|^2\right)^\beta dA\right)\to H^\infty/b_\lambda^n H^\infty}\,c_2(\beta).$$

Now, we recall that

$$L_a^2\left(\left(1-|z|^2\right)^\beta dA\right)=l_a^2\left(\frac{1}{(k+1)^{\frac{\beta+1}{2}}}\right),\ \beta>-1.$$

As a consequence,

$$\|T\|_{L_a^2\left(\left(1-|z|^2\right)^\beta dA\right)\to H^\infty/b_\lambda^n H^\infty}=c\left(\sigma,\,l_a^2\left(\frac{1}{(k+1)^{\frac{\beta+1}{2}}}\right),H^\infty\right),$$

and, applying Theorem 1.6.0,

$$\|TP_\gamma\|_{L^2\left(\left(1-|z|^2\right)^\beta dA\right)\to H^\infty/b_\lambda^n H^\infty}\leq c_2(\beta)A^{'}\left(\frac{\beta+1}{2},\,2\right)\left(\frac{n}{1-|\lambda|}\right)^{\frac{2\frac{\beta+1}{2}+1}{2}}=$$

$$=c_2(\beta)A^{'}\left(\frac{\beta+1}{2},\,2\right)\left(\frac{n}{1-|\lambda|}\right)^{\frac{\beta+2}{2}}.$$

We finish the reasoning applying Riesz-Thorin Theorem, (see [Tr] for example), to the operator TP_γ. If $p\in[1,\,2]$, there exists $0\leq\theta\leq 1$ such that

$$\frac{1}{p}=(1-\theta)\frac{1}{1}+\theta\frac{1}{2}=1-\frac{\theta}{2},$$

and then,

$$\left[L_a^1\left(\left(1-|z|^2\right)^\beta dA\right),\,L_a^2\left(\left(1-|z|^2\right)^\beta dA\right)\right]_\theta=L_a^p\left(\left(1-|z|^2\right)^\beta dA\right),$$

and

$$\|TP_\gamma\|_{L^p\left(\left(1-|z|^2\right)^\beta dA\right)\to H^\infty/b_\lambda^n H^\infty}\leq$$

$$\leq\left(\|TP_\gamma\|_{L^1\left(\left(1-|z|^2\right)^\beta dA\right)\to H^\infty/b_\lambda^n H^\infty}\right)^{1-\theta}\left(\|TP_\gamma\|_{L^2\left(\left(1-|z|^2\right)^\beta dA\right)\to H^\infty/b_\lambda^n H^\infty}\right)^{\theta}\leq$$

$$\leq\left(c_1(\beta)A^{'}(\beta,\,1)\left(\frac{n}{1-|\lambda|}\right)^{\beta+2}\right)^{1-\theta}\left(c_2(\beta)A^{'}\left(\frac{\beta+1}{2},\,2\right)\left(\frac{n}{1-|\lambda|}\right)^{\frac{\beta+2}{2}}\right)^{\theta}=$$

$$=\left(c_1(\beta)A^{'}(\beta,\,1)\right)^{1-\theta}\left(c_2(\beta)A^{'}\left(\frac{\beta+1}{2},\,2\right)\right)^{\theta}\left(\frac{n}{1-|\lambda|}\right)^{(\beta+2)(1-\theta)+\theta\frac{\beta+2}{2}}.$$

Now, since $\theta = 2(1-\frac{1}{p})$, $(\beta+2)(1-\theta)+\theta\frac{\beta+2}{2} = \beta-(1-\frac{1}{p})\beta+2-2+\frac{2}{p} = \frac{\beta+2}{p}$, and

$$\|T\|_{L^p_a\left(\left(1-|z|^2\right)^\beta dA\right)\to H^\infty/b^n_\lambda H^\infty} \leq \|TP_\gamma\|_{L^p\left(\left(1-|z|^2\right)^\beta dA\right)\to H^\infty/b^n_\lambda H^\infty},$$

we complete the proof. $\square$

1.12. About the links with Carleson interpolation

In this section, we compare the method used in Sections 1.3, 1.4 and 1.6, with those resulting from Carleson-type interpolation. Especially, we are interested in the case of circular sequences σ and radial sequences σ. Recall that given a (finite) set $\sigma = \{\lambda_1, ..., \lambda_n\} \subset \mathbb{D}$, the interpolation constant $C_I(\sigma)$ is defined by

$$C_I(\sigma) = sup_{\|a\|_{l^\infty}\leq 1} inf\left(\| g \|_\infty \colon\, g \in H^\infty,\, g_{|\sigma} = a\right).$$

Theorem. 1.12.0 *Let X be a Banach space, $X \subset Hol(\mathbb{D})$. Then, for all sequences $\sigma = \{\lambda_1, ..., \lambda_n\}$ of distinct points in the unit disc $\mathbb{D}$,*

$$max_{1\leq i\leq n}\|\varphi_{\lambda_i}\| \leq c(\sigma,\, X,\, H^\infty) \leq C_I(\sigma).max_{1\leq i\leq n}\|\varphi_{\lambda_i}\|,$$

where $C_I(\sigma)$ stands for the interpolation constant.

Proof. Let $f \in X$. By definition of $C_I(\sigma)$, there exist a $g \in H^\infty$ such that

$$f(\lambda_i) = g(\lambda_i)\ \forall i = 1..n,$$

with

$$\| g \|_\infty \leq C_I(\sigma) max_i |f(\lambda_i)| \leq$$

$$\leq C_I(\sigma) max_i \|\varphi_{\lambda_i}\| \, \|f\|_X.$$

Now, taking the supremum over all $f \in X$ such that $\|f\|_X \leq 1$, we get the right hand side inequality. The left hand side one is proved in Theorem 1.1.3, (1). $\square$

Comments 1.12.1

Theorem 1.12.0 tells us that, for σ with a "reasonable" interpolation constant $C_I(\sigma)$, the quantity $c(\sigma,\, X,\, H^\infty)$ behaves as $max_i\|\varphi_{\lambda_i}\|$. However, for "tight" sequences σ, the constant $C_I(\sigma)$ is so large that the estimate in question contains almost no information. On the other hand, an advantage of the estimate of Theorem 1.12.0 is that it does not contain $\#\sigma = n$ explicitly. Therefore, for well-separated sequences σ, Theorem 1.12.0 should give a better estimate than those of Corollary 1.3.2, and of Theorem 1.6.0.

Now, how does the interpolation constant $C_I(\sigma)$ behave in terms of the caracteristic r and n of σ? In what follows we try to answer that question when σ is a $r-circular$ sequence. In that case, we recall the definition of the constant α:

$$\alpha = \frac{min_{i\neq j}|\lambda_i-\lambda_j|}{1-r} = \frac{ra}{1-r}.$$

Example. 1.12.2 *Two points sets. Let* $\sigma = \{\lambda_1, \lambda_2\}$, $\lambda_i \in \mathbb{D}$, $\lambda_1 \neq \lambda_2$. *Then,*

$$\frac{1}{|b_{\lambda_1}(\lambda_2)|} \leq C_I(\sigma) \leq \frac{2}{|b_{\lambda_1}(\lambda_2)|},$$

and Theorem 1.12.0 implies

$$c(\sigma,\, X,\, H^\infty) \leq \frac{2}{|b_{\lambda_1}(\lambda_2)|} max_{i=1,2} \|\varphi_{\lambda_i}\|,$$

whereas a straightforward estimate gives

$$c(\sigma,\, X,\, H^\infty) \leq \|\varphi_{\lambda_1}\| + max_{|\lambda| \leq r} \|\varphi_{\lambda,1}\| (1 + |\lambda_1|),$$

where $r = max(|\lambda_1|, |\lambda_2|)$ *and the functional* $\varphi_{\lambda,1}$ *is defined in* 1.1. *The difference is that the first upper bound blows up when* $\lambda_1 \to \lambda_2$, *whereas the second one is still well-bounded.*

Indeed, for an H^∞-function f solving the interpolation $f(\lambda_1) = 1$, $f(\lambda_2) = -1$, we have

$$2 = |f(\lambda_1) - f(\lambda_2)| \leq 2\|f\|_\infty |b_{\lambda_1}(\lambda_2)|,$$

(indeed, the function $g = \frac{f(\lambda_1) - f}{b_{\lambda_1}}$ is holomorphic in $\mathbb{D}$ and its $H^\infty-$ norm on $\mathbb{T}$ is equal to $\|f(\lambda_1) - f\|_\infty$, (which is less or equal than $2\|f\|_\infty$), since the Blaschke factor b_{λ_1} has modulus 1 on the torus $\mathbb{T}$). Hence, $\|f\|_\infty \geq \frac{1}{|b_{\lambda_1}(\lambda_2)|}$, which shows $C_I(\sigma) \geq \frac{1}{|b_{\lambda_1}(\lambda_2)|}$.

On the other hand, setting

$$f = a_1 \frac{b_{\lambda_2}}{b_{\lambda_2}(\lambda_1)} + a_2 \frac{b_{\lambda_1}}{b_{\lambda_1}(\lambda_2)},$$

for arbitrary $a_1,\, a_1 \in \mathbb{C}$, we get $\|f\|_\infty \leq \frac{|a_1| + |a_2|}{|b_{\lambda_1}(\lambda_2)|} \leq \frac{max(|a_1|, |a_2|)}{|b_{\lambda_1}(\lambda_2)|}$. This implies $C_I(\sigma) \leq \frac{2}{|b_{\lambda_1}(\lambda_2)|}$.

For the second estimate stated in the example, taking $f \in X$ we set

$$g = f(\lambda_1) + \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1}(z - \lambda_1),$$

and we get

$$\|g\|_\infty \leq |f(\lambda_1)| + \left|\frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1}\right| (1 + |\lambda_1|) \leq$$
$$\leq \|\varphi_{\lambda_1}\| + max_{\lambda \in [\lambda_1, \lambda_2]} \|\varphi_{\lambda,1}\| (1 + |\lambda_1|),$$

and the result follows.

□

Example. 1.12.3 *Circular sequences. Let* $0 < r < 1$ *and* $\sigma = \{\lambda_1, \lambda_2, ..., \lambda_n\}$, $\lambda_i \neq \lambda_j$, $|\lambda_i| = r$ *for every* i, *and let* $\alpha = \frac{min_{i \neq j}|\lambda_i - \lambda_j|}{1 - r}$. *Then,* $\frac{1}{\alpha} \leq C_I(\sigma) \leq 8e^{K'\left(1 + \frac{K}{\alpha^3}\right)}$, *where* $K,\, K' > 0$ *are absolute constants. Therefore,*

$$c(\sigma,\, X,\, H^\infty) \leq \left(8e^{K'\left(1 + \frac{K}{\alpha^3}\right)}\right) .max_{|\lambda| = r} \|\varphi_\lambda\|$$

for every $r-$ *circular set* σ *(an estimate does not depending on* n *explicitly). In particular, there exists an increasing function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that, for* n *uniformly distincts points* $\lambda_1, ..., \lambda_n$, $|\lambda_i| = r$, $|\lambda_i - \lambda_{i+1}| = 2r sin\left(\frac{\pi}{2n}\right)$, *we have*

(1) $c(\sigma, H^2, H^\infty) \leq \varphi\left(\frac{n(1-r)}{r}\right)\frac{1}{(1-r)^{\frac{1}{2}}}$ *, for every* n *and* r*,* $0 < r < 1$ *and in particular, for* $n \leq [r(1-r)^{-1}]$ *we obtain*

$$c(\sigma, H^2, H^\infty) \leq c\frac{1}{(1-r)^{\frac{1}{2}}},$$

whereas our specific Corollary 1.3.2, (which is sharp over all n *elements sequences* σ*), gives*

$$c(\sigma, H^2, H^\infty) \leq c\frac{1}{(1-r)}$$

only.

(2) $c(\sigma, L^2_a, H^\infty) \leq \varphi\left(\frac{n(1-r)}{r}\right)\frac{1}{(1-r)}$ *, for every* n *and* r*,* $0 < r < 1$ *and in particular, for* $n \leq [r(1-r)^{-1}]$ *we obtain*

$$c(\sigma, L^2_a, H^\infty) \leq c\frac{1}{(1-r)},$$

whereas our specific Theorem 1.6.0, (which, again, is sharp over all n *elements sequences* σ*), gives*

$$c(\sigma, L^2_a, H^\infty) \leq c\frac{1}{(1-r)^2}$$

only.

In order to explain the statements of this example, we observe first that the Carleson interpolation constant $C_I(\sigma)$, for $r-circular$ sets σ, essentially depends on α only. Indeed, as is known, the separation constant

$$\Delta = inf_{1\leq j,\, k\leq n,\, j\neq k}|b_{\lambda_j}(\lambda_k)|,$$

is of the order of $min(\alpha, 1)$, and the Carleson measure density for $\mu = \sum_{i=1}^n \left(1-|\lambda_i|^2\right)\delta_{\lambda_i}$ also depends on α only. All together, $C_I(\sigma)$ is bounded if and only if α is separated from 0; see [N1] p.158 for the details of this reasoning. In fact, we can show that

$$\frac{\alpha}{1+\alpha r} \leq \Delta \leq \alpha$$

and

$$\frac{1}{\alpha} \leq C_I(\sigma) \leq e^{K'\left(1+\frac{K}{\alpha^3}\right)},$$

(as claimed as above), where $K,\, K' > 0$ are absolute constants, see Appendix 1.12.5 for details.

Now, checking point (1) for n equidistant points on the circle $|z| = r$, $\lambda_j = re^{\frac{2i\pi j}{n}}, j = 1, 2, ..., n$, one obtains $|\lambda_i - \lambda_{i+1}| = 2rsin\left(\frac{\pi}{2n}\right) \geq \frac{2r}{n}$, and hence $\alpha \geq \frac{2r}{n(1-r)}$. The above estimate for $C_I(\sigma)$ entails that we can take $\varphi(t) = 8e^{K'\left(1+Kt^3\right)}$ and then,

$$C_I(\sigma) \leq 8e^{K'\left(1+\frac{K}{\alpha^3}\right)} \leq \varphi\left(\frac{n(1-r)}{r}\right).$$

Since, for the space H^2, we have $\|\varphi_\lambda\| = (1-|\lambda|^2)^{-\frac{1}{2}}$, the upper estimate for $c(\sigma, H^2, H^\infty)$ follows. Since for the space L^2_a, we have $\|\varphi_\lambda\| = (1-|\lambda|^2)$, the same reasoning works for $c(\sigma, L^2_a, H^\infty)$.

$\square$

Example. 1.12.4 *Radial sequences.* Now we compare our two estimates of the interpolation constant $c(\sigma, X, H^\infty)$ (through the Carleson interpolation, and by the preceding general and specific methods) for special (geometric) sequences on the radius of the unit disc $\mathbb{D}$, say on the radius $[0, 1)$. Let $0 < \rho < 1$, $p \in (0, \infty)$and

$$\lambda_j = 1 - \rho^{j+p},\ j = 0,\ ...,\ n,$$

so that the distances $1 - \lambda_j = \rho^j \rho^p$ form a geometric progression; the starting point is $\lambda_0 = 1 - \rho^p$. Let

$$r = max_{0 \le j \le n} \lambda_j = \lambda_k = 1 - \rho^{n+p},$$

and $\delta = \delta(B) = min_{0 \le k \le n} \left| B_k\left(\lambda_k\right) \right|$, where $B_k = \frac{B}{b_{\lambda_k}}$. It is known that $\frac{1}{\delta} \le C_I(\sigma) \le \frac{8}{\delta^2}$. (The left hand side inequality is easy: if $f \in H^\infty$, $f\left(\lambda_k\right) = 1$, $f\left(\lambda_j\right) = 0$ for $j \ne k$, then $f = B_k g$ and $\|f\|_\infty = \|g\|_\infty \ge \left|g\left(\lambda_k\right)\right| = \frac{1}{\left|B_k\left(\lambda_k\right)\right|}$, and hence $C_I(\sigma) \ge \frac{1}{\left|B_k\left(\lambda_k\right)\right|}$ for every $k = 0, 1, 2, ..., n$. The right hand side inequality is a theorem by P. Jones and S. Vinogradov, see ([N1], p 189). So, we need to know the asymptotic behaviour of $\delta = \delta(B)$ when $n \to \infty$, or $\rho \to 1$, or $\rho \to 0$, or $p \to \infty$, or $p \to 0$.

Claim. Let $\sigma = \left\{1 - \rho^{p+k}\right\}_{k=1}^n$, $0 < \rho < 1$, $p > 0$. *The estimate of* $c(\sigma, H^2, H^\infty)$ *via the Carleson constant* $C_I(\sigma)$ *(using Theorem 1.12.0) is comparable with or better than the estimates from Corollary 1.3.2 (for* $X = H^2$*) and Theorem 1.6.0 (for* $X = L_a^2$ *and* $X = L_a^2\left(\left(1 - |z|^2\right)^\beta\right)$*) for sufficently small values of* ρ *(as* $\rho \to 0$*) and/or for a fixed* ρ *and* $n \to \infty$*. In all other cases, as for* $p \to \infty$ *(which means* $\lambda_1 \to 1$*), or* $\rho \to 1$*, or* $n \to \infty$ *and* $\rho \to 1$*, it is worse.*

In order to justify that claim, we use the following upper bound for $\delta(B) = min_{0 \le k \le n} \left|B_k\left(\lambda_k\right)\right|$, assuming (for the notation convenience) the n is an even integer $n = 2k$ and computing $B_k\left(\lambda_k\right)$,

$$\left|B_k\left(\lambda_k\right)\right| = \Pi_{j=1}^{k-1} \frac{\lambda_k - \lambda_j}{1 - \lambda_j \lambda_k} . \Pi_{j=k+1}^{2k} \frac{\lambda_j - \lambda_k}{1 - \lambda_j \lambda_k} =$$

$$= \Pi_{j=1}^{k-1} \frac{1 - \rho^{k-j}}{1 + \rho^{k-j} - \rho^{k+p}} . \Pi_{j=k+1}^{2k} \frac{1 - \rho^{j-k}}{1 + \rho^{j-k} - \rho^{j+p}} =$$

$$= \Pi_{s=1}^{k} \frac{1 - \rho^s}{1 + \rho^s\left(1 - \rho^{p+k-s}\right)} . \Pi_{s=1}^{k} \frac{1 - \rho^s}{1 + \rho^s\left(1 - \rho^{p+k}\right)} \le$$

$$\le \left(\Pi_{s=1}^{k} \frac{1 - \rho^s}{1 + \rho^s\left(1 - \rho^{p+k-s}\right)}\right)^2 \le \left(\Pi_{s=1}^{k} \frac{1 - \rho^s}{1 + \rho^s\left(1 - \rho^{p}\right)}\right)^2 =: A(n,\ \rho,\ p).$$

For a lower bound, we proceed as in [N1] p.160 and get

$$\left|B_k\left(\lambda_k\right)\right| = \Pi_{s=1}^{k} \frac{1 - \rho^s}{1 + \rho^s\left(1 - \rho^{p+k-s}\right)} . \Pi_{s=1}^{n-k} \frac{1 - \rho^s}{1 + \rho^s\left(1 - \rho^{p+k}\right)} \ge$$

$$\ge \left(\Pi_{s=1}^{n} \frac{1 - \rho^s}{1 + \rho^s\left(1 - \rho^{p+n}\right)}\right)^2 =: C(n,\ \rho,\ p)$$

for every $k = 0,\ 1,\ ...,\ n$. Hence,

$$C(n,\ \rho,\ p) \le \delta(B) \le A(n,\ \rho,\ p).$$

On the other hand, using Corollary 1.3.4 (for $X = H^2$)

$$c(\sigma,\, H^2,\, H^\infty) \le \left(\sum_{j=1}^{n} \frac{1+|\lambda_j|}{1-|\lambda_j|}\right)^{\frac{1}{2}} \le \left(\sum_{j=1}^{n} \frac{2}{\rho^{j+p}}\right)^{\frac{1}{2}} =$$

$$= \left(\frac{2}{\rho^{n+p}}\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} \rho^{n-j}\right)^{\frac{1}{2}} = \left(\frac{2}{1-r}\right)^{\frac{1}{2}} \left(\frac{1-\rho^n}{1-\rho}\right)^{\frac{1}{2}} =: D(n,\, \rho,\, p).$$

Now, we can compare the behaviour of $D(n,\, \rho,\, p)$ and $C_I(\sigma).max_j \left\|\varphi_{\lambda_j}\right\|_{H^2}$ for every parameter $n,\, \rho,\, p$.

1.12.4 (a) *Sparse sequences* σ *(*$\rho \to 0$*, or at least* $0 < \rho \le \epsilon < 1$*).*

If $\rho \to 0$, one has $lim_{\rho\to 0} C(n,\, \rho,\, p) = 1$, and hence $\overline{lim}_{\rho\to 0} C_I\left(\sigma_{n,\,\rho,\,p}\right) \le 8$. So, asymptotically, Theorem 1.12.0 implies

$$c(\sigma_{n,\,\rho,\,p},\, H^2,\, H^\infty) \le (8+\epsilon)\left(\frac{2}{1-r}\right)^{\frac{1}{2}},$$

and Corollary 1.3.4 gives slightly better but comparable estimate,

$$c(\sigma_{n,\,\rho,\,p},\, H^2,\, H^\infty) \le (1+\epsilon)\left(\frac{2}{1-r}\right)^{\frac{1}{2}}.$$

In our definition, if $p > 0$ is fixed and $\rho \to 0$ then $\lambda_1 = \lambda_1(\rho,\, p) \to 1$. In order to keep λ_1 at a fixed position we can set $p = p(\rho) = \frac{c}{log\left(\frac{1}{\rho}\right)}$. Then $\lambda_1 = 1 - \rho^p = 1 - e^{-c}$, $c > 0$. Still, $lim_{\rho\to 0} C\left(n,\, \rho,\, p(\rho)\right) = 1$.

1.12.4 (b) *Condensed sequences* σ *(*$\rho \to 1$*).* In this case, $lim_{\rho\to 0} D(n,\, \rho,\, p) = \left(\frac{2}{1-r}\right)^{\frac{1}{2}} \sqrt{n+1}$, and hence using Corollary 1.3.4 we cannot get better than the general estimate of Corollary 1.3.5, $c(\sigma,\, H^2,\, H^\infty) \le \left(\sqrt{n+1}+\epsilon\right)\left(\frac{2}{1-r}\right)^{\frac{1}{2}}$. To the contrary, $A(n,\, \rho,\, p) \sim_{\rho\to 1} \frac{\left(\frac{n}{2}\right)!}{2^{\frac{n}{2}}}(1-\rho)^{\frac{n}{2}}$, and therefore $C_I(\sigma) \ge \delta^{-1} \ge \left(A(n,\, \rho,\, p)\right)^{-1}$ which blows up as $\frac{const}{(1-\rho)^n}$. So, as it can be predicted, in this case the Carleson interpolation is worse for our problem. Fixing $\lambda_1 = 1 - \rho^p$ at an arbitrary position $\left(p = \frac{c}{log\left(\frac{1}{\rho}\right)}\right)$ will not change the conclusion.

1.12.4 (c) *Long sequences (*$n \to \infty$*).* With fixed ρ and p, let $n \to \infty$. Then, by Corollary 1.3.4,

$$c(\sigma,\, H^2,\, H^\infty) \le \left(\frac{2}{1-r}\right)^{\frac{1}{2}} \left(\frac{1}{1-\rho}\right)^{\frac{1}{2}}.$$

(Observe, however, that is also not constant $1 - r = \rho^{n+p}$). In its turn, Theorem 1.12.0 gives

$$c(\sigma,\, H^2,\, H^\infty) \le \frac{8}{\delta^2} \frac{1}{(1-r)^{\frac{1}{2}}} \sim_{n\to\infty} \left(\Pi_{s=1}^{\infty} \frac{1-\rho^s}{1+\rho^s}\right)^{-4} \frac{8}{(1-r)^{\frac{1}{2}}},$$

because $lim_n C(n,\, \rho,\, p) = lim_n A(n,\, \rho,\, p) = \left(\Pi_{s=1}^{n} \frac{1-\rho^s}{1+\rho^s}\right)^{-4}$ for every ρ, $0 < \rho < 1$. Of course, the latter estimate is much worse than the former one, because $\Pi_{s=1}^{\infty} \frac{1+\rho^s}{1-\rho^s} \sim \frac{\sqrt{1-\rho}}{2\sqrt{\pi}} exp\left(\frac{3\pi^2}{12} \frac{1}{1-\rho}\right)$ as

$\rho \to 1$. Indeed, setting $\varphi(\rho) = \Pi_{s=1}^{\infty} \frac{1}{1-\rho^s}$ for all $\rho \in [0,\, 1[$, we have (see [Ne] p.22),

$$\varphi(\rho) = \sqrt{\frac{1-\rho}{2\pi}} exp\left(\frac{\pi^2}{12}\frac{1+\rho}{1-\rho}\right)[1 + O(1-\rho)].$$

Now, setting $\psi(\rho) = \Pi_{s=1}^{\infty} \frac{1}{1+\rho^s}$ we get $(\varphi\psi)(\rho) = \frac{1}{\Pi_{k\geq 1}(1-\rho^{2k})} = \varphi\left(\rho^2\right)$ and,

$$\Pi_{s=1}^{\infty} \frac{1+\rho^s}{1-\rho^s} = \frac{\varphi(\rho)}{\psi(\rho)} = \varphi(\rho)\frac{\varphi(\rho)}{\varphi\left(\rho^2\right)} = \frac{(\varphi(\rho))^2}{\varphi\left(\rho^2\right)} =$$

$$= \frac{1-\rho}{2\pi} exp\left(\frac{\pi^2}{6}\frac{1+1}{1-\rho}\right)\sqrt{\frac{2\pi}{1-\rho^2}} exp\left(-\frac{\pi^2}{12}\frac{1+1}{(1-\rho)\,(1+1)}\right)[1+o(1)] =$$

$$= \frac{\sqrt{1-\rho}}{2\sqrt{\pi}} exp\left(\frac{3\pi^2}{12}\frac{1}{1-\rho}\right)[1+o(1)], \text{ as } \rho \to 1.$$

Appendix 1.12.5

Let $\sigma = \{\lambda_1,\, ...,\, \lambda_n\}$ be a $r - circular$ sequence, $|\lambda_i| = r\, \forall i = 1...n$, $0 \leq r < 1$; here we show the links between the constants $\Delta = \Delta(\sigma) = inf_{i\neq j}\left|b_{\lambda_i}(\lambda_j)\right|$, and $\alpha = \frac{min_{i\neq j}|\lambda_i - \lambda_j|}{1-r}$, and establish an estimate for the Carleson interpolation constant $C_I(\sigma)$.

Lemma. 1.12.6 *In the above notation, we have*

$$\frac{\alpha}{1+\alpha r} \leq \Delta \leq \alpha.$$

Proof. The right hand side inequality is clear, since

$$\left|b_{\lambda_i}(\lambda_j)\right| = \frac{|\lambda_i - \lambda_j|}{\left|1 - \overline{\lambda_i}\lambda_j\right|} \leq \frac{|\lambda_i - \lambda_j|}{1-r^2} \leq \frac{|\lambda_i - \lambda_j|}{1-r}.$$

For the left hand side one, we have

$$\left|b_{\lambda_i}(\lambda_j)\right| = \frac{|\lambda_i - \lambda_j|}{\left|1 - \overline{\lambda_i}\left(\lambda_i - (\lambda_i - \lambda_j)\right)\right|} = \frac{|\lambda_i - \lambda_j|}{\left|1 - r^2 + \overline{\lambda_i}\left(\lambda_i - \lambda_j\right)\right|} \geq$$

$$\geq \frac{|\lambda_i - \lambda_j|}{1 - r^2 + r\,|\lambda_i - \lambda_j|} \geq \frac{|\lambda_i - \lambda_j|}{1 - r + r\,|\lambda_i - \lambda_j|} = \frac{1}{\frac{1-r}{|\lambda_i - \lambda_j|} + r} \geq$$

$$\geq \frac{1}{\alpha^{-1} + r} = \frac{\alpha}{1 + r\alpha}.$$

□

Lemma. 1.12.7 *In the above notation, we have the following estimate for the Carleson interpolation constant $C_I(\sigma)$: there exists numerical constants K, $K^{'} > 0$ such that*

$$C_I(\sigma) \leq 8e^{K^{'}\left(1+\frac{K}{\alpha^3}\right)}.$$

Proof. We recall that,

$$\delta = \delta\left(B_\sigma\right) = inf_{1\leq k\leq n}\Pi_{j,\, j\neq k}\left|\frac{\lambda_k - \lambda_j}{1 - \bar{\lambda}_j\lambda_k}\right| = inf_{1\leq k\leq n}\Pi_{j,\, j\neq k}\left|b_{\lambda_j}(\lambda_k)\right|.$$

We have

$$\delta \geq e^{-\frac{3E}{\Delta^2}},$$

where $\Delta = \Delta(\sigma)$ still stands for the separation constant of σ , $\Delta = inf_{1\leq j,\, k\leq n,\, j\neq k}|b_{\lambda_j}(\lambda_k)|$, and E is the embedding constant in Carleson's embedding theorem. Using a theorem by P. Jones and S. Vinogradov, see [N1] p.189, we have

$$C_I(\sigma) \leq \frac{8}{\delta^2}.$$

From [N1] p.158, we have

$$\delta \geq e^{-\frac{C^2}{2\Delta^2}},$$

where C is the constant defined in Carleson's imbedding theorem (Imbedding Theorem see [N1] p.151). We know moreover that C anf E are linked by the following inequality:

$$\frac{1}{147}E \leq C^2 \leq 6E.$$

(see [N1] p.153). As a consequence,

$$\delta \geq e^{-\frac{C^2}{2\Delta^2}} \geq e^{-\frac{3E}{\Delta^2}}.$$

Moreover,

$$E = sup_{l>0}\frac{\mu(Q_l)}{l}$$

where,

$$\mu = \sum_{j=1..n}(1-|\lambda_j|^2)\delta_{\lambda_j},$$

(see [N1] p.153). We recall that $a = \frac{1}{r}min_{i\neq j}|\lambda_i - \lambda_j|$ and we notice that

$$\mu(Q_l) \leq \left\{ \begin{array}{cc} 0 & if\, l < 1-r \\ (1-r^2) & if\, 1-r \leq l \leq ra \\ (1-r^2).(number\ of\ \lambda_j\ belonging\, to\, Q_l) & if\, l \geq ra \end{array} \right. .$$

Indeed,
if $l < 1-r$ then none of the λ_j belong to Q_l,
and if $1-r \leq l \leq ra$, one of the λ_j belongs at most to Q_l.
As a result,

$$E \leq max\left(sup_{1-r\leq l\leq ra}\frac{\mu(Q_l)}{l},\ sup_{l\geq ra}\frac{\mu(Q_l)}{l}\right).$$

But,

$$sup_{1-r\leq l\leq ra}\frac{\mu(Q_l)}{l} \leq (1+r)(1-r) \times sup_{1-r\leq l\leq ra}\frac{1}{l}$$

$$\leq 2(1-r)\frac{1}{1-r} = 2.$$

On the other hand,

$$sup_{l\geq ra}\frac{\mu(Q_l)}{l}\leq(1+r)(1-r)\times A\frac{\frac{l}{ra}}{l}$$
$$\leq 2A\frac{1-r}{ra}=2A\frac{1}{\alpha}$$

where A is a numerical constant and
$$\alpha=\frac{ra}{1-r}.$$

Finally,
$$E\leq 2max(1,\ A\frac{1}{\alpha})\leq 2\left(1+\frac{A}{\alpha}\right),$$

and, using Lemma 1.12.6, we get
$$\frac{E}{\Delta^2}\leq 2\left(\frac{1}{\alpha}+r\right)^2\left(1+\frac{A}{\alpha}\right)\leq$$
$$\leq 2\left(\frac{1}{\alpha}+1\right)^2\left(1+\frac{A}{\alpha}\right)=2\left(1+\frac{2+A}{\alpha}+\frac{2A+1}{\alpha^2}+\frac{A}{\alpha^3}\right)\leq$$
$$\leq K^{'}\left(1+\frac{K}{\alpha^3}\right),$$

where $K^{'}$, $K>0$ are numerical constants. As a result, $8\delta^{-2}\leq 8e^{\frac{6E}{\Delta^2}}\leq 8e^{6K^{'}\left(1+\frac{K}{\alpha^3}\right)}$, which gives
$$C_I(\sigma)\leq 8e^{6K^{'}\left(1+\frac{K}{\alpha^3}\right)},$$

and completes the proof. □

$\square$

2. Lower bounds for $C_{n,r}(X,\, H^\infty)$

2.1. The case $X = l_a^2\,(w_k)$

Here, we consider the weighted spaces $l_A^2(w_k)$ of polynomial growth and the problem of lower estimates for the one point special case $\sigma_{\lambda,n} = \{\lambda, \lambda, ..., \lambda\}$, ($n$ times) $\lambda \in \mathbb{D}$. Recall the definition of the semi-free interpolation constant

$$c(\sigma_{\lambda,n}, H, H^\infty) = \; sup\left\{\|f\|_{H^\infty/b_\lambda^n H^\infty} \,:\, f \in H,\, \|f\|_H \leq 1\right\},$$

where $\|f\|_{H^\infty/b_\lambda^n H^\infty} = inf\,\{\|f + b_\lambda^n g\|_\infty \,:\, g \in H\}$. In particular, our aim is to prove the sharpness of the upper estimate for the quantity

$$C_{n,\,r}\left(l_a^2\left(\frac{1}{(k+1)^{\frac{N-1}{2}}}\right),\, H^\infty\right),$$

(where $N \geq 1$ is an integer), in Theorem 1.6.0.

Theorem. 2.1.0 *Let $N \geq 1$ be an integer. Then,*

$$c\left(\sigma_{\lambda,n},\, l_A^2\left(\frac{1}{(k+1)^{\frac{N-1}{2}}}\right),\, H^\infty\right) \geq a_N\left(\frac{n}{1-|\lambda|}\right)^{\frac{N}{2}}$$

for a positive constant a_N depending on N only. In particular,

$$a_N\left(\frac{n}{1-r}\right)^{\frac{N}{2}} \leq C_{n,\,r}\left(l_a^2\left(\frac{1}{(k+1)^{\frac{N-1}{2}}}\right),\, H^\infty\right) \leq A\left(\frac{n}{1-r}\right)^{\frac{N}{2}},$$

for all $n \geq 1$, $0 \leq r < 1$, where $A = A\left(\frac{N-1}{2}\right)$ is a constant defined in Theorem 1.6.0.

(1) We first recall some properties of spaces $X = \, l_a^p(w_k)$. As it is mentionned in the Introduction,

$$l_a^p(w_k) = \left\{f = \sum_{k\geq 0} \hat{f}(k)z^k \,:\, \|f\|^p = \sum_{k\geq 0} |\hat{f}(k)|^p w_k^p < \infty\right\},$$

with a weight w satisfying $w_k > 0$ for every $k \geq 0$ and $\overline{lim}_k(1/w_k^{1/k}) = \, 1$. The latter condition implies that $l_a^p(w_k)$ is continuously embedded into the space of holomorphic functions $Hol(\mathbb{D})$ on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (and not on a larger disc, i.e. $l^p_a(w_k)$ does not contained in $Hol(r\mathbb{D})$ for every $r > 1$). In this section, we study the case $p = 2$, so that $l^2_a(w_k)$ is a reproducing kernel Hilbert space on the disc $\mathbb{D}$. The reproducing kernel of $l^2_a(w_k)$, by definition, is a $l^2_a(w_k)$-valued function $\lambda \longmapsto k^w_\lambda$, $\lambda \in \mathbb{D}$, such that $(f, k^w_\lambda) = f(\lambda)$ for every $f \in l^2_a(w_k)$, where $(.,.)$ means the scalar product $(f, g) = \sum_{k\geq 0} \hat{h}(k)\overline{\hat{g}(k)}w_k^2$. Since one has $f(\lambda) = \sum_{k\geq 0} \hat{f}(k)\lambda^k \frac{1}{w_k^2} w_k^2$ $(\lambda \in \mathbb{D})$, it follows that

$$k^w_\lambda(z) = \sum_{k\geq 0} \frac{\overline{\lambda}^k z^k}{w_k^2},\ z \in \mathbb{D}.$$

In particular, for the Hardy space $H^2 = l^2_a(1)$, we get the Szegö kernel

$$k_\lambda(z) = (1 - \overline{\lambda}z)^{-1},$$

for the Bergman space $L^2_a(\mathbb{D}) = l^2_a\left(\frac{1}{(k+1)^{\frac{1}{2}}}\right)$ - the Bergman kernel $k_\lambda(z) = (1 - \overline{\lambda}z)^{-2}$.

(2) Conversely, following the Aronszajn theory of RKHS (see, for example [A] or [N2] p.317), given a positive definit function $(\lambda, z) \longmapsto k(\lambda, z)$ on $\mathbb{D} \times \mathbb{D}$ (i.e. such that $\sum_{i,j} \overline{a}_i a_j k(\lambda_i, \lambda_j) > 0$ for all finite subsets $(\lambda_i) \subset \mathbb{D}$ and all non-zero families of complex numbers (a_i)) one can define the corresponding Hilbert spaces $H(k)$ as the completion of finite linear combinations $\sum_i \overline{a}_i k(\lambda_i, \cdot)$ endowed with the norm

$$\|\sum_i \overline{a}_i k(\lambda_i, \cdot)\|^2 = \sum_{i,j} \overline{a}_i a_j k(\lambda_i, \lambda_j).$$

When k is holomorphic with respect to the second variable and antiholomorphic with respect to the first one, we obtain a RKHS of holomorphic functions $H(k)$ embedded into $Hol(\mathbb{D})$.

For functions k of the form $k(\lambda, z) = K(\overline{\lambda}z)$, where $K \in Hol(\mathbb{D})$, the positive definitness is equivalent to $\hat{K}(j) > 0$ for every $j \geq 0$, where $\hat{K}(j)$ stands for Taylor coefficients, and in this case we have $H(k) = l^2_a(w_j)$, where $w_j = 1/\sqrt{\hat{K}(j)}$, $j \geq 0$. In particular, for $K(w) = (1 - w)^{-\beta}$, $k_\lambda(z) = (1 - \overline{\lambda}z)^{-\beta}$, $\beta > 0$, we have $\hat{K}(j) = \binom{\beta+j-1}{\beta-1}$ (binomial coefficients), and hence $w_j = \left(\frac{j!}{\beta(\beta+1)...(\beta+j-1)}\right)^{\frac{1}{2}}$. Indeed, deriving $\frac{1}{1-z}$, we get by induction

$$(1 - z)^{-\beta} = \frac{1}{(\beta - 1)!} \sum_{j\geq 0} (j + \beta - 1)...(j + 1)z^k = \sum_{j\geq 0} \binom{\beta+j-1}{\beta-1} z^j.$$

Clearly, $w_j \simeq 1/j^{\frac{\beta-1}{2}}$, where $a_j \simeq b_j$ means that there exist constants $c_1 > 0$, $c_2 > 0$ such that $c_1 a_j \leq b_j \leq c_2 a_j$ for every j. Therefore, $H(k) = l^2_a\left(\frac{1}{(k+1)^{\frac{\beta-1}{2}}}\right)$ (a topological identity: the spaces are the same and the norms are equivalent).

(3) Reproducing kernel Hilbert spaces containing H^2. We will use the previous observations for the following composed reproducing kernels (Aronszajn-deBranges, see [N2] p.320): given a reproducing kernel k and an entire function $\varphi = \sum_{j\geq 0} \hat{\varphi}(j)z^j$ with $\hat{\varphi}(j) \geq 0$ for every $j \geq 0$, the function $\varphi \circ k$ is also positive definit and the corresponding RKHS

$$H(\varphi \circ k) =: \varphi(H(k))$$

satisfies the following. For every $f \in H(k)$ we have $\varphi\circ f \in \varphi(H(k))$ and $\|\varphi\circ f\|^2_{\varphi(H(k))} \le \varphi(\|f\|^2_{H(k)})$ (see [N2] p.320). In particular, if φ is a polynomial of degree N and k is the Szegö kernel then $\varphi \circ k_\lambda(z) = \sum_{j\ge 0} c_j \overline{\lambda}^j z^j$ with $c_k \simeq (k+1)^{N-1}$, and hence

$$\varphi(H^2) = l^2_a\left(\frac{1}{(k+1)^{\frac{N-1}{2}}}\right)$$

(a topological identity: the spaces are the same and the norms are equivalent). The link between spaces of type $l^2_a\left(\frac{1}{(k+1)^{\frac{N-1}{2}}}\right)$ (already mentionned in Section 1.6) and of type $\varphi(H^2) = H_\varphi$ being established, we give the following result.

Lemma 2.1.1 *Let $\varphi(z) = \sum_{k=0}^N a_k z^k$, $a_k \ge 0$ $(a_N > 0)$, and $H_\varphi = \varphi(H^2)$ be the reproducing kernel Hilbert space corresponding to the kernel $\varphi\left(\frac{1}{1-\overline{\lambda}z}\right)$. Then, there exists a constant $a = a(\varphi) > 0$ such that*

$$c(\sigma_{\lambda,n}, H_\varphi, H^\infty) \ge a\left(\frac{n}{1-|\lambda|}\right)^{\frac{N}{2}}.$$

Proof. 1) We set

$$Q_n = \sum_{k=0}^{n-1} b_\lambda^k \frac{(1-|\lambda|^2)^{1/2}}{1-\overline{\lambda}z},\ H_n = \varphi\circ Q_n,$$

$$\Psi = bH_n.$$

Then $\|Q_n\|_2^2 = n$, and hence by the Aronszajn-deBranges inequality, see [N2] p.320, point (k) of Exercise 6.5.2, with $\varphi(z) = z^N$ and $K(\lambda, z) = k_\lambda(z) = \frac{1}{1-\overline{\lambda}z}$, and noticing that $H(\varphi\circ K) = H_\varphi$,

$$\|\Psi\|^2_{H_\varphi} \le b^2\varphi\left(\|Q_n\|_2^2\right) = b^2\varphi(n).$$

Let $b > 0$ such that $b^2\varphi(n) = 1$.

2) Since the spaces H_φ and H^∞ are rotation invariant, we have $c(\sigma_{\lambda,n}, H_\varphi, H^\infty) = c(\sigma_{\mu,n}, H_\varphi, H^\infty)$ for every λ, μ with $|\lambda| = |\mu| = r$. Let $\lambda = -r$. To get a lower estimate for $\|\Psi\|_{H_\varphi/b^n_\lambda H_\varphi}$ consider G such that $\Psi - G \in b^n_\lambda Hol(\mathbb{D})$, i.e. such that $bH_n\circ b_\lambda - G\circ b_\lambda \in z^n Hol(\mathbb{D})$.

3) First, we show that

$$\psi =: \Psi\circ b_\lambda = bH_n\circ b_\lambda$$

is a polynomial (of degree nN) with positive coefficients. Note that

$$Q_n \circ b_\lambda = \sum_{k=0}^{n-1} z^k \frac{(1-|\lambda|^2)^{1/2}}{1-\overline{\lambda} b_\lambda(z)} =$$

$$= \left(1-|\lambda|^2\right)^{-\frac{1}{2}} \left(1 + (1-\overline{\lambda}) \sum_{k=1}^{n-1} z^k - \overline{\lambda} z^n\right) =$$

$$= (1-r^2)^{-1/2} \left(1 + (1+r) \sum_{k=1}^{n-1} z^k + r z^n\right) =: (1-r^2)^{-1/2} \psi_1.$$

Hence, $\psi = \Psi \circ b_\lambda = bH_n \circ b_\lambda = b\varphi \circ \left((1-r^2)^{-\frac{1}{2}} \psi_1\right)$ and

$$\varphi \circ \psi_1 = \sum_{k=0}^{N} a_k \psi_1^k(z).$$

(In fact, we can simply assume that $\varphi \circ \psi_1 = \psi_1^N(z)$ since $H_\varphi = l_a^2 \left(\frac{1}{(k+1)^{\frac{N-1}{2}}}\right) = H_{z^N}$). Now, it is clear that ψ is a polynomial of degree Nn such that

$$\psi(1) = \sum_{j=0}^{Nn} \hat{\psi}(j) = b\varphi\left((1-r^2)^{-1/2}(1+r)n\right) = b\varphi\left(\sqrt{\frac{1+r}{1-r}} n\right) > 0.$$

4) Next, we show that there exists a constant $c = c(\varphi) > 0$ (for example, $c = \alpha/2^{2N}(N-1)!$, α is a numerical constant) such that

$$\sum^{m}(\psi) =: \sum_{j=0}^{m} \hat{\psi}(j) \geq c \sum_{j=0}^{Nn} \hat{\psi}(j) = c\psi(1),$$

where $m \geq 1$ is such that $2m = n$ if n is even and $2m - 1 = n$ if n is odd.

Indeed, setting

$$S_n = \sum_{j=0}^{n} z^j,$$

we have

$$\sum^{m}\left(\psi_1^k\right) = \sum^{m}\left(\left(1 + (1+r)\sum_{k=1}^{n-1} z^k + r z^n\right)^k\right) \geq \sum^{m}\left(S_{n-1}^k\right).$$

Next, we obtain

$$\sum^{m}\left(S_{n-1}^k\right) = \sum^{m}\left(\left(\frac{1-z^n}{1-z}\right)^k\right) =$$

$$= \sum^{m} \left(\sum_{j=0}^{k} C_k^j \frac{1}{(1-z)^j} \cdot \left(\frac{-z^n}{1-z} \right)^{k-j} \right) = \sum^{m} \left(\frac{1}{(1-z)^k} \right) =$$

$$= \sum^{m} \left(\sum_{j \geq 0} C_{k+j-1}^j z^j \right) = \sum_{j=0}^{m} C_{k+j-1}^j \geq \sum_{j=0}^{m} \frac{(j+1)^{k-1}}{(k-1)!} \geq$$

$$\geq \alpha \frac{m^k}{(k-1)!},$$

where $\alpha > 0$ is a numerical constant. Finally,

$$\sum^{m} (\psi_1^k) \geq \alpha \frac{m^k}{(k-1)!} \geq \alpha \frac{(n/2)^k}{(k-1)!} =$$

$$= \frac{\alpha}{2^k (k-1)!} \cdot \frac{((1+r)n)^k}{(1+r)^k} = \frac{\alpha}{2^k (1+r)^k (k-1)!} \cdot (\psi_1(1))^k \geq$$

$$\geq \frac{\alpha}{2^N (1+r)^N (N-1)!} \cdot (\psi_1(1))^k.$$

Summing up these inequalities in $\sum^m(\psi) = b\sum^m(\varphi \circ \psi_1) = b\sum_{k=0}^N a_k \left(1-r^2\right)^{-k/2} \sum^m(\psi_1^k)$ (or simply taking $k = N$, if we already supposed $\varphi = z^N$), we obtain the result claimed.

5) Now, using point 4) and the preceding Fejer kernel argument and denoting $F_n = \Phi_m + z^m \Phi_m$, where Φ_k stands for the k-th Fejer kernel, we get

$$\|\Psi\|_{H^\infty / b_\lambda^n H^\infty} = \|\psi\|_{H^\infty / z^n H^\infty} \geq \frac{1}{2} \|\psi * F_n\|_\infty \geq \frac{1}{2} \sum_{j=0}^{m} \hat{\psi}(j) \geq$$

$$\geq \frac{c}{2} \psi(1) = \frac{c}{2} b \varphi \left(\sqrt{\frac{1+r}{1-r}} n \right) = \frac{c}{2} \cdot \frac{\varphi \left(\sqrt{\frac{1+r}{1-r}} n \right)}{(\varphi(n))^{1/2}} \geq$$

(assuming that $\varphi = z^N$)

$$\geq a \left(\frac{n}{1-r} \right)^{\frac{N}{2}}.$$

$\square$

Proof of Theorem 2.1.0. In order to prove the left hand side inequality, it suffices to apply Lemma 2.1.1 with $\varphi(z) = z^N$. Indeed, in this case $H_\varphi = l_a^2 \left(\frac{1}{(k+1)^{\frac{N-1}{2}}} \right) = H_{z^N}$. The right hand side inequality is a straightforward consequence of Theorem 1.6.0.

$\square$

In the case of the Hardy space H^2, recall that $H^2 = l_a^2\left(\frac{1}{(k+1)^{\frac{N-1}{2}}}\right)$, with $N=1$ (or equivalently $H^2 = H_\varphi$, with $\varphi(z)=z$), we can explicit the constant $a=a_1$ of Theorem 2.1.0 (or $a=a(\varphi)$ of Lemma 2.1.1).

Theorem. 2.1.2 *We have*

$$\frac{1}{4\sqrt{2}}\frac{\sqrt{n}}{\sqrt{1-r}} \leq C_{n,r}\left(H^2,\, H^\infty\right) \leq \sqrt{2}\frac{\sqrt{n}}{\sqrt{1-r}},$$

$\forall n \geq 1$, $\forall r \in [0,\, 1[$.

In the same way, recall that the Bergman space L_a^2 verifies $L_a^2 = l_a^2\left(\frac{1}{(k+1)^{\frac{N-1}{2}}}\right)$, with $N=2$ (or $L_a^2 = H_\varphi$, with $\varphi(z)=z^2$), we can also give explicitly the constant $a=a_2=a(\varphi)$ of Theorem 2.1.0.

Theorem. 2.1.3 *We have,*

$$\frac{1}{32}\frac{n}{1-r} \leq C_{n,r}\left(L_a^2,\, H^\infty\right) \leq \sqrt{14}\frac{n}{1-r}$$

$\forall n \geq 1$, $\forall r \in [0,\, 1[$.

2.2. The case $X = H^p$

The aim of this section is to prove the sharpness (for even p) of the upper estimate, found in Theorem 1.5.0, of the quantity $C_{n,r}\left(H^p, H^\infty\right)$. This is the subject of the following theorem.

Theorem. 2.2.0 *Let* $p \in 2\mathbb{Z}_+$*, then*

$$c\left(\sigma_{\lambda,n},\, H^p, H^\infty\right) \geq \frac{1}{32^{\frac{1}{p}}}\left(\frac{n}{1-|\lambda|}\right)^{\frac{1}{p}},$$

for every $\lambda \in \mathbb{D}$ *and every integer* $n \geq 1$*, where* $\sigma_{\lambda,\,n} = \{\lambda,\, \lambda,\, ...,\, \lambda\}$ *and hence*

$$\frac{1}{32^{\frac{1}{p}}}\left(\frac{n}{1-r}\right)^{\frac{1}{p}} \leq C_{n,r}\left(H^p, H^\infty\right) \leq A_p\left(\frac{n}{1-r}\right)^{\frac{1}{p}},$$

for all $n \geq 1$, $0 \leq r < 1$, *where* A_p *is a constant depending only on* p *which is defined in Theorem 1.5.0.*

We first prove the following lemma.

Lemma. 2.2.1 *Let* p,q *such that* $\frac{p}{q} \in \mathbb{Z}_+$*, then* $c\left(\sigma,\, H^p, H^\infty\right) \geq c\left(\sigma,\, H^q, H^\infty\right)^{\frac{q}{p}}$ *for every sequence* σ *of* $\mathbb{D}$*.*

Proof. **Step 1.** Recalling that

$$c\left(\sigma,\, H^p,\, H^\infty\right) = sup_{\|f\|_p \leq 1} inf\left\{\|g\|_\infty : \, g \in Y,\, g_{|\sigma} = f_{|\sigma}\right\},$$

we first prove that

$$c\left(\sigma,\, H^p,\, H^\infty\right) = sup_{\|f\|_p \leq 1,\, f\, exterior}\, inf\left\{\|g\|_\infty : \, g \in Y,\, g_{|\sigma} = f_{|\sigma}\right\}.$$

Indeed, we clearly have the inequality

$$sup_{\|f\|_p\leq 1,\, f\, exterior}\, inf\left\{\|g\|_\infty : g\in Y,\, g_{|\sigma}=f_{|\sigma}\right\}\leq c\left(\sigma,\, H^p,\, H^\infty\right),$$

and if the inequality were strict, that is to say

$$sup_{\|f\|_p\leq 1,\, f\, exterior}\, inf\left\{\|g\|_\infty : g\in Y,\, g_{|\sigma}=f_{|\sigma}\right\}< sup_{\|f\|_p\leq 1} inf\left\{\|g\|_\infty : g\in Y,\, g_{|\sigma}=f_{|\sigma}\right\},$$

then we could write that there exists $\epsilon>0$ such that for every $f=f_i.f_e\in H^p$ (where f_i stands for the inner function corresponding to f and f_e to the exterior one) with $\|f\|_p\leq 1$ (which also implies that $\|f_e\|_p\leq 1$, since $\|f_e\|_p=\|f\|_p$), there exists a function $g\in H^\infty$ verifying both $\|g\|_\infty\leq(1-\epsilon)c\left(\sigma,\, H^p,\, H^\infty\right)$ and $g_{|\sigma}=f_{e|\sigma}$. This entails that $f_{|\sigma}=(f_ig)_{|\sigma}$ and since $\|f_ig\|_\infty=\|g\|_\infty\leq(1-\epsilon)c\left(\sigma,\, H^p,\, H^\infty\right)$, we get that $c\left(\sigma,\, H^p,\, H^\infty\right)\leq(1-\epsilon)c\left(\sigma,\, H^p,\, H^\infty\right)$, which is a contradiction and proves the equality of Step 1.

Step 2. Using the result of Step 1, we get that $\forall\epsilon>0$ there exists an exterior function $f_e\in H^q$ with $\|f_e\|_p\leq 1$ and such that

$$inf\left\{\|g\|_\infty : g\in Y,\, g_{|\sigma}=f_{e|\sigma}\right\}\geq c\left(\sigma,\, H^q, H^\infty\right)-\epsilon.$$

Now let $F=f_e^{\frac{q}{p}}\in H^p$, then $\|F\|_p^p=\|f_e\|_q^q\leq 1$. We suppose that there exists $g\in H^\infty$ such that $g_{|\sigma}=F_{|\sigma}$ with

$$\|g\|_\infty<\left(c\left(\sigma,\, H^q, H^\infty\right)-\epsilon\right)^{\frac{q}{p}}.$$

Then, since $g\left(\lambda_i\right)=F\left(\lambda_i\right)=f_e\left(\lambda_i\right)^{\frac{q}{p}}$ for all $i=1..n$, we have $g\left(\lambda_i\right)^{\frac{p}{q}}=f_e\left(\lambda_i\right)$ and $g^{\frac{p}{q}}\in H^\infty$ since $\frac{p}{q}\in\mathbb{Z}_+$. We also have

$$\left\|g^{\frac{p}{q}}\right\|_\infty=\|g\|_\infty^{\frac{p}{q}}<\left(c\left(\sigma,\, H^q, H^\infty\right)-\epsilon\right)^{\frac{q}{p}},$$

which is a contradiction. As a result, we have

$$\|g\|_\infty\geq\left(c\left(\sigma,\, H^q, H^\infty\right)-\epsilon\right)^{\frac{q}{p}},$$

for all $g\in H^\infty$ such that $g_{|\sigma}=F_{|\sigma}$, which gives

$$c\left(\sigma,\, H^p, H^\infty\right)\geq\left(c\left(\sigma,\, H^q, H^\infty\right)-\epsilon\right)^{\frac{q}{p}},$$

and since that inequality is true for every $\epsilon>0$, we get the result. □

Proof of Theorem 2.2.0. We first prove the left hand side inequality. Writing $p=2.\frac{p}{2}$, we apply Lemma 2.2.1 with $q=2$ and this gives

$$c\left(\sigma_{\lambda,n},\, H^p, H^\infty\right)\geq c\left(\sigma_{\lambda,n},\, H^2, H^\infty\right)^{\frac{2}{p}}\geq\frac{1}{32^{\frac{1}{p}}}\left(\frac{n}{1-|\lambda|}\right)^{\frac{2}{p}}$$

for all integer $n\geq 1$. The last inequality being a consequence of Theorem 2.1.2. The right hand side inequality is proved in Theorem 1.5.0. □

3. Toeplitz condition numbers as an H^∞ interpolation problem

Let H be a Hilbert space of finite dimension n and T an invertible operator acting on H such that $\| T \| \leq 1$. We are interested in estimating the norm of the inverse of T :

$$\| T^{-1} \| .$$

More precisely, given a family $\mathcal{F}$ of $n-$dimensional operators and a $T \in \mathcal{F}$, we set

$$r_{min}(T) = min_{i=1..n}|\lambda_i| > 0,$$

where $\{\lambda_1, ..., \lambda_n\} = \sigma(T)$ is the spectrum of T. We are looking for "the best possible" majorant $\Phi_n(r)$ such that

$$\left\|T^{-1}\right\| \leq \Phi_n(r)$$

for every $T \in \mathcal{F}$, $\|T\| \leq 1$. This leads to define the following bound $c_n(\mathcal{F}, r)$, where $0 < r < 1$,

$$c_n(\mathcal{F}, r) = sup\left\{\left\|T^{-1}\right\| : \ T \in \mathcal{F}, \ \|T\| \leq 1, \ r_{min}(T) \geq r\right\}.$$

The following classical result is attributed to Kronecker (XIX c.)

Theorem. 3.0 (Kronecker)

Let $\mathcal{F}$ be the set of all n-dimensional operators defined on an euclidean space. Then

$$c_n(r) := c_n(\mathcal{F}, r) = \frac{1}{r^n}$$

Since obviously the upper bound in $c_n(r)$ is attained (by a compactness argument), a natural question arises: how to describe the extremal matrices T such that $\|T\| \leq 1$, $r_{min}(T) \geq r$ and $\|T^{-1}\| = \frac{1}{r^n}$. The answer is contained in N. Nikolski [N3] in the following form: the case of equality

$$\left\|T^{-1}\right\| = \frac{1}{r^n}$$

occurs for a matrix T with $\| T \| = 1$ if and only if:

(1) either $r = 1$ and then T is an arbitrary unitary matrix.

(2) or $r < 1$, and then the eigenvalues $\lambda_j(T)$ of T are such that

$$|\lambda_j(T)| = r$$

and given $\sigma = \{\lambda_1, ..., \lambda_n\}$ on the circle, there exists a unique extremal matrix T (up to a unitary equivalence) with the spectrum $\{\lambda_1, ..., \lambda_n\}$ having the form

$$T = U + K$$

where K is a rank one matrix, U is unitary and U and K are both given explicitly. (In fact, T is nothing but the so-called model operator corresponding to the Blaschke product $B = \Pi_{j=1}^{n} b_{\lambda_j}$, see [N2] for definitions).

For numerical analysis, the interest is in some classes of structured matrices such as Toeplitz, Hankel etc.... In that note, we are going to focus on the Toeplitz structure. Recall that T is a Toeplitz matrix if and only if there exists a sequence $(a_k)_{k=-n+1}^{k=n-1}$ such that

$$T = T_a = \begin{pmatrix} a_0 & a_{-1} & . & . & a_{-n+1} \\ a_1 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & a_{-1} \\ a_{n-1} & . & . & a_1 & a_0 \end{pmatrix},$$

and that T is an analytic Toeplitz matrix if and only if there exists a sequence $(a_k)_{k=0}^{k=n-1}$ such that

$$T = T_a = \begin{pmatrix} a_0 & 0 & . & . & 0 \\ a_1 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & 0 \\ a_{n-1} & . & . & a_1 & a_0 \end{pmatrix}.$$

We denote by $\mathcal{T}_n$ the set of Toeplitz matrices of size n, and $\mathcal{T}_n^a$ will be the set of analytic Toeplitz matrices of size n. This leads to the following questions.

How behave the constants $c_n(\mathcal{T}_n, r)$ and $c_n(\mathcal{T}_n^a, r)$ when $n \to \infty$ and/or $r \to 0$? Are they uniformly comparable with the Kronecker bound $c_n(r)$? Are there exist Toeplitz matrices among extremal matrices described above? The answers seem not to be obvious, at least the obvious candidates like $T = \frac{\lambda + J_n}{\|\lambda + J_n\|}$, where J_n is the $n-$dimensional Jordan matrix, do not lead to the needed uniform (in n and r) equivalence. For short, we denote

$$t_n(r) = c_n(\mathcal{T}_n, r)$$

and

$$t_n^a(r) = c_n(\mathcal{T}_n^a, r).$$

Obviously we have,

$$t_n^a(r) \leq t_n(r) \leq c_n(r) = \frac{1}{r^n}.$$

The following theorem (see [Z]) answers the above questions.

Theorem. 3.1 *1) For all* $r \in]0, 1[$ *and* $n \geq 1$,

$$\frac{1}{2} \leq r^n t_n^a(r) \leq r^n c_n(r) = 1$$

2) For every $n \geq 1$

$$lim_{r\to 0} r^n t_n^a(r) = lim_{r\to 1} r^n t_n^a(r) = 1$$

and for every $0 < r \leq 1$

$$lim_{n\to\infty} r^n t_n^a(r) = 1.$$

The proof of the theorem is given in Section 3.2 below.

3.1. Operator M_n and its commutant

Let $M_n : (\mathbb{C}^n, <.,.>) \longrightarrow (\mathbb{C}^n, <.,.>)$ be the nilpotent Jordan Block of size n

$$M_n = \begin{pmatrix} 0 & & & & \\ 1 & . & & & \\ & . & . & & \\ & & . & . & \\ & & & 1 & 0 \end{pmatrix}.$$

It is well known that the commutant $\{M_n\}^{'} = \{A \in \mathcal{M}_n(\mathbb{C}) : AM_n = M_nA\}$ of M_n verifies

$$\{M_n\}^{'} = \{p\,(M_n) : p \in Pol_+\},$$

where Pol_+ is the space of analytic polynomials. On the other hand, we can state this fact in the following way. Let

$$K_{z^n} = \left(z^n H^2\right)^{\perp} = Lin\left(1,\ z, ...,\ z^{n-1}\right),$$

H^2 being the standard Hardy space in the disc $\mathbb{D} = \{z : |z| < 1\}$, and

$$M_{z^n} : K_{z^n} \to K_{z^n}$$

such that

$$M_{z^n} f = P_{z^n}(zf),\ \forall f \in K_{z^n}.$$

Then the matrix of M_{z^n} in the orthonormal basis of K_{z^n}, $B_n = \{1,\ z, ...,\ z^{n-1}\}$ is exactly M_n, and hence

$$\{p\,(M_n)\,,\, p \in Pol_+\} = \{M_n\}^{'} = \{M_{z^n}\}^{'}$$

The following straightforward link between $n \times n$ analytic Toeplitz matrices and $\{M_n\}'$ is well known.

Lemma. 3.1.0 $\mathcal{T}_n^a = \{M_n\}^{'}$.

Proof. Let

$$\phi(z) = \sum_{k\geq 0} \hat{\phi}(k) z^k.$$

Then,

$$\phi\left(M_n\right)=\sum_{k=0}^{n-1}\hat{\phi}(k)M_n^k=\begin{pmatrix} \hat{\phi}(0) & & & \\ \hat{\phi}(1) & . & & \\ . & . & . & \\ . & & . & . \\ \hat{\phi}(n-1) & . & . & \hat{\phi}(1) & \hat{\phi}(0) \end{pmatrix}.$$

Conversely, if $A=\begin{pmatrix} a_0 & 0 & . & . & 0 \\ a_1 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & 0 \\ a_{n-1} & . & . & a_1 & a_0 \end{pmatrix}\in\mathcal{T}_n^a$ then $A=\left(\sum_{k=0}^{n-1}a_kz^k\right)\left(M_n\right).$ □

We also need the Schur-Caratheodory interpolation theorem (1912), which also can be considered as a partial case of the commutant lifting theorem of Sarason and Sz-Nagy-Foias (1968) see [N2] p.230 Theorem 3.1.11.

Proposition. 3.1.1 *The following are equivalent.*
i) T is an $n\times n$ analytic Toeplitz matrix.
ii) There exists $g\in H^\infty$ such that $T=g\left(M_n\right)$.
Moreover

$$\| T \|= inf\left\{\| g \|_\infty :\ g\in H^\infty(\mathbb{D})\,,\, g\left(M_n\right)=T\right\}$$

$$= min\left\{\| g \|_\infty :\ g\in H^\infty(\mathbb{D})\,,\, g\left(M_n\right)=T\right\},$$

where $\| g \|_\infty= sup_{z\in\mathbb{T}}|g(z)|$.

3.2. Proof of Theorem 3.0

Lemma. 3.2.0 *Let T be an invertible analytic Toeplitz matrix of size $n\times n$ (which means that there exists $f\in Pol_+\subset H^\infty$ such that $T=f\left(M_n\right)$). Then*

$$\left\|T^{-1}\right\|=inf\left\{\| g \|_\infty :\ g,\, h\in H^\infty,\ fg+z^nh=1\right\}.$$

Proof. Since T^{-1}belongs also to $\{M_n\}'$, there exists $g\in Pol_+\subset H^\infty$ such that $T^{-1}=g\left(M_n\right)$. This implies in particular that

$$\left(fg\right)\left(M_n\right)=I_n,$$

which means that $fg-1$ annihilates M_n. That means that

$$fg-1$$

is a multiple of z^n in H^∞. Conversely, if $g\in H^\infty$ verifies the above Bezout equation with $h\in H^\infty$ then

$$g\left(M_n\right)=T^{-1}.$$

But by Proposition 3.1.1, we have

$$\left\|T^{-1}\right\| = inf\left\{\| g \|_\infty \colon g \in H^\infty,\, g\left(M_n\right) = T^{-1}\right\},$$

and hence

$$\left\|T^{-1}\right\| = inf\left\{\| g \|_\infty \colon g,\, h \in H^\infty,\, fg + z^n h = 1\right\}.$$

□

Proof of Theorem 3.0. First, we prove that for every $r \in]0,1[$ there exists an analytic $n \times n$ Toeplitz matrix T_r such that

1) $\|T_r\| \leq 1$,

2) $\sigma\left(T_r\right) = \{r\}$,

3) $\|T_r^{-1}\| \geq \frac{1}{r^n} - 1$.

Indeed, let

$$b_r(z) = \frac{r - z}{1 - rz} \in H^\infty$$

be the Blaschke factor corresponding to r. The H^∞ calculus of M_n tells us that the operator

$$T_r := b_r\left(M_n\right)$$

satisfies property 1):

$$\|T_r\| \leq \|b_r\|_\infty = 1.$$

On the other hand, by the spectral mapping theorem

$$\sigma\left(T_r\right) = \left\{b_r\left(\sigma\left(M_n\right)\right)\right\} = \left\{b_r(0)\right\} = \{r\}.$$

In particular this proves that T_r is invertible. Finally, using Lemma 3.2.0, we get

$$\left\|T_r^{-1}\right\| = inf\left\{\| g \|_\infty \colon g,\, h \in H^\infty,\, b_r g + z^n h = 1\right\} =$$

$$= inf\left\{\left\|\frac{1 - z^n h}{b_r}\right\|_\infty \colon h \in H^\infty,\, r^n h(r) = 1\right\} =$$

$$= inf\left\{\| 1 - z^n h \|_\infty \colon h \in H^\infty,\, r^n h(r) = 1\right\}.$$

But if $h \in H^\infty$and $r^n h(r) = 1$, we have

$$\| 1 - z^n h \|_\infty \geq \| h \|_\infty - 1$$

and

$$\| h \|_\infty \geq |h(r)| = \frac{1}{r^n},$$

which gives

$$\| 1 - z^n h \|_\infty \geq \frac{1}{r^n} - 1.$$

Therefore

$$\left\|T_r^{-1}\right\| \geq \frac{1}{r^n} - 1,$$

which completes the proof of property 3) of T_r.

Now we obtain

$$1 - r^n \leq r^n \left\|T_r^{-1}\right\| \leq r^n t_n^a(r) \leq r^n t_n(r) \leq r^n c_n(r) = 1$$

for every $r \in]0,1[$. On the other hand, we have $\|T_r^{-1}\| \, \|T_r\| \geq 1$ and hence

$$\left\|T_r^{-1}\right\| \geq \frac{1}{\|T_r\|} \geq 1.$$

As a result for all $r \in]0,1[$,

$$r^n \left\|T_r^{-1}\right\| \geq r^n,$$

and combining with the previous estimate, we obtain

$$\frac{1}{2} \leq max(r^n, 1 - r^n) \leq r^n \left\|T_r^{-1}\right\| \leq r^n t_n^a(r) \leq r^n t_n(r) \leq r^n c_n(r) = 1,$$

which completes the first claim of the theorem. The second claim follows from the previous inequalities. □

Remark. It should be mention that we have not obtained an explicit formula for $t_n^a(r)$. Regarding the description of extremal matrices (for the quantity $c_n(r)$) mentioned in the Introduction, it seems likely that $t_n^a(r) < c_n(r) = \frac{1}{r^n}$. In the same spirit, it would be of interest to know the limits $lim_{r\to 1}\left(inf_{n\geq 1} r^n t_n^a(r)\right)$ and $lim_{n\to\infty}\left(inf_{0<r<1} r^n t_n^a(r)\right)$.

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73

74

Résumé de la thèse:

La thèse est consacrée à une étude d'interpolation complexe "semi-libre" dans le sens suivant: étant donné un ensemble fini σ du disque unité $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ et une fonction f holomorphe dans $\mathbb{D}$ appartenant à une certaine classe X, on cherche g dans une autre classe Y (plus petite que X) qui minimise la norme de g dans Y parmi toutes les fonctions g satisfaisant la condition $g_{|\sigma} = f_{|\sigma}$. Plus précisément, nous nous intéressons à la constante d'interpolation suivante

$$c\left(\sigma,\, X,\, Y\right) = sup_{f \in X,\, \|f\|_X \leq 1} inf \left\{ \|g\|_Y : \, g_{|\sigma} = f_{|\sigma} \right\}.$$

Dans la thèse, nous étudions le cas où $Y = H^\infty$ et où l'espace des contraintes X est choisi parmi les espaces suivants: les espaces de Hardy, les espaces de Bergman pondérés à poids radial ou encore les espaces de fonctions holomorphes dans $\mathbb{D}$ ayant leurs coefficients de Taylor dans $l^p(w)$ (w étant un poids). La thèse contient également certaines applications au conditionnement des matrices de Toeplitz.

Mots-clés:

interpolation
interpolation de Nevanlinna-Pick
interpolation de Carathéodory-Schur
interpolation de Carleson
espaces de Hardy
espaces de Bergman à poids
inégalité type Bernstein
conditionnement
matrices de Toepitz